Group Strategy-Proofness in Private Good Economies

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December 2014

Barcelona GSE Working Paper Series

Working Paper nº 801

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by

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December 19th, 2014

[1] This paper extends a previous one entitled "Group strategy-proofness in private good economies without money: matching, division and house allocation" which was published as WP773 Barcelona GSE and SSRN-id2467988.

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Abstract: We observe that many salient rules to allocate private goods are not only (partially) strategy-proof, but also (partially) group strategy-proof, in appropriate domains of definition. That is so for solutions to matching, division, cost sharing, house allocation and auctions, in spite of the substantive disparity between these cases. In a general framework that encompasses all of them, we remark that those strategy-proof rules share a common set of properties, which together imply their group strategy-proofness. Hence, the equivalence between the two forms of strategy-proofness is due to the underlying common structure of all these rules, irrespectively of their specific application.

Journal of Economic Literature Classification Numbers: C78, D71, D78.

Keywords: Matching, Division, House allocation, Cost sharing, Auctions, Strategy-proofness, Richness, Group strategy-proofness, Joint monotonicity, Respectfulness, Non-bossiness.

1 Introduction

In many contexts where satisfactory strategy-proof mechanisms do not exist, asking for more becomes redundant. But in domains where that basic incentive property can be met, it becomes natural to investigate whether there exist mechanisms that are not only immune to manipulation by individuals, but can also resist manipulation by groups of coordinated agents.[1] This is particularly relevant because individual strategy-proofness per se is a rather fragile property, unless one can also preclude manipulations of the social outcome by potential coalitions, especially if these may be sufficiently small and easy to coordinate.

In this paper we study the incentive properties of mechanisms to allocate private goods among selfish agents. We start from the striking observation that many well known individually strategy-proof mechanisms are also group strategy-proof, even if the latter is in principle a much stronger condition than the former. And this happens in situations that are formally modeled in rather different manners, including matching, division, cost sharing, house allocation and auctions. We want to determine whether this coincidence between the two a priori different incentive properties arises in each case for specific reasons, or whether there is a common ground for all of them.

In fact, strategy-proofness is a property of social choice functions, rather than a property of mechanisms. Agents that operate under a given mechanism may be using very general strategies, and the idea of truthtelling is not directly applicable to them. However, mechanisms where agents are endowed with dominant strategies are naturally associated with social choice functions representing the direct mechanism where agents' messages are preferences, and indeed these functions are strategy-proof. We make this remark for precision, though in what follows we shall abide with tradition and keep talking about strategy-proof mechanisms when this does not lead to confusion. Actually, the remark is important for another, more important reason. Thinking of dominant strategy mechanisms in terms of their direct representation as social choice functions provides us with a common language to describe models that take very different forms, but share the same feature of being both individual and group strategy-proof in their respective setups. Some of these mechanisms operate in worlds where monetary transfers are excluded, like many-to-one matching (Gale and Shapley, 1962, see also Roth and Sotomayor, 1990), division problems (Sprumont, 1991) and house allocation in its simplest form (Shapley and Scarf, 1974). Others address problems where money transfers are possible: sharing the costs of public goods (Serizawa,

[1] We study that question for economies with public goods in Barberà, Berga, and Moreno (2010).

1999), matching with contracts (Hatfield and Kojima, 2009), house allocation with prices (Miyagawa, 2001) or auctions (Vickrey, 1961). Under some of these mechanisms only subsets of agents have the right incentives, while for others all of them do. For example, in matching models or in auctions, strategy-proofness in its different forms may be only satisfied for members of one side of the market, while in housing problems these properties apply to all participants. And there are many other differences among the models we are interested in. Yet, we identify properties that are sufficient to precipitate the group strategy-proofness of those social choice functions that are individually strategy-proof. All of our examples satisfy these properties, once specialized to fit the characteristics of each model. Hence, our equivalence result does not only clarify the intriguing connection between the two properties, but also proves that their link is independent of whether individual and group strategy-proofness hold for all agents or only for some.

The paper proceeds as follows. We first present a general model (Section 2), provide examples of a variety of well known allocation problems where (partially) group strategy-proof mechanisms can be defined, and briefly discuss how each one of them fits the general framework. In Section 3, we describe specific (sometimes partially) group strategy-proof mechanisms that are standard solutions to these different allocation problems, and we identify three conditions that are satisfied by all these mechanisms: one is on the domain and the others are defined on the function itself. Then we prove (Section 4) our main result regarding the relationship between individual and group strategy-proofness, and show that the conditions involved are independent and non-redundant. In Section 5 we discuss the connection between our results and some related ones in the literature on implementation where individual and group incentives are discussed. Comments and conclusions (Section 6) close the paper.

2 The model and some leading interpretations

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$. Let B_i be the set of possible consequences for i, $i \in N$. Let $A \subseteq B_1 \times ... \times B_n$ be the set of feasible combinations of consequences for agents and $a = (a_1, ..., a_n) \in A$. A is our set of alternatives.

Each agent i has preferences denoted by R_i on B_i. As usual, we denote by P_i and I_i the strict and the indifference part of R_i, respectively. For any $a \in A$ and $R_i \in \mathcal{R}_i$, the *strict lower contour set of* R_i *at* a_i is $\overline{L}(R_i, a_i) = \{b_i \in B_i : a_i P_i b_i\}$ and the *strict upper contour set of* R_i at a_i is $\overline{U}(R_i, a_i) = \{b_i \in A : b_i P_i a_i\}$.

Let $\widetilde{\mathcal{R}}_i$ be the set of complete, reflexive, and transitive orderings on B_i. From preferences on B_i we can induce preferences on A as follows: For any $a, b \in A$, aR_ib if and only if $a_iR_ib_i$. That is, we assume that, when evaluating different alternatives, agents are selfish. Note that, abusing notation we use the same symbol R_i to denote preferences on A and on B_i.

Let $\mathcal{R}_i \subseteq \widetilde{\mathcal{R}}_i$ be *the set of admissible preferences for agent* $i \in N$. A *preference profile,* denoted by $R_N = (R_1, ..., R_n)$, is an element of $\times_{i\in N}\mathcal{R}_i$. We will write $R_N = (R_C, R_{N\setminus C}) \in \times_{i\in N}\mathcal{R}_i$ when we want to stress the role of coalition C in N.

A *social choice function* (or a rule) is a function $f : \times_{i\in N}\mathcal{R}_i \to A$.

Let us briefly discuss how the different allocation problems that we mentioned in the introduction actually fit the general framework we just defined. Since we are just trying to illustrate the fact that the model encompasses many special cases, we do not seek full generality. Rather, we try to describe the specific formulation of these problems as they appear in representative papers within each of the fields we mean to cover, and refer to these papers at each point.

Matching. (Gale and Shapley, 1962, Roth and Sotomayor, 1990, Kelso and Crawford, 1982, Hatfield and Milgrom, 2005) In the simplest case of many-to-one matching, workers in a set W must be allocated to firms in a set F of potential employers, each one able to hire a given amount of workers[2]. There, agents are all the workers and all the firms involved. Alternatives are matchings of workers and firms. Consequences for an agent are what the agent may get from some matching. Preferences are initially defined over consequences, but extend to alternatives (matchings) in a natural way, when agents are selfish. Admissible profiles are lists of preferences over alternatives, one for each agent. All rankings of firms by workers are possible, and rankings by firms of sets of workers are only restricted by the requirement of responsiveness.[3] The resulting admissible profiles constitute the domain of social choice functions selecting one matching for each profile.

In more complex matching models, workers can be offered different employment conditions in the form of alternative contracts, whose set may be denoted by X. Contracts

[2]A special case of the many-to-one model we have described is that where firms have only one opening. It corresponds to the case where matchings are one-to-one, usually called the marriage market.

