

The Dynamics of Optimal Risk Sharing*

Patrick Bolton
Columbia Business School

Christopher Harris
Faculty of Economics, Cambridge University

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Abstract

In this paper we study a dynamic contracting problem of optimal risk-sharing between a principal and an agent who invest in a common constant-returns-to-scale risky venture. Investment flow-returns follow a geometric Brownian motion and the two agents' risk-preferences are represented by additively separable utility functions exhibiting constant relative risk-aversion (CRRA). Principal and agent have different coefficients of relative risk-aversion. In any time period they invest their wealth in the risky venture and optimally share the underlying return risk and termination risk. When the project ends the two individuals divide the accumulated proceeds as specified in the risk-sharing contract and consume their final accumulated wealth. The paper characterizes risk-sharing formulae that approximate the optimal risk-sharing rule.

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1 Introduction

This paper considers an optimal risk-sharing problem between two individual investors who pool their resources to invest in a common constant-returns-to-scale risky venture. Investment flow-returns are assumed to follow a geometric Brownian motion and the two agents risk-preferences are represented by additively separable utility functions exhibiting constant relative risk-aversion (CRRA). The two individuals have different coefficients of relative risk-aversion and may each start with a different finite wealth endowment. The venture may end at any time with positive probability and when it ends the two individuals consume their final accumulated wealth.

To keep the analysis tractable we have stripped out of the model many features which would make it more realistic. Thus, our model allows for only two agents, only one risky asset, and agents only consume at the end. In addition, we simplify the formulation of the optimal contracting problem by letting one individual, the principal, make a take-it-or-leave-it contract offer to the other, the agent. Even so, the analysis of this optimal contracting problem is sufficiently complex that we are only able to characterize approximately optimal risk-sharing formulae. For reasonable parameter configurations, however, our formulae are a good fit for the optimal risk-sharing rule.

Optimal risk-sharing between two parties has first been analyzed by Borch (1962) in the context of a reinsurance problem. He considers an optimal contract to share risk between an insurance and a re-insurance company (or between two insurance companies). While his framework is more general in many respects than the one we have just described, he only goes as far as characterizing a necessary condition for optimal co-insurance between two risk-averse agents, the well known Borch condition.

In this paper we push the analysis further and derive explicit risk-sharing formulae that approximate the optimal risk-sharing rule. We are able to do this by formulating the risk-sharing problem as a recursive problem and by deriving formulae for optimally sharing incremental output risk. As we show in section 4, the optimal contracting problem where the principal agrees to make a payment to the agent when the venture ends contingent on the sample-path of output of the venture, can be reformulated as an equivalent recursive contracting problem where instead the principal offers the agent a sequence of spot contracts, each specifying three variables: *i*) a share S of spot investment returns, *ii*) a fixed transfer F to the agent and, *iii*) a final transfer B to be paid to the agent in the event of termination of the venture. We then derive relatively simple formulae for S and B that approximate the optimal risk-sharing spot contract. Thus, a central contribution of this paper is to derive (approximate) formulae for optimal risk-sharing for the CRRA case.

The recursive contracting problem has an intuitive interpretation as a short-run contracting problem, which takes the following form. At any given time the principal makes an offer of a spot contract to the agent. Under any such contract the two parties pool their investments in the risky venture. The agent is free to reject any offer and to invest on his own in the venture without sharing risk with the principal. In that case, the principal too will be forced to invest on her own without sharing risks. Therefore, the principal must always guarantee the agent at least her autarky payoff. When the flow returns on the investment are realized they are divided among the parties according to the contract and, as long as the venture is not terminated, the parties start over again by reinvesting their proceeds and sharing risk with a new spot contract. And so on until the venture is terminated, at which point

both parties consume their accumulated wealth.

As each party's aversion to risk and capacity to insure the other party varies with its wealth the optimal shares S and B vary with the underlying wealth distribution. Thus, one advantage of our recursive contracting-problem formulation is that it brings out explicitly the underlying dynamics of the risk-sharing problem. These dynamics can be understood as follows. Whenever the two parties engage in risk-sharing, the optimal spot contract will specify a division of realized flow returns that is different from each party's share of investment in the venture, and when this is the case next period's wealth distribution will change. If, say, the principal insures the agent, by taking on a bigger share of risk, then he will grow richer relative to the agent when there is a high investment return, or poorer when there is a low investment return. Either way, the wealth distribution changes and consequently each party's attitude towards risk and capacity to insure changes following the realization of investment returns. This change in each party's capacity to insure introduces an endogenous risk with respect to the cost of insurance and forward-looking agents will take this risk into account when they determine their optimal spot contract.

To gain insight into how this risk with respect to the future price of insurance can affect optimal risk-sharing, consider the extreme case in which the agent is risk-neutral and the principal is risk-averse. It is well known that optimal risk-sharing in a one-shot insurance contracting problem in this case would require that the agent perfectly insure the principal. But if the agent were to do this repeatedly then she would be sure to go bankrupt at some point and then the principal would no longer be able to get any insurance at all. Foreseeing this, the principal may then want to hold back from getting perfect insurance. Only when the agent is relatively wealthy

would the principal seek perfect insurance. When the agent is relatively poor the principal may optimally limit the amount of insurance he gets to preserve future insurance opportunities.

When both parties are risk averse there is unfortunately no simple translation of this insight. However, by assuming that both principal and the agent are close to myopic, and taking approximations to each party's payoff function around the myopic optimum, we are able to derive closed-form solutions for the optimal spot contract. These provide remarkably accurate qualitative predictions for the optimal contract even when both agents are far from being myopic.

We begin by showing that the optimal *myopic* share of risk for the agent is given by the well known ratio of the principal's coefficient of absolute risk aversion divided by the sum of both parties' coefficients of absolute risk aversion. We then proceed to characterize deviations from the optimal myopic rule to take account of dynamic considerations. These involve the following corrections: *i*) for any given wealth distribution, when both parties are fairly risk tolerant it is optimal for the *less* risk-averse party to take on less risk in the dynamic-contracting problem than the myopic rule would specify. This is because in such a case, preserving future risk-sharing options is the dominant consideration; *ii*) when both parties are fairly risk averse, optimal risk-sharing in the dynamic-contracting problem requires that the less risk averse party take on *more* risk than the myopic rule would specify, and *iii*) when one party, say the agent, is fairly risk tolerant but the other party is fairly risk averse, then a party takes on more risk when it is relatively wealthy and less risk (in order to preserve future risk-sharing opportunities) when it is relatively poor.

Although our model is highly stylized, it may be relevant to a number

of applications. We have already mentioned reinsurance as one application. Insurance companies are obviously wealth constrained and they rely on each other to share common risk. Our analysis sheds light on how these companies should structure their risk sharing to take account of the endogenous risk with respect to the future price of insurance. Another application, which has been our initial motivation for this paper, is portfolio or fund management contracts. In practice the contract between a representative client and a fund manager often takes the simple form of a share of portfolio returns for the client equal to the client's share of investments in the fund minus a management fee, which is equal to a small percentage of the funds under management. We recognize that the main concern in portfolio management generally is the manager's incentive to run the fund in the client's best interest. Still, we believe that our analysis may be relevant if there are also dynamic risk-sharing considerations involved in the long-term relation between the client and the manager.

Our paper also relates to a small literature in asset pricing that considers intertemporal models with heterogeneous investors in terms of risk preferences, in particular Dumas (1989), Wang (1996) and Hara (2005). Dumas (1989) introduces two classes of agents with different coefficients of relative risk aversion in an otherwise standard economy with aggregate shocks, investment and consumption. Similarly, Wang (1996) studies allocation of risk in a pure exchange economy with again two classes of agents with heterogeneous CRRA risk-preferences. Finally, Hara (2005) considers a continuous-time model where consumers have heterogeneous risk-preferences and shows how this model can be reduced to a static model with a representative agent.

There is by now accumulating evidence that consumers indeed differ substantially in their risk-preferences and also that CRRA risk-preferences are

a good description of consumers' revealed risk preferences. Indeed, Barsky, Juster, Kimball, and Shapiro (1997) in their experimental study on risk-taking decisions found that 5% of subjects' behavior is consistent with a coefficient of relative risk aversion of 33 or higher, and for another 5% with a coefficient of 1.3, while the median coefficient in their study is 7. Similarly, Guiso and Paiella (2001), Chiappori and Paiella (2006), and Chiappori and Salanié (2006) among others find evidence of heterogeneous risk preferences in households' actual portfolio allocations. In addition, using panel data on individual portfolio allocations between risky and riskless assets, Chiappori and Paiella (2006) are able to determine that the elasticity of the risky asset share to wealth in their sample is small and statistically insignificant, which is consistent with behavior obtained from CRRA risk preferences.

