Dynamic Mechanism Design:
Revenue Equivalence, Profit Maximization, and Information Disclosure

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Motivation

- Mechanism Design: auctions, taxation, etc...

- Standard model: one-time information, one-time decisions

- Many real-world settings
  - Information arrives over time (serially correlated)
  - Sequence of decisions
  - Non-time-separable technology/preferences
Examples

- Sequential procurement auctions
  - bidders acquire information, invest, learn by doing...
  - intertemporal capacity constraints

- New “experience goods”
  - valuation dynamics driven by consumption ("experimentation")
  - price discrimination by menu of price paths

- Advance sales (e.g., flight tickets)
  - buyers receive information, make investments over time
  - price discrimination on early info. by menu of price-refund options
State of the Literature

● Efficient dynamic mechanisms:
  ● Athey-Segal, Bergemann-Valimaki ...

● Special cases of profit-maximization: typically one agent, Markov process
  ● Baron-Besanko: two-period monopoly regulation
  ● Courty-Li: two-period advance ticket sales
  ● Eso-Szentes: two-period, one decision
  ● Battaglini: infinite horizon with 2 types in each period

● Hanging questions:
  ● Necessary + sufficient conditions for incentive compatibility with many agents, many periods, non-Markov processes, continuous types
  ● Properties of profit-maximizing mechanisms
  ● Important technical assumptions
What’s Different about Dynamic Mechanisms?

- How to derive transfers, payoffs from nonmonetary allocations ("revenue equivalence")?

  → Must control for multi-period contingent deviations
Payoff Non-equivalence with Discrete Future Types

- What assumptions on type-process are needed?

Example

Payoff: $\theta_2 x_2 - p_1 - p_2$

- $2^{nd}$ period consumption: $x_2 \in \{0, 1\}$, no consumption in $1^{st}$ period
- Types: $\theta_2 \in \{H, L\}$ and $\theta_1 = \text{Pr}\{\theta_2 = H\} \in [0, 1]$
- Mechanism:
- 1st period: nothing
- 2nd period: post price $q$, with $L \leq q \leq H$
- Allocation $x_2(H) = 1$, $x_2(L) = 0$ for any $\theta_1$, regardless of $q$!
- Equilibrium payoff: $V(\theta_1) = \theta_1(H - q)$

Revenue Equivalence at $t = 1$ fails because of disconnected type space at $t = 2$ (despite connected type-space at $t = 1$)
Payoff Non-equivalence with Discontinuous Transitions

Example (continued)

- Payoff: $\theta_2 x_2 - p_1 - p_2$
- Types: $\theta_1, \theta_2 \in [0, 1]$ with

$$f_2(\theta_2|\theta_1) = \begin{cases} 
1 & \text{if } \theta_1 < \frac{1}{2} \\
2\theta_2 & \text{if } \theta_1 \geq \frac{1}{2}
\end{cases}$$

- Mechanism:
  - 1st period: advance contract with posted price $q$ with $q \in \left(\frac{1}{2}, \frac{2}{3}\right)$
  - 2nd period: execute contract
- Allocation $x_2(\theta_1) = 1$ iff $\theta_1 \geq \frac{1}{2}$ regardless of $\theta_2$, regardless of $q$!
- Eq. payoff: $V(\theta_1) = 0$ if $\theta_1 < \frac{1}{2}$, and $V(\theta_1) = \frac{2}{3} - q$ if $\theta_1 \geq \frac{1}{2}$
- E.g., if $V(0) = 0$, then $V(1) \in \left[0, \frac{1}{6}\right]$ 

- Revenue Equivalence at $t = 1$ fails because of discontinuous transitions
Results of this Paper

- Incentive compatibility $\Rightarrow$ Formula expressing agents' eq. payoffs
  - Summarizes “first-order” multi-period IC (cf. Mirrlees)
  - Technical "smoothness" conditions for this to hold

- Sufficient conditions for “global” incentive compatibility

- In quasilinear multi-agent environments, with statistically independent types across agents:
  - Revenue Equivalence Theorem
  - Principal’s expected profits $=$ expected “dynamic virtual surplus”
  - Profit-maximizing mechanisms
  - Dynamics of distortions

