Further Topics
Equilibrium Semantics, Ehrenfeucht-Fraisse and Model Consistency Game

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Contenido

Introduction

- We continue using games to explore IF logic.
- A usefull characterization of game theoretic semantics for IF logic is one based on Skolem functions.

Introduction

- Recall the definition of game theoretic semantics

Definition (Game Theoretic Semantics)

Given a sentence ϕ, we say:

$$\mathfrak{M} \models^{+} \phi \textit{ iff Eloise has a winning strategy in } G(\mathfrak{M}, \phi) \tag{1}$$

$$\mathfrak{M} \models^{-} \phi \textit{ iff Abelard has a winning strategy in } G(\mathfrak{M}, \phi) \tag{2}$$

Contenido

Skolem Semantics

Definition

The Skolem form or Skolemization of an IF formla ψ is defined recursively by:

1. Atomic, negation and connective forms are standard (distributes).
2. $Sk((\exists x/W)\phi) = Subs(Sk(\phi), x, f_x(y_1, ..., y_n))$
3. $Sk((\forall x/W)\phi) = \forall x Sk(\phi)$

where $(y_1, ..., y_n)$ are all quantified variables in the scope of which $(\exists x/W)$ occurs and f_x is a new function symbol not present in the original language.

Skolem Semantics

Example

Consider the sentence $\forall x \forall y \exists u/y \exists v/x R(x, y, u, v)$ Then the Skolem form is:

$$\forall x \forall y R(x, y, f_u(x), f_v(y, f_u(x))) \tag{3}$$

Skolem Semantics

Example (Matching Pennies)

$$\forall x (\exists y / \{x\}) x = y \tag{4}$$

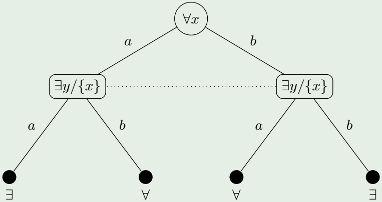

Figure 4.1 The semantic game for $\forall x (\exists y / \{x\}) x = y$ in the structure $\mathbb{M} = \{a, b\}$

The Skolem form is $(\forall x) x = c$ where c is a constant.

Skolem Semantics

- An alternative definition of truth for IF sentences is based on Skolem functions.

Definition

Let ϕ be an IF sentence. ϕ is true in $\mathfrak{M}$ if and only if there exist functions $f_{x_1}^{\mathfrak{M}}, ..., f_{x_n}^{\mathfrak{M}}$ such that:

$$\mathfrak{M}, f_{x_1}^{\mathfrak{M}}, ..., f_{x_n}^{\mathfrak{M}} \models \phi \tag{5}$$

which is equivalent to:

$$\mathfrak{M} \models \exists f_{x_1}^{\mathfrak{M}}, ..., \exists f_{x_n}^{\mathfrak{M}} \phi \tag{6}$$

Skolem Semantics

- The equivalence between game theoretic semantics and Skolem semantics asserts that GTS truth definition $\models^+$ is equivalent to Skolem semantics.
- It follows that not being true in terms of Skolem definition is equivalent to not beeing true in GTS which is not the same as being false (satisfy relation $\models^-$).
- This explains why the Skolem form of matching pennies is not true in terms of Skolem semantics and why it is not true in GTS.
- To characterize falsum relation we use Kreisle counterexamples. Given ϕ an IF sentence, let $\neg\phi$ stand for the formula with $\neg$ pushed all the way down to atomic formulas. The Skolem form of this sentence we call Kreisel counterexample.
- Kreisel counterexamples characterize falsity.

Skolem Semantics

Example

The Skolem and Kreisel counterexample of matching pennies show that matching pennies is neither true not false in GTS: It is not tru in terms of Skolem semantics and it is not false in terms of Kreisel semantics.

- As noted previously, truth in GTS is charcterized by:

$$\mathfrak{M} \models \exists f_{x_1}^{\mathfrak{M}}, ..., \exists f_{x_n}^{\mathfrak{M}} \phi \tag{7}$$

- This explains that IF logic is equivalent to existential second order logic Σ_1^1. This is the fragment of second order logic with sentences of the form: $\exists X_1, ..., \exists X_n \phi$ where ϕ is first order.

Contenido

Equilibrium Semantics

- In a zero sum game let winners payoff be 1 and loser 0. Then we have a constant sum game.

Definition

Let $\epsilon > 0$ let ϕ be an IF sentence and $\mathfrak{M}$ a **finte** structure. The truth value of ϕ on $\mathfrak{M}$, $\Gamma(\phi, \mathfrak{M})$ is the value of the semantic game in normal form (minimax value). We define the satisfaction relation $\mathfrak{M} \models_\epsilon \phi$ if $\Gamma(\phi, \mathfrak{M}) \geq \epsilon$.

- $\epsilon = 1$ characterizes truth in GTS.

Equilibrium Semantics

Example

Consider again the matching pennies formula interpreted in a structure with n elements. A uniform distribution on the universe of this structure is a Nash equilibrium hence the value of the associated semantic game is $\frac{1}{n}$. The asymptotic value value approaches falsehood.

- Nevertherless, interpreting value in IF logic as degree of belief is unapropiate.