[3]In matching, where agents have preferences over sets, responsiveness requires the following. Assume that worker s is preferred to worker t, when comparing them as singletons. Then, for any two sets sharing the same workers, except that one contains s and the other contains t, the former is preferred to the latter. We concentrate on that domain of preferences because, as we shall see, it guarantees that the Gale-Shapley algorithm, originally designed for one-to-one matchings, can be used in that more general case and provide us with a rule satisfying the good properties we are after.

may simply specify a salary (Kelso and Crawford, 1982) or be more complicated (Hatfield and Milgrom, 2005). Now matchings involve the further specification of what contract is held between each of the workers and the firm she is matched to. Again, these matchings are the alternatives. The consequences for each agent are given by what agent from the other side they are matched to, under what contract. Preferences are then defined on consequences, reflecting the joint valuation of the individual match and the corresponding contract. All rankings of the empty set and contracts where the worker is present are possible. Firms' preferences satisfy substitutability and the law of aggregate demand.[4] These preferences are naturally extended to alternatives under the selfishness assumption, and their set constitutes the domain of the relevant social choice problems.

Division. (Sprumont, 1991) A set of individuals must share a task. An allocation is a vector of shares, indicating what proportion of the total task is assigned to each individual. The consequence of an allocation for an agent is just the share of the task she is assigned. Each agent is assumed to have single-peaked preferences on consequences, which can be extended to allocations. These extended preferences are the domain of social choice rules for the division problem.

Cost sharing of a public good. (Serizawa, 1999) A set of individuals must select an amount of public good and share its production cost. An allocation is a vector indicating the amount of the public good, and the payments for each agent, with the added requirement that the sum of these payments must equal the cost of producing the public good. The consequences of the overall allocation for an agent are just the total amount of the public good, along with her personalized payment. The agent's preferences on alternatives are again the natural extension under selfishness of their preferences on consequences, which are assumed to be representable by some continuous, strictly quasi-concave utility function, that is decreasing with cost and increasing in the amount of public good.

House allocation. (Shapley and Scarf, 1974, Pápai, 2000, Miyagawa, 2001) A set of individuals must be allocated a maximum of one house each, out of a set of houses, and an eventual payment in the form of a divisible private good ("money").[5] An allocation fully

[4] See Hatfield and Milgrom (2005) for the definition of these two properties over firms/hospitals preferences. According to their Theorem 11, these preferences ensure the worker/doctor optimal deferred acceptance mechanism to be strategy-proof. In Theorem 7 if hospital h's preferences are quasi-linear and satisfy the substitutes condition, then they satisfy the law of aggregate demand.

[5] Since we are using these models for motivational purposes, we stick to the simplest version of the house allocation model. More complex cases allow for more than one house to be allocated to the same agents, the existence of property rights and other possible variations.

specifies who gets what house, if any, and a payment for each individual. The consequence of an allocation for each agent are given by the house she gets and how much she pays.

Individuals have preferences over houses and money. For the same amount of payment, they are typically assumed to prefer any house to none, and their preferences over houses are otherwise unrestricted. Their preferences are assumed to be increasing in money. Houses are not agents, as they are not endowed with preferences. So now individuals are the agents, alternatives are allocations of at most one house to each agent, plus their individual payments, and social choice functions are defined over the domain of all quasi-linear preferences with respect to money, and unrestricted rankings of houses.

The case where no monetary transfers are allowed was the one to be studied first. Then allocations just specify who gets what house, consequences for agents are simply the house they get, and preferences over houses are typically unrestricted.

Auctions. There is a seller and a number of buyers. The seller wants to give away one good in exchange for a monetary transfer. An auction is given by a set of rules that determines what actions must the seller and the eventual buyers use in order to arrive at a transaction, and what positive or negative transfers will be made to each partner involved.[6] Agents are the seller and the buyers. Alternatives are full specifications of who gets the good, and what net transfers are made. Consequences for each agent are whether or not they get the good, and the net transfer she obtains. Obtaining the good is typically assumed to be better than not, at the same level of transfers, and preferences are assumed to be separable and monotonic in money. Extending the preferences of selfish agents from consequences to alternatives provides us with the domain of preferences for the social choice function that assigns the result of an auction to each preference profile.

After these examples, let us close the section by defining the two basic incentive properties that we are interested in.

Definition 1 *A social choice function f on $\times_{i\in N}\mathcal{R}_i$ is manipulable at $R_N \in \times_{i\in N}\mathcal{R}_i$ by coalition $C \subseteq N$ if there exists $R'_C \in \times_{i\in C}\mathcal{R}_i$ ($R'_i \neq R_i$ for any $i \in C$) such that $f(R'_C, R_{N\setminus C})P_i f(R_N)$ for all $i \in C$. For $H \subseteq N$, a social choice function is H-group strategy-proof if it is not manipulable by any coalition $C \subseteq H$, and it is H-individually strategy-proof if it is not manipulable by any singleton $\{i\} \subseteq H$.*

The requirement of H-individual strategy-proofness demands that revealing one's preferences should be a dominant strategy for all agents in H. Likewise, H-group strategy-

[6] Here again, as in the preceding examples, we stik to a simple formulation, for illustrative purposes.

proofness requires that no subset of agents in H can coordinate their actions and obtain a better result by just declaring their preferences.

Remark that, formally, the former is a much weaker condition than the latter. That's why we want to know under what cases this gap closes, and both conditions become equivalent.

It is also worth insisting that our conditions are defined for any subset H of agents, while their standard versions apply to the whole set of individuals involved in the allocation problem ($H = N$). This is because in some important problems, standard versions of those properties are essentially impossible to obtain, and yet some attractive and meaningful allocation methods satisfy them for at least some part of the agents. This is the case in models where there are two differentiated types of agents, or sides of the market. For example, as we shall see, in classical matching models the Gale-Shapley mechanism only provides one side of the market with dominant strategies to reveal the truth. By relaxing the standard definitions and allowing for partial notions of manipulation and of strategy-proofness, we can approach matching problems and mechanisms to others that do no present the same asymmetries among agents.

Let us also remark that what we call group strategy-proofness requires that agents in a deviating coalition must all gain from participating in it. Because of that, our condition is sometimes qualified as "weak" group strategy-proofness, by contrast with a "strong" requirement that would also preclude gains for some agents deviating along with others who remain indifferent among the initial and the final result. Of course, the distinction would make no difference if agents held strict preferences over alternatives. However, selfishness implies that all agents are indifferent among any alternative that assigns the same consequences for them. Thus, it is worth to point out that these are two different versions of group strategy-proofness, and that we concentrate on the one we find more natural and easier to be satisfied.

3 Some representative mechanisms and their common properties

In this Section we present two conditions on social choice functions and one on their domain that jointly guarantee the equivalence between individual and group strategy-proofness. These conditions are quite natural, and they do not come out of the blue. Rather, we have identified them as being a common denominator for a variety of mechanisms that different strands of literature have proposed to solve the different kinds of allocation problems we have described in the preceding section. What was already known is that, taken one by one, these mechanisms were not only individually but also group strategy-proof. Therefore, in order to motivate our conditions, we shall briefly describe some of the most salient in the literature, consider the social choice functions or direct mechanisms associated with them, and extract the relevant conditions that apply to them all, briefly arguing why this is the case.