- Applications: sequential auctions, mechanisms for selling new goods, etc.
Environment (as seen by one agent)

- In each period $t = 1, \ldots, T$
  - Agent privately observes $\theta_t \in \Theta_t \subset \mathbb{R}$
  - Decision $y_t \in Y_t$
- Histories:
  \[ y^t = (y_1, \ldots, y_t) \in Y^t = \prod_{\tau=1}^{t} Y_{\tau}, \]
  \[ \theta^t = (\theta_1, \ldots, \theta_t) \in \Theta^t = \prod_{\tau=1}^{t} \Theta_{\tau} \]
  full histories: $y = y^T \in Y = Y^T$, $\theta = \theta^T \in \Theta = \Theta^T$
- Technology:
  \[ \tilde{\theta}_t \sim F_t(\cdot | \theta^{t-1}, y^{t-1}) \]
  allows learning-by-doing, information acquisition, etc.
- Agent’s payoff: $U(\theta, y)$
Mechanisms

- Revelation principle (Myerson 86) ⇒ direct mechanisms:
  - In each period $t$
    - Agent observes $\theta_t \in \Theta_t$
    - Agent submits report $m_t \in \Theta_t$
    - Mechanism draws $y_t \in Y_t$ from probability distribution $\Omega_t(\cdot | m^t, y^{t-1})$
      - Randomization allows e.g. dependence on other agents’ messages

- (Randomized direct) mechanism:

$$\Omega = \left\{ \Omega_t : \Theta^t \times Y^{t-1} \to \Delta(Y_t) \right\}_{t=1}^T$$

- Agent’s reporting strategy:

$$\sigma = \left\{ \sigma_t : \Theta^t \times \Theta^{t-1} \times Y^{t-1} \to \Theta_t \right\}_{t=1}^T$$

- Truthful strategy:

$$\sigma_t(\theta^t, m^{t-1}, y^{t-1}) \equiv \theta_t \quad \text{for all } t, \quad \text{all } (\theta^t, m^{t-1}, y^{t-1})$$
Stochastic Process and Expected Payoffs

- Histories:

\[ H = \{(\theta^s, m^t, y^u) : s \geq t \geq u \geq t - 1\} \]

- Technology \( F \), mechanism \( \Omega \), strategy \( \sigma \), and history \( h \in H \) \( \implies \) probability measure \( \mu[\Omega, \sigma]|h \) on \( \Theta \times \Theta \times Y \)
  - \( \mu[\Omega]|h \) if \( \sigma \) is truthful
  - \( \mu[\Omega, \sigma] \) if \( h \) is null history

\[ \mathbb{E}^{\mu[\Omega, \sigma]|h}[U(\tilde{\theta}, \tilde{y})] = \text{resulting exp. payoff} \]

- Value function:

\[ V(h) = \sup_{\sigma} \mathbb{E}^{\mu[\Omega, \sigma]|h}[U(\tilde{\theta}, \tilde{y})] \]
Incentive Compatibility

### Definition

Mechanism $\Omega$ is *incentive compatible at history* $h$ (IC at $h$) if

$$\mathbb{E}^{\mu[\Omega]|h}[U(\tilde{\theta}, \tilde{y})] = V(h)$$

- Focus on ex ante rationality:

### Definition

Mechanism $\Omega$ is *ex-ante incentive compatible* (ex-ante IC) if it is IC at $\emptyset$

- Ex-ante IC implies IC at truthful histories (i.e., on eqpath) with $\mu[\Omega]$-prob. 1
First-Order IC in Static Model (Miryreles, Myerson)

- Assume $T = 1$
- Mechanism $\Omega$ is IC at each $\theta$:

$$V(\theta) \equiv \sup_{m \in \Theta} \int_Y U(\theta, y) d\Omega(y|m) = \int_Y U(\theta, y) d\Omega(y|\theta)$$

- Envelope Theorem:

$$V'(\theta) = \int_Y \frac{\partial U(\theta, y)}{\partial \theta} d\Omega(y|\theta)$$

- Quasilinear setting:

