

Financially Constrained Arbitrage and Cross-Market Contagion ^{*}

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Abstract

We propose a discrete time infinite horizon equilibrium model of financial markets in which arbitrageurs have valuable investment opportunities but face financial constraints. The investment opportunities, varying in horizon and volatility, are provided by pairs of similar assets trading at different prices in segmented markets. By exploiting these opportunities, arbitrageurs reduce the segmentation of markets, providing liquidity to other investors. The financial constraints arise from the arbitrageurs' need to collateralize separately their positions in each asset. We characterize the optimal investment policy of arbitrageurs and derive implications for asset prices.

Keywords: Financial constraints, arbitrage, liquidity, contagion.

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1. Introduction

Financially Constrained Arbitrage

- Arbitrageurs:
 - Hedge funds, dealers, I-banks,...
 - Exploit price wedge between similar (portfolios of) assets.
 - Bring asset prices close to fundamentals.
 - Provide liquidity to other investors.
- Textbook:
 - Costless arbitrage.
 - \Rightarrow Absence of Arbitrage Opportunities.
- Reality:
 - Margin requirements + Access to external capital costly and limited.
 - If/when this is first order, there should be consequences for asset prices and liquidity.

The 1998 Financial Crisis

- Arbitrage.
 - Many hedge funds bet on the convergence of prices of similar-payoff assets.
 - During the crisis, prices diverged.
 - Hedge funds experienced heavy losses + distress \Rightarrow Liquidated positions deemed profitable.
- Asset prices and liquidity.
 - Prices were pushed away from fundamentals.
 - Liquidity dried up.
 - Contagion: Losses-liquidations chain propagated, affecting prices and liquidity in other markets.
- Public policy.
 - Liquidation of LTCM's large positions could disrupt financial markets.
 - Concerns about systemic risk \Rightarrow US Fed coordinated LTCM's rescue.

Some Questions

- Arbitrageurs' strategy
 - What is the optimal investment strategy of an arbitrageur with financial constraints?
 - Financial constraints create a need for risk management.
 - How is this resolved when there are multiple arbitrage opportunities with different characteristics?
 - How does an arbitrageur's optimal policy vary with his wealth? Asset characteristics?
- Asset prices and liquidity
 - Financial constraints \Rightarrow Wealth effects \Rightarrow Price and liquidity linkage across markets.
 - Which convergence spreads are more sensitive to changes in arbitrageurs' capital?
 - How much of time-variation in convergence spreads is explained by contagion vs. fundamentals?
 - Is diversification of arbitrageurs effective despite contagion effects?
- Welfare and public policy
 - Do contagion and liquidity linkages across markets have welfare implications?
 - What are possible policy responses? Capital adequacy for arbitrageurs? Margin requirements?

Main Contributions (1)

- Model of financial markets in which some investors (arbitrageurs) face:
 - Better investment opportunities than others: Market segmentation.
 - Financial constraints: Margin constraints.
- Fairly tractable and flexible framework (e.g. closed for solutions):
 - Infinite horizon + Stationarity.
 - Model intertemporal aspects of arbitrage.
 - Multiple investment opportunities with different characteristics: volatility, horizon, market size...
 - Arbitrage: Riskfree or risky (fundamental risk and/or preference risk).

Main Contributions (2)

- Riskfree arbitrage case (for now).
- Equilibrium solved in closed form with intuitive properties:
 - Equalization of one-period excess returns per unit of collateral across arbitrage opportunities.
 - Risk premia are larger for arbitrage opportunities with longer time to convergence.
 - The effect of maturity is greater for arbitrage opportunities with more volatility.
- Effect of fundamental shocks:
 - Effect on arbitrageurs' wealth and on risk premia.
 - Effect is stronger for arbitrage opportunities with longer time to convergence.
- Effect of integration of (arbitrage) markets:
 - Integration \Rightarrow Contagion, i.e., correlation between otherwise unrelated assets.
 - Lowers the variance of (global) arbitrageurs' portfolio.

Related Literature (Incomplete)

- Shleifer and Vishny (1997)
- Basak and Croitoru (2000,2006)
- Xiong (2001), Kyle and Xiong (2001)
- Gromb and Vayanos (2002)
- Liu and Longstaff (2004)
- Pavlova and Rigobon (2005)
- Gorton and He (2006)
- Krishnamurthy and He (2007)
- Brunnermeier and Pedersen (2007)
- Kondor (2007a, 2007b)
- Anshuman and Viswanathan (2007)
- Hendershott, Moulton and Seasholes (2007)

The paper proceeds as follows. Section 2 presents the model. For the case of riskfree arbitrage, Section 3 derives general results, Section 4 studies the steady state equilibrium and Section 5 studies the effect of small one-off shocks. Section 6 (to be written) sets up the problem of risky arbitrage. Section 7 concludes. The Appendix contains mathematical proofs.

2. The Model

The model has an infinite number of discrete time periods indexed by $t \in \mathbb{Z}$. Section 2.1 describes the universe of assets which consists of a riskless asset and many risky assets. These assets can be traded by two classes of agents: outside investors and arbitrageurs, described in Section 2.2 and Section 2.3 respectively. On the one hand, arbitrageurs have better investment opportunities than outside investors. Indeed, while all agents can invest in the riskless asset, only arbitrageurs can invest in all risky assets, while each outside investor can invest in only one specific risky asset. On the other hand, arbitrageurs face financial constraints in that they must open and collateralize margin accounts asset by asset.

2.1. Assets

The riskless asset has an exogenous return r .

In each period $t \in \mathbb{Z}$, the set of risky assets is \mathcal{I} . Asset $i \in \mathcal{I}$ is in zero net supply and pays a random stream of dividends from period $h(i) - M$ until period $h(i)$, where $h(i)$ is the asset's horizon and $M > 0$ the exogenous (time to) maturity common to all the assets.¹ For any period $t \in \{h(i) - M, \dots, h(i)\}$, asset i 's dividend is:

$$\delta_{i,t} \equiv \bar{\delta} + \epsilon_{i,t}, \quad (1)$$

where $\bar{\delta}$ is a constant, and $\epsilon_{i,t}$ is a zero-mean random variable revealed in period t , and symmetrically distributed around zero over the finite support $[-\bar{\epsilon}_i, +\bar{\epsilon}_i]$.² More specifically, we assume that the variables $(\epsilon_{i,t}/\bar{\epsilon}_i)$ are identically and symmetrically distributed over $[-1, +1]$.

Assets come in pairs. For any asset $i \in \mathcal{I}$, we denote $-i$ the other asset in i 's pair and assume $h(-i) = h(i)$. Moreover, $\epsilon_{i,t}$ and $\epsilon_{i',t'}$ are independently distributed unless $i' = -i$ and $t = t'$. Instead, $\epsilon_{i,t}$ and $\epsilon_{-i,t}$ are assumed to be identically distributed ($\bar{\epsilon}_i = \bar{\epsilon}_{-i}$) and correlated. This specification nests the case in which the dividends paid by assets i and $-i$ are correlated but different ($\epsilon_{i,t} \neq \epsilon_{-i,t}$) and that in which they are identical ($\epsilon_{i,t} = \epsilon_{-i,t}$). These cases will correspond to arbitrage opportunities with and without fundamental risk respectively.

We denote by $p_{i,t}$ the ex-dividend price of asset i in period t . Since asset i pays no dividends after period h , we have $p_{i,t} = 0$ for all $t \geq h(i)$.

2.2. Outside Investors

2.2.1. Market Segmentation

For the outside investors, the markets for all risky assets are segmented. Specifically, each outside investor can form portfolios of only two assets: the riskless asset and a single and specific

¹The assumptions of exogenous riskless return, and zero net supply assets are for simplicity. In particular, the zero net supply assumption ensures that arbitrageurs hold opposite positions in the two risky assets. The assumption that all asset pairs come into existence with the same maturity M is only but a simple way of ensuring that in each period assets with a variety of maturities exist. It will also ensure the model's stationarity.

²The bounded support assumption plays a role for the financial constraint (see below).

risky asset. The outside investors who can invest in risky asset i maturing in period $h(i)$ form overlapping two-period generations, living in periods t and $t + 1$ for $t \in \{h(i) - M, \dots, h(i) - 1\}$. We refer to the generation living in periods t and $t + 1$ as (i, t) -investors.³

Market segmentation is taken as given, i.e., (i, t) -investors are assumed to face prohibitively large transaction costs for investing in any other risky asset than asset i . These costs can be due to physical factors (e.g., distance), information asymmetries or institutional constraints. Market segmentation is a realistic assumption in many contexts. In an international context, for example, it is well known that investors mainly hold domestic assets.

2.2.2. Endowment Shocks

We assume that (i, t) -investors are competitive and form a continuum with measure μ_i . They have initial wealth $w_{i,t}$ in period t and maximize the expected utility of wealth in period $t + 1$, $w_{i,t+1}$, which we assume to be exponential utility, i.e.,

$$-\exp(-\alpha w_{i,t+1}) \quad \text{with } \alpha > 0. \quad (2)$$

In period t , each (i, t) -investor anticipates that in period $t + 1$ he will receive an endowment correlated with the dividend $\delta_{i,t+1}$ paid by asset i . Specifically, the endowment is equal to

$$u_{i,t} \cdot \epsilon_{i,t+1}. \quad (3)$$

where $u_{i,t}$ is a coefficient revealed in period t .