Equilibrium Semantics

Example

Consider the following formula ϕ_{even}:

$$\forall x \forall y (\exists u/y)(\exists v/x, u)((x = y \rightarrow u = v) \wedge (u = y \rightarrow v = x) \wedge u \neq x)$$

In Skolem form, $\exists f, g$ simplifies to:

$$\forall x \forall y ((x = y \rightarrow f(x) = g(y)) \wedge (f(x) = y \rightarrow g(y) = x) \wedge f(x) \neq x)$$

that simplifies to: $\forall x (f(f(x)) = x \wedge (f(x) \neq x))$.
That is, f is an involution. A finite structure has an even number of elements if and only if there is an involution that does not have a fixed point.
This is one more example of IF sentence that expresses a property that cannot be expressed in first order logic (note the sentence has no perfect recall).

Equilibrium Semantics

Example

When the previous formula is intepreted in a circular graph it can be proved that for n odd the value is: $1 - \frac{1}{2n}$. Hence the higher the odd number of elements the closer to 1 it is. Thus, value is not a reasonable metric for belief.

Equilibrium Semantics

Theorem

Every rational number in $(0,1)$ is realizable as the value of an IF sentence on every structure with at least two elements.

- Let M be a set with at least n objects and C a subset with precisely n objects. Consider the game:
  1. Abelard picks $m < n$ objects from M.
  2. Eloise picks c from M not knowing Abelard choice.
  3. Eloise wins if and only if at least one of the following holds: Abelard has chosen two equal objects, Abelard has chosen ouside C or Eloise has chosen an element already chosen by Abelard.
- It is a weakly dominated strategy that Eloise chooses outside C. This games has value $\frac{m}{n}$.
- The aregument can now be extended to the case of a set M with only two elements. Players now pick strings of elements.

Equilibrium Semantics

- It is an interesting fact that from a model theoretic point of view for all $\epsilon, \epsilon', \in (0, 1)$, rationals, $\models_\epsilon$ and $\models_{\epsilon'}$ are the same. If the value of sentence is at least ϵ, there is an in sentence such that its value in the same struture is at least ϵ'.

Contenido

Separation Game

- This game is an extremly useful and versatile tool in logic.
- It is usefull for characterizing isomrphisms among countable structures (a structural property) and elementary equivalence (a semantic property).
- We need three basic concepts, substructure, minimal generated substructure and partial ismorphism.

Definition (Back and Forth Sets)

Let $P \subseteq Part(\mathfrak{M}, \mathfrak{N})$ be any non-empty sets of partial isomorphisms. We say P is back and forth set for $\mathfrak{M}$ and $\mathfrak{N}$ if:

$$\forall f \in P \forall m \in M \exists g \in P (f \subseteq g \wedge m \in dom(g)) \tag{8}$$

$$\forall f \in P \forall n \in N \exists g \in P (f \subseteq g \wedge n \in rng(g)) \tag{9}$$

Separation Game

- Two structures are said to be partially isomorphic if there is a Back and Forth set for them (in symbols $\mathfrak{M} \simeq_p \mathfrak{N}$). The relation $\simeq_p$ is an equivalence relation and charcaterizes up to isomorphism, countable structures. The proposition fails for uncountable structures.

Theorem

$\mathfrak{M} \simeq_p \mathfrak{N}$ *if and only if* $\mathfrak{M} \approxeq_p \mathfrak{N}$

Separation Game

Theorem (Cantor)

Any two dense linear orders without endpoints are isomoprhic.

- By weakening the concept of back and forth sets to that of back and forth sequences it si possible to give a characterization up to elementary equivalence of countable structures.

Contenido

Model Existence Game

- Notice that in the semantic game , the only place where we make reference to the structure that defines the game is when defining the winning condition.
- Let T be a set of L-sentences in negation normal form (NNF).
- Let C be a countable set of new constants.

Model Existence Game

- The model existence game $MEG(T, L)$ is the game $G(A, W)$ in which players follow the rules in next figure and W is such that no sequence $(x_0, y_0, x_1, y_1, ...) \in W$ has a $L \cup C$-atomic sentence ϕ such that both ϕ and $\neg\phi$ are in $\{y_0, y_1, ...\}$.
- The idea of the game is to have I chanllenge II by picking $\phi \in T$ and running through all subformulas trying to make II play contradictory sentences.

Model Existence Game

x_n	y_n	Explanation
φ		**I** enquires about $\varphi \in T$.
	φ	**II** confirms.
$\approx tt$		**I** enquires about an equation.
	$\approx tt$	**II** confirms.
$\varphi(t')$		**I** chooses played $\varphi(t)$ and $\approx tt'$ with φ basic and enquires about substituting t' for t in φ.
	$\varphi(t')$	**II** confirms.
φ_i		**I** tests a played $\varphi_0 \wedge \varphi_1$ by choosing $i \in \{0, 1\}$.
	φ_i	**II** confirms.
$\varphi_0 \vee \varphi_1$		**I** enquires about a played disjunction.
	φ_i	**II** makes a choice of $i \in \{0, 1\}$
$\varphi(c)$		**I** tests a played $\forall x \varphi(x)$ by choosing $c \in C$.
	$\varphi(c)$	**II** confirms.
$\exists x \varphi(x)$		**I** enquires about a played existential statement.
	$\varphi(c)$	**II** makes a choice of $c \in C$

Model Existence Game

Theorem (Model Existence Theorem)

Suppose L is countable and T is a set of L-sentences. Then the following are equivalent:

1. *T is satisfiable by an L-structure.*
2. *Player II has a winning strategy in $MEG(T, L)$.*

- This theorem has many applications including a proof of the compactness theorem.