The paper is organized as follows. Section 2 describes the model. Section 3 derives the payoff of an agent under autarky. Section 4 formulates the bilateral contracting problem and derives the Bellman equation for the principal. Section 5 uses asymptotic expansions to derive risk-sharing formulae which approximate the optimal risk-sharing rules. Section 6 shows how the two-dimensional Bellman equation for the principal can be reduced to a one-dimensional equation. This equation can then be solved numerically. Section 7 presents numerical solutions for the optimal risk-sharing rules, and highlights how well the formulae derived in section 5 predict the qualitative shape of the optimal risk-sharing rules. Section 8 offers some concluding comments. Finally, the Appendix contains some technical proofs.

2 The Basic Model

We consider a dynamic-contracting problem between a principal and an agent. The agent's risk preferences are represented by the increasing concave

utility function $u = u(w)$, where w is the agent's wealth. The principal's preferences are represented by the increasing concave utility function $U = U(W)$, where W denotes the principal's wealth. For expositional clarity we begin by taking general functional forms for u and U , but ultimately we shall assume that each utility function takes the constant relative risk aversion (CRRA) form:

$$u(w) = \frac{w^{1-r} - 1}{1-r} \equiv C_r(w)$$

with $r > 0$, and

$$U(W) = \frac{W^{1-R} - 1}{1-R} \equiv C_R(W)$$

with $R > 0$. The agent's initial wealth is $\bar{w} \geq 0$, and the principal's initial wealth is $\bar{W} \geq 0$.

At any given time t the agent can invest her own and the principal's wealth in a portfolio, which for an investment x yields flow returns:

$$dx = (\mu dt + \sigma dZ)x,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and Z is a standard Wiener process (i.e. the dZ are independently and identically distributed increments with mean zero and variance dt).

The contractual relation between the two parties is open ended and there is a constant probability per unit time β that the relationship ends. A key simplifying assumption is that the two parties only consume their wealth once the relationship ends.

3 Autarky for the Agent

Consider first the case in which the agent invests on his own. His value function for this case will provide his reservation value in the spot-contracting

problem with the principal described below. If we denote this value function by $v = v(w)$, and if we take it that v is twice continuously differentiable, then the expected change in the agent's value will be

$$\mathbb{E}[dv] = \left(\frac{\partial v}{\partial w} \mu w + \frac{1}{2} \frac{\partial^2 v}{\partial w^2} \sigma^2 w^2 + \beta (u(w) - v) \right) dt.$$

Hence the Bellman equation for the agent under autarky is:

$$0 = \frac{\partial v}{\partial w} \mu w + \frac{1}{2} \frac{\partial^2 v}{\partial w^2} \sigma^2 w^2 + \beta (u(w) - v). \quad (\text{Autarky})$$

Notice that we can solve this equation explicitly when, as we shall ultimately assume, $u = C_r$. In that case, we expect the value function v to inherit the functional form C_r . That is, we expect that $v = C_r(\rho_A w)$, where ρ_A represents the agent's risk-adjusted rate of return.

Pursuing this lead, we get

$$\begin{aligned} v &= C_r(\rho_A w) \\ &= C_r(\rho_A) + C_r(w) + (1 - r) C_r(\rho_A) C_r(w) \\ &= \psi_A + \gamma_A C_r(w), \end{aligned}$$

where

$$\psi_A = C_r(\rho_A)$$

and

$$\gamma_A = 1 + (1 - r) C_r(\rho_A).$$

Furthermore,

$$\frac{\partial v}{\partial w} = \gamma_A C_r'(w)$$

and

$$\frac{\partial^2 v}{\partial w^2} = \gamma_A C_r''(w) = -\gamma_A \frac{r}{w} C_r'(w).$$

Substituting for v in the Bellman equation, we therefore obtain

$$0 = (\gamma_A \mu - \frac{1}{2} \gamma_A r \sigma^2 - \beta \psi_A) w C'_r(w).$$

Moreover, dividing through by $w C'_r(w)$ and putting $\gamma_A = 1 + (1-r) \psi_A$ yields

$$\psi_A = \frac{\mu - \frac{1}{2} r \sigma^2}{\beta_r}$$

where

$$\beta_r = \beta - (1-r) (\mu - \frac{1}{2} r \sigma^2),$$

and hence,

$$\gamma_A = \frac{\beta}{\beta_r}.$$

A natural interpretation for β_r is as the ‘effective’ discount rate at which flow payoffs are discounted. Thus, the following condition must hold if our problem is to be well posed:

Condition 1 $\beta_r > 0$.

This condition ensures that the agent’s expected payoff from investing is finite.

4 Bilateral Contracting

Consider now the case of bilateral contracting.

4.1 The long-run contracting problem

Suppose that the principal offers the agent a long-run contract proposing to pool and invest their total initial wealth $X_0 = (\bar{w} + \bar{W})$ in the risky venture until maturity, in exchange for a (bounded) payment contingent on the time-path of total wealth up to maturity $X_{|[0,T]}$. We denote this payment

by $q(X_{|[0,T]})$. Given our assumptions on the process determining investment returns, the time-path of total wealth $X_t = W_t + w_t$ is then given by $\frac{dX}{X} = \mu dt + \sigma dZ$, and the termination time T is exponentially distributed with rate β .

The principal's and agent's respective expected payoffs under this contract are then

$$E[U(X_T - q(X_{|[0,T]})])]$$

and

$$E[u(q(X_{|[0,T]}))].$$

Without loss of generality, an optimal long-run contract for the principal holds the agent down to his reservation payoff:

$$E[u(q(X_{|[0,T]}))] = v(\bar{w}), \tag{1}$$

where $v(\bar{w})$ is the agent's autarky payoff. The long-run contracting problem is therefore given by:

$$\left\{ \begin{array}{l} \max_q E[U(X - q(X_{|[0,T]})]) \\ \text{subject to:} \\ E[u(q(X_{|[0,T]}))] = v(\bar{w}) \\ \frac{dX}{X} = \mu dt + \sigma dZ, \quad \text{and} \\ X_0 = \bar{W} + \bar{w} \end{array} \right. \tag{2}$$

To proceed further with the analysis of this contracting problem we reformulate the long-run contracting problem as a recursive (short-run) contracting problem.

4.2 The recursive (short-run) contracting problem

We can transform the contracting problem (2) into a recursive problem by noting that the agent's expected utility under the long-run contract, $q(X_{[0,T]})$, follows a martingale and by appealing to the martingale representation theorem. More precisely, for any bounded payment schedule $q(X_{[0,T]})$ that satisfies (1) we can define the martingale

$$u_t = E[u(q(X_{[0,T]})) \mid \mathcal{F}_t]$$

where $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration representing the information available up to time t . By the martingale representation theorem, there exists a process (ζ, η, θ) where $\zeta + \beta\theta = 0$, such that

$$du_t = \zeta dt + \eta dZ + \theta d\chi \quad \text{and} \quad u_0 = v(\bar{w}).$$

We can also define a certainty-equivalent wealth process w_t for the agent—where $v(w_t) = u_t$ for $t \in [0, T]$, with $w_T = q(X_{[0,T]})$ —and a wealth process for the principal W_t , where $W_t = X_t - w_t$. The agent's certainty-equivalent wealth process is then defined as (f, s, b) such that:

$$\begin{aligned} dw_t &= fX_t dt + s dX + bX_t d\chi \\ &= fX_t dt + sX_t(\mu dt + \sigma dZ) + bX_t d\chi \\ &= X_t((f + s\mu)dt + s\sigma dZ + b d\chi), \end{aligned}$$

and $w_0 = \bar{w}$.

By Ito's lemma, we then have:

$$\begin{aligned} du_t &= v'(w_{t-})X_t((f + s\mu)dt + \sigma dZ) + \\ &\quad \frac{1}{2}v''(w_{t-})s^2\sigma^2X_t^2dt + (u(w_{t-} + bX_t) - v(w_{t-}))d\chi \end{aligned}$$

or, rearranging,

$$\begin{aligned} du_t &= (X_t(f + s\mu)v'(w_{t-}) + \frac{1}{2}s^2\sigma^2X_t^2v''(w_{t-}))dt + \\ &\quad + s\sigma X_tv'(w_{t-})dZ + (u(w_{t-} + bX_t) - v(w_{t-}))d\chi \end{aligned}$$