The coefficient $u_{i,t}$, which can be positive or negative, measures the extent to which the endowment covaries with $\delta_{i,t+1}$. If $u_{i,t}$ is large and positive, the shock and the dividend are highly positively correlated, and thus the willingness of (i, t) -investors to hold asset i in period t is low. Conversely, if $u_{i,t}$ is large and negative, the shock and the dividend are highly negatively correlated, and thus the (i, t) -investors are keen to hold asset i in period t for insurance purposes. Therefore, we refer to $u_{i,t}$ as (i, t) -investors' supply shock in period t , to emphasize that their demand for asset i in period t decreases with $u_{i,t}$.⁴

This specification nests the case in which $u_{i,t}$ is effectively stochastic and that in which it is deterministic, which corresponds to arbitrage with and without preference risk respectively.

We assume that (i, t) -investors and $(-i, t)$ -investors are identical (i.e., $\mu_i = \mu_{-i}$)⁵ but for the fact that they incur opposite supply shocks, i.e.,

$$\forall i \in \mathcal{I}, \quad u_{i,t} = -u_{-i,t}. \quad (4)$$

Because (i, t) -investors and $(-i, t)$ -investors incur different shocks, they have different propensities to hold risky assets i and $-i$ respectively. However, they cannot realize the potential gains from trade due to market segmentation. This creates a role for arbitrageurs.

³The assumption that outside investors form overlapping generations is practical but not crucial.

⁴To be consistent with the zero net supply assumption, the endowments can be interpreted as positions in a different but correlated asset, e.g. labor income. This specification of endowments is quite standard in the market microstructure literature (see O'Hara, 1995).

⁵For shocks to be the only difference between (i, t) -investors and $(-i, t)$ -investors, one should also assume $w_{i,t} = w_{-i,t}$. However, exponential utilities make this assumption useless for the analysis.

2.2.3. The Outside Investors' Problem

Each (i, t) -investor chooses $y_{i,t}$, her holding of asset i in period t , to maximize their expected utility of period $t + 1$ wealth. Their optimization problem, $\mathcal{P}_{i,t}$, is

$$\max_{y_{i,t}} -E_t \exp(-\alpha w_{i,t+1}), \quad (5)$$

subject to the dynamic budget constraint. In period t , each (i, t) -investor invests $y_{i,t}p_{i,t}$ in asset i and the rest of his wealth, $(w_{i,t} - y_{i,t}p_{i,t})$, in the riskfree asset. By period $t + 1$, the first position is worth $y_{i,t}(\delta_{i,t+1} + p_{i,t+1})$ while the second has grown to $(1 + r)(w_{i,t} - y_{i,t}p_{i,t})$. Moreover, the investor has received an endowment $u_{i,t}\epsilon_{i,t+1}$. Therefore, the (i, t) -investors' dynamic budget constraint can be written as follows

$$w_{i,t+1} = y_{i,t}(\delta_{i,t+1} + p_{i,t+1}) + (1 + r)(w_{i,t} - y_{i,t}p_{i,t}) + u_{i,t}\epsilon_{i,t+1} \quad (6)$$

$$= (1 + r)w_{i,t} + y_{i,t}\Phi_{i,t} + (y_{i,t} + u_{i,t})\epsilon_{i,t+1}. \quad (7)$$

where

$$\Phi_{i,t} \equiv (\bar{\delta} + p_{i,t+1}) - (1 + r)p_{i,t} \quad (8)$$

is the gain (i, t) -investors would realize between periods t and $t + 1$ over and above the riskfree return if dividends were certain (i.e., if $\bar{\epsilon}_i = 0$). This gain can be risky due to the possible uncertainty of price $p_{i,t+1}$. The term $(y_{i,t} + u_{i,t})\epsilon_{i,t+1}$ represents the risk borne by the (i, t) -investors between periods t and $t + 1$ due to the uncertainty of the dividend $\delta_{i,t+1}$. It is the sum of the uncertain component of the dividend and of the uncertain period $t + 1$ endowment.

2.3. Arbitrageurs

2.3.1. Utility

Arbitrageurs are infinitely lived, i.e., they live in all periods $t \in \mathbb{Z}$. They are competitive and form a continuum with measure 1. We denote their wealth in period t as W_t .⁶

We assume that in any period t , arbitrageurs consume a fraction g of their wealth, with $(1 - g) < \frac{1}{1+r}$ so that arbitrageurs' total wealth remains bounded. The arbitrageurs' utility is

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} U(g \cdot W_s), \quad (9)$$

where U is a well-behaved utility function and $\beta < 1$ a discount factor. When utility is logarithmic, the arbitrageurs' assumed linear consumption rule is optimal.

⁶By fixing the measure of the arbitrageurs, we rule out entry into the arbitrage industry. This seems a reasonable assumption at least for understanding short-run market behavior. However, an alternative interpretation of the maturity M during which an asset pair pays dividends is the length of time it takes for enough new arbitrageurs or new arbitrage capital to enter and eliminate the arbitrage opportunity.

2.3.2. Financial Constraints

Unlike other investors, arbitrageurs can invest in all risky assets as well as in the riskless asset. Therefore, an arbitrageur's dynamic budget constraint can be written as

$$W_{t+1} = (1 - g) \left[(1 + r) \left(W_t - \sum_{i \in \mathcal{I}} x_{i,t} p_{i,t} \right) + \sum_{i \in \mathcal{I}} x_{i,t} (\delta_{i,t+1} + p_{i,t+1}) \right], \quad (10)$$

where the first term inside the bracket is the payoff from the riskless asset and the second term the payoff from the risky assets. This expression can be rewritten as

$$W_{t+1} = (1 - g) \left[(1 + r)W_t + \sum_{i \in \mathcal{I}} x_{i,t} (\epsilon_{i,t+1} + \Phi_{i,t}) \right]. \quad (11)$$

Arbitrageurs face financial constraints: They must hold separate and fully collateralized margin accounts for each risky asset they invest in.^{7,8} The margin account for a given asset consists of a position in that asset and of riskless collateral, but cross-margining is ruled out, i.e., positions in other accounts cannot be used as collateral. Suppose, for example, that an arbitrageur wants to borrow to long a share of asset i . He must post in his i -account both the share of asset i and some additional collateral in the form of a position in the riskfree asset. Similarly, if an arbitrageur shorts asset i , he must post as collateral in his i -account both the sale's cash proceeds and some additional cash to cover the cost of buying asset i next period. In both cases, he cannot deposit any other risky asset than asset i in the i -account.⁹

Second, the margin account must be fully collateralized, i.e., the arbitrageur must post as extra collateral a position in the riskless asset sufficient to guarantee his counterparty the riskfree rate of return. The full collateralization assumption ensures that arbitrageurs never default. This allows us to avoid modeling the margin accounts' custodians. (For example, they can be (i, t) -investors.) It is because of this assumption that we consider dividends with bounded supports. With an unbounded support, the maximum loss of a position would be infinite any borrowing would require arbitrageurs to post an infinite amount of collateral. Therefore, arbitrageurs would not have access to outside finance: They would be able to long assets only to the extent that they can pay with their own wealth and they would not be able to short assets at all.¹⁰

⁷In one sense, our financial constraint is endogenous in that it depends on the properties of the price process. The notion that margin requirements are endogenously chosen to prevent default has appeared in the general equilibrium literature (see, e.g., Geanakoplos, 2003).

⁸In the model, W_t is the arbitrageurs' own wealth. An alternative interpretation is that W_t is the pool of capital from which the arbitrageurs can raise freely, i.e., without financial friction.

⁹The no cross-margining assumption is related to that of market segmentation. Indeed, the same frictions that prevent (i, t) -investors from investing in any other risky asset can also prevent the custodians of arbitrageurs' i -accounts from accepting any other risky asset as collateral. These custodians can be financial exchanges in the case of futures contracts, or brokers/dealers in that of stocks or bonds. The no cross-margining assumption is quite realistic in both cases. For example, futures exchanges generally accept as collateral only positions in contracts traded within the exchange, and dealers generally accept only positions in assets they are dealing in. In practice, arbitrageurs sometimes avoid cross-margining to avoid front running by single counterparties to whom their trades would reveal information (Ko, 2007).

¹⁰We do not consider financial constraints for outside investors. That is, we assume that either these investors do not face constraints or that their constraints are not binding. The latter situation will arise if the outside investors' initial wealth is large enough. Indeed, with exponential utility, optimal holdings of the risky asset are

Denoting by $x_{i,t}$ an arbitrageur's position in asset i in period t , the arbitrageur must add as extra collateral a position $z_{i,t}$ in the riskless asset that by period $t+1$ will exceed the maximum shortfall relative to the the riskfree rate of return:

$$(1+r)x_{i,t}p_{i,t} \leq \min_{p_{i,t+1}, \delta_{i,t+1}} \{x_{i,t}(p_{i,t+1} + \delta_{i,t+1})\} + (1+r)z_{i,t}, \quad (12)$$

which can be rewritten as

$$\frac{1}{(1+r)} \max_{\epsilon_{i,t+1}, \Phi_{i,t}} \{x_{i,t}(\epsilon_{i,t+1} - \Phi_{i,t})\} \leq z_{i,t}. \quad (13)$$

Since the total collateral an arbitrageur can post in all positions is bounded by his wealth, the positions an arbitrageur can take are restricted by his wealth as follows:

$$\frac{1}{(1+r)} \sum_{i \in \mathcal{I}} \max_{\epsilon_{i,t+1}, \Phi_{i,t}} \{x_{i,t}(\epsilon_{i,t+1} - \Phi_{i,t})\} \leq W_t. \quad (14)$$

2.3.3. The Arbitrageurs' Problem

The arbitrageurs' optimization problem in period t , \mathcal{P}_t , is

$$\max_{x_{i,s}} E_t \sum_{s=t}^{\infty} \beta^{s-t} U(g \cdot W_s), \quad (15)$$

subject to the dynamic budget constraint (11) and the financial constraint (14).