  - $U(\theta, (x, p)) = u(\theta, x) + p$

  ⇒ Revenue Equivalence, characterization of optimal mechanisms
First-Order IC in Dynamic Model: Heuristic Derivation

- Mechanism $\Omega$ is IC at (truthful) history $h = (\theta^t, \theta^{t-1}, y^{t-1})$:

$$V(h) = \mathbb{E}^{\mu[\Omega]|h}[U(\tilde{\theta}, \tilde{y})]$$

$$= \int U(\theta, y) \prod_{\tau=t}^{T} [d\Omega_{\tau}(y_\tau|m^\tau, y^{\tau-1})dF_{\tau+1}(\theta_{\tau+1}|\theta^\tau, y^\tau)]_{m=\theta}$$

- Differentiate wrt current type $\theta_t$:
  1. in $U(\theta, y) \Rightarrow \mathbb{E}^{\mu[\Omega]|h}[\partial U(\tilde{\theta}, \tilde{y})/\partial \theta_t]$
  2. in $F_{\tau+1}(\theta_{\tau+1}|\theta^\tau, y^\tau) \Rightarrow$ integrate by parts, differ. within integral:

$$-\mathbb{E}^{\mu[\Omega]|h} \left[ \int \frac{\partial V((\tilde{\theta}^\tau, \theta_{\tau+1}), \tilde{y}^\tau)}{\partial \theta_{\tau+1}} \frac{\partial F_{\tau+1}(\theta_{\tau+1}|\tilde{\theta}^\tau, \tilde{y}^\tau)}{\partial \theta_t} d\theta_{\tau+1} \right]$$

- Derivatives wrt report $m_t = \theta_t$: vanish by (appropriate version of) Envelope Thm
Technical Assumptions

- Don’t want to impose “smoothness” on mechanism
- “Smooth” environment needed to iterate Envelope Thm backward
- Ensure one can differentiate totally and under expectations
  - Need new assumptions on kernels $F_t$
Technical Assumptions

1. \( \Theta_t = (\theta_t, \bar{\theta}_t) \) with \(-\infty \leq \theta_t \leq \bar{\theta}_t \leq +\infty \)

2. \( \partial U(\theta, y)/\partial \theta_t \) exists and bounded uniformly in \((\theta, y)\)

3. “Full Support”: \( F_t(\theta_t|\theta^{t-1}, y^{t-1}) \) strictly increasing in \( \theta_t \)

4. \( \int |\theta_t| dF_t(\theta_t|\theta^{t-1}, y^{t-1}) < +\infty \)

5. For \( \tau < t \), \( \partial F_t(\theta_t|\theta^{t-1}, y^{t-1})/\partial \theta_\tau \) exists and bounded in abs. value by an integrable function \( B_t(\theta_t) \)

6. \( F_t(\cdot|\theta^{t-1}, y^{t-1}) \) continuous in \( \theta^{t-1} \) in total variation metric

7. \( F_t(\cdot|\theta^{t-1}, y^{t-1}) \) abs. continuous, with density \( f_t(\cdot|\theta^{t-1}, y^{t-1}) \) (only to simplify formulas)
Payoff via FOC: Formal Result

Theorem

Under Assumptions 1-7, if $\Omega$ is IC at $h^{t-1} = (\theta^{t-1}, \theta^{t-1}, y^{t-1})$, then $V(\theta, h^{t-1})$ is Lipschitz continuous in $\theta_t$, and for a.e. $\theta_t$,

$$
\frac{\partial V(\theta_t, h^{t-1})}{\partial \theta_t} = \mathbb{E}^{\mu[\Omega]}(\theta_t, h^{t-1}) \left[ \sum_{\tau=t}^{T} J_t(\tilde{\theta}, \tilde{y}) \frac{\partial U(\tilde{\theta}, \tilde{y})}{\partial \theta_\tau} \right]
$$

(IC-FOC)

where

$$
J_t(\theta, y) = \sum_{K \in \mathbb{N}, l \in \mathbb{N}^K: t = l_0 < ... < l_K = \tau} I^{l_{K-1}}(\theta, y)
$$

“Total information index”

$$
I_t(\theta, y) = -\frac{\partial F_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1})/\partial \theta_t}{f_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1})}
$$