We begin by considering the Gale-Shapley deferred acceptance mechanism, which is the best known method to arrive at a single result for each specification of the preferences of agents in matching problems. The basic version of the mechanism for the one-to-one case[7] can be adapted easily, under some additional substitutability assumptions on the preferences of firms, to cover the many-to-one case and also the more complex situations of matching with contracts.[8]

The use of the Gale-Shapley mechanism at each preference profile generates a social

[7]The deferred acceptance algorithm, for one- to-one matching starts with all workers applying to their preferred firms, and firms tentatively accepting the one worker they prefer among those that applied for it. If that leaves some workers unmatched, these are then asked to apply to their second best firms. Once their applications get in, firms may accept these new applicants if they are better for them than the ones they retained in the first round, and reject the previously accepted ones. That leads to a new matching and to a new list of unmatched workers. Again, if some workers remain unmatched, they may now re-apply to those firms that are the best among those they did not yet apply to in preceding rounds. The process continues until no further changes occur. In the one-to-one case this method always leads to a unique alternative. Now, to extend the use of the Gale-Shapley mechanism to our many-to-one case, simply re-define a new set of firms, so that each original one, with capacity for workers, becomes one of different firms with capacity one. Let all these firms still have the same preferences over singletons as before. Define the preferences of workers in such a way that they still preserve the ranking among firms that come from different original ones, and let all of them rank the small firms coming from the same one in an arbitrary, common order. Then, run the Gale-Shapley mechanism for this well-defined one-to-one matching problem, and finally assign to the "real" initial firms all the workers that are matched with any of its small, instrumental firms into which it was divided. Similarly, matching with contract problems under substitutable preferences can be solved by applying the algorithm where now proposals involve not only who gets with whom, but also under what specific contract.

[8]The matching thus obtained is stable: that is, it is individually rational and no two individuals, one from each side of the market, can improve upon it by forming a new pair and leaving their present match. Discussing the stability of matchings is the main concern of that literature, but since we concentrate here on incentive properties, we shall not insist on that point.

choice function. We will abuse terminology and call it the Gale-Shapley social choice function from now on. In a similar vein, a variation of the mechanism can be applied to more complex problems of matching with contracts, and will also fall within the class of social choice functions that we consider (Hatfield and Milgrom, 2005).

Turn next to the uniform rule[9] for the division problem (Sprumont, 1991). Again, given a profile of single-peaked preferences over shares of the job, the rule determines a unique alternative, that is, a vector of shares. Our social choice function is induced by assigning this unique proposal to each preference profile.

For the housing problem without money transfers (Shapley and Scarf, 1974), the top trading cycle mechanism also determines a unique alternative, this time an allocation of houses based on the strict preferences of agents over them.[10] It thus generates what we call, abusing terminology, the top trading cycle social choice function. All linear orders of the houses by agents are allowed, and their natural extension to allocations under the selfishness assumption constitute the domain of definition of the social choice function.

If money transfers are allowed (Miyagawa, 2001), a variant of the above is again a group strategy-proof mechanism, generating the corresponding social choice function. But now individuals express their preferences over the pairs of houses and the (eventually personalized) prices they must pay for them. These preferences must be monotonic in money, and may now present indifferences. The relevant social choice function is derived from a natural adaptation of the top trading cycle, that includes a tie breaking rule among indifferent alternatives and provides well defined assignments for each preference profile. And again, an extension of the preferences on pairs of houses and prices to full allocations provides the domain of preferences over alternatives.

[9] Here is how this rule works. Ask agents for their preferred share of the job. If the sum of the desired shares exceeds one, find a number λ with the following property. If all agents who demand less than λ are allowed to have their preferred share, and all others are required to accept λ, then the total assignment adds up to one. If the sum of the desired shares is short of one, find a number λ' having the following property. If all agents who demand more than λ' are allowed to have their preferred share, and all others are required to accept λ', then the total assignment adds up to one.

These values for λ or λ' always exist, and thus the rule based on them determines an assignment of shares that is always feasible. It therefore defines a social choice function on the set of admissible profiles.

[10] Here is how the method works. Ask agents to point at their preferred house. There will always be some set of agents (maybe several) whose demands form a cycle. Agents who point at their own house, form a cycle by themselves. Give those agents in the cycle their preferred houses and remove them. Now ask the remaining set of agents for their preferred houses over the remaining ones, and proceed likewise until all houses are assigned.

The provision and cost sharing of a public good is also within our scope, to the extent that agents must contribute their personalized payments, in addition to enjoying the public good. Serizawa (2005) describes several families of group strategy-proof mechanisms. Individual consequences are pairs of public good amounts and money, and individual preferences are assumed to be convex and increasing in both arguments.[11]

The extension of these preferences to full allocations will be the domain of the induced social choice function.

Finally, notice that Vickrey's second price auction (Vickrey, 1961) is also an example of a rule that is not only strategy-proof for the buyers, but also (weakly) group strategy-proof. Given the popularity of the rule, we refrain from further elaboration. However, let us notice that we are not claiming that proving (weak) group strategy-proofness, in that case of auctions or in any other, closes any further discussion about other forms of disturbing the good functioning of a mechanism, by either agents or groups. Bribery may be natural in some cases. Its threat in different contexts has been discussed, and specifically for second price auctions, where alternative implementation methods have been considered in view of this possibility (Che and Kim, 2006). We do not deny this additional threat. However, we want to point out that it lies outside the models under consideration, and its analysis requires further modelling efforts. At any rate, group strategy-proofness is a first line of resistance for a mechanism, since it guarantees that coalitions that want to make jointly beneficial moves will have to resort to actions that are outside the admissible rules.

The above mechanisms are well known and paradigmatic within their respective parts of the literature. There, each one is described in the usual language of the field. But since all of them clearly generate social choice functions, that allows us to examine their common traits in a unified manner. The first remarkable coincidence is that all these rules are not only individually but also group strategy-proof. But there are also other common characteristics, that we emphasize in turn.

[11] These mechanisms have their origins in Moulin (1994), and they come in several variants of it. For the purposes of illustration, we choose a very simple one among those that Serizawa proves to be, in addition to group strategy-proof, also budget balancing, symmetric and individually rational. In that simple version, each agent demands an amount of the public good. The minimum amount that is demanded is produced, and all agents are charged the same share of the total cost. Notice that the properties predicated on that mechanism only hold if, in addition to the assumptions on preferences, the cost function has the appropriate shape to guarantee that, once the the cost sharing rule is fixed, the ranking of agents for the amount of the public good is single-peaked. Our statements will, as always, be conditional to the mechanism being well defined on environments where our stated property of group strategy-proofness does hold.

We start by the domains of definition. They are clearly different, and it is not even the case that the restrictions that make sense when alternatives are matchings would also be applicable to the case of division, or vice-versa. Yet, here are general requirements on the profiles of preferences over consequences that all three domains we consider do satisfy.

Definition 2 *A set of individual preferences $\mathcal{R}_i$ is* ***rich*** *if for any $R_i, \widetilde{R}_i \in \mathcal{R}_i$, $a_i, b_i \in B_i$ such that $b_i P_i a_i$,*
there exists $R'_i \in \mathcal{R}_i$ such that $\overline{U}(R'_i, b_i) \subseteq \overline{U}(R_i, b_i) \cap \overline{U}(\widetilde{R}_i, b_i)$,$\overline{L}(\widetilde{R}_i, b_i) \subseteq \overline{L}(R'_i, b_i)$, $\overline{U}(R_i, a_i) = \overline{U}(R'_i, a_i)$ and $\overline{L}(R_i, a_i) = \overline{L}(R'_i, a_i)$.

Definition 3 *Let $H \subseteq N$. A domain of preferences $\times_{i \in N} \mathcal{R}_i$ is H****-rich*** *if for any $i \in H$, $\mathcal{R}_i$ is rich.*

This is a condition guaranteeing that the domain is rich enough, in the sense that, should it contain certain preferences, then some others must also belong to it. Specifically, here is what it demands under selfishness. Take any two preferences R and $\widetilde{R}$ in the domain, and any two alternatives a and b such that b is preferred to a by R. Then there must be another preference R' such that (1) the strict upper contour set of b in R' is in the intersection of the strict upper contour sets of b in R and $\widetilde{R}$, (2) the lower contour set of b in $\widetilde{R}$ is a subset of the one in R', and (3) a stays in the same position than it held in R, relative to all other alternatives.