2.4. Equilibrium

Definition 1 *A competitive equilibrium consists of prices $p_{i,t}$, asset holdings of the (i, t) -investors, $y_{i,t}$, and of the arbitrageurs, $x_{i,t}$, such that*

- *given the prices, $y_{i,t}$ solves problem $\mathcal{P}_{i,t}$, and $\{x_{i,s}\}_{s \geq t}$ solve problem \mathcal{P}_t ,*
- *and the markets for all risky assets clear:*

$$\mu_i y_{i,t} + x_{i,t} = 0. \quad (16)$$

Because of our model's symmetry, we can show the existence of a competitive equilibrium that is symmetric in the following sense.

Definition 2 *A competitive equilibrium is symmetric if for any two assets $(i, -i)$ in a pair*

independent of wealth, and so are capital gains. Moreover, since asset payoffs and supply shocks have bounded support, capital gains are also bounded. Therefore, for large enough initial wealth, capital losses are always smaller than wealth, and the financial constraint is not binding. Note that the initial wealth of the outside investors need not exceed that of the arbitrageurs. Indeed, if the measures μ_i of the outside investors are large enough, the arbitrageurs' positions are much larger than those of the outside investors, and thus require more collateral.

- the one-period excess returns are opposites, i.e.,

$$\Phi_{-i,t} = -\Phi_{i,t}, \quad (17)$$

- the arbitrageurs' positions in the two assets are opposites, i.e.,

$$x_{-i,t} = -x_{i,t}, \quad (18)$$

- and so are the positions of the outside investors, i.e.,

$$y_{-i,t} = -y_{i,t}. \quad (19)$$

Intuitively, one-period excess returns are opposites because assets are in zero net supply and the supply shocks of the (i, t) -investors and $(-i, t)$ -investors are opposites. The arbitrageurs' positions are opposites because the risk premia are opposites. Note that arbitrageurs act as intermediaries. Suppose, for example, that (i, t) -investors receive a positive supply shock, in which case $(-i, t)$ -investors receive a negative shock. Then arbitrageurs buy asset i from the (i, t) -investors, who are willing to sell, and sell asset $-i$ to the $(-i, t)$ -investors, who are willing to buy. Through this transaction arbitrageurs realize a profit, while at the same time providing liquidity to the other investors. Finally, the positions of the outside investors are opposites because the arbitrageurs' positions are opposites, and markets must clear.

Note that by symmetry, the midpoint between prices $p_{i,t}$ and $p_{-i,t}$ is the value they would both take absent dividend risk, i.e.,

$$\frac{p_{-i,t} + p_{i,t}}{2} = \sum_{s=t+1}^{h(i)} \frac{\bar{\delta}}{(1+r)^{s-t}} = \frac{\bar{\delta}}{r} \left(1 - \frac{1}{(1+r)^{h(i)-t}} \right). \quad (20)$$

Therefore, the risk premium for asset $i \in \mathcal{I}$ in period $t \in \mathbb{Z}$ defined as

$$\phi_{i,t} \equiv \frac{\bar{\delta}}{r} \left(1 - \frac{1}{(1+r)^{h(i)-t}} \right) - p_{i,t} \quad (21)$$

is also one-half of the price wedge between assets $-i$ and i , i.e.,

$$\phi_{i,t} = \frac{p_{-i,t} - p_{i,t}}{2}. \quad (22)$$

Since assets i and $-i$ pay no dividends after period $h(i)$, we have $\phi_{i,t} = 0$ for all $t \geq h(i)$.

3. Riskless Arbitrage: General Results

We start with the case in which there is no fundamental risk and no preference risk. The assumption of no fundamental risk means that the two assets i and $-i$ of any given pair pay identical dividends, i.e., $\delta_{i,t} = \delta_{-i,t}$ so that $\epsilon_{i,t} = \epsilon_{-i,t}$. The assumption of no preference risk means that $u_{i,t}$ is deterministic. For simplicity, we assume that $u_{i,t}$ is constant over time, i.e., $u_{i,t} \equiv u_i$. Finally, we define the subset $\mathcal{A} \equiv \{i \in \mathcal{I}, u_i > 0\}$.

In the riskless arbitrage case, we can show the existence of a symmetric equilibrium in which risk premia $\phi_{i,t}$, outside investors' positions $y_{i,t}$, and arbitrageurs' positions $x_{i,t}$ and total wealth

W_t are deterministic. The reason is as follows. Since there is no fundamental risk and the arbitrageurs' positions in assets i and $-i$ are opposites, the arbitrageurs' wealth W_t does not depend on the dividend $\delta_{i,t}$. Therefore, $\phi_{i,t}$ and $x_{i,t}$ are also independent of the dividend. This means that $\phi_{i,t}$, $x_{i,t}$ and W_t can be stochastic only because of the supply shocks, but these are deterministic in the riskless arbitrage case.¹¹

Sections 3.1-3.3 characterize properties of a symmetric equilibrium in which $\phi_{i,t}$, $y_{i,t}$, $x_{i,t}$ and W_t are deterministic. A proof that such an equilibrium exists is in Section 3.3.

3.1. Outside Investors

Before studying the (i, t) -investors' problem, we begin with a technical result.

Lemma 1 *The function f defined by $E_t \left[\exp\left(-\frac{\alpha y \epsilon_i}{\bar{\epsilon}_i}\right) \right] \equiv \exp[\alpha f(y)]$ is positive and strictly convex. Moreover, it satisfies $f(y) = f(-y)$, $f'(y) \in (-1, 1)$ and $\lim_{y \rightarrow \infty} f'(y) = 1$.*

Notice that f is indeed the same for all variables $\epsilon_{i,t}$ because of our assumption that the variables $(\epsilon_{i,t}/\bar{\epsilon}_i)$ are identically distributed.

We can now study the optimization problem $\mathcal{P}_{i,t}$ of the (i, t) -investors for any asset $i \in \mathcal{A}$. Equation (7), the budget constraint of the (i, t) -investors, can be rewritten as

$$w_{i,t+1} = (1+r)w_{i,t} + y_{i,t}\Phi_{i,t} + (y_{i,t} + u_i)\bar{\epsilon}_i \left(\frac{\epsilon_{i,t+1}}{\bar{\epsilon}_i} \right). \quad (23)$$

The gain $\Phi_{i,t}$ being deterministic, using this expression and the definition of f , problem $\mathcal{P}_{i,t}$ becomes

$$\max_{y_{i,t}} -E_t \exp[-\alpha((1+r)w_{i,t} + y_{i,t}\Phi_{i,t} - f((y_{i,t} + u_i)\bar{\epsilon}_i))], \quad (24)$$

or, since $w_{i,t}$ is a constant as of period t , to

$$\max_{y_{i,t}} y_{i,t}\Phi_{i,t} - f((y_{i,t} + u_i)\bar{\epsilon}_i). \quad (25)$$

The first term is the gain over and above the riskless rate of return the investor would realize between periods t and $t+1$ if dividends were certain. This term is certain. The second term, $f((y_{i,t} + u_i)\bar{\epsilon}_i)$, is the cost of bearing risk between periods t and $t+1$ due to the uncertainty of dividends and of the supply shock. This inventory cost increases with the position in the risky asset, $y_{i,t}$, with the magnitude of the supply shock, u_i , and with the volatility of dividends, $\bar{\epsilon}_i$. At the optimum, the marginal inventory cost must equal the expected capital gain per unit of risky asset, i.e., the first-order condition is

$$\bar{\epsilon}_i f'((y_{i,t} + u_i)\bar{\epsilon}_i) = \Phi_{i,t}. \quad (26)$$

Therefore, the (i, t) -investors' demand for asset i is

$$\mu_i y_{i,t} = -\mu_i \left(u_i - \frac{1}{\bar{\epsilon}_i} (f')^{-1} \left(\frac{\Phi_{i,t}}{\bar{\epsilon}_i} \right) \right). \quad (27)$$

¹¹Note that the asset prices $p_{i,t}$ are also deterministic. This property is less robust as it depends on the news in period t being only about period t dividends.

By symmetry, the optimization problem of the $(-i, t)$ -investors yields the same first order condition as that of the (i, t) -investors. Indeed, the risk premia, the supply shocks, and investors' positions for assets i and $-i$ are opposites, and $f'(y) = -f'(-y)$.

3.2. Arbitrageurs

Next we study the arbitrageurs' optimization problem \mathcal{P} . By symmetry, the arbitrageurs' dynamic budget constraint in period t , equation (11), can be written as

$$W_{t+1} = (1 - g) \left[(1 + r)W_t + 2 \sum_{i \in \mathcal{A}} x_{i,t} \Phi_{i,t} \right]. \quad (28)$$

The second term within the brackets is the sum of the arbitrageurs' gains between periods t and $t + 1$ over and above the riskfree rate of return. Because the arbitrageurs' positions in assets i and $-i$ are opposites, they effectively eliminate dividend risk. Therefore, these gains are an excess return between periods t and $t + 1$.