“Direct information index”
Example: AR(k) Process

\[ \theta_t = \sum_{l=1}^{k} \phi_l \theta_{t-l} + \varepsilon_t \]

- \( \varepsilon_t \sim G_t \), independent across \( t \); \( \theta_t \) public for \( t \leq 0 \)

- \[ F_\tau(\theta_\tau|\theta_{\tau-1}^\tau, y_{\tau-1}^\tau) = G_\tau \left( \theta_\tau - \sum_{l=1}^{k} \phi_l \theta_{\tau-l} \right) \]

- \[ I_\tau^t (\theta, y) = -\frac{\partial F_\tau(\theta_\tau|\theta_{\tau-1}^\tau, y_{\tau-1}^\tau)/\partial \theta_t}{f_\tau(\theta_\tau|\theta_{\tau-1}^\tau, y_{\tau-1}^\tau)} = \phi_{\tau-t} \]

- \[ J_\tau^t (\theta, y) = \sum_{K \in \mathbb{N}, l \in \mathbb{N}^K : t = l_0 < \ldots < l_K = \tau} \prod_{k=1}^{K} \phi_{l_k - l_{k-1}} \quad \text{“impulse response” constants} \]

- AR(1):

\[ I_\tau^t (\theta, y) = \begin{cases} 
\phi_1 & \text{if } \tau = t + 1 \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad J_\tau^t (\theta, y) = (\phi_1)^{\tau-t}. \]
Alternative Approach: Independent-shock Representation

- \( \theta_t = z(\varepsilon^t; y^{t-1}) \) where \( \varepsilon_t \sim G_t \), support in \( \mathbb{R} \), independent across \( t \)
- E.g. AR(k): \( \theta_t = \sum_{l=1}^{k} \phi_l \theta_{t-l} + \varepsilon_t \)
- Two representations are equivalent: Given mechanism \( \Omega \) for \( F \), there exists \( \hat{\Omega} \) for \( (G, z) \) that induces same distribution on \( \Theta \times Y \) as \( \Omega \). And vice versa.
- Alternative route: have agent report \( (\varepsilon_t)_{t=1}^T \) \( \longrightarrow \) mechanism \( \hat{\Omega} \)
- Redefine utility in terms of \( \varepsilon \): \( \hat{U}(\varepsilon, y) \equiv U(z(\varepsilon; y), y) \)
- With serially independent shocks, IC-FOC formula simplifies to

\[
\frac{\partial \hat{V}(\varepsilon_t, h^{t-1})}{\partial \varepsilon_t} = \mathbb{E}_{\mu[\hat{\Omega}]}(\varepsilon_t, h^{t-1}) \left[ \frac{\partial \hat{U}(\tilde{\varepsilon}, \tilde{y})}{\partial \varepsilon_t} \right]
\]

where \( h^{t-1} = (\varepsilon^{t-1}, \varepsilon^{t-1}, y^{t-1}) \)

- Simpler proof: sufficient to consider period-\( t \) deviations
Independent Shocks: Results

**Theorem**

Any $F$ admits “canonical” independent-shock representation in which for all $t$, $\tilde{\varepsilon}_t \sim U(0, 1)$.

- Proof by induction on $t$ using "prob. integral transform thm":

  $$z_t(\varepsilon^t; y^{t-1}) = F_{t-1}^{-1}(\varepsilon_t| z^{t-1}(\varepsilon^{t-1}; y^{t-2}), y^{t-1})$$

- Given model specified in terms of $F$, two routes to payoff equivalence:
  1. Work with $F$ and impose Assumptions 1-7 from above
  2. Convert $F$ into independent shocks $(G, z)$ and identify assumptions on $F, U$ that ensure $\hat{U}$ is “smooth”

- Turns out that assumptions required for 1 and 2 are not nested:
  1. Rules out "shifting atoms" (e.g., fully persistent types)
  2. Rules out "growing atoms" but allows for shifting atoms
Independent Shocks: Assumptions for IC-FOC

- New conditions:
  
  (a) \( U(\cdot, y) \) equi-Lipschitz and \textit{continuously} differentiable in \( \theta \)

  (b) \( F_{t-1}^{-1}(\varepsilon|\cdot, y^{t-1}) \) equi-Lipschitz and \textit{continuously} diff in \( \theta^{t-1} \)

  (c) \( F_{t-1}^{-1}(\cdot|\theta^{t-1}, y^{t-1}) \) equi-Lipschitz and \textit{continuously} diff. in \( \varepsilon \).