The condition has an easy implication when the preferences are such that each agent has always a best consequence, and all possible consequences can be best. This is the case, for example, when there is a finite set of objects and preferences are linear orders of objects, like in matching, or when, even if infinite, in the case of single-peaked preferences over shares. Then, given any R, $\widetilde{R}$, a, b we can check if there exists an R' where a stays at the same position than in R but b ranks at the top.[12] The reason why we do not postulate our domain restrictions in these simpler terms is that, in our present formulation, we can also include problems where the notion of a best element is not well defined, but our condition holds and is even easy to check for. This is the case, for example, in allocation problems where we admit unbounded monetary compensations.

Now, we claim that the standard domains of definition for all the proposed examples of group strategy-proof rules are rich, either for all agents or for at least some subset H of them.

[12] This gives an easy way to check for our domain condition in the case of well defined maximal elements.

In many-to-one matching models no restrictions are imposed on the preferences of the set of workers, either when they just rank firms or firms with contracts, other than being linear orders on the set of consequences, and eventually respecting monotonicity with respect to monetary compensations. Additional restrictions, like responsiveness, substitutability or the law of demand may be imposed on the firms' preferences, and these will not allow such preferences to be rich. Therefore, when it comes to make statements about matching models, we shall restrict attention to the incentives of the worker's side of the market, for which the condition is clearly respected in the models we have referred to.

As for division problems, the family of all single-peaked preferences is again rich because given any $a_i, b_i \in B_i$ such that $b_i P_i a_i$, one can trivially find another single-peaked preference $R_i' \in \mathcal{R}_i$ with b_i being the best consequence of R_i' and such that $\overline{U}(R_i, a_i) = \overline{U}(R_i', a_i)$, $\overline{L}(R_i, a_i) = \overline{L}(R_i', a_i)$.

In housing problems, no restrictions are imposed on the agents' preferences over houses, and in the case of additional payments the monotonicity of preferences with respect to money also allow the domains of definition to satisfy our richness domain condition.

Similar considerations apply to the rest of examples, and we leave to the reader to check that in all cases the condition is satisfied.

Next we identify two more conditions on social choice functions that are satisfied by our leading examples. Such additional conditions are not needed to prove our desired equivalence in models with public goods only (see Barberà, Berga, and Moreno, 2010). There, only domain restrictions are relevant. However, in the presence of private goods and selfish agents, we need to impose some additional requirements. In that case, the sets of allocations that are indifferent to others for any given agent become very large. Since violations of strategy-proofness require strict improvements by manipulators, it is difficult to distinguish between functions whose only differences occur within indifference sets, and our conditions are both reasonable and technically helpful.

The first one is a restricted version of non-bossiness. That condition was first proposed by Satterthwaite and Sonnenschein (1981) and requires that no agent can change the consequences for other agents unless the consequences for herself are also changed. The condition admits many variants, and has been defended from different angles. A recent paper by Thomson (2014) is quite critical to its use, and to the justifications that have been proposed for using it as a means to narrow down the class of mechanisms to be considered. We agree that it is a strong condition. For example, in many examples where preferences of agents over consequences are strict (even if not over alternatives!), then strategy-proofness and non-bossiness implies strong group strategy-proofness. That led Thomson (2014) to ask for further scrutiny of the connections between these properties. For our purposes, two remarks are relevant. One is that, in domains where the preferences of agents on consequences are not strict, then strategy-proofness and non-bossiness does no longer imply group strategy-proofness (see Example 4 and Footnote 14 below). And the models we consider, allowing for money transfers, are such that preferences on consequences are not strict. So, even if we did require non-bossiness, our concern about the equivalence between individual and group strategy-proofness would still be open for some of our mechanisms. The second one is that we do not wish to restrict attention to non-bossy mechanisms, because this strong condition would indeed exclude interesting mechanisms where equivalence applies. In particular, the Gale-Shapley social choice function violates non-bossiness, even if we restrict attention to agents in only one side of the market.[13]

The following concept of H-respectfulness is a mild version of non-bossiness, where the requirement is limited to changes in consequences induced by a limited type of preference changes, and is only predicated for some of the agents.

Definition 4 *Let $H \subseteq N$ be a subset of agents. A social choice function f on $\times_{i\in N}\mathcal{R}_i$ is H-**respectful** if*

$$f_i(R_N) I_i f_i(R'_i, R_{N\backslash\{i\}}) \textit{ implies } f_j(R_N) I_j f_j(R'_i, R_{N\backslash\{i\}}),\ \forall j \in H\backslash\{i\},$$

for each $i \in H$, $R_N \in \times_{i\in N}\mathcal{R}_i$, and $R'_i \in \mathcal{R}_i$ such that $\overline{U}(R_i, f_i(R_N)) = \overline{U}(R'_i, f_i(R_N))$ and $\overline{L}(R_i, f_i(R_N)) = \overline{L}(R'_i, f_i(R_N))$.

Notice that the Gale-Shapley social choice function is W-respectful. This is because when a worker is indifferent between two allocations, it must be receiving the same firm in both. But then, the condition requires that this agent's preferences before and after the change must have the same firms in their upper and lower contour sets, and thus rank the firm that she is getting in both cases in the same rank as before. Under these conditions, the results of the Gale-Shapley algorithm do not change, and therefore all other agents, in addition to the one who has changed preferences, stay with the same partner in both matches, as required.

Some of the mechanisms we have used as examples are indeed well known to be non-bossy and usually defined so that agents have (almost) strict preferences on the consequences

[13] Kojima (2010) proves that stability is incompatible with non-bossiness. Since the Gale-Shapley mechanism produces stable outcomes it must be bossy.

they may get. This is the case for the uniform rule and for the top trading cycle. In that case, there is no need to check further, since non-bossiness is a stronger condition than respectfulness in those domains.

Again, we leave the reader to check that our condition applies for the rest of cases.

The second condition is a limited form of monotonicity, that we call H-joint monotonicity. The condition needs to only hold for some group H, and then requires the following. Take a preference profile, and a subset of agents in H. Suppose that the preferences of all agents in the subset do change, and that the consequences they were getting under the previous profile become now better choices than before for all of them. Then, these agents should get an outcome at least as good as the previous one.

Formally,

Definition 5 *Let $H \subseteq N$ be a subset of agents. A social choice function f satisfies H-**joint monotonicity** on $\times_{i \in N} \mathcal{R}_i$ if for any $R_N \in \times_{i \in N} \mathcal{R}_i$, $C \subseteq H$, and $R'_C \in \times_{i \in C} \mathcal{R}_i$ ($R'_i \neq R_i$ for any $i \in C$) such that $\overline{L}(R_i, f_i(R_N)) \subseteq \overline{L}(R'_i, f_i(R_N))$ and $\overline{U}(R_i, f_i(R_N)) \supseteq \overline{U}(R'_i, f_i(R_N))$ for each $i \in C$, then $f_i(R'_C, R_{N \setminus C}) R'_i f_i(R_N)$ for each $i \in C$.*

Let us hint at the proofs that our mechanisms also meet this condition. In the case of house allocation without money transfers, it is clearly satisfied by the top trading cycle, since any agent endowed with some house at a certain profile will retain that house if it eventually becomes higher in her ranking. Notice that our argument here applies to each agent individually, but that clearly implies our condition which is only predicated for groups, by repeated application of individual changes and the fact that the mechanism is non-bossy and individual preferences on consequences are strict.

A similar argument applies to the uniform rule for the division problem: if an agent is assigned a share of a task, and that share becomes even better for her, it will persist. Again, non-bossiness allows us to go from the individual to the group condition, using the additional fact that the mechanism is efficient and that this does not permit an allocation change for an agent to switch the allocation of any other from one share to the (at most) other that is indifferent to it. Hence, N-joint monotonicity is satisfied.