We have assumed $u_i > 0$ for all $i \in \mathcal{A}$. Therefore, arbitrageurs long asset i , i.e., $x_i \geq 0$, and asset i yields a positive one-period excess return, i.e., $\Phi_{i,t} \geq 0$. The arbitrageurs' financial constraint (14) takes the form

$$\frac{2}{(1 + r)} \sum_{i \in \mathcal{A}} x_{i,t} (\bar{\epsilon}_i - \Phi_{i,t}) \leq W_t. \quad (29)$$

Proposition 1 *All else equal, the arbitrageurs' financial constraint becomes*

- *more severe when the arbitrageurs' wealth W_t decreases,*
- *more severe when the volatility of any asset $\bar{\epsilon}_i$ increases,*
- *and less severe when the one-period excess return $\Phi_{i,t}$ of any asset increases.*

The intuition is as follows. First, with less wealth the arbitrageurs can post less collateral, and therefore invest less. Second, the maximum loss a position in an asset can experience increases with that asset's dividend volatility. Hence arbitrageurs must post more collateral to take a position, which ties up more capital. Finally, the higher the one-period excess return $\Phi_{i,t}$ of asset i , the lower the maximum loss of a position in asset i can experience. Hence the arbitrageurs must post less collateral, which frees up capital. The latter effect means that all else equal, outside finance is easier to raise when arbitrage opportunities are more profitable.

Next we derive each arbitrageur's optimal investment policy.

Proposition 2 *In any given period t , an arbitrageur's optimal investment policy is as follows.*

- *If for all $i \in \mathcal{I}$, $\Phi_{i,t} = 0$, the arbitrageurs are indifferent between all investment policies.*

- Otherwise, each arbitrageur's optimal strategy is to invest only in arbitrage opportunities

$$(i, -i) \text{ with } i \in \arg \max_{j \in \mathcal{A}} \left(\frac{\Phi_{j,t}}{\bar{\epsilon}_j} \right) \quad (30)$$

up until his financial constraint (29) binds.

The intuition is as follows. From expression (28), each arbitrageur's optimization problem in any given period is to maximize the sum of one-period excess returns subject to the financial constraint (29). The solution is simple. If these excess returns are zero for all opportunities, the arbitrageur is indifferent between all feasible investment policies. If however some pairs give strictly positive one-period excess returns, the arbitrageur should focus on those asset pairs giving the highest excess returns per unit of collateral, and invest up to the financial constraint in any subset of these pairs. Establishing the "return on collateral" per each asset pair is also simple. Indeed, the asset pair $(i, -i)$ yields a one-period excess return $\Phi_{i,t}$ per leg of the spread trade but requires the arbitrageur to post some additional collateral in the form of a position in the riskfree asset. This position in period t must grow in period $t+1$ to cover the maximum loss per leg of the position. This maximum loss being $(\bar{\epsilon}_i - \Phi_{i,t})$, he needs to post $(\bar{\epsilon}_i - \Phi_{i,t}) / (1+r)$ of additional collateral per leg. In other words, investing one dollar of collateral in arbitrage opportunity $(i, -i)$ yield a "return on collateral" of

$$(1+r) + (1+r) \times \frac{\Phi_{i,t}}{\bar{\epsilon}_i - \Phi_{i,t}}. \quad (31)$$

This can be rewritten as

$$\frac{(1+r)}{\left(1 - \frac{\Phi_{i,t}}{\bar{\epsilon}_i}\right)}, \quad (32)$$

which increases with $(\Phi_{i,t}/\bar{\epsilon}_i)$. Finally, since at least some of the arbitrage opportunities offer arbitrageurs a certain return that strictly exceeds the riskfree rate, the arbitrageur should invest as much as possible, i.e., "max out" his financial constraint.

3.3. Equilibrium

Proposition 3 *For any given period t , there exists $\Pi_t \in [0, 1)$ such that arbitrageurs are only invested in opportunities $(i, -i)$ satisfying $f'(u_i \bar{\epsilon}_i) > \Pi_t$. Moreover,*

- All opportunities in which arbitrageurs invest offer the same one-period excess return per unit of volatility, equal to Π_t , i.e.,

$$\frac{\Phi_{i,t}}{\bar{\epsilon}_i} = \Pi_t, \quad (33)$$

while opportunities in which they do not invest offer lower returns.

- When arbitrageurs' positions are non-zero, they are given by

$$x_{i,t} = \mu_i \left(u_i - \frac{(f')^{-1}(\Pi_t)}{\bar{\epsilon}_i} \right). \quad (34)$$

That all opportunities in which arbitrageurs invest offer the same one-period excess return per unit of volatility is a consequence of equilibrium: if returns differed, arbitrageurs would focus only on those opportunities with the highest returns (Proposition 2), a contradiction. Arbitrageurs do not invest in opportunities with $f'(u_i \bar{\epsilon}_i) \leq \Pi_t$ since investing in those opportunities would push their return below Π_t .

Proposition 3 has a number of implications for the cross-section of profitability and premia associated with different arbitrage opportunities.

Corollary 1 *In any given period t , holding volatility constant, arbitrage opportunities with a longer maturity have higher risk premia, i.e., for all $(i, j) \in \mathcal{A}^2$*

$$\bar{\epsilon}_i = \bar{\epsilon}_j \quad \text{and} \quad h(i) > h(j) \quad \Rightarrow \quad \phi_{i,t} > \phi_{j,t} > 0. \quad (35)$$

The intuition is as follows. From Proposition 3, in equilibrium, in any given period, arbitrage opportunities with the same dividend volatility must offer arbitrageurs the same one-period excess return, i.e.,

$$\Phi_{i,t} = \bar{\epsilon}_i \Pi_t. \quad (36)$$

Moreover, risk premia equal the present value of future one-period excess returns, i.e.,

$$\phi_{i,t} = \sum_{s=t}^{h(i)-1} \frac{\Phi_{i,s}}{(1+r)^{s+1-t}} = \bar{\epsilon}_i \sum_{s=t}^{h(i)-1} \frac{\Pi_s}{(1+r)^{s+1-t}}. \quad (37)$$

Therefore

$$\phi_{i,t} - \phi_{j,t} = \bar{\epsilon}_i \sum_{s=h(j)}^{h(i)-1} \frac{\Pi_h}{(1+r)^{h+1-t}} > 0.$$

Corollary 2 *In any given period t , holding horizon constant, arbitrage opportunities with higher volatility offer higher one-period excess returns, i.e., for all $(i, j) \in \mathcal{A}^2$*

$$h(i) = h(j) \quad \text{and} \quad \bar{\epsilon}_i > \bar{\epsilon}_j \quad \Rightarrow \quad \Phi_{i,t} > \Phi_{j,t}. \quad (38)$$

and have higher risk premia, i.e., for all $(i, j) \in \mathcal{A}^2$

$$h(i) = h(j) \quad \text{and} \quad \bar{\epsilon}_i > \bar{\epsilon}_j \quad \Rightarrow \quad \phi_{i,t} > \phi_{j,t}. \quad (39)$$

Moreover, the effect of volatility on risk premia is larger for opportunities with longer maturity, i.e., for all (i, i) and (i', j') as above with $h(i) > h(i')$, $\bar{\epsilon}_i = \bar{\epsilon}_{i'}$ and $\bar{\epsilon}_j = \bar{\epsilon}_{j'}$,

$$\phi_{i,t} - \phi_{i',t} > \phi_{j,t} - \phi_{j',t}. \quad (40)$$

The intuition is that with higher dividend volatility, more collateral is needed for a given position in the arbitrage opportunity. Therefore, in equilibrium, arbitrage opportunities involving assets with more volatile dividends must offer a greater reward to arbitrageurs, i.e., a higher

one-period excess return. Since risk premia are equal to the present value of future one-period excess returns, i.e.,

$$\phi_{i,t} = \bar{\epsilon}_i \sum_{s=t}^{h(i)-1} \frac{\Pi_s}{(1+r)^{s-t+1}}, \quad (41)$$

they are higher for opportunities with higher dividend volatility $\bar{\epsilon}_i$. Moreover, since the present value increases with maturity, it is more sensitive to volatility for opportunities with longer maturity.

Corollary 3 *In equilibrium in any given period t and for any asset $i \in \mathcal{A}$, the slope of the (i, t) -investors' demand decreases with dividend volatility, i.e.,*

$$\frac{\partial (\mu_i y_{i,t})}{\partial \Phi_{i,t}} = \frac{\mu_i}{\bar{\epsilon}_i^2} \left((f')^{-1} \right)' (\Pi_t)$$

decreases with $\bar{\epsilon}_i$.

The intuition is that when dividend volatility is greater, investors must be compensated more (with a larger gain $\Phi_{i,t}$) to absorb a given quantity of risky asset.

Given Proposition 3, we can rewrite the arbitrageurs' dynamic budget constraint (28) as

$$W_{t+1} = (1-g) \left[(1+r)W_t + 2\Pi_t \sum_{i \in \mathcal{A}} \bar{\epsilon}_i x_{i,t} \right], \quad (42)$$

and their financial constraint (29) as

$$\frac{2(1-\Pi_t)}{(1+r)} \sum_{i \in \mathcal{A}} \bar{\epsilon}_i x_{i,t} \leq W_t. \quad (43)$$

Using equations (42) and (43), as well as the market-clearing equation (16) and outside investors' first-order condition (26), we can fully solve for equilibrium.