Thus, a process (f, s, b) for the agent's certainty-equivalent wealth w_t exists, where:

$$\begin{aligned} \zeta &= X_t(f + s\mu)v'(w_{t-}) + \frac{1}{2}s^2\sigma^2X_t^2v''(w_{t-}), \\ \eta &= s\sigma X_tv'(w_{t-}), \\ \theta &= u(w_{t-} + bX_t) - v(w_{t-}), \end{aligned}$$

and

$$\begin{aligned} 0 &= X_t(f + s\mu)v'(w_{t-}) + \frac{1}{2}s^2\sigma^2X_t^2v''(w_{t-}) + \\ &\quad \beta(u(w_{t-} + bX_t) - v(w_{t-})), \end{aligned}$$

so that $\zeta + \beta\theta = 0$. Or, rearranging, the process (f, s, b) for w_t is given by:

$$\begin{aligned} s &= \frac{\eta}{\sigma X_tv'(w_{t-})}, \\ f &= \frac{\zeta - \frac{1}{2}s^2\sigma^2X_t^2v''(w_{t-})}{X_tv'(w_{t-})} - s\mu, \end{aligned}$$

and

$$b = \frac{u^{-1}(\theta + v(w_{t-})) - w_{t-}}{X_t}.$$

Similarly, we can define a process for the principal's wealth W_t . Indeed, from $W_t = X_t - w_t$ and

$$W_0 = X_0 - w_0 = (\overline{W} + \overline{w}) - \overline{w} = \overline{W}$$

we can deduce that:

$$dW_t = dX_t - dw_t = -fX_t dt + (1-s)X_t dX - bX_t d\chi,$$

and

$$W_T = X_T - w_T = X_T - q(X_{[0,T]}).$$

Therefore, our original long-run contracting problem (2) can be reformulated as the following equivalent contracting problem, namely:

$$\left\{ \begin{array}{l} \max_{f,s,b} E[U(W_T)] \\ \text{subject to:} \\ dW = -fX_t dt + (1-s)X_t dX - bX_t d\chi \\ dw = fX_t dt + sX_t dX + bX_t d\chi, \quad \text{and} \\ 0 = (f + s\mu)X_t v'(w_{t-}) + \frac{1}{2}s^2\sigma^2 X_t^2 v''(w_{t-}) + \\ \beta(u(w_{t-} + bX_t) - v(w_{t-})) \end{array} \right. \quad (3)$$

Interestingly, this contracting problem can be interpreted as a short-run contracting problem where at each time t , the principal makes a take-it-or-leave-it offer of a spot contract $\{f, s, b\}$ to the agent. If the agent accepts, then he receives:

1. a non-contingent flow transfer $F = f(W + w)$, which is an upfront payment for his participation in the risk-sharing arrangement;
2. a contingent flow transfer $S = s(dW + dw)$, which is his share in the total returns on investment; and
3. a contingent lump-sum transfer

$$B = \left\{ \begin{array}{ll} b(W + w) & \text{if the venture terminates} \\ 0 & \text{if the venture does not terminate} \end{array} \right\},$$

which is an insurance payment in the event that he loses the investment opportunity as a result of termination of the venture.

If the agent rejects, then both parties invest under autarky for the current period.

Finally, if we substitute for $X_t = W_t + w_t$, it becomes immediately apparent that the contracting problem (3) has a recursive structure. The certainty equivalent wealths processes of the principal and agent are then given by:

$$dW = -f(W_{t-} + w_{t-})dt + (1 - s)(dW_{t-} + dw_{t-}) - b(W_{t-} + w_{t-})d\chi$$

$$dw = f(W_{t-} + w_{t-})dt + s(dW_{t-} + dw_{t-}) + b(W_{t-} + w_{t-})d\chi.$$

Furthermore, if we denote the principal's value function by

$$V(W_{t-}, w_{t-}) = \max_{f,s,b} E[U(W_T) \mid \mathcal{F}_t],$$

and assume that V is twice continuously differentiable, then the expected change in the principal's value will be

$$\begin{aligned} E[dV] &= \left(\frac{\partial V}{\partial W} (-f + (1 - s)\mu)(W + w) + \frac{\partial V}{\partial w} (f + s\mu)(W + w) \right. \\ &\quad + \frac{1}{2} \sigma^2 \left(\frac{\partial^2 V}{\partial W^2} (1 - s)^2 + 2 \frac{\partial^2 V}{\partial W \partial w} (1 - s)s + \frac{\partial^2 V}{\partial w^2} s^2 \right) (W + w)^2 \\ &\quad \left. + \beta (U(W - b(W + w)) - V) \right) dt. \end{aligned}$$

(where we have suppressed dependence of W and w on t)

Therefore, the Bellman equation for the Principal associated with the contracting problem (3) takes the form

$$\begin{aligned} 0 &= \max_{f,s,b} \left\{ \frac{\partial V}{\partial W} (-f + (1 - s)\mu)(W + w) + \frac{\partial V}{\partial w} (f + s\mu)(W + w) \right. \\ &\quad + \frac{1}{2} \sigma^2 \left(\frac{\partial^2 V}{\partial W^2} (1 - s)^2 + 2 \frac{\partial^2 V}{\partial W \partial w} (1 - s)s + \frac{\partial^2 V}{\partial w^2} s^2 \right) (W + w)^2 \\ &\quad \left. + \beta (U(W - b(W + w)) - V) \right\} \quad (\text{Bellman 1}) \end{aligned}$$

subject to

$$\begin{aligned}
0 &= \frac{\partial v}{\partial w} (f + s\mu) (W + w) + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial w^2} s^2 (W + w)^2 \\
&\quad + \beta (u(w + b(W + w)) - v).
\end{aligned} \tag{IR 1}$$

This formulation emphasizes the original dynamics in terms of the respective wealth endowments, W and w .

It turns out, however, that in this problem a more natural formulation is in terms of the principal's wealth W and the difference in the wealth of the two parties: $T = (W - w)$. The principal's Bellman equation in terms of these new variables then takes the form

$$\begin{aligned}
0 &= \max_{f,s,b} \left\{ \frac{\partial V}{\partial W} \mu (W + w) - \frac{\partial V}{\partial T} (f + s\mu) (W + w) \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{\partial^2 V}{\partial W^2} - 2 \frac{\partial^2 V}{\partial W \partial T} s + \frac{\partial^2 V}{\partial T^2} s^2 \right) \sigma^2 (W + w)^2 \right. \\
&\quad \left. + \beta (U(W - b(W + w)) - V) \right\}
\end{aligned} \tag{Bellman 2}$$

subject to

$$\begin{aligned}
0 &= \frac{\partial v}{\partial w} (f + s\mu) (W + w) + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial w^2} s^2 (W + w)^2 \\
&\quad + \beta (u(w + b(W + w)) - v).
\end{aligned} \tag{IR 2}$$

Finally, from the agent's individual-rationality constraint (IR 2), we obtain

$$(f + s\mu) (W + w) = -\frac{1}{2} \frac{\partial^2 v / \partial w^2}{\partial v / \partial w} s^2 \sigma^2 (W + w)^2 - \beta \frac{u(w + b(W + w)) - v}{\partial v / \partial w}.$$

Hence, eliminating $(f + s\mu) (W + w)$ from the principal's objective (Bellman 2), denoting the partial derivatives of V by V_W , V_T , V_{WW} and V_{WT} , denoting

the partial derivatives of v by v_w and v_{ww} , and rearranging, we obtain the final form of the principal's Bellman equation, namely:

$$\begin{aligned}
0 = \max_{s,b} & \left\{ V_W \mu (W + w) \right. && \text{(Bellman 3)} \\
& + \frac{1}{2} \left(V_{WW} - 2 V_{WT} s + \left(V_{TT} + \frac{v_{ww}}{v_w} V_T \right) s^2 \right) \sigma^2 (W + w)^2 \\
& \left. + \beta \left((U(W - b(W + w)) - V) + \frac{V_T}{v_w} (u(w + b(W + w)) - v) \right) \right\}.
\end{aligned}$$

4.3 A first characterization of the optimal risk-sharing rule s_*

Differentiating the principal's objective with respect to s , we immediately obtain our first characterization of optimal risk-sharing:

Lemma 1 *The optimal dynamic risk-sharing rule s_* is given by*

$$s_* = \frac{-\frac{V_{TT}}{V_T} - \frac{V_{wT}}{V_T}}{-\frac{V_{TT}}{V_T} - \frac{v_{ww}}{v_w}}. \quad (4)$$

Proof. The objective is quadratic in s . Hence, using the fact that the maximizer of an expression of the form $a s^2 - 2 b s$ is $\frac{b}{a}$, we obtain:

$$s_* = \frac{V_{WT}}{V_{TT} + \frac{v_{ww}}{v_w} V_T}.$$

Dividing through by $-V_T$ and noting that

$$V_{WT} = V_{TT} + V_{wT}$$

we obtain the desired expression. ■

This expression for the optimal *dynamic* risk-sharing rule s_* summarizes the main economic issues underlying our dynamic risk-sharing problem. In order to understand it better, it is helpful to compare it with the optimal *static* risk-sharing rule

$$s_*^S = \frac{-\frac{U_{WW}}{U_W}}{-\frac{U_{WW}}{U_W} - \frac{u_{ww}}{u_w}}.$$

Compared with this rule, the optimal dynamic rule exhibits three complications. First, the exogenous utility functions U and u are replaced with the endogenous value functions V and v . Second, risk aversion is evaluated not with respect to own wealth, but with respect to the difference in wealth, $T = W - w$ for the principal and $t = w - W$ for the agent¹. Finally, there is an additional term $-\frac{V_{wT}}{V_T}$ in the numerator. This term captures the idea that current changes in the agent's wealth have implications for the price at which the principal will be able to obtain insurance in the future.