**Theorem**

Suppose \((U, F)\) satisfies assumptions (1)-(2) + (a)-(c). Then \( \hat{U}(\varepsilon, y) \) is equi-Lipschitz continuous and differentiable in \( \varepsilon \). It follows that if \( \hat{\Omega} \) is IC at history \( h^{t-1} = (\varepsilon^{t-1}, \varepsilon^{t-1}, y^{t-1}) \), then

\[
\frac{\partial \hat{V}(\varepsilon_t, h^{t-1})}{\partial \varepsilon_t} = \mathbb{E}_{\mu[\hat{\Omega}]}(\varepsilon_t, h^{t-1}) \left[ \frac{\partial \hat{U}(\varepsilon, \tilde{y})}{\partial \varepsilon_t} \right] \quad \text{a.e.}
\]
Quasilinear Settings with multiple agents

- Agents $i = 1, ..., N$

- $(x_t, p_t)$, where $p_t \in \mathbb{R}^N$, $x_t = (x_{1t}, ..., x_{Nt}) \in X_t \subset \prod X_{it}$

- $U_i(\theta, (x, p)) = u_i(\theta, x) + \sum_t p_{it}$

- Assumption: $F_{it}(\theta_{it}|\theta_{t-1}, (x_{t-1}, p_{t-1})) = F_{it}(\theta_{it}|\theta_{i-1}^{t-1}, x_{i-1}^{t-1})$

- **Independent Types**: $\tilde{\theta}_{i,t} \sim F_{i,t}(\cdot|\theta_{i-1}^{t-1}, x_{i-1}^{t-1})$, independent across $i$

- BNE

- Revelation Principle: *truthful + minimal disclosure*
  - postponed payments

- Deterministic direct mechanisms: $\langle \chi_t : \Theta^t \to X_t \rangle_{t=1}^T \psi : \Theta \to \mathbb{R}^N$

- $\mu_i[\chi, \psi|(\theta_i^s, m_i^t, x_i^u)]$: process as viewed by $i$
Payoff Equivalence

- IC-FOC: For all $t$, all $h^{t-1}_i = (\theta^{t-1}_i, \theta^{t-1}_i, x^{t-1}_i)$

$$\frac{\partial V_i(\theta_{it}, h^{t-1}_i)}{\partial \theta_{it}} = \mathbb{E}^{\mu_i}[\chi, \psi](\theta_{it}, h^{t-1}_i) \left[ \sum_{\tau=t}^{T} J^{\tau}_{it}(\tilde{\theta}, \tilde{x}) \frac{\partial u_i(\tilde{\theta}, \tilde{x})}{\partial \theta_{i\tau}} \right]$$

- Pins down $V_i(\theta_{it}, h^{t-1}_i)$ as function of $\chi$ and $\theta_{it}$ up to $K_i(h^{t-1}_i)$
- Iterated expectations $\rightarrow$ get rid of dependence of $K_i(h^{t-1}_i)$ on $h^{t-1}_i$

**Theorem**

Let $(\chi, \psi)$ and $(\chi, \hat{\psi})$ be any two ex-ante IC mechanisms that implement same $\chi$. For all $t, i$, with prob. 1,

$$\mathbb{E}^{\mu}[\chi, \psi][U_i(\tilde{\theta}, \tilde{y}) \mid \theta^t_i] - \mathbb{E}^{\mu}[\chi, \hat{\psi}][U_i(\tilde{\theta}, \tilde{y}) \mid \theta^t_i] = K_i$$

- Single agent $\Rightarrow$ $\chi$ pins down payoff and transfer
- Many agents $\Rightarrow$ expectation of payoff and transfer over others’ types pinned down as function of own type
- E.g., different dynamic mechanisms implementing efficiency (Athey-Segal, Bergemann-Valimaki,...) are “equivalent” in this sense
Participation Constraint and Relaxed Problem