Proposition 4 *A symmetric equilibrium exists in which $(\phi_{i,t}, y_{i,t}, x_{i,t}, W_t)$ are deterministic.*

4. Riskless Arbitrage: Steady-State Equilibrium

We now specialize the model to make it stationary and study its steady state equilibrium. We assume that in each period t , N pairs of assets $(i, -i)$ are born and indexed by $n(i) \in \{1, \dots, N\}$. The parameters associated with asset $i \in \mathcal{A}$ only depend on the index of that asset's pair, i.e.,

$$u_i \equiv u_{n(i)}, \quad \mu_i = \mu_{n(i)}, \quad \text{and} \quad \bar{\epsilon}_i \equiv \bar{\epsilon}_{n(i)}. \quad (44)$$

For all periods t and all assets i , we define $m(i, t) \equiv h(i) - t$. In a steady-state equilibrium, risk premia, risky asset positions, and arbitrageurs' wealth are independent of time t , i.e.,

$$\Phi_{i,t} \equiv \Phi_{n(i), m(i,t)}, \quad \phi_{i,t} \equiv \phi_{n(i), m(i,t)}, \quad x_{i,t} \equiv x_{n(i)}, \quad \Pi_t \equiv \Pi, \quad \text{and} \quad W_t \equiv W. \quad (45)$$

We now show that in the riskless arbitrage case a steady-state equilibrium exists. This equilibrium has a simple structure and can be used as a basis for studying non-steady-state dynamics and risky arbitrage.

Equation (34) takes the form

$$x_n = \mu_n \left(u_n - \frac{(f')^{-1}(\Pi)}{\bar{\epsilon}_n} \right). \quad (46)$$

The arbitrageurs' dynamic budget constraint (42) takes the form

$$W = (1 - g) \left[(1 + r)W + 2M\Pi \sum_{n=1}^N \bar{\epsilon}_n x_n \right], \quad (47)$$

and their financial constraint (43) takes the form

$$\frac{2M(1 - \Pi)}{(1 + r)} \sum_{n=1}^N \bar{\epsilon}_n x_n \leq W. \quad (48)$$

Proposition 5 *In a steady state symmetric equilibrium of the riskless arbitrage case, the following hold in any period t .*

- *The arbitrageurs' financial constraint is binding and*

$$\Pi = 1 - (1 - g)(1 + r) > 0. \quad (49)$$

- *The one-period excess returns are*

$$\Phi_{n,m} = \bar{\epsilon}_n \cdot \Pi, \quad (50)$$

and the risk premia are:

$$\phi_{n,m} = \frac{1}{r} \left(1 - \frac{1}{(1 + r)^m} \right) \bar{\epsilon}_n \Pi. \quad (51)$$

- *The arbitrageurs' positions are:*

$$x_n = \mu_n \left(u_n - \frac{(f')^{-1}(\Pi)}{\bar{\epsilon}_n} \right). \quad (52)$$

- *The arbitrageurs' total wealth is:*

$$W = 2(1 - g)M \left[\sum_{n=1}^N \mu_n u_n \bar{\epsilon}_n - \left(\sum_{n=1}^N \mu_n \right) (f')^{-1}(\Pi) \right]. \quad (53)$$

To understand why the return on each spread trade must be strictly positive, suppose instead that $\Pi = 0$. In that case, there are no arbitrage opportunities and arbitrageurs have no means to increase their wealth at a rate exceeding the riskfree rate r . Since the arbitrageurs' wealth is depleted over time at a rate g exceeding r , their wealth must be zero in a steady state. But

if arbitrageurs have no wealth, they cannot invest in the spread trades and therefore cannot eliminate arbitrage opportunities. This contradicts the premise that $\Pi = 0$.

Since $\Pi > 0$ (Proposition 2), the arbitrageurs' financial constraint must be binding, i.e., equation (48) holds with equality. Expression (49), obtained from equations (47) and (48), has a simple intuition. In the steady state, the return on collateral, given by expression (32), is $(1+r)/(1-\Pi)$. Therefore, one dollar of arbitrageur wealth in period t grows to $(1-g)(1+r)/(1-\Pi)$ by period $t+1$, which in steady state must be equal to one. This gives the expression of Π , from which the rest of the proposition's expressions are derived.

This result has several implications for the cross-section of profitability and premia associated with different arbitrage opportunities.

Corollary 4 *An increase in the dividend volatility of one asset pair (i.e., a tightening of the financial constraint for these assets) has the following effects.*

- *For these assets, the one-period excess returns and the risk premia increase, i.e.,*

$$\forall(n, m), \quad \frac{\partial \Phi_{n,m}}{\partial \bar{\epsilon}_n} > 0, \quad \text{and} \quad \frac{\partial \phi_{n,m}}{\partial \bar{\epsilon}_n} > 0, \quad (54)$$

while the arbitrageurs' positions increase, i.e.,

$$\forall n, \quad \frac{\partial x_n}{\partial \bar{\epsilon}_n} > 0. \quad (55)$$

- *The other assets are unaffected, i.e.,*

$$\forall n \neq n', \forall m, \quad \frac{\partial x_{n'}}{\partial \bar{\epsilon}_n} = \frac{\partial \phi_{n',m}}{\partial \bar{\epsilon}_n} = 0. \quad (56)$$

The intuition is as follows. First, arbitrage opportunities with higher dividend volatility must provide greater compensation to arbitrageurs in the form of higher excess returns. These also translate into higher risk premia. Second, greater dividend volatility implies that the outside investors' demand is more inelastic. This in turn means that arbitrageurs can take larger positions without narrowing the price wedge between the assets too much.

The fact that a change in an asset (pair)'s volatility has no effect on the premia and arbitrageurs' holdings of other (pairs of) assets has also a simple but important intuition. First, recall that in the steady state, the overall return on collateral must exactly offset the depletion of the arbitrageurs' wealth. Second, in equilibrium, the return on collateral must be the same for all arbitrage opportunities in which the arbitrageur invests. Therefore, the return on collateral *for each arbitrage opportunity* must exactly offset the depletion of the arbitrageurs' wealth.

Note that this reasoning applies equally to any change in an asset's characteristics: The steady state equilibrium risk premium and arbitrageurs' holdings of a given asset are independent of other assets' characteristics.

Corollary 5 *An increase in the withdrawal rate g :*

- increases the risk premia (i.e., $\frac{\partial \phi_{n,m}}{\partial g} > 0$),
- reduces the asset positions (i.e., $\frac{\partial x_n}{\partial g} < 0$),
- and reduces the steady-state fund size (i.e., $\frac{\partial W}{\partial g} < 0$).

The intuition is as follows. As g increases, the arbitrage trades must be more profitable in equilibrium for arbitrageurs to maintain a constant wealth level. Therefore, Π must increase. This translates directly into an increase of the risk premia. For this to be the case, however, the equilibrium positions taken by the arbitrageurs must decrease (f being convex, $(f')^{-1}$ is increasing). Indeed these are given by Π and the outside investors' demand functions which are unaffected by a change in g .

4.1. Integration vs. Segmentation of Arbitrage Markets

We next use our model to study the effects of arbitrageur diversification. We compare two polar cases: integration, where all arbitrageurs hold the same portfolio, thus being invested in all opportunities, versus segmentation, where each arbitrageur is invested only in opportunities indexed by a specific n .

Corollary 6 *At the steady state equilibrium, all assets' risk premia are the same under segmentation and integration.*

This result is a simple consequence of our earlier remark that in steady state, equilibrium variables (premium and holdings) for one asset pair do not depend on characteristics of other (pairs of) assets. This result simply extends this remark to note that they do not even depend on the existence of other pairs of assets. Note that this result holds true even if arbitrageurs do not invest in certain arbitrage opportunities under integration. Indeed, under segmentation, arbitrageurs specialized in these arbitrage opportunity have zero wealth in steady state (because their return on collateral does not offset the arbitrageurs' wealth depletion) and therefore do not invest in these arbitrage opportunities either.

When arbitrage is riskless, the integration and segmentation cases produce the same outcome: risk premia are identical, and so are arbitrageurs' total wealth and aggregate position in each opportunity. As we will see, however, differences between integration and segmentation arise when arbitrage is risky.

5. Riskless Arbitrage: Unanticipated One-Off Shocks

We next consider the impact of shocks to arbitrageurs' wealth arising because of either fundamental or preference risk. To remain within the confines of the riskless arbitrage case, we assume that these shocks are unanticipated and are not followed by future shocks. Studying the impact of these unanticipated one-off shocks provides insights into the mechanisms operating in the risky arbitrage case.

5.1. Fundamental Shocks

Suppose that we are in the steady state of the riskless arbitrage case, and in a given period t the two assets in the pair $(i, -i)$ do not pay exactly the same dividend. That is, the dividends for assets i and $-i$ are

$$\epsilon_{i,t} + \eta_{i,t} \quad \text{and} \quad \epsilon_{i,t} - \eta_{i,t}, \quad (57)$$

where $\eta_{i,t} \neq 0$ is a small shock. Our goal is to understand how this shock affects the wealth of arbitrageurs and the risk premia.

The arbitrageurs' budget constraint between periods t and $t + 1$ is

$$W_{t+1} = (1 - g) \left[(1 + r)W_t + 2 \sum_{j \in \mathcal{A}} x_{j,t} [(1 + r)\phi_{j,t} - \phi_{j,t+1}] + 2x_{i,t}\eta_{i,t} \right]. \quad (58)$$

Note that the shock $\eta_{i,t}$ does not affect the arbitrageurs' pre-shock wealth W_t , their holdings of the risky assets x_j , and the risk-premia $\phi_{j,t}$, which are all at their steady state values in period t , i.e., $W_t = W$, $x_{j,t} = x_{n(j)}$ and $\phi_{j,t} = \phi_{n(j),h(j)-t}$. However, the shock has both a direct and an indirect effect on the arbitrageurs' post-shock wealth. The direct effect is the change in value of arbitrageurs' positions, holding prices constant: it corresponds to the term $2x_{n(i)}\eta_{i,t}$. The indirect effect is a feedback effect: the change in arbitrageurs' wealth affects prices and this feeds back into wealth. The feedback amplifies the direct effect of the shock. For example, if the direct effect reduces arbitrageur wealth, then arbitrageurs reduce their positions, risk premia increase, and this further reduces wealth. The amplification effect works both through the change in risk premia of the assets $(i, -i)$ generating the shock, and through the change in premia of the other assets. The latter channel represents cross-market contagion.