4.4 A first characterization of the optimal termination transfer b_*

We can also obtain a condition akin to the familiar Borch rule for the optimal transfer b_* by differentiating the principal's objective in (Bellman 3) with respect to b :

Lemma 2 *The optimal transfer b_* is such that:*

$$\frac{U'(W - b(W + w))}{u'(w + b(W + w))} = \frac{V_T}{v_w} \quad (5)$$

The optimal final transfer is set so that the ratio of the principal's marginal utility of wealth to the agent's marginal utility of wealth in the event

¹Since the agent's value function v does not depend on W we have $v_t = v_w$, $v_{tt} = v_{ww}$ and $-\frac{v_{tt}}{v_t} = -\frac{v_{ww}}{v_w}$.

that the venture terminates is equal to the ratio of the principal's marginal utility of transfers to the agent's marginal utility of transfers. Note the close analogy of this condition with Borch's rule, which suggests the interpretation of V_T and v_w as the respective welfare weights of the principal and agent in a welfare maximization problem.

5 Asymptotic Expansions

A first approach to characterizing the optimal sharing rule is to consider what happens when the 'discount factor' β is large, so that future returns are heavily discounted. The effects at work in this extreme case are likely to be also present for lower values of β . Moreover, for large values of β , we are able to approximate the value function V and the optimal risk-sharing rule.

The idea is to find asymptotic expansions for V, v, b and s in powers of $\frac{1}{\beta}$. More precisely, we determine sequences of functions $\{V^{(0)}, V^{(1)}\}$, $\{v^{(0)}, v^{(1)}\}$, $\{b^{(0)}, b^{(1)}\}$ and $\{s^{(0)}, s^{(1)}\}$ such that the value functions V and v , and optimal risk-sharing rules, b and s , can be approximated by polynomials of the form

$$V^{(0)} + \frac{1}{\beta}V^{(1)} \text{ and } v^{(0)} + \frac{1}{\beta}v^{(1)}$$

and

$$b^{(0)} + \frac{1}{\beta}b^{(1)} \text{ and } s^{(0)} + \frac{1}{\beta}s^{(1)}$$

when β is large.

5.1 Order 0 expansions

We begin by determining the functions $V^{(0)}$, $v^{(0)}$ and $b^{(0)}$. In order to do this we substitute the series $(V^{(0)} + \frac{1}{\beta}V^{(1)})$, $(v^{(0)} + \frac{1}{\beta}v^{(1)})$ and $(b^{(0)} + \frac{1}{\beta}b^{(1)})$

for respectively V, v and b in the equations for the principal's and agent's payoffs. We then equate terms of order 0 in $\frac{1}{\beta}$.

Thus, equating terms of order 0 in (IR 2) we obtain that

$$0 = u(w + b^{(0)}(W + w)) - v^{(0)}$$

and equating terms of order 0 in (Autarky) that

$$0 = u(w) - v^{(0)}.$$

Similarly, equating terms of order 0 in (Bellman 2) we obtain that

$$0 = \max U(W - b^{(0)}(W + w)) - V^{(0)}.$$

Hence:

Lemma 3 $b^{(0)} = 0, v^{(0)} = u$, and $V^{(0)} = U$.

Similarly, equating terms of order 0 in (4) we obtain:

Proposition 4 *The order 0, or 'myopic', risk-sharing rule is $s_*^{(0)} = \frac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{v''}{v}}$.*

Proof. The optimal risk-sharing rule is such that:

$$s_* = \frac{-\frac{V_{TT}}{V_T} - \frac{V_{wT}}{V_T}}{-\frac{V_{TT}}{V_T} - \frac{v_{ww}}{v_w}},$$

or, multiplying by V_T and noting that $V_{TT} = V_{WT} - V_{wT}$,

$$s_* = \frac{V_{WT}}{V_{TT} + \frac{v_{ww}}{v_w} V_T}.$$

Substituting the series $V = V^{(0)} + \frac{1}{\beta} V^{(1)}$ and $v = v^{(0)} + \frac{1}{\beta} v^{(1)}$ into this formula (and ignoring terms of order $(\frac{1}{\beta})^2$ and higher) we have

$$s_* = \frac{V_{WT}^{(0)} + \frac{1}{\beta} V_{WT}^{(1)}}{V_{TT}^{(0)} + \frac{1}{\beta} V_{TT}^{(1)} + \frac{v_{ww}^{(0)}}{v_w^{(0)}} \left(V_T^{(0)} + \frac{1}{\beta} V_T^{(1)} \right)}.$$

Rearranging we obtain

$$s_* = \left(\frac{V_{WT}^{(0)}}{V_{TT}^{(0)} + \frac{v_{ww}^{(0)}}{v_w^{(0)}} V_T^{(0)}} \right) + \frac{1}{\beta} \left(\frac{V_{WT}^{(0)}}{V_{TT}^{(0)} + \frac{v_{ww}^{(0)}}{v_w^{(0)}} V_T^{(0)}} \right) \left(\frac{V_{WT}^{(1)}}{V_{WT}^{(0)}} - \frac{V_{TT}^{(1)} + \frac{v_{ww}^{(0)}}{v_w^{(0)}} V_T^{(1)}}{V_{TT}^{(0)} + \frac{v_{ww}^{(0)}}{v_w^{(0)}} V_T^{(0)}} \right).$$

Hence, equating terms of order 0,

$$s_*^{(0)} = \frac{V_{WT}^{(0)}}{V_{TT}^{(0)} + \frac{v_{ww}^{(0)}}{v_w^{(0)}} V_T^{(0)}} = \frac{U''}{U'' - \frac{r}{w} U'},$$

where $r = -\frac{wv_{ww}^{(0)}}{v_w^{(0)}} = -\frac{wu''}{u'}$. ■

The optimal myopic risk-sharing rule, thus, takes a particularly simple and intuitive form. It is the familiar ratio of the principal's coefficient of absolute risk aversion and the sum of the two parties coefficients of absolute risk aversion, which one would expect to obtain if both contracting parties had constant absolute risk averse (or CARA) risk preferences.

Contrasting this rule with our previous general formula

$$s_* = \frac{-\frac{V_{TT}}{V_T} - \frac{V_{wT}}{V_T}}{-\frac{V_{TT}}{V_T} - \frac{v_{ww}}{v_w}},$$

we observe that when the contracting parties are myopic, the principal's payoff is independent of the agent's wealth w , so that $\frac{V_{wT}}{V_T} = 0$ and the principal's *endogenous* absolute risk-aversion coefficient reduces to $-\frac{V_{TT}}{V_T} = -\frac{V_{WW}}{V_W}$. This is not surprising, as a myopic principal would not care about

the risk with respect to changes in the future price of insurance driven by changes in the agent's future wealth.

Denoting the principal's share of total wealth by $y = \frac{W}{W+w}$ we can also rewrite the formula for the myopic sharing rule as follows:

$$s_*^{(0)} = \frac{R(1-y)}{R(1-y) + ry}.$$

This latter formula already provides useful insights into how risk-sharing between the two parties varies with the share of the principal's investment in the portfolio y . At one extreme, the principal's wealth is negligible compared to the agent's (i.e. $y = 0$) and we have $s_*^{(0)} = 1$. In other words, as intuition suggests, the agent takes on all the risk in the portfolio. At the other extreme, the agent's wealth is negligible compared with the principal's (i.e. $y = 1$) and we have $s_*^{(0)} = 0$. Note that, when both $R > 0$ and $r > 0$, we have $s_*^{(0)} \in [0, 1]$. In other words, when both contracting parties are risk averse, it is optimal for them to share risk. Now it becomes clear why we have assumed that both principal and agent are risk-averse. Although our analysis could allow for one or both of the parties to be risk-loving, it would then require the introduction of gambling, a complication which we prefer to rule out at this stage.

A central theme of our analysis will be to compare the values of $s_*^{(0)}$ and s_* . We will be interested in determining under what circumstances the preservation of the option of future gains from trade from co-insurance will lead the principal to hold back and underinsure relative to what is desirable in a one-shot contract.