As for the Gale-Shapley social choice function, it will satisfy W-joint monotonicity by the following reasoning. Start from a profile R_N, and consider another profile R'_N where all agents in a subset $C \subseteq W$ consider that $f_i(R_N)$ is better ranked in R'_N that it was in R_N. Notice that $f(R_N)$ will also be stable at profile R'_N. Moreover, since the Gale-Shapley social choice function selects the W-best stable matching, $f(R_N)$ has this property in R_N, and clearly it will also have it in R'_N. Thus, $f(R_N) = f(R'_N)$.

Remark that, in contrast with our preceding reasoning for the top trading cycle and the uniform rule, our argument here involves the simultaneous consideration of preference changes for all agents in W. Indeed, the repeated use of single agent changes could be a problem here, since the Gale-Shapley mechanism is bossy and that may compromise a chain of reasoning where one must control for the consequences of one agent's preference change on the assignment received by others.

Before we close the section, two remarks are in order.

First, notice that in some of our models it makes sense to speak about an agent's most preferred alternatives. For example, in a simple matching model, those are the ones where agents are matched with their preferred mate. Or, in a division problem, the allocations where agent's get their peak on the set of shares. However, there are other cases where "best" alternatives are not defined. This happens, for example, when one of the components of an allocation is money, and the amount that an agent can get of that good is not bounded.

Our definitions of H-richness and H-joint monotonicity have been chosen so that the existence of best elements is unnecessary. However, they can take a very easy form when such maximal elements exist (see Barberà, Berga, and Moreno, 2014).

The second remark refers to the meaning of group strategy-proofness in some of our examples.

The Vickrey mechanism is (weakly) group strategy-proof for the buyers. However, many authors have remarked that this property does not guarantee immunity to all forms of coalitional manipulation, since the winner of the auction would be inclined to bribe the second price bidder to bid lower than her true value, in order to get a better price. This is certainly a possibility, but one outside the original setting, whose full analysis requires to properly model a new game. Hence, without denying the theoretical and practical importance of other forms of manipulation, we think it is important to recognize that these examples are also covered by our theorem, and to vindicate (weak) group strategy-proofness as an attractive property per se.

4 Individual versus group strategy-proofness

We have observed that, in spite of several important formal differences regarding the space of alternatives and the domains of preferences, our reference social choice functions share a number of properties. In particular, all of them are group strategy-proof. The following theorem proves that the equivalence between that property and individual strategy-proofness is not a lucky coincidence, but a result of the fact that any strategy-proof social choice function satisfying the remaining common requisites that we just exhibited, for individuals in a set H must also be immune to manipulation by subsets of H.

Theorem 1 If f is H-joint monotonic, H-respectful and defined on an H-rich domain, then it is H-strategy-proof if and only if it is H-group strategy-proof.

Proof Obviously, H-group strategy-proofness implies H-strategy-proofness. To prove the converse, suppose by contradiction that there exists $R_N \in \times_{i \in N} \mathcal{R}_i$, $C \subseteq H$, $\widetilde{R}_C \in \times_{i \in C} \mathcal{R}_i$ such that for any agent $i \in C$, $f_i(\widetilde{R}_C, R_{N \backslash C}) P_i f_i(R_N)$. Let $b = f(\widetilde{R}_C, R_{N \backslash C})$ and $a = f(R_N)$. Without loss of generality, let $C = \{1, ..., c\}$.

First, we recursively define the preferences R'_i for i in C. For $i = 1$, let R'_1 be such that $\overline{U}(R'_1, b_1) \subseteq \overline{U}(R_1, b_1) \cap \overline{U}(\widetilde{R}_1, b_1)$, $\overline{L}(R_1, a_1) = \overline{L}(R'_1, a_1)$, and $\overline{U}(R_1, a_1) = \overline{U}(R'_1, a_1)$. By H-richness, such a preference R'_1 is in $\mathcal{R}_1$. For $k \in C \backslash \{1\}$, define $a'_k = f_k(R'_{\{1,...,k-1\}}, R_{N \backslash \{1,...,k-1\}})$ and R'_k to be such that $\overline{U}(R'_k, b_k) \subseteq \overline{U}(R_k, b_k) \cap \overline{U}(\widetilde{R}_k, b_k)$, $\overline{L}(R_k, a'_k) = \overline{L}(R'_k, a'_k)$, and $\overline{U}(R_k, a'_k) = \overline{U}(R'_k, a'_k)$. By H-richness, such a preference R'_k is in $\mathcal{R}_k$.

Let us now change one by one the preferences of each agent in C from R_i to R'_i. By strategy-proofness applied to agent 1: $f_1(R'_1, R_{N \backslash \{1\}}) I_1 f_1(R_N) = a_1$, otherwise agent 1 would manipulate f at (R_N) via R'_1 or at $(R'_1, R_{N \backslash \{1\}})$ via R_1. By H-respectfulness, $f_j(R'_1, R_{N \backslash \{1\}}) I_j f_j(R_N) = a_j$ for each agent $j \in H \backslash \{1\}$. Hence, for $j \in H \backslash \{1\}$, by transitivity of R_j, $b_j P_j f_j(R'_1, R_{N \backslash \{1\}})$. And by transitivity of R'_1, since $\overline{L}(R_1, a_1) = \overline{L}(R'_1, a_1)$ and $\overline{U}(R_1, a_1) = \overline{U}(R'_1, a_1)$ then $b_1 P'_1 f_1(R'_1, R_{N \backslash \{1\}})$.

A similar argument applies when the profile changes from $(R'_{\{1,...,k-1\}}, R_{N \backslash \{1,...,k-1\}})$ to $(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}})$, for $k = 2, ..., c$. Again, by strategy-proofness applied to agent k, $f_k(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}}) I_k f_k(R'_{\{1,...,k-1\}}, R_{N \backslash \{1,...,k-1\}}) = a'_k$.

By H-respectfulness, $f_j(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}}) I_j f_j(R'_{\{1,...,k-1\}}, R_{N \backslash \{1,...,k-1\}})$ for each agent $j \in H \backslash \{1, ..., k\}$ and $f_l(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}}) I'_l f_l(R'_{\{1,...,k-1\}}, R_{N \backslash \{1,...,k-1\}})$ for each agent $l \in \{1, ..., k-1\}$. Therefore, by transitivity of R_j, $b_j P_j f_j(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}})$ for each agent $j \in H \backslash \{1, ..., k\}$. By transitivity of R'_l, $b_l P'_l f_l(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}})$ for each agent $l \in \{1, ..., k-1\}$. And by transitivity of R'_k, since $\overline{L}(R_k, a'_k) = \overline{L}(R'_k, a'_k)$, $\overline{U}(R_k, a'_k) = \overline{U}(R'_k, a'_k)$ then $b_k P'_k f_k(R'_{\{1,...,k\}}, R_{N \backslash \{1,...,k\}})$.

Repeating the same argument for each agent in C we obtain that

(1) $b_j P'_j f_j(R'_C, R_{N \backslash C})$ for each agent $j \in C$.

We could have also arrived at the profile $(R'_C, R_{N \backslash C})$ from $(\widetilde{R}_C, R_{N \backslash C})$. Notice that by construction of the R'_i we have that for each $i \in C$, $\overline{L}(\widetilde{R}_i, f_i(\widetilde{R}_N)) \subseteq \overline{L}(R'_i, f_i(\widetilde{R}_N))$ and

$\overline{U}(\widetilde{R}_i, f_i(\widetilde{R}_N)) \supseteq \overline{U}(R'_i, f_i(\widetilde{R}_N))$. Thus, we can apply H-joint monotonicity and obtain that (2) $f_i(R'_C, R_{N\setminus C}) R'_i f_i(\widetilde{R}_C, R_{N\setminus C}) = b_i$ for each $i \in C$.