Differentiating expression (58) with respect to $\eta_{i,t}$, we find

$$\frac{\partial W_{t+1}}{\partial \eta_{i,t}} = 2(1 - g) \left[\sum_{j \in \mathcal{A}} x_{n(j)} \left(-\frac{\partial \phi_{j,t+1}}{\partial \eta_{i,t}} \right) + x_{n(i)} \right] \quad (59)$$

$$= 2(1 - g) \left[\sum_{j \in \mathcal{A}} x_{n(j)} \left(-\frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} \right) \left(\frac{\partial W_{t+1}}{\partial \eta_{i,t}} \right) + x_{n(i)} \right], \quad (60)$$

where the first term inside the square bracket is the indirect effect and the second term is the direct effect. Therefore, the effect of the shock $\eta_{i,t}$ on arbitrageur wealth is

$$\frac{\partial W_{t+1}}{\partial \eta_{i,t}} = \frac{2(1 - g)x_{n(i)}}{1 + 2(1 - g) \sum_{j \in \mathcal{A}} x_{n(j)} \left(\frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} \right)}. \quad (61)$$

To fully compute this expression, we need to evaluate the term in the denominator, i.e., how a change in wealth in period $t + 1$ affects risk premia in that period. Since there are no shocks after period $t + 1$, we can compute the effect of wealth on risk premia using the equations of the riskless arbitrage case, i.e., the market-clearing equation (16), the first-order condition (26) of outside investors, the arbitrageurs' budget constraint (42), and financial constraint (43). These equations determine the effect of wealth on risk premia in closed form. Closed-form solutions can

also be derived for the effect of wealth on one-period excess returns and arbitrageurs' positions. Proposition 6 reports the closed-form solutions, using the notation

$$Z \equiv \frac{1-g}{\left[\frac{\sum_{n'=1}^N \mu_{n'} u_{n'} \bar{\epsilon}_{n'}}{\sum_{n'=1}^N \mu_{n'}} - (f')^{-1}(\Pi) \right] f''(Y) + (1-g)(1+r)}. \quad (62)$$

Note that $Z < 1$.

Proposition 6 *An increase in arbitrageur wealth in period $t+1$, relative to its steady-state value, has the following effects*

- *The one-period excess return per unit of volatility decreases*

$$\frac{\partial \Pi_{t+1}}{\partial W_{t+1}} = -\frac{(1+r)f''((f')^{-1}(\Pi))Z}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} < 0. \quad (63)$$

- *Risk premia decrease*

$$\forall j \in \mathcal{A}, \quad \frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} = -\frac{f''((f')^{-1}(\Pi))}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \frac{Z - Z^{h(j)-t}}{1-Z} \bar{\epsilon}_j < 0. \quad (64)$$

- *Arbitrageurs' positions increase*

$$\forall j \in \mathcal{A}, \quad \frac{\partial x_{j,t+1}}{\partial W_{t+1}} = \frac{(1+r)Z}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \left(\frac{\mu_j}{\bar{\epsilon}_j} \right) > 0. \quad (65)$$

- *Arbitrageurs' wealth in future periods increases*

$$\forall s \geq t+1, \quad \frac{\partial W_s}{\partial W_{t+1}} = [(1+r)Z]^{s-t-1} \quad (66)$$

Intuitively, an increase in wealth relaxes the arbitrageurs' financial constraint, allowing them to invest more aggressively in all arbitrage opportunities. Therefore, arbitrageurs' positions increase, and this reduces one-period excess returns. It also reduces risk premia, since these are equal the present value of future one-period excess returns. Since this present value increases with maturity, the effect of wealth is larger on the risk premia of opportunities with longer maturity. Likewise, since one-period excess returns increase with volatility, the effect of wealth is larger on the risk premia of more volatile opportunities.

Corollary 7 *An increase in arbitrageur wealth in period $t+1$, relative to its steady-state value,*

- *has a stronger (more negative) effect on the risk premia of arbitrage opportunities with longer maturity, i.e.,*

$$\bar{\epsilon}_j = \bar{\epsilon}_{j'} \quad \text{and} \quad h(j) > h(j') \quad \Rightarrow \quad \frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} < \frac{\partial \phi_{j',t+1}}{\partial W_{t+1}} < 0. \quad (67)$$

- has a stronger (more negative) effect on the risk premia of arbitrage opportunities with higher volatility, i.e.,

$$h(j) = h(j') \quad \text{and} \quad \bar{\epsilon}_j > \bar{\epsilon}_{j'} \quad \Rightarrow \quad \frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} < \frac{\partial \phi_{j',t+1}}{\partial W_{t+1}} < 0. \quad (68)$$

Using the closed-form solutions of Proposition 6, we can return to equation (61) and compute the effect of a fundamental shock on arbitrageur wealth and risk premia.

Proposition 7 *Following a small unanticipated one-off fundamental shock in period t ,*

- the arbitrageurs' wealth changes by:

$$\frac{\partial W_{t+1}}{\partial \eta_{i,t}} = \frac{2(1-g)\mu_i \left(u_i - \frac{(f')^{-1}(\Pi)}{\bar{\epsilon}_i} \right)}{1 - \frac{1-g}{M} \left(1 - \frac{rZ}{1-Z} \right) \left(M - \frac{1-Z^M}{1-Z} \right)}, \quad (69)$$

- and the risk premium of asset $j \in \mathcal{A}$ changes by:

$$\frac{\partial \phi_{j,t+1}}{\partial \eta_{i,t}} = - \frac{f''((f')^{-1}(\Pi))\mu_i \left(u_i - \frac{(f')^{-1}(\Pi)}{\bar{\epsilon}_i} \right)}{M \left(\sum_{n'=1}^N \mu_{n'} \right) \left[1 - \frac{1-g}{M} \left(1 - \frac{rZ}{1-Z} \right) \left(M - \frac{1-Z^M}{1-Z} \right) \right]} \frac{Z - Z^{h(j)-t}}{1-Z} \bar{\epsilon}_j. \quad (70)$$

Proposition 7 implies that a negative fundamental shock to one asset pair leads to an increase in the risk premia of all assets. The intuition for this contagion effect is similar to that for the effect of financial constraints in a conglomerate involved in multiple unrelated lines of business. A negative shock to one business tightens the conglomerate's financial constraint and restricts its investment capacity in all lines of business. In the case at hand, investment has a feedback effect on asset prices, and therefore, financial constraints create a link between the prices of otherwise unrelated assets.

5.1.1. Integration vs. Segmentation of Arbitrage Markets

We now return to our study of the effects of arbitrageur diversification. As before, we compare two polar cases: integration, where all arbitrageurs can invest in all opportunities, and segmentation, where each arbitrageur can invest only in opportunities indexed by a specific n .

We have seen that in the riskfree arbitrage case, steady state equilibrium variables do not depend on whether arbitrage markets are segmented (Corollary 6). Differences between integration and segmentation arise when arbitrage is risky. We consider risky arbitrage in Section 6, but analyzing the impact of unanticipated one-off shocks gives a flavor of the results.

When arbitrage is risky, an individual arbitrageur can benefit from investing in all opportunities since this increases diversification. At the same time, diversification by arbitrageurs as a whole is limited because of contagion: When all arbitrageurs are invested in all opportunities, shocks to one asset are transmitted to all assets. Our goal is to examine whether diversification

is effective despite contagion. A first step in answering this question is to compare the effects of unanticipated one-off fundamental shocks in the cases of integration and segmentation.

Suppose that in period t , unanticipated one-off fundamental shocks can occur to all asset pairs $(i, -i)$. Suppose also that these shocks, $\eta_{i,t}$, are normal with mean zero, variance $\lambda \bar{\epsilon}_i^2$ for λ small, and are independent across opportunities. We can then define the variance of arbitrageurs' total wealth induced by these shocks as

$$Var_{\eta}(W) \equiv \lambda \sum_{i \in \mathcal{A}} \left[\frac{\partial W_{t+1}}{\partial \eta_{i,t}} \right]^2 \bar{\epsilon}_i^2 \quad (71)$$

and the variance of risk premia as

$$Var_{\eta}(\phi_{j,t+1}) \equiv \lambda \sum_{i \in \mathcal{A}} \left[\frac{\partial \phi_{j,t+1}}{\partial \eta_{i,t}} \right]^2 \bar{\epsilon}_i^2. \quad (72)$$

In the case of integration, the variances $Var_{\eta}(W)$ and $Var_{\eta}(\phi_{j,t+1})$ can be computed using Proposition 7. The formulas in Proposition 7 can also be extended to cover the case of segmentation. Proposition 8 compares integration and segmentation when asset pairs are symmetric, i.e., $(\mu_i, u_i, \bar{\epsilon}_i)$ is independent of i , and Proposition 9 considers the asymmetric case.