5.2 Order 1 expansions

Returning to the Bellman equation (Bellman 3) and equating terms of order 1 in $\frac{1}{\beta}$, we obtain

$$\begin{aligned} V^{(1)} &= V_W^{(0)} \mu (W + w) - V_T^{(0)} \left(\mu - \frac{1}{2} r \sigma^2 \right) w \\ &\quad + \frac{1}{2} \left(V_{WW}^{(0)} - 2 V_{WT}^{(0)} s^{(0)} + \left(V_{TT}^{(0)} - \frac{r}{w} V_T^{(0)} \right) (s^{(0)})^2 \right) \sigma^2 (W + w)^2 \}. \end{aligned}$$

From this equation we get an insightful characterization of the first-order gains to be obtained by departing from the myopic sharing rule. Letting,

$$R = -\frac{W U''}{U'} \quad \text{and} \quad r = -\frac{w u''}{u'}$$

denote the coefficients of relative risk aversion of the principal and agent, we obtain the following simple expression for the first-order gains from dynamic risk-sharing:

Proposition 5 *The first-order gains from optimal dynamic risk-sharing are:*

$$V^{(1)} = \left(\left(\mu - \frac{1}{2} R \sigma^2 \right) W + \frac{1}{2} \frac{(R - r)^2 \sigma^2}{\frac{R}{W} + \frac{r}{w}} \right) U'$$

Proof. From the (Autarky) equation we obtain that

$$\frac{v^{(1)}}{v_w^{(0)}} = \left(\mu - \frac{1}{2} r \sigma^2 \right) w$$

Substituting in

$$\begin{aligned} V^{(1)} &= V_W^{(0)} \mu (W + w) - V_T^{(0)} \left(\mu - \frac{1}{2} r \sigma^2 \right) w \\ &\quad + \frac{1}{2} \left(V_{WW}^{(0)} - 2 V_{WT}^{(0)} s^{(0)} + \left(V_{TT}^{(0)} - \frac{r}{w} V_T^{(0)} \right) (s^{(0)})^2 \right) \sigma^2 (W + w)^2 \}. \end{aligned}$$

and using the facts that: 1) the maximum of an expression of the form $-2bs + as^2$ is $-\frac{b^2}{a}$; 2) $V^{(0)} = U$; 3)

$$\frac{r}{w} = -\frac{u''}{u'} \text{ and } \frac{-\frac{U''}{U'}}{-\frac{U''}{U'} + -\frac{u''}{u'}} = s^{(0)}$$

we obtain that

$$V^{(1)} = V_W^{(0)} \mu (W + w) - V_T^{(0)} \frac{v^{(1)}}{v_w^{(0)}} + \frac{1}{2} \left(V_{WW}^{(0)} - V_{WT}^{(0)} s^{(0)} \right) \sigma^2 (W + w)^2.$$

Next, substituting for

$$-V_T^{(0)} \frac{v^{(1)}}{v_w^{(0)}} = -U' \left(\mu - \frac{1}{2} r \sigma^2 \right) w$$

we have

$$\begin{aligned} V^{(1)} &= U' \mu (W + w) - U' \left(\mu - \frac{1}{2} r \sigma^2 \right) w + \frac{1}{2} \left(U'' - U'' s^{(0)} \right) \sigma^2 (W + w)^2 \\ &= U' \left(\mu (W + w) - \left(\mu - \frac{1}{2} r \sigma^2 \right) w - \frac{1}{2} \frac{R}{W} \left(1 - s^{(0)} \right) \sigma^2 (W + w)^2 \right) \end{aligned}$$

Substituting for $s^{(0)} = \frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}}$ and rearranging we then obtain

$$V^{(1)} = U' \left(\left(\mu - \frac{1}{2} R \sigma^2 \right) W + \frac{1}{2} R \sigma^2 W + \frac{1}{2} r \sigma^2 w - \frac{1}{2} \frac{\frac{R}{W} \frac{r}{w}}{\frac{R}{W} + \frac{r}{w}} \sigma^2 (W + w)^2 \right)$$

Rearranging further we obtain the desired expression. ■

The equation for $V^{(1)}$ provides a lot of intuition into our dynamic-contracting problem. It has the following interpretation:

1. U' is the principal's marginal utility of wealth;
2. $\left(\mu - \frac{1}{2} R \sigma^2 \right) W$ is the monetary value of the investment return to the principal under autarky;

3. $\frac{1}{2} \frac{(R-r)^2 \sigma^2}{\frac{R}{W} + \frac{r}{w}}$ is the monetary value of the gains from trade, which are entirely appropriated by the principal.

Thus, we learn from the equation for $V^{(1)}$ that the gains from departing from the myopic problem are given by the sum of the investment gains under autarky and the gains from trade under optimal contracting.

5.2.1 First-order approximation to the optimal risk-sharing rule

s_*

In light of proposition 5, we find the following approximation to the value function of the principal:

$$V(W, w) = U \left(W + \frac{1}{\beta} \left((\mu - \frac{1}{2} R \sigma^2) W + \frac{1}{2} \frac{(R-r)^2 \sigma^2}{\frac{R}{W} + \frac{r}{w}} \right) \right).$$

Or, in more compact notation,

$$V = U \left(W + \frac{1}{\beta} (P + G) \right),$$

where,

$$P = (\mu - \frac{1}{2} R \sigma^2) W$$

and

$$G = \frac{1}{2} \frac{(R-r)^2 \sigma^2}{\frac{R}{W} + \frac{r}{w}}.$$

As we have explained, P represents the monetary value of the principal's investment under autarky, and G the monetary value of the gains from trade.

Similarly, since the agent appropriates no gains from trade, we can write:

$$v = u \left(w + \frac{1}{\beta} A \right),$$

where

$$A = (\mu - \frac{1}{2} r \sigma^2) w$$

is the monetary value of the agent's investment under autarky.

Differentiating V with respect to T , and v with respect to w , we then obtain

$$V_T = U' \left(W + \frac{1}{\beta} (P + G) \right) \left(1 + \frac{1}{\beta} (P_T + G_T) \right), \quad (6)$$

and

$$v_w = u' \left(w + \frac{1}{\beta} A \right) \left(1 + \frac{1}{\beta} A_w \right). \quad (7)$$

Further differentiating V_T with respect to T and W we also obtain

$$\begin{aligned} V_{TT} &= U'' \left(W + \frac{1}{\beta} (P + G) \right) \left(1 + \frac{1}{\beta} (P_T + G_T) \right)^2 + \\ &U' \left(W + \frac{1}{\beta} (P + G) \right) \frac{1}{\beta} G_{TT} \end{aligned}$$

or

$$V_{TT} = U' \left(W + \frac{1}{\beta} (P + G) \right) \left[\frac{1}{\beta} G_{TT} - \frac{R}{W} \left(1 + \frac{1}{\beta} (P_T + G_T) \right)^2 \right],$$

as $P_{TT} = 0$, and

$$V_{TW} = U' \left(W + \frac{1}{\beta} (P + G) \right) \left[\frac{1}{\beta} G_{TW} - \frac{R}{W} \left(1 + \frac{1}{\beta} (P_T + G_T) \right) \left(1 + \frac{1}{\beta} (P_W + G_W) \right) \right],$$

as $P_{TW} = 0$.

Finally, differentiating v_w with respect to w we get

$$v_{ww} = u' \left(w + \frac{1}{\beta} A \right) - \frac{r}{w} \left(1 + \frac{1}{\beta} A_w \right)^2.$$

We can now substitute these expressions into the general formula for s_* and, following straightforward but tedious algebra, obtain the following closed-form expression for $s_*^{(1)}$:

Proposition 6 *The first-order correction to the optimal myopic sharing rule is given by*

$$s_*^{(1)} = \frac{Rr\sigma^2W^2w^2}{2(Rw+rW)^5} (R-r)^3 ((R-2)w + (r-2)W).$$

Proof. See the appendix. ■

If we divide $s^{(1)}$ through by $(W + w)$ we can rewrite the formula as a function of the principal's share $y = \frac{W}{W+w}$ of total wealth :

$$s_*^{(1)} = \frac{rR(1-y)^2y^2\sigma^2}{2(R(1-y) + ry)^5}(R-r)^3(R(1-y) + ry - 2).$$

To summarize, we have shown that:

Proposition 7 *When β is large the optimal sharing rule is approximated by the following formula:*

$$s_* = \frac{R(1-y)}{R(1-y) + ry} \left[1 + \frac{1}{\beta} \frac{r(1-y)^2y^2\sigma^2(R-r)^3(R(1-y) + ry - 2)}{2(R(1-y) + ry)^4} \right]. \quad (8)$$

As this formula highlights, the agent's share s_* in the returns on total wealth can be expected to be small when the agent's wealth is small (i.e. when y is near to 1). This suggests that a more transparent formula is obtained by replacing s with

$$z = \frac{s}{1-y},$$

so that

$$z_* = \frac{R}{R(1-y) + ry} \left[1 + \frac{1}{\beta} \frac{r(1-y)^2y^2\sigma^2(R-r)^3(R(1-y) + ry - 2)}{2(R(1-y) + ry)^4} \right]. \quad (9)$$

In particular, we then have $z_*(1) = \frac{R}{r}$ and $z_*(0) = 1$. This normalization of s simply involves measuring the agent's share of total output relative to the agent's wealth w instead of relative to total wealth $W + w$.