- Agents can quit in any period

- Agents can post bonds ⇒ only 1\textsuperscript{st}-period participation constraints bind:
  \[ V_i (\theta_i) \geq 0 \quad \text{(IR}_i (\theta_i)) \]

- “Relaxed Program”: max profits subject to IC-FOC and IR\(_i (\theta_i)\)

  - Sufficient conditions for “IC-FOC + IR\(_i (\theta_i)\) ⇒ IR\(_i\)”:  
    \[ \frac{\partial u_i (\theta, x)}{\partial \theta_i} \geq 0 \text{ and } I_{it} (\theta, x) \geq 0 \quad (\Rightarrow J_{it} (\theta, x) \geq 0) \]
    ⇒ by IC-FOC, \[ \frac{\partial V_i (\theta_i)}{\partial \theta_i} \geq 0 \]
    ⇒ only IR\(_i (\theta_i)\) binds

  - Sufficient conditions for “IC-FOC ⇒ IC” — later
Information Rents

- Let

\[ \eta_{i1}(\theta_{i1}) \equiv \frac{f_{i1}(\theta_{i1})}{1 - F_{i1}(\theta_{i1})} \]

- Agent \( i \)'s ex-ante expected information rent (using IC-FOC)

\[
\mathbb{E} \left[ V_i(\tilde{\theta}_{i1}) \right] = \mathbb{E} \left[ \frac{1}{\eta_{i1}(\tilde{\theta}_{i1})} \frac{\partial V_i(\tilde{\theta}_{i1})}{\partial \theta_{i1}} \right] = \mathbb{E} \left[ \frac{1}{\eta_{i1}(\tilde{\theta}_{i1})} \sum_{\tau=1}^{T} J_{\tilde{i} \tau}^\tau(\tilde{\theta}, \tilde{x}) \frac{\partial u_i(\tilde{\theta}, \tilde{x})}{\partial \theta_{i\tau}} \right]
\]
Principal $\rightarrow$ agent 0

**Theorem**

Let $\mathcal{X}^*$ denote set of allocation rules that maximize "expected virtual surplus"

$$
\mathbb{E} \left[ \sum_{i=0}^{N} u_i(\tilde{\theta}, \chi(\tilde{\theta})) - \sum_{i=1}^{N} \frac{1}{\eta_{i1}(\tilde{\theta}_{i1})} \sum_{t=1}^{T} J_{i1}^t(\tilde{\theta}_{i1}, \chi(\tilde{\theta})) \frac{\partial u_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial \theta_{it}} \right],
$$

and arise in an IC and IR mechanism $(\chi, \psi)$. If $\mathcal{X}^*$ is non-empty, then $\mathcal{X}^*$ is set of profit-maximizing allocation rules.
Intuition for Allocative Distortions

- Assume $N = 1$, $u_i(\theta, x) = \sum_t u_{it}(\theta, x_t)$, $i = 0, 1$, $J_1^t(\theta)$

- Maximize virtual surplus for each $t, \theta$:

$$\max_{x_t \in X_t} \left[ u_{0t}(\theta, x) + u_{1t}(\theta, x) - \frac{J_1^t(\theta)}{\eta_1(\theta_1)} \frac{\partial u_{1t}(\theta, x)}{\partial \theta_t} \right]$$

- Distort $x_t$ to reduce info. rents based on $\theta_1$ and its effect on period $t$
- E.g., for $t > 1$: If $\theta_t = \bar{\theta}_t$ or $= \theta_t$, then $F_t(\theta_t|\theta^{t-1}) \equiv 1$ or $\equiv 0$ 
  $\Rightarrow J_1^t(\theta) \equiv 0 \Rightarrow$ implement efficient $x_t$

- $F_\tau(\theta_\tau|\theta^{\tau-1}, x^{\tau-1})$ decreasing in $\theta^{\tau-1}$ (FOSD) $\Rightarrow I_t^\tau, J_t^\tau \geq 0 \Rightarrow$
  distort $x_t$ to reduce $\frac{\partial u_{1t}(\theta, x_t)}{\partial \theta_\tau}$

- E.g. $\frac{\partial^2 u_{1t}(\theta, x_t)}{\partial \theta_t \partial x_t} > 0$ (SCP) $\Rightarrow$ distort $x_t$ below efficient level

- Note: distortion in $x_t$ is nonmonotonic in $\theta_t$ for $t > 1$ (unlike in static model, or in Battaglini)
Conditions for Downward Distortions

- $\mathcal{X}$: set of all (measurable) allocation rules. $\mathcal{X}^0$: set of allocation rules solving Relaxed Program. $\mathcal{X}^E$: set of allocation rules maximizing expected total surplus.