Proposition 8 *If for all $i \in \mathcal{A}$, $(\mu_i, u_i, \bar{\epsilon}_i)$ is independent of i , then*

- *The variance of arbitrageurs' total wealth is identical under integration and segmentation.*
- *The variance of risk premia is lower under integration.*

The intuition for the effect on the variance of arbitrageurs' wealth is as follows. First, a shock's *direct effect* on arbitrageurs' wealth is the same under segmentation and integration. This is because the arbitrageurs' steady state equilibrium positions are the same in both situations (Corollary 6). Second, a shock's *indirect effect* on arbitrageurs' wealth is also the same in both situations. Indeed, by symmetry between the arbitrage opportunities, the change in wealth due to the direct effect leads arbitrageurs to change their positions in all opportunities by the same amount. The investors' demand for all assets is equally sensitive to excess returns and are the same under segmentation and integration. Therefore, the sum of the changes in excess returns is the same under both segmentation and integration: Under integration, each opportunity reacts less but there are more such opportunities. The direct and indirect effects being the same under segmentation and integration, the overall effect on arbitrageurs' wealth is the same in both situations. The wealth effect of a given shock being identical under integration and segmentation, the variance of arbitrageurs' total wealth is also identical. Said differently, arbitrageur diversification does not reduce the variance of arbitrageurs' total wealth. Indeed, while each arbitrage opportunity has lower variance, opportunities also become correlated. As a result, the variance of arbitrageurs' total wealth, i.e., of a diversified investment across all opportunities, stays constant.

The intuition for the effect on the variance of risk premia is a simple implication. Compared to segmentation, the same wealth shock is "spread" across more arbitrage opportunities under

integration. Therefore the risk premium of each opportunity reacts less to the average shock and the variance of risk premia is lower under integration.

Note that while the variance of arbitrageurs' total wealth is the same under integration and segmentation, integration benefits arbitrageurs because variations in their total wealth are better shared among them. Since integration lowers the variance of risk premia, it also benefits the outside investors.

Proposition 9 *If for all $i \in \mathcal{A}$, $(\mu_i, u_i, \bar{\epsilon}_i)$ depends on $n(i)$, then*

- *The variance of arbitrageurs' total wealth is lower under integration.*
- *The average variance of risk premia over all asset pairs is lower under integration. However, for asset pairs with low $(u_i, \bar{\epsilon}_i)$, the variance of risk premia can be higher under integration.*

The intuition is as follows.

Under segmentation, fundamental shocks have large effects for opportunities where $(u_i, \bar{\epsilon}_i)$ are large. There are two reasons for this. First, arbitrageurs' positions are larger for these opportunities (Corollary 4), leading to a larger direct effect. Second, investor demand for assets with a larger volatility $\bar{\epsilon}_i$ is less sensitive to the expected excess return $\Phi_{i,t}$ (Corollary 3), i.e., the expected return needs to change more for the investors to absorb the change in arbitrageurs' position. This leads to a greater indirect effect.

Integration reduces this "mismatch". Fundamental shocks have the same direct effect as under segmentation but a lower indirect effect. Shocks to opportunities where $(u_i, \bar{\epsilon}_i)$ are large are "absorbed" by markets with more elastic demand (lower $\bar{\epsilon}_i$). The reduced indirect effect leads to a lower reaction of wealth to the shock.

Integration dampens these effects because arbitrageurs can transfer capital from other opportunities (e.g., with small $(u_i, \bar{\epsilon}_i)$) to smooth the shocks. At the same time, the contagion generated by these transfers can raise the variance of risk premia of opportunities with small $(u_i, \bar{\epsilon}_i)$. The average variance of risk premia over all asset pairs decreases, however, consistent with the symmetric case.

The variance of arbitrageurs' total wealth decreases under integration because the reduction in variance for opportunities with large $(u_i, \bar{\epsilon}_i)$ dominates the increase for opportunities with small $(u_i, \bar{\epsilon}_i)$. Intuitively, amplification effects exhibit convexity: their strength increases with arbitrageurs' positions at an increasing rate.

5.2. Preference Shocks (To be written)

Suppose that we are in the steady state of the riskless arbitrage case, and in a given period t the supply shock for the two assets in the pair $(i, -i)$ becomes $u_{i,t} \neq u_i$.

6. Risky Arbitrage (To be written)

We next consider the case where there is fundamental risk, i.e., $\epsilon_{i,t} \neq \epsilon_{-i,t}$, and preference risk, i.e., $u_{i,t}$ is stochastic. More precisely, we assume that the dividends for assets i and $-i$ are

$$\epsilon_{i,t} + \eta_{i,t} \quad \text{and} \quad \epsilon_{i,t} - \eta_{i,t}, \quad (73)$$

where $\eta_{i,t} \neq 0$ is a small shock and the supply shock for (i, t) -investors is

$$u_{i,t} = u_i + \rho(u_{i,t} - u_i) + \omega_{i,t}, \quad (74)$$

where the three sequences $\{\epsilon_{i,t}, \eta_{i,t}, \omega_{i,t}\}$ are zero-mean, i.i.d., and independent of each other. We denote the finite support of $\epsilon_{i,t}$ by $[-\bar{\epsilon}_i, +\bar{\epsilon}_i]$, $\eta_{i,t}$ by $[-\bar{\eta}_i, +\bar{\eta}_i]$, and $\omega_{i,t}$ by $[-\bar{\omega}_i, +\bar{\omega}_i]$. The shocks $\eta_{i,t}$ represent the fundamental risk of the arbitrage, and the shocks $\omega_{i,t}$ represent the preference risk.

To derive closed-form solutions, we consider the case where arbitrageurs face small uncertainty. Since the arbitrageurs' uncertainty is described by the shocks $\eta_{i,t}$ and $\omega_{i,t}$, we take $\bar{\eta}_i$ to zero, holding $\Omega \equiv \bar{\omega}_i/\bar{\eta}_i$ constant. For $\bar{\eta}_i = 0$ we are back in the riskless arbitrage case.

We consider Taylor expansions of the risk premia and the arbitrageurs' positions in order $\bar{\eta}_i$. We set

$$\phi_{i,t} = \phi_{i,t}^0 + \bar{\eta}_i \phi_{i,t}^1 + o(\bar{\eta}_i), \quad (75)$$

$$x_{i,t} = x_{i,t}^0 + \bar{\eta}_i x_{i,t}^1 + o(\bar{\eta}_i), \quad (76)$$

where $o(\bar{\eta}_i)$ denotes terms of order smaller than $\bar{\eta}_i$. The terms with superscript zero are the values in the riskless arbitrage case ($\bar{\eta}_i = 0$). The terms with superscript one are new terms introduced by uncertainty. We are interested in the following questions:

- What is the volatility of the price wedge (i.e., of the risk premium $\phi_{i,t}$) that is generated by fundamental shocks? What is the volatility generated by preference shocks? How is volatility influenced by the presence of arbitrageurs and by changes in their wealth? To answer these questions, we need to know how $\phi_{i,t}$ depends on the arbitrageurs' wealth W_t and the preference shock $u_{i,t}$. For small uncertainty, we can focus on the term of order zero, $\phi_{i,t}^0$.
- How are the arbitrageurs' positions influenced by uncertainty? How do positions differ across opportunities, depending on the risk of each opportunity and on the arbitrageurs' wealth? To answer these questions, we need to consider the terms of order one in the function $x_{i,t}$. Indeed, the terms of order zero correspond to the riskless arbitrage case, and are equal across all opportunities. The terms of order one can differ across opportunities, depending on the opportunities' risks. Moreover, the difference can depend on the arbitrageurs' wealth.

7. Conclusion

- Financial market model where some investors (arbitrageurs)

- have better investment opportunities than others.
- face financial constraints.
- Tractable equilibrium model with explicit theory of financial constraints.
- Work in progress:
 - Uncertainty.
 - * Preference risk (random supply shocks) and/or fundamental risk ($\epsilon_{t,A_{h,n}} \neq \epsilon_{t,B_{h,n}}$).
 - * Role for diversification \Rightarrow Arbitrageurs may not max out the constraint.
 - * Contagion vs. diversification.
 - * Arbitrageurs may exacerbate price volatility.
 - Welfare analysis.
 - Partial collateralization, VAR, etc.
 - Partial cross-margining.

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APPENDIX

Proof of Lemma 1: By assumption, the variables $\left(\frac{\epsilon_{i,t}}{\epsilon_i}\right)$ are identically and symmetrically distributed over $[-1, +1]$. Therefore f is indeed identical for all these variables. Let ϵ denote one such variable.

To show that f is positive, we use Jensen’s inequality, which is strict since ϵ is stochastic, and the fact that $E(\epsilon) = 0$:

$$\exp(\alpha f(y)) = E \exp(-\alpha y \epsilon) > \exp [E(-\alpha y \epsilon)] = \exp [-\alpha y E(\epsilon)] = 1. \quad (77)$$

To show that f is strictly convex, we compute its second derivative. We have

$$f(y) = \frac{1}{\alpha} \log [E \exp(-\alpha y \epsilon)]. \quad (78)$$

Therefore,

$$f'(y) = -\frac{E(\epsilon \exp(-\alpha y \epsilon))}{E \exp(-\alpha y \epsilon)}, \quad (79)$$

and

$$f''(y) = \alpha \frac{E(\epsilon^2 \exp(-\alpha y \epsilon)) E \exp(-\alpha y \epsilon) - [E(\epsilon \exp(-\alpha y \epsilon))]^2}{[E \exp(-\alpha y \epsilon)]^2}. \quad (80)$$

That $f''(y) > 0$ follows from the Cauchy-Schwarz inequality

$$E(GH)^2 \leq E(G^2)E(H^2), \quad (81)$$

for the functions $G = \epsilon \exp(-\alpha y \epsilon/2)$ and $H = \exp(-\alpha y \epsilon/2)$. The Cauchy-Schwarz inequality is strict since ϵ is stochastic, and thus G and H are not proportional.