As can be readily inferred from the formula, when both parties are fairly risk tolerant – that is, when $R < 2$ and $r < 2$, so that $R(1-y) + ry < 2$ for

all $y \in (0, 1)$ – then it is optimal for the more risk-averse party to take on more risk in the dynamic contracting problem than in the myopic problem for any given wealth distribution.

In contrast, when both parties are fairly averse to risk – that is, when $R > 2$ and $r > 2$, so that $R(1 - y) + ry > 2$ for all $y \in (0, 1)$ – optimal risk sharing in the dynamic contracting problem involves more risk-taking by the less risk-averse party than is optimal in the myopic problem.

Finally, when one party is fairly risk tolerant (say, $r < 2$) but the other is fairly risk averse (say, $R > 2$) then the more risk-averse party takes on more risk in the dynamic contracting problem than in the myopic problem if and only if she is relatively wealthy (i.e. if and only if $R(1 - y) + ry < 2$ or $y > \frac{R-2}{R-r}$).

We summarize these differences between the myopic and dynamic solutions in the figure below, which is drawn for the situation where the principal is the more risk averse of the two parties.

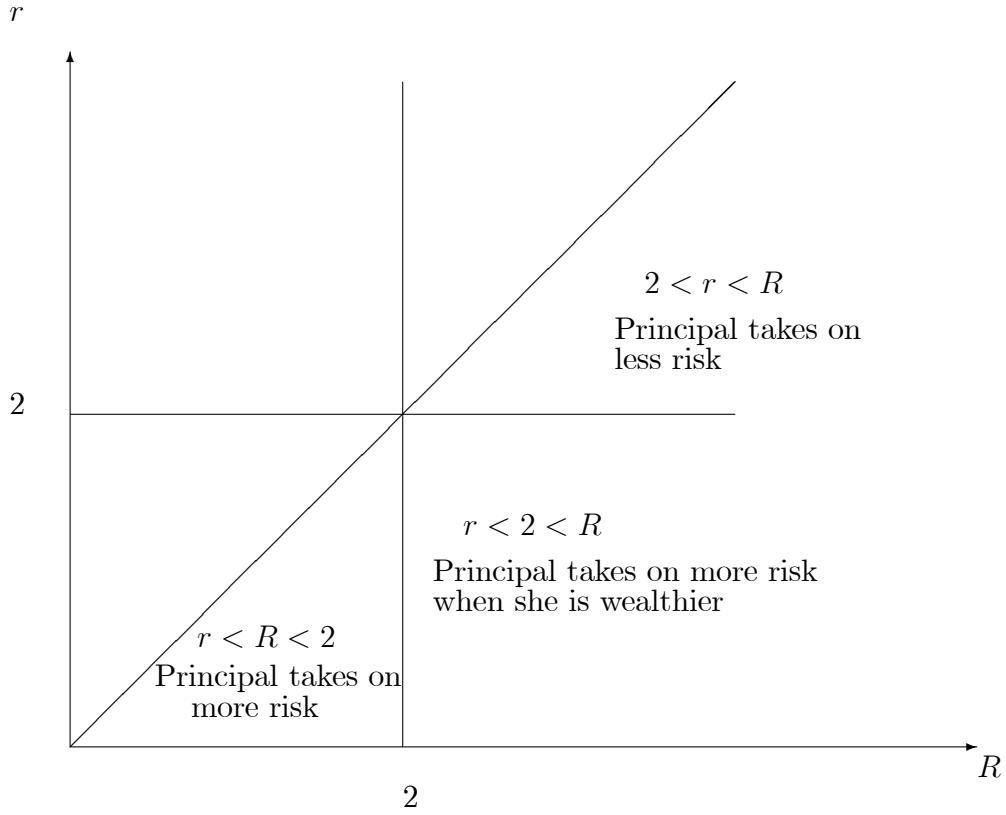


Figure 1: First-order corrections to myopic risk-sharing

As seems intuitive, the formula for z_* confirms that the difference between the myopic and dynamic solutions is small when the underlying risk of the portfolio is small (i.e. σ^2 is small) or when the two contracting parties have similar coefficients of relative risk aversion, (i.e. $|R - r|$ is small).

The formula also shows that the difference between the two solutions goes to zero as the wealth differences between the two principals grows (that is when y tends to either 0 or 1).

Finally, the formula shows that when both principal and agent have the same coefficient of risk aversion, (i.e. $r = R$) then $z_* = 1$. As intuition suggests, there are no benefits from risk-sharing in this case!

5.2.2 First-order approximation to the optimal termination payment b_*

Proceeding as above we can also derive a formula for the termination transfer that approximates the optimal termination payment b_* . Starting from the generalized Borch rule

$$\frac{U'(W - b(W + w))}{u'(w + b(W + w))} = \frac{V_T}{v_w}, \quad (10)$$

which implicitly defines the optimal transfer b_* , we can substitute for V_T and v_w using the expressions obtained in (6) and (7) to obtain a closed-form expression for $b_*^{(1)}$ as follows.

First, when we substitute for V_T and v_w , take first-order expansions of $U'(W - b(W + w))$ and $u'(w + b(W + w))$ and note that $b_*^{(0)} = 0$, we obtain

$$\frac{U' - \frac{1}{\beta}(W + w)b_*^{(1)}U''}{u' + \frac{1}{\beta}(W + w)b_*^{(1)}u''} = \frac{\left(U' + \frac{1}{\beta}(P + G)U''\right)\left(1 + \frac{1}{\beta}\left(\frac{\partial P}{\partial T} + \frac{\partial G}{\partial T}\right)\right)}{\left(u' + \frac{1}{\beta}A u''\right)\left(1 + \frac{1}{\beta}\frac{\partial A}{\partial w}\right)}.$$

Next, we divide the numerators by U' and the denominators by u' to get

$$\frac{1 + \frac{1}{\beta}(W + w)b_*^{(1)}\frac{R}{W}}{1 - \frac{1}{\beta}(W + w)b_*^{(1)}\frac{r}{w}} = \frac{\left(1 - \frac{1}{\beta}(P + G)\frac{R}{W}\right)\left(1 + \frac{1}{\beta}\left(\frac{\partial P}{\partial T} + \frac{\partial G}{\partial T}\right)\right)}{\left(1 - \frac{1}{\beta}A\frac{r}{w}\right)\left(1 + \frac{1}{\beta}\frac{\partial A}{\partial w}\right)}.$$

Therefore, when we equate terms of order 1 we have

$$\left(\frac{R}{W} + \frac{r}{w}\right)(W + w)b_*^{(1)} = \frac{r}{w}A - \frac{\partial A}{\partial w} - \frac{R}{W}(P + G) + \left(\frac{\partial P}{\partial T} + \frac{\partial G}{\partial T}\right).$$

Finally, rearranging, we obtain the formula

$$(W + w)b_*^{(1)} = \frac{\frac{r}{w}}{\frac{R}{W} + \frac{r}{w}}A - \frac{1}{\frac{R}{W} + \frac{r}{w}}\frac{\partial A}{\partial w} - \frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}}(P + G) + \frac{1}{\frac{R}{W} + \frac{r}{w}}\left(\frac{\partial P}{\partial T} + \frac{\partial G}{\partial T}\right). \quad (11)$$

This formula also has a clear economic interpretation:

1. When the venture ends, the agent loses his investment return A . The term

$$\frac{\frac{r}{w}}{\frac{R}{W} + \frac{r}{w}} A$$

thus tells us that the principal partially insures the agent by taking a share

$$\frac{\frac{r}{w}}{\frac{R}{W} + \frac{r}{w}}$$

in this loss.

2. The agent must, however, pay for the insurance that he receives from the principal. Doing so reduces his wealth and therefore his investment return. The term

$$-\frac{1}{\frac{R}{W} + \frac{r}{w}} \frac{\partial A}{\partial w}$$

tells us that the principal reduces her share in the agent's loss to reflect the opportunity cost to the agent of paying for the insurance.

3. Similarly, when the venture ends, the principal loses both her investment return P and the gains from trade G . The term

$$-\frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}} (P + G)$$

tells us that the agent takes a share

$$\frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}}$$

in this loss.

4. But the principal must pay for the insurance that she receives from the agent and doing so reduces her wealth and investment return. The term

$$\frac{1}{\frac{R}{W} + \frac{r}{w}} \left(\frac{\partial P}{\partial T} + \frac{\partial G}{\partial T} \right)$$

thus is the amount by which the agent reduces his share in the principal's loss to reflect the opportunity cost to the principal of paying for the insurance.