By (1) and (2) we get the desired contradiction. ■

Our main purpose in this paper is achieved: we have proven that the coincidence between the two incentive properties in the apparently diverse worlds of matching, division, cost sharing of a public good, house allocation and auctions is the consequence of a shared structure that goes beyond the details of each particular model.

The following examples show that the result is robust in that all assumptions we use are needed. We also note that H-group strategy-proofness does imply neither H-respectful nor H-joint monotonicity (see Examples 5 and 6).

Example 1 *Violation of Theorem 1 when the domain is not H-rich. Let* $W = \{1, 2, 3\}$ *and* $F = \{4, 5, 6\}$. *For each* $j \in F$, $B_j = W \cup \{j\}$ *and for agent* $3 \in W$, $B_3 = F \cup \{3\}$ *any linear order on* B_l *is admissible. For agents* 1 *and* $2 \in W$, $B_i = F \cup \{i\}$, *the set of admissible preferences are as follows:*

R_1^1	R_1^2	R_1^3	R_2^1	R_2^2	R_2^3
4	5	6	4	5	6
5	4	4	5	4	5
6	6	5	6	6	4
1	1	1	2	2	2

Define the following rule f *where neither firms nor agent 3 play any role, that is:*

$f(\cdot, R_3, R_F)$	R_2^1	R_2^2	R_2^3
R_1^1	$4, 5, 6, 1, 2, 3$	$4, 5, 6, 1, 2, 3$	$5, 6, 4, 3, 1, 2$
R_1^2	$5, 4, 6, 2, 1, 3$	$5, 4, 6, 2, 1, 3$	$5, 6, 4, 3, 1, 2$
R_1^3	$6, 4, 5, 2, 3, 1$	$6, 4, 5, 2, 3, 1$	$5, 4, 6, 2, 1, 3$

For $H = W$, *the domain is not H-rich (let* $R_1 = R_1^3$, $\widetilde{R}_1 = R_1^2$, $b_i = 4$ *and* $a_i = 5$, *there does not exist any* R'_1 *satisfying the required condition). And the above rule is H-respectful, H-joint monotonic, H-strategy-proof, but it is not H-group strategy-proof (agents* 1 *and* 2 *deviates from* (R_1^3, R_2^3) *to* (R_1^1, R_2^1) *and they are strictly better off).*

Example 2 *Let* $H = W$. *A social choice function defined on a H-rich domain (of strict preferences over consequences) that is H-respectful, H-joint monotonic but not H-strategy-proof. The much celebrated Boston mechanism provides an example. In a first round, each* *student applies to his (reported) top choice and each school admits applicants one at a time according to its preferences until either capacity is exhausted or there are no more students who ranked it first. In Round k, each unmatched student applies to his kth choice and schools with remaining capacity admits applicants one at a time according to its preferences until either the remaining capacity is exhausted or there are no more students who ranked it kth.*

Example 3 *Let $H = W$. A social choice function defined on a H-rich domain that is H-joint monotonic, H-strategy-proof but neither H-respectful nor H-group strategy-proof . Let $W = \{1, 2\}$ and $F = \{3, 4\}$. For each $j \in F$, $B_j = W \cup \{j\}$ and any linear order on B_j is admissible. For each $i \in W$, $B_i = F \cup \{i\}$ and any linear order on B_i is admissible. The latter is as in the following table:*

R_i^1	R_i^2	R_i^3	R_i^4	R_i^5	R_i^6
3	4	3	4	i	i
4	3	i	i	3	4
i	i	4	3	4	3

Define the following rule f where firms do not play any role, that is for any $R_F \in \mathcal{R}_F$:

$f(\cdot, R_F)$	R_2^1	R_2^2	R_2^3	R_2^4	R_2^5	R_2^6
R_1^1	$3, 4, 1, 2$	$3, 4, 1, 2$	$3, 2, 1, 4$	$3, 4, 1, 2$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^2	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^3	$3, 4, 1, 2$	$3, 4, 1, 2$	$3, 2, 1, 4$	$3, 4, 1, 2$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^4	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 3, 2, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$	$4, 2, 3, 1$
R_1^5	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$4, 3, 2, 1$	$4, 3, 2, 1$
R_1^6	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$1, 3, 2, 4$	$4, 3, 2, 1$	$4, 3, 2, 1$

.

For any $R_F \in \mathcal{R}_F$, note that $f(R_1^5, R_2^5, R_F) = (4, 3, 2, 1)$. Observe that $f_1(R_1^5, R_2^5, R_F) = f_1(R_1^3, R_2^5, R_F) = 4$, $\overline{U}(R_1^5, 4) = \overline{U}(R_1^3, 4)$, $\overline{L}(R_1^5, 4) = \overline{L}(R_1^3, 4)$ but $f_2(R_1^5, R_2^5, R_F) \neq f_2(R_1^3, R_2^5, R_F)$, violating H-respectfulness. Coalition $C = \{1, 2\} \subseteq H$ manipulates f at (R_1^5, R_2^5, R_F) via (R_1^1, R_2^3) thus violating H-group strategy-proofness.

Example 4 *Let $H = N$. A social choice function defined on a H-rich domain that is H-strategy-proof, H-respectful, but neither H-joint monotonic nor H-group strategy-proof. Let $N = \{1, 2\}$, agents's consequences $B_1 = B_2 = \{a, b, c, d\}$, and the set of admissible* *preferences of each agent over consequences be the same:* $\mathcal{R}_1 = \mathcal{R}_2 = \{R, R'\}$, *where* $R : cIdPaIb$ *and* $R' : aI'bP'cI'd$. *Observe that* $\mathcal{R}_1 \times \mathcal{R}_2$ *is an* N*-rich domain. Let* f *be the social choice function defined as follows:* $f(R_1, R_2) = (b, b)$, $f(R_1, R_2') = (c, a)$, $f(R_1', R_2) = (a, c)$, *and* $f(R_1', R_2') = (d, d)$. *Observe that* f *is strategy-proof and* N*-respectful (in a vacuous way).*[14] *However,* f *violates* N*-group strategy-proofness:* $f(R_1', R_2') = (d, d)$ *but both agents would be strictly better off obtaining* b *which would be their assignment if their preferences were* (R_1, R_2). *Furthermore,* f *violates* N*-joint monotonicity: when going from profile* (R_1', R_2') *to profile* (R_1, R_2) *we have that for each* $i \in N$, $\overline{L}(R_i', d) \subseteq \overline{L}(R_i, d)$ *and* $\overline{U}(R_i, d) \subseteq \overline{U}(R_i', d)$, *but* $f_i(R_1', R_2') = dP_ib = f_i(R_1, R_2)$ *which contradicts* N*-joint monotonicity.*

Example 5 *Let* $H = W$. *A social choice function defined on a* H*-rich domain that is* H*-respectful,* H*-strategy-proof but not* H*-joint monotonic. Let* $W = \{1, 2, 3\}$ *and* $F = \{4, 5, 6\}$. *For each* $i \in W$, $B_i = F \cup \{i\}$ *and any linear order on* B_i *is admissible. For each* $j \in F$, $B_j = W \cup \{j\}$ *and any linear order on* B_j *is admissible. For any* $R_1 \in \mathcal{R}_1$, *let* $\tau^2(R_1)$ *be the most preferred alternative on* $B_1 \backslash \tau(R_1)$ *of* R_1. *Define the following rule* f *where neither firms nor agents in* $W \backslash \{1\}$ *play any role:*

	$\forall R_{-1},\ f(R_N)$
If $\tau(R_1) = 5$ *then*	$5, 6, 4, 3, 1, 2$
If $\tau(R_1) = 6$ *then*	$6, 5, 4, 3, 2, 1$
If $\tau(R_1) = 1$ *then*	$1, 5, 4, 3, 2, 6$
If $\tau(R_1) = 4$ *and* $\tau^2(R_1) = 5$ *then*	$5, 4, 6, 2, 1, 3$
If $\tau(R_1) = 4$ *and* $\tau^2(R_1) = 6$ *then*	$6, 4, 5, 2, 3, 1$
If $\tau(R_1) = 4$ *and* $\tau^2(R_1) = 1$ *then*	$1, 4, 5, 2, 3, 6$

For $H = W$, *the above rule is* H*-respectful,* H*-strategy-proof (it is also* H*-group strategy-proof), but it is not* H*-joint monotonic. To see this consider the following preference profile:*

$\widehat{R}_1$	$\widehat{R}_2$	$\widehat{R}_3$
4	5	4
1	4	5
5	6	6
6	2	3

[14] Note that f also satisfies non-bossiness.