To show that $f(y) = f(-y)$, we use the symmetry of ϵ 's probability distribution around zero:

$$\exp(\alpha f(y)) = E \exp(-\alpha y \epsilon) = E \exp(\alpha y \epsilon) = \exp(\alpha f(-y)). \quad (82)$$

Finally, to show that $\lim_{y \rightarrow \infty} f'(y) = 1$, we note that

$$|f'(y) - 1| = \left| -\frac{E[\epsilon \exp(-\alpha y \epsilon)]}{E \exp(-\alpha y \epsilon)} - 1 \right| = \left| \frac{E[(\epsilon + 1) \exp(-\alpha y \epsilon)]}{E \exp(-\alpha y \epsilon)} \right|. \quad (83)$$

To show that the last term goes to zero when y goes to ∞ , we fix $\eta > 0$. We have

$$\left| \frac{E[(\epsilon + 1) \exp(-\alpha y \epsilon) \mathbf{1}_{\epsilon \leq -1 + \eta}]}{E \exp(-\alpha y \epsilon)} \right| \leq \eta \left| \frac{E[\exp(-\alpha y \epsilon) \mathbf{1}_{\epsilon \leq -1 + \eta}]}{E \exp(-\alpha y \epsilon)} \right| \leq \eta. \quad (84)$$

Moreover, for y large enough,

$$\left| \frac{E[(\epsilon + 1) \exp(-\alpha y \epsilon) \mathbf{1}_{\epsilon > -1 + \eta}]}{E \exp(-\alpha y \epsilon)} \right| \leq \eta. \quad (85)$$

■

Proof of Proposition 7: Denote $Y \equiv (f')^{-1}(\Pi)$. Using (34), the arbitrageurs' financial constraint (assumed to be binding since we are close to the steady state) can be rewritten as

$$\frac{2(1 - \Pi_s)}{(1 + r)} \sum_{j \in \mathcal{A}} \mu_j \left(u_j \bar{\epsilon}_j - (f')^{-1}(\Pi_s) \right) = W_s. \quad (86)$$

Differentiating with respect to W_s for Π_s at its steady state value yields

$$-\frac{2}{(1 + r)} \left[\frac{\partial \Pi_s}{\partial W_s} \sum_{j \in \mathcal{A}} \mu_j (u_j \bar{\epsilon}_j - Y) + (1 - \Pi) \sum_{j \in \mathcal{A}} \frac{\mu_j}{f''(Y)} \frac{\partial \Pi_s}{\partial W_s} \right] = 1, \quad (87)$$

which yields (63):

$$\frac{\partial \Pi_s}{\partial W_s} = -\frac{(1 + r)}{2} \frac{1}{\sum_{j \in \mathcal{A}} \mu_j (u_j \bar{\epsilon}_j - Y) + \frac{(1 - \Pi)}{f''(Y)} \sum_{j \in \mathcal{A}} \mu_j} = -\frac{(1 + r) f''(Y) Z}{2(1 - g) M \sum_{n'=1}^N \mu_{n'}}. \quad (88)$$

Differentiating (34) with respect to W_s for Π_s at its steady state value yields (65):

$$\frac{\partial x_{j,s}}{\partial W_s} = -\frac{\mu_j}{\bar{\epsilon}_j f''(Y)} \frac{\partial \Pi_s}{\partial W_s} = \frac{(1+r)Z}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \frac{\mu_j}{\bar{\epsilon}_j}. \quad (89)$$

Assuming that the arbitrageurs's financial constraints (43) is binding in period s and plugging the expression of W_s into that of the arbitrageurs dynamic budget constraint (42) yields:

$$W_{s+1} = 2(1-g) \sum_{j \in \mathcal{A}} \bar{\epsilon}_j x_{j,s}. \quad (90)$$

Differentiating this expression with respect to W_s yields

$$\frac{\partial W_{s+1}}{\partial W_s} = 2(1-g) \sum_{j \in \mathcal{A}} \bar{\epsilon}_j \frac{\partial x_{j,s}}{\partial W_s} = 2(1-g) \sum_{j \in \mathcal{A}} \frac{(1+r)Z\mu_j}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} = (1+r)Z, \quad (91)$$

which implies (66):

$$\forall s \geq t+1, \quad \frac{\partial W_s}{\partial W_{t+1}} = \prod_{\theta=t+1}^{s-1} \frac{\partial W_{\theta+1}}{\partial W_\theta} = [(1+r)Z]^{s-(t+1)}. \quad (92)$$

Using (37), we get (64) since for all $j \in \mathcal{A}$ such that $h(j) \geq t+2$ we have:

$$\frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} = \bar{\epsilon}_j \sum_{\theta=1}^{h(j)-(t+1)} \frac{1}{(1+r)^\theta} \frac{\partial \Pi_{t+\theta}}{\partial W_{t+1}} = \bar{\epsilon}_j \sum_{\theta=1}^{h(j)-(t+1)} \frac{1}{(1+r)^\theta} \frac{\partial \Pi_{t+\theta}}{\partial W_{t+\theta}} \frac{\partial W_{t+\theta}}{\partial W_{t+1}} \quad (93)$$

$$= \bar{\epsilon}_j \sum_{\theta=1}^{h(j)-(t+1)} \frac{1}{(1+r)^\theta} \left(-\frac{(1+r)f''(Y)Z}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \right) [(1+r)Z]^{\theta-1} \quad (94)$$

$$= -\frac{f''(Y)\bar{\epsilon}_j}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \sum_{\theta=1}^{h(j)-(t+1)} Z^\theta = -\frac{f''(Y)\bar{\epsilon}_j}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \times \frac{Z - Z^{h(j)-t}}{1-Z} \quad (95)$$

This expression also holds for $h(j) = t+1$ since in that case $\phi_{j,t+1} = 0$. From this we have:

$$\sum_{j \in \mathcal{A}} x_{n(j)} \left(\frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} \right) = -\frac{f''(Y)}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \sum_{j \in \mathcal{A}} \frac{Z - Z^{h(j)-t}}{1-Z} x_{n(j)} \bar{\epsilon}_{n(j)} \quad (96)$$

$$= -\frac{f''(Y)}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \sum_{n=1}^N \mu_n (u_n \bar{\epsilon}_n - Y) \sum_{m=1}^M \frac{Z - Z^m}{1-Z} \quad (97)$$

$$= -\frac{f''(Y)}{2(1-g)M} \left(\frac{\sum_{n=1}^N \mu_n u_n \bar{\epsilon}_n}{\sum_{n=1}^N \mu_n} - Y \right) \left(\frac{Z \left(M - \sum_{m=0}^{M-1} Z^m \right)}{1-Z} \right) \quad (98)$$

$$= -\frac{f''(Y)Z}{2(1-g)M(1-Z)} \left(\frac{\sum_{n=1}^N \mu_n u_n \bar{\epsilon}_n}{\sum_{n=1}^N \mu_n} - Y \right) \left(M - \frac{1-Z^M}{1-Z} \right) \quad (99)$$

$$= -\frac{(1-g)(1-(1+r)Z)}{2(1-g)M(1-Z)} \left(M - \frac{1-Z^M}{1-Z} \right) \quad (100)$$

$$= -\frac{1}{2M} \left(1 - \frac{rZ}{1-Z} \right) \left(M - \frac{1-Z^M}{1-Z} \right) \quad (101)$$

We can now write (61) as:

$$\frac{\partial W_{t+1}}{\partial \eta_{i,t}} = \frac{2(1-g)x_{n(i)}}{1 - \frac{(1-g)}{M} \left(1 - \frac{rZ}{1-Z}\right) \left(M - \frac{1-Z^M}{1-Z}\right)} = \frac{2(1-g)\mu_i \left(u_i - \frac{Y}{\bar{\epsilon}_i}\right)}{1 - \frac{(1-g)}{M} \left(1 - \frac{rZ}{1-Z}\right) \left(M - \frac{1-Z^M}{1-Z}\right)}. \quad (102)$$

Using this expression we get

$$\begin{aligned} \frac{\partial \phi_{j,t+1}}{\partial \eta_{i,t}} &= \left(\frac{\partial \phi_{j,t+1}}{\partial W_{t+1}} \right) \left(\frac{\partial W_{t+1}}{\partial \eta_{i,t}} \right) \\ &= -\frac{f''(Y)\bar{\epsilon}_j}{2(1-g)M \sum_{n'=1}^N \mu_{n'}} \times \frac{Z - Z^{h(j)-t}}{1-Z} \times \frac{2(1-g)\mu_i \left(u_i - \frac{Y}{\bar{\epsilon}_i}\right)}{1 - \frac{(1-g)}{M} \left(1 - \frac{rZ}{1-Z}\right) \left(M - \frac{1-Z^M}{1-Z}\right)} \\ &= -\frac{f''(Y)\mu_i \left(u_i - \frac{Y}{\bar{\epsilon}_i}\right)}{M \left(\sum_{n'=1}^N \mu_{n'}\right) \left[1 - \frac{(1-g)}{M} \left(1 - \frac{rZ}{1-Z}\right) \left(M - \frac{1-Z^M}{1-Z}\right)\right]} \times \frac{Z - Z^{h(j)-t}}{1-Z} \bar{\epsilon}_j. \end{aligned}$$

■