The final step is to substitute for the expressions of $A, P, G, \frac{\partial P}{\partial T}, \frac{\partial G}{\partial T}$ and $\frac{\partial A}{\partial w}$ in (11). We then obtain the following formula for $b_*^{(1)}$:

Proposition 8 *To a first approximation the optimal termination payment is given by*

$$b_*^{(1)} = \frac{1}{W+w} \frac{r-R}{\frac{R}{W} + \frac{r}{w}} \left(\mu + \frac{1}{2}(1-r-R)\sigma^2 \right. \\ \left. + \frac{1}{2}(r-R)\sigma^2 \left(\frac{\frac{R}{W^2} - \frac{r}{w^2}}{\left(\frac{R}{W} + \frac{r}{w}\right)^2} - \frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}} \right) \right)$$

Proof. Substituting for $A, P, G, \frac{\partial P}{\partial T}, \frac{\partial G}{\partial T}$ and $\frac{\partial A}{\partial w}$ in (11) and factorizing $\frac{1}{\frac{R}{W} + \frac{r}{w}}$ we obtain

$$(W+w)b_*^{(1)} = \left(\frac{1}{\frac{R}{W} + \frac{r}{w}} \right) \left[r(\mu - \frac{1}{2}r\sigma^2) - R(\mu - \frac{1}{2}R\sigma^2) - \right. \\ \left. \frac{1}{2}(R-r)^2\sigma^2 \frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}} + (\mu - \frac{1}{2}R\sigma^2) + \right. \\ \left. \frac{1}{2}(R-r)^2\sigma^2 \left[\frac{\frac{R}{W^2} - \frac{r}{w^2}}{\left(\frac{R}{W} + \frac{r}{w}\right)^2} - (\mu - \frac{1}{2}r\sigma^2) \right] \right]. \quad (12)$$

or, rearranging we obtain the desired expression. ■

We can again rewrite the formula as a function of $y = \frac{W}{W+w}$ and obtain further:

$$b_*^{(1)}(y) = \frac{y(1-y)}{R(1-y) + ry} (R-r) \left(\frac{1}{2}\sigma^2 B(y) - \mu \right)$$

where,

$$B(y) = r \frac{(R-r)^2 y^2 - 3R(R-r)y + R(2R-1)}{(R(1-y) + ry)^2}$$

Finally, observe that the transfer $b_*^{(1)}$ is small when either the principal or the agent has very little wealth (i.e. when y is close to 0 or 1). Therefore, as for the formula for $s_*^{(1)}$, this suggests the normalization:

$$g = \frac{b}{y(1-y)},$$

so that

$$g_*^{(1)}(y) = \frac{(R-r)(\frac{1}{2}\sigma^2 B(y) - \mu)}{R(1-y) + ry}. \quad (13)$$

It is easy to see from the formula in (13) that the shape of $g_*^{(1)}(y)$ is essentially determined by the shape of $B(y)$, and fortunately $B(y)$ is a relatively simple function. In particular, its derivative takes the following simple expression:

$$B'(y) = \frac{Rr(R-r)}{((1-y)R + yr)^3} ((1-y)R + yr - 2).$$

So that,

$$B'(0) = \frac{r}{R^2} (R-r)(R-2)$$

and

$$B'(1) = \frac{R}{r^2} (R-r)(r-2).$$

From these expressions for $B'(y)$, we are able to characterize the shape of $B(y)$ as follows. First, as can be easily inferred from the formula in (13), the principal insures the agent ($g_*^{(1)}(y) > 0$) when $\frac{1}{2}\sigma^2 B(y) > \mu$ (low μ) and the agent insures the principal ($g_*^{(1)}(y) < 0$) when $\frac{1}{2}\sigma^2 B(y) < \mu$ (high μ). Furthermore:

1. When $R > r > 2$, $B(y)$ is increasing in y . In other words, the principal provides better insurance against termination to the agent (or demands less insurance from the agent) the wealthier she is relative to the agent.
2. When $2 > R > r$, $B(y)$ is decreasing in y . The principal then provides worse insurance to the agent the wealthier she is.
3. When $R > 2 > r$, $B'(0) > 0$ and $B'(1) < 0$ so that $B(y)$ is hump-shaped and reaches a maximum value for some $y \in (0, 1)$. In this case, the principal provides maximum insurance to the agent at some interior wealth share, and for some values of μ the insurance pattern may be such that the agent insures the principal for y close to zero or one and the principal insures the agent at more even wealth distributions.²

The qualitative predictions on the shape of the optimal termination payment obtained from the formula in (13) are extremely accurate, as can be seen from the numerical solutions we report on the back of the paper.

Remarkably, even though the approximation is theoretically accurate only when β is large, the formula still provides an accurate qualitative prediction for values of β as low as 0.05.

Having obtained approximately optimal closed-form solutions for the risk-sharing contract when β is large, we now turn to a numerical analysis of the optimal contract. To solve the Bellman equation (Bellman 3) numerically, it is helpful to reduce it from a two-dimensional equation in (W, w) to a

²We are also able to determine from the formula for $B(y)$ when $B(1) > B(0)$. It turns out that this depends on whether the following inequality holds:

$$\frac{1}{\frac{1}{2}(\frac{1}{R} + \frac{1}{r})} \equiv H(R, r) > 2$$

or, in other words, whether the harmonic mean risk aversion is greater than 2 (or mean risk tolerance is less than 1/2).

one-dimensional equation in $y = \frac{W}{W+w}$. We proceed with this transformation in the next section. Although the ultimate purpose of this reduction is to simplify the numerical analysis, the one-dimensional reduction also provides new insights into the structure of our contracting problem, and yields key analytic results. Nonetheless, a reader eager to see the numerical solutions, and how they compare with our formulae for z^* and g^* , may want to skip the next section on a first reading.

6 The One-dimensional Bellman Equation

Asymptotic expansions in $\frac{1}{\beta}$ have yielded a formula for dynamically optimal risk sharing when $\frac{1}{\beta}$ is small. But how good an approximation is this formula for the general case? To answer this question we need to compute numerical solutions to the Bellman equation and the optimal contract, and compare the solutions with the approximation given by the formula.

Solving for the optimal contract numerically, however, requires further simplification of our two-dimensional Bellman equation:

$$\begin{aligned}
0 = \max_{s,b} & \left\{ V_W \mu(W+w) \right. \\
& + \frac{1}{2} \left(V_{WW} - 2V_{WT}s + \left(V_{TT} + \frac{v_{ww}}{v_w} V_T \right) s^2 \right) \sigma^2 (W+w)^2 \\
& \left. + \beta \left((U(W-b(W+w)) - V) + \frac{V_T}{v_w} (u(w+b(W+w)) - v) \right) \right\}.
\end{aligned}$$

(Bellman 3)

In this section we show how this two-dimensional equation in (W, w) -space can be reduced to a pair of one-dimensional equations in y -space. In the process of reducing the dimensionality of the Bellman equation, we also gain further economic insights into our dynamic-contracting problem.

6.1 Substituting for V

As both principal and agent have *CRRA* risk preferences, and since the returns to their investment follow a geometric Brownian motion, it is natural to write the principal's value function V in the form

$$V(W, w) = C_R(\rho(W, w) W),$$

where $\rho(W, w)$ represents a certainty-equivalent rate of return under optimal risk-sharing.

Furthermore, since investment exhibits constant returns to scale, one would expect that, if both W and w are increased by the same factor λ , then ρ will be unchanged. More formally, one would expect that

$$\rho(\lambda W, \lambda w) = \rho(W, w) \text{ for all } \lambda > 0.$$

In particular,

$$\rho(W, w) = \rho\left(\frac{W}{W+w}, \frac{w}{W+w}\right) = \rho(y, 1 - y).$$

In other words, ρ should only depend on the wealth shares of the two parties, and not on their wealth levels.

This in turn implies that

$$\begin{aligned} V(W, w) &= C_R(\rho(y, 1 - y) W) \\ &= C_R(W) + W^{1-R} C_R(\rho(y, 1 - y)) \\ &= C_R(W) + W C'_R(W) \psi(y), \end{aligned}$$

where

$$\psi(y) \equiv C_R(\rho(y, 1 - y)).$$

We shall use this formula to substitute for V in the two-dimensional Bellman equation.

6.2 Some preliminaries

To carry out the substitution we need formulae for the first and second derivatives of V in terms of ψ and ψ' .

6.2.1 First derivatives of V

We begin by deriving expressions for the first derivatives of V .

Lemma 9 *We have*

$$V_W = (1 + (1 - R)\psi + y(1 - y)\psi') C'_R$$

and

$$V_T = (1 + (1 - R)\psi + y\psi') C'_R,$$

where we have suppressed the dependence of V on W and w , the dependence of C'_R on W and the dependence of ψ on y .