For any $R_F \in \mathcal{R}_F$, note that $f(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3, R_F) = (1,4,5,2,3,6)$. Suppose that the preferences of agents in $H = W$ change and the consequences they have obtained are now the most preferred ones

$\widetilde{R}_1$	$\widetilde{R}_2$	$\widetilde{R}_3$
1	4	5
4	5	4
5	6	6
6	2	3

For any $R_F \in \mathcal{R}_F$, note that $f(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, R_F) = (1,5,4,3,2,6)$. Observe that $f_1(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, R_F) = f_1(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3, R_F) = 1$ but $f_i(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3, R_F) \neq f_i(\widehat{R}_1, \widehat{R}_2, \widehat{R}_3, R_F)$ for $i = 2, 3$, violating H-joint monotonicity.

Example 6 *Let $H = W$. A social choice function defined on a H-rich domain that is H-group strategy-proof but not H-respectful. Let $W = \{1,2,3\}$ and $F = \{4,5,6\}$. For each $i \in W$, $B_i = F \cup \{i\}$ and any linear order on B_i is admissible. For each $j \in F$, $B_j = W \cup \{j\}$ and any linear order on B_j is admissible. For any $R_1 \in \mathcal{R}_1$, let $\tau^2(R_1)$ be the most preferred alternative on $B_1 \backslash \tau(R_1)$ of R_1 and $\tau^3(R_1)$ be the worst alternative on B_1 of R_1. Define the following rule f where worker 1 dictates as follows: $f_1(R_N) = \tau(R_1)$, $f_2(R_N) = \tau^2(R_1)$, and $f_3(R_N) = \tau^3(R_1)$. This rule f is H-group strategy-proof but f is not H-respectful: Let R_1 and R_1' such that $4P_1 5 P_1 6$ and $4P_1' 6 P_1' 5$. By definition of f, for any $R_{N\backslash\{1\}}$, $f_1(R_1, R_{N\backslash\{1\}}) = f_1(R_1', R_{N\backslash\{1\}}) = 4$, $f_2(R_1, R_{N\backslash\{1\}}) = 5 \neq f_2(R_1', R_{N\backslash\{1\}}) = 6$ and $f_3(R_1, R_{N\backslash\{1\}}) = 6 \neq f_3(R_1', R_{N\backslash\{1\}}) = 5$, $\overline{L}(R_1,\ f_1(R_1, R_{N\backslash\{1\}})) = \overline{L}(R_1',\ f_1(R_1, R_{N\backslash\{1\}}))$, and $\overline{U}(R_1,\ f_1(R_1, R_{N\backslash\{1\}})) = \overline{U}(R_1',\ f_1(R_1, R_{N\backslash\{1\}})) = \varnothing$.*

5 Connection with implementation literature

We are not the first to discuss the connection between individual and group incentives in the private good environments. Yet, our results apply to many mechanisms that were not covered by previous remarks and theorems in the literature.

An appropriate term of reference is a classical result by Dasgupta, Hammond, and Maskin (1979), which is re-stated in the following form in Maskin and Sjöström (2002, Theorem 7).

The authors first define the concept of "improvement". If $u_i(a,\theta) \geq u_i(b,\theta)$ and $u_i(a,\theta') \leq u_i(b,\theta')$ and at least one inequality is strict, then b improves with respect to

a for agent i as the state changes from θ to θ'.

Definition. Rich domain: For any $a, b \in A$ and any $\theta, \theta' \in \Theta$, if, for all $i \in N$, b does not improve with respect to a for when the state changes from θ to θ', then there exists $\theta'' \in \Theta$ such that $L_i(a, \theta) \subseteq L_i(a, \theta'')$ and $L_i(b, \theta') \subseteq L_i(b, \theta'')$ for all $i \in N$.

Theorem 7 [Dasgupta, Hammond, and Maskin (1979)]. Suppose f is a monotonic social choice function, the domain is rich, and the preference domain has a product structure $\mathcal{R}(\Theta) = \times_{i=1}^{n} \mathcal{R}_i$. Then f is coalitionally (or group) strategy-proof.

Indeed, this result bears a clear resemblance to ours, as it starts from some assumptions on the domain of preferences and the properties of a mechanism, to conclude that it is coalitionally (or group) strategy-proof. There are, however, some important differences in scope.

The main difference is that Maskin monotonicity is assumed in the preceding statement, while we do not require this condition. This makes a big difference, because that condition implies, along with the richness of the domain, that the corresponding social choice function must be non-bossy. And that immediately excludes several of the mechanisms we have covered in our theorem, and in particular all those considered in matching theory.

Indeed, in our search for a common ground for the classical and attractive mechanisms that we have highlighted, we have identified the condition of H-respectfulness, a milder condition than the property of non-bossiness that has been often used in the mechanism design literature since its introduction by Satterthwaite and Sonnenschein (1981). As we have already remarked, non-bossiness and individual strategy-proofness sometimes may imply strong group strategy-proofness, but not always.[15] But we did not want to take that shortcut even for the limited number of cases where it would apply. This is because, on the one hand, we wanted to stay in a world where comparisons between the classical solution for matching and those for other allocation problems did coexist. And that required a weakening of non-bossiness in that framework, because the Gale-Shapley social choice function is bossy. On the other hand, notice that our notion of group strategy-proofness is a mild one, where in order for a group to manipulate, all agents in it must strictly gain. Indeed, we think that this is an attractive property per se, and the natural one in our context where the set of alternatives is discrete.

It is fair to say that both results are not comparable, since the domain conditions that

[15] It does, for example, in the case of exchange economies (see Barberà and Jackson, 1995). But not in other cases, as shown by our Example 4.

we propose, and the richness condition used in Dasgupta, Hammond, and Maskin (1979) do not imply each other.

Another interesting connection with the implementation literature is provided by the analysis of cost sharing rules proposed by Moulin and Shenker (1992).[16] These authors define a mechanism under which agents do not have dominant strategies, but where the successive elimination of strictly dominated strategies (that converges to the, hence unique, Nash equilibrium) leads to a unique outcome at each preference profile. As a result, the unique equilibrium outcomes of these Nash equilibrium at each preference profile do generate a social choice function that is itself strategy-proof, and also group strategy proof, since the domain of definition of the social choice function, and the social choice function itself satisfy the conditions of our theorem. Similarly, one can associate a social choice function to each preference profile when the amendments rule is used to determine the winner of a sequential majority vote (Miller, 1977, Barberà and Gerber, 2014), and sophisticated voting is allowed. When voters' preferences are restricted to be single-peaked, the induced social choice function will be strategy-proof, and also group strategy-proof, even if the result of a mechanism that has no dominant strategies for voters.

These cases lead us to remark that our theorem, by covering results where the social choice function is not derived from a dominant strategy mechanism, does apply to other cases and opens the way to analyze more complex relationships than those we have emphasized here.