**Theorem**

Suppose each $X_t$ is lattice and

(i) decisions don’t affect types: $F_{i,t} (\theta_{it} | \theta_{i-1}^t)$

(ii) FOSD: $F_{i,t} (\theta_{it} | \theta_{i-1}^t)$ nondecreasing in $\theta_{i-1}^t$

(iii) SCP: $u_i (\theta, x)$ supermodular in $(x, \theta_i)$

(iv) $u_i (\theta, x)$ supermodular in $x$

(v) $\frac{\partial u_i (\theta, x)}{\partial \theta_{it}}$ submodular in $x$

Then $\mathcal{X}^0 \leq \mathcal{X}^E$ in strong set order.

- Proof: Topkis Thm applied to

$$g(\chi, z) \equiv \mathbb{E} \left[ \sum_{i=0}^{N} u_i (\tilde{\theta}, \chi(\tilde{\theta})) + z \sum_{i=1}^{N} \frac{1}{\eta_{i1}(\tilde{\theta}_{i1})} \sum_{t=1}^{T} J_{i1}^t (\tilde{\theta}_{i}) \frac{\partial u_i (\tilde{\theta}, \chi(\tilde{\theta}))}{\partial \theta_{it}} \right]$$
Sufficient Condition for Implementable Allocation Rules

- Characterization hard due to multidimensional strategies, decisions

**Theorem**

*Suppose mechanism \((\chi, \psi)\) is IC at any (possibly non-truthful) period \(t + 1\) history. If for all \(i\), all \((\theta_i^t, x_i^{t-1})\)*

\[
\mathbb{E}_{\mu_i}[\chi, \psi]\theta_i^t, (\theta_i^{t-1}, m_{it}), x_i^{t-1} \sum_{\tau=t}^{T} J_{it}(\tilde{\theta}, \tilde{x}_i) \frac{\partial w_i(\tilde{\theta}_i, \tilde{x}_i)}{\partial \theta_{it}}.
\]

is nondecreasing in \(m_{it}\), then there exists transfer rule \(\hat{\psi}\) s.t. mechanism \((\chi, \hat{\psi})\) is IC at (a) any truthful period-\(t\) history, (b) at any period \(t + 1\) history.

- Markov process: IC at truthful histories \(\iff\) IC at all histories,
  - can iterate backward to show that \(\chi\) is implementable in mechanism that is IC at all histories
  - truthful strategies form weak PBE (with beliefs that other agents are truthful at all histories)
Sufficient Condition - Intuition

- IC at all period $t+1$ histories $\Rightarrow$ suffices to prevent single lie $m_{it}$

- $\Psi_t (\theta_{it}, m_{it})$: agent $i$’s expected utility at history $(\theta_{i-1}^t, \theta_{i-1}^t, x_{i-1}^t)$

- Think of $m_{it}$ as 1-dimensional “allocation” chosen by agent $i$

- Condition says that $\partial \Psi_t (\theta_{it}, m_{it}) / \partial \theta_{it}$ (evaluated using IC-FOC at period $t+1$ histories) is nondecreasing in $m_{it}$, $\rightarrow$ i.e., $\Psi_t$ has SCP

- $\Rightarrow$ monotonic “allocation rule” $m_{it} (\theta_{it})$ is implementable (using transfers constructed from IC-FOC)
A Set of (Stronger) Sufficient Conditions

1. Decisions don’t affect types: \( F(\theta_{it}|\theta_{i}^{t-1}) \)

2. FOSD: \( F(\theta_{it}|\theta_{i}^{t-1}) \) is nonincreasing in \( \theta_{i}^{t-1} \) \( \Rightarrow J_{it}^\tau (\theta) \geq 0 \)