6 Conclusions

We have proven that the coincidence between individual and group strategy-proofness in diverse worlds, where these properties may be satisfied for all agents, or only by a subset of them, is the consequence of a shared structure that goes beyond the details of each

[16] A set of individuals must share the cost of production of a private good, that each one of them consumes in personalized amounts. An allocation is a vector indicating the consumption of each agent and her payment for the private good, satisfying the requirement that the sum of the agent's payments equals the cost of producing their aggregate consumption.

The consequences for each agent are just what she consumes and what she pays. Alternatives are full allocation vectors. Individual preferences over consequences are non-decreasing in their consumption of the good, non-increasing in their payments, nowhere locally satiated, continuous and convex. Preferences over alternatives are obtained as the natural extensions, based on the selfishness assumption, and are the domains for social choice funtions choosing one allocation for each given problem.

particular model.

We have motivated our choice of framework by showing that it encompasses classical mechanisms to solve a variety of allocation problems. But the result goes well beyond the examples we have used. Even within the framework of those problems we have focused on, there are many other mechanisms that satisfy group strategy-proofness for the same reasons we just discussed. This is the case, for the division problem, of the non-anonymous and non-neutral social choice functions induced by sequential mechanisms, as described in Barberà, Jackson, and Neme (1997), or in Massó and Neme (2001). Solutions to housing problems as the one discussed by Pápai (2000) are also within the scope of our results. And so are mechanisms for school choice (Abdulkadiroglu and Sönmez, 2003).

In all these cases, functions satisfying our performance and domain requirements are presented, and group strategy-proofness holds. But this is often presented as a lucky consequence of the basic quest for individual strategy-proofness, while we have emphasized here that group strategy-proofness can and should be a fundamental and attainable objective per se.

Let us also remark that the designer's choice of mechanism may be guided by other criteria than those we have emphasized here. This is particularly true in the case of matching, where the basic concern has been to find procedures that guarantee stability. Our emphasis on different forms of strategy-proofness and their relationship has sidestepped that main concern, though we have identified conditions that are mild enough to still admit the deferred acceptance procedure proposed by Gale and Shapley as part of our universe. Hence, our results also apply to possible mechanisms that do not meet the requirement of stability. For example, you may think of a segmented society with two culturally differentiated groups h and l, match their men M_h and M_l, and their W_h, W_l, according to the Gale-Shapley deferred acceptance mechanism. However, the men M_l are the proposers to W_l in group l, while the women W_h in h are the proposers to M_h. The composed matchings for these two segments of society result in an overall matching that clearly does not guarantee overall stability, but that fits well in our context. It is easy to see that the induced social choice function will be strategy-proof relative to the union of M_l and W_h, and satisfy all of our conditions relative to that set as well.

Our main message is a plea for consideration of group strategy-proofness as an extremely attractive property that may be attained in contexts of relevance, then avoiding the fragility of individually strategy-proof rules when those are possibly manipulable by small and easy to coordinate groups.

In more general terms, our results apply to social choice functions that may be associated to mechanisms through different considerations, and do not require any implementability condition. However, mechanisms that are directly implementable provide an interesting instance of application.

Ackowledgements

S. Barberà acknowledges support from the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2011-0075), and grants "Consolidated Group-C" ECO2008-04756 and FEDER, and SGR2014-515. D. Berga acknowledges the support of the Spanish government (grants ECO2010-16353 and ECO2013-45395-R) and of Generalitat de Catalunya (SGR2009-0189 and 2014-SGR-1360). B. Moreno acknowledges financial support from Junta de Andalucía (grants SEJ4941 and SEJ-5980) and the Spanish Ministry of Science and Technology (grant ECO2011-29355).

References

[1] A. Abdulkadiroglu, and T. Sönmez. "School Choice: A Mechanism Design Approach". *American Economic Review* **93,** (3): 729-747 (2003).

[2] S. Barberà, D. Berga, and B. Moreno. "Individual versus group strategy-proofness: when do they coincide?". *Journal of Economic Theory* **145**: 1648-1674 (2010).

[3] S. Barberà, D. Berga, and B. Moreno. "Group strategy-proofness in private good economies without money: matching, division and house allocation". Barcelona GSE Working Paper Series, Working Paper n^o 773 (2014).

[4] S. Barberà, and A. Gerber. "Sequential Voting and Agenda Manipulation". Barcelona GSE Working Paper No. 782 (2014).

[5] S. Barberà, and M. Jackson. "Strategy-proof Exchange". *Econometrica* **63**, (1): 51-87 (1995).

[6] S. Barberà, M. Jackson, and A. Neme. "Strategy-Proof Allotment Rules". *Games and Economic Behavior* **18**: 1-21 (1997).

[7] Y. Che, and J. Kim. "Robustly colusion-proof implementation". *Econometrica* **74** (4): 1063-1107 (2006).

[8] P. Dasgupta, P. Hammond, and E. Maskin. "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility". *The Review of Economic Studies* **46** (2): 185-216 (1979).

[9] D. Gale, and L. Shapley. "College Admissions and the Stability of Marriage". *American Mathematical Monthly* **69**: 9-15 (1962).

[10] J. W. Hatfield, and F. Kojima. "Group Incentive Compatibility for Matching with Contracts". *Games and Economic Behavior* **67**: 745-749 (2009).

[11] J. W. Hatfield, and P. R. Milgrom. "Matching with Contracts". American Economic Review **95** (4): 913-935 (2005).

[12] A. S. Kelso, Jr and V. P. Crawford. "Job Matching, Coalition Formation and Gross Substitutes". Econometrica **50** (6): 1483-1504 (1982).

[13] F. Kojima. "Impossibility of stable and nonbossy matching mechanisms". *Economics Letters* **107** (1): 69–70 (2010).

[14] E. Maskin, and T. Sjöström. "Implementation Theory". In *Handbook of Social Choice and Welfare*, Volume 1, Edited by K.J Arrow, A.K. Sen and K. Suzumura, Elsevier Science (2002).

[15] J. Massó, and A. Neme. "A Maximal Domain of Preferences for the Division Problem". *Games and Economic Behavior* **37**: 367-387 (2001).

[16] N. R. Miller. "Graph-Theoretical Approaches to the Theory of Voting". *American Journal of Political Science* **21**: 769-803 (1977).

[17] E. Miyagawa. "House allocation with transfers". *Journal of Economic Theory* **100**: 329-355 (2001).

[18] H. Moulin. "Serial Cost Sharing of excludable public goods". *Review of Economic Studies* **61**: 305-325 (1994).

[19] H. Moulin, and S. Shenker. "Serial Cost Sharing". *Econometrica* **60** (5): 1009-1037 (1992).

[20] S. Pápai. "Strategyproof Assignment by Hierarchical Exchange". *Econometrica* **68** (6): 1403–1433 (2000).

[21] A. Roth, and M. Sotomayor. *Two-sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Cambridge University Press, Cambridge, England. [Econometrica Society Monographs No. 18] (1990).

[22] M. A. Satterthwaite, and H. Sonnenschein. "Strategy-Proof Allocation Mechanisms at Diferentiable Points". *Review of Economic Studies* **48**: 587-597 (1981).

[23] S. Serizawa. "Strategy-Proof and Symmetric Social Choice Functions for Public Good Economies". *Econometrica* **67** (1): 121-145 (1999).

[24] L. Shapley, and H. Scarf. "On Cores and Indivisibilities". *Journal of Mathematical Economics* **1**: 23-37 (1974).

[25] Y. Sprumont. "The division problem with single-peaked preferences: A characterization of the uniform allocation rule". *Econometrica* **59:** 509-519 (1991).

[26] W. Thomson. "Non-bossiness". *Mimeo.* University of Rochester (2014).

[27] W. Vickrey. "Counterspeculation, Auctions, and Competitive Sealed Tenders". *Journal of Finance* **16**: 8-37 (1961).