3. SCP: \( \partial u_i (\theta, x_i) / \partial \theta_{it} \) nondecreasing in \( x_i \)

4. \( \chi_i (\theta) \) nondecreasing in \( \theta_i \) (“Strong” Monotonicity)

(1)-(4) imply monotonicity condition in theorem

- \( \chi \) implementable with mechanism that is IC even if \( i \) is shown \( \theta_{-i} \) (both past and future)
Application: Linear AR(k) values

\[ u_i(\theta, x) = \sum_{t=1}^{T} \theta_{it} x_{it} - c_i(x_i^T); \quad X_t \subset \mathbb{R}^N; \]

\[ \theta_{it} = \sum_{l=1}^{k} \phi_{il} \theta_{i,t-l} + \varepsilon_{it} \text{ for } t > 1. \]

- Total information indices \( J_{i1}^t(\theta, x) = J_{i1}^t \) “impulse responses constant”
- Expected virtual surplus:

\[
\mathbb{E} \left[ u_0(\tilde{\theta}, x) - \sum_{i=1}^{N} \sum_{t=1}^{T} J_{i1}^t \eta_{i1}^{-1}(\tilde{\theta}_{i1}) x_{it} + \sum_{i=1}^{N} u_i(\tilde{\theta}, x) \right]
\]

- Agent i’s “info rents”

- Optimal mechanism: “Handicapped” efficient mechanism (with extra costs \( J_{i1}^t \eta_{i1}^{-1}(\theta_{i1}) \) of giving objects to agents)
- Incentives from \( t = 2 \) onward ensured using e.g. “Team Transfers” (Athey-Segal) following truthtelling in \( t = 1 \)
- Incentives at \( t = 1 \) must be checked application-by-application
Auctions with AR(k) values

- Time-separable payoffs:  \( u_i (\theta, x) = \sum_{t=1}^{T} \theta_{it} x_{it} \) (thus \( c_i (x_i) = 0 \))

- Can maximize virtual surplus separately for each \( t, \theta \):

\[
\chi_t (\theta) \in \arg \max_{x \in X_t} \left( \theta_{0t} x_{0t} + \sum_{i=1}^{N} \left( \theta_{it} - \frac{J_{i1}^t}{\eta_{i1} (\theta_{i1})} \right) x_{it} \right)
\]

- \( \chi_t (\theta) \) depends only on 1\(^{st}\)-period types and current types!

- Implementation: Each \( i \) makes a 1\(^{st}\)-period payment determining his “handicap.” Then each period, a “handicapped” VCG auction is played

- Truthtelling is IC at any \( h_i^t, \ t \geq 2 \) (actually ex post IC)

- Assume \( \phi_{il} \geq 0 \) \( (\Rightarrow J_{i1}^t \geq 0 \) and \( \eta_{i1}' (\cdot) \geq 0 \) \( \Rightarrow \chi_{it} (\theta) \)
  nondecreasing in \( \theta_{i1} \) \( \Rightarrow \) IC at \( t = 1 \) as well
Other Applications

- Agents learn values by consuming – experimentation
- Principal or agents have intertemporal costs/capacity constraints
  - In all these settings profit-maximizing mechanisms can again be viewed as “handicapped” version of corresponding efficient mechanism
- Non-quasilinear payoffs: wealth effects, cash constraints, or intertemporal consumption smoothing/risk sharing
  - “Bonding” is not optimal/feasible $\Rightarrow$ participation constraints may bind in all periods
  - $\Rightarrow$ 1$^{st}$ period is not as prominent $\Rightarrow$ analysis more difficult
  - cf. Hendel-Lizzeri paper on optimal long-term life insurance contracts with consumption smoothing
Summary

- Methodological contributions:
  - “Smoothness” conditions for environment (not mechanisms)
  - Formula for payoffs via IC-FOC from incentive compatibility
  - Revenue equivalence
  - Profit-maximizing mechanisms

- Sufficient conditions for IC

- Applications
  - Handicapped-efficient mechanisms
  - Optimal sequential auctions
  - Experimentation