Proof. Writing the formula for V in the form

$$V(W, w) = C_R(W) + W C'_R(W) \psi(y),$$

we obtain

$$\begin{aligned} V_W &= C'_R + C'_R \psi + W C''_R \psi + W C'_R \psi' \frac{\partial y}{\partial W} \\ &= C'_R + C'_R \psi - R C'_R \psi + W C'_R \psi' \frac{1 - y}{W + w} \end{aligned}$$

(where we have used the facts that $W C''_R = -R C'_R$ and $\frac{\partial y}{\partial W} = \frac{1 - y}{W + w}$)

$$= (1 + (1 - R)\psi + y(1 - y)\psi') C'_R.$$

Similarly,

$$V_w = W C'_R \psi' \frac{\partial y}{\partial w} = W C'_R \psi' \frac{-y}{W + w} = -y^2 \psi' C'_R.$$

Finally, $V_T = V_W - V_w$. ■

We have already seen that the shadow price of transfers V_T plays an important role in the risk-sharing formula obtained from the two-dimensional Bellman equation. One might therefore anticipate that the analogue of V_T in the one-dimensional model, namely

$$\gamma = 1 + (1 - R) \psi + y \psi' = \frac{V_T}{C'_R}$$

will play a comparable role in the one-dimensional model. This is indeed the case, and it is therefore helpful to rewrite V_W and V_T in terms of ψ and γ instead of ψ and ψ' .

Lemma 10 *We have*

$$V_W = ((1 - y) \gamma + y(1 + (1 - R) \psi)) C'_R$$

and

$$V_T = \gamma C'_R.$$

Proof. These formulae follow immediately from those of Lemma 9 on noting that $y \psi' = \gamma - (1 + (1 - R) \psi)$ and rearranging. ■

Corollary 11 *We have*

$$\frac{v_{ww}}{v_w} V_T = -\frac{r}{1 - y} \gamma \frac{C'_R}{W + w}.$$

Proof. Indeed,

$$\frac{v_{ww}}{v_w} V_T = -\frac{r}{w} \gamma C'_R = -\frac{r(W + w)}{w} \gamma \frac{C'_R}{W + w} = -\frac{r}{1 - y} \gamma \frac{C'_R}{W + w},$$

as required. ■

Remark 12 *Economic intuition tells us that we must have $\gamma > 0$ for all $y \in [0, 1]$.*

6.2.2 Second derivatives of V

For our substitution we also need formulae for the second derivatives of V in terms of ψ , γ and γ' .

Lemma 13 *We have*

$$\begin{aligned} V_{WW} &= \left((1-y)^2 \gamma' - \frac{R}{y} \gamma + Ry(\gamma - 1 - (1-R)\psi) \right) \frac{C'_R}{W+w}, \\ V_{WT} &= \left((1-y) \gamma' - \frac{R}{y} \gamma \right) \frac{C'_R}{W+w}, \\ V_{TT} &= \left(\gamma' - \frac{R}{y} \gamma \right) \frac{C'_R}{W+w}. \end{aligned}$$

Proof: See the Appendix. ■

It should now be apparent from the formulae for the first and second derivatives of V that a key step in reducing the two-dimensional Bellman equation for V into a one-dimensional equation for ψ is to divide the Bellman equation through by $(W+w)C'_R$, as this factor will cancel from the terms in μ and σ^2 .

But, before we can complete our substitution, we need to find out what happens when the terms in β are divided through by $(W+w)C'_R$. Two observations are helpful to this end. First, recall from Section 3 that

$$v(w) = \psi_A + \gamma_A C_r(w),$$

where $\psi_A = C_r(\rho_A)$ and $\gamma_A = 1 + (1 - r) \psi_A = \rho_A C'_r(\rho_A)$. Second, as we have observed earlier, it is also natural to introduce the normalization,

$$g = \frac{b}{y(1 - y)}.$$

With these substitutions we obtain the following expressions:

Lemma 14

$$\frac{U(W - b(W + w)) - V}{(W + w) C'_R} = y (C_R(1 - (1 - y)g) - \psi)$$

and

$$\frac{V_T}{v_w} \frac{u(w + b(W + w)) - v}{(W + w) C'_R} = (1 - y) \frac{\gamma}{\gamma_A} (C_r(1 + yg) - \psi_A).$$

Proof: See the Appendix. ■

6.3 The one-dimensional equation for ψ

To summarize, the idea is to make the substitution

$$V(W, w) = C_R(W) + W C'_R(W) \psi(y)$$

in the Bellman equation for V , and then to divide the equation through by the strictly positive quantity

$$(W + w) C'_R(W).$$

In this way, we obtain a pair of one-dimensional equations for ψ and γ .

To abridge our expressions for ψ and γ it is also helpful to introduce the following piece of notation:

$$\Phi(g, y, \gamma) = \beta \frac{C_R(1 - (1 - y)g)}{1 - y} + \gamma \beta_r \frac{C_r(1 + yg)}{y}$$

for all $g \in (-y^{-1}, (1-y)^{-1})$, $y \in (0, 1)$ and $\gamma \in (0, \infty)$.³ The results of the substitution can then be summarized as follows.

Lemma 15 *The two dimensional equation for V , namely (Bellman 3), is equivalent to a pair of one-dimensional equations for γ and ψ , namely*

$$0 = \max_{s,g} \left\{ y(1-y) \Phi(g, y, \gamma) + y \left((\mu - \frac{1}{2} R \sigma^2) - \beta_R \psi \right) \right. \\ \left. + \frac{1}{2} (Ry + r(1-y)) \sigma^2 \gamma + \frac{1}{2} \left((1-y)^2 \gamma' - \frac{R}{y} \gamma \right) \sigma^2 \right. \\ \left. - \left((1-y) \gamma' - \frac{R}{y} \gamma \right) s \sigma^2 + \frac{1}{2} \left(\gamma' - \frac{R}{y} \gamma - \frac{r}{1-y} \gamma \right) s^2 \sigma^2 \right\}, (14)$$

and

$$\gamma = 1 + (1-R) \psi + y \psi'. \quad (15)$$

Proof: See the Appendix. ■

Differentiating the principal's payoff in (14) with respect to s , we obtain yet another characterization of the optimal risk-sharing rule:

Corollary 16 *The optimal risk-sharing rule takes the form*

$$s_* = \frac{\left(\frac{R}{y} - \frac{\gamma'}{\gamma} \right) + y \frac{\gamma'}{\gamma}}{\left(\frac{R}{y} - \frac{\gamma'}{\gamma} \right) + \frac{r}{1-y}}.$$

3

Notice in fact that $C_R(1) = C_r(1) = 0$ and $C'_R(1) = C'_r(1) = 1$. The function Φ therefore extends continuously to the cases $y = 0$ and $y = 1$. Indeed:

$$\Phi(g, 0, \gamma) = \beta C_R(1-g) + \gamma \beta_r g$$

for $g \in (-\infty, 1)$ and $\gamma \in (0, \infty)$; and

$$\Phi(g, 1, \gamma) = -\beta g + \gamma \beta_r C_r(1+g)$$

for $g \in (-1, +\infty)$ and $\gamma \in (0, \infty)$.

From this formula, we gain the additional insight that the optimal sharing rule departs from the myopic sharing rule

$$s_*^{(0)} = \frac{\frac{R}{y}}{\frac{R}{y} + \frac{r}{1-y}},$$

in two ways. First, the coefficient of absolute risk aversion of the principal, namely $\frac{R}{y}$, is replaced by $\frac{R}{y} - \frac{\gamma'}{\gamma}$. Secondly, the numerator includes the extra term $y \frac{\gamma'}{\gamma}$. Thus, to the extent that the shadow-cost of transfers decreases with the share of the principal's wealth ($\gamma' < 0$) the principal is effectively more risk-averse in the dynamic risk-sharing problem, and as a result the agent takes on less risk in the dynamic problem. Unfortunately, however, the shadow-cost of transfers does not always decrease monotonically with the principal's share of wealth, which is why the optimal share of risk for the agent in the dynamic problem may be higher or lower than in the myopic problem.

6.4 Some more normalizations

In deriving the one-dimensional equation for ψ we have made the natural normalization of replacing the transfer b with

$$g = \frac{b}{y(1-y)}.$$

This normalization has the added advantage that it ensures that our numerical solutions are stable.

As we have highlighted in the previous section, another natural normalization is to replace s with

$$z = \frac{s}{1-y}.$$

Along with this normalization we also introduce the helpful notation

$$Z = \frac{1-s}{y}.$$

Finally, we shall make another normalization, which is somewhat less obvious, but is suggested by the following considerations. In the absence of risk sharing (i.e. when $g = 0$ and $s = 1 - y$), we have

$$\psi = \psi_P = \frac{\mu - \frac{1}{2} R \sigma^2}{\beta_R}$$

and

$$\gamma = \gamma_P = \frac{\beta}{\beta_R},$$

where

$$\beta_R = \beta - (1 - R) (\mu - \frac{1}{2} R \sigma^2).$$

It therefore makes sense to write

$$\psi = \frac{\mu - \frac{1}{2} R \sigma^2 + (1 - y) \chi}{\beta_R},$$

where χ can be thought of as a measure of the gains from risk sharing. Thus, our final normalization involves replacing ψ with

$$\chi = \frac{\beta_R \psi - (\mu - \frac{1}{2} R \sigma^2)}{1 - y}$$

to obtain the following expression for the principal's payoff.

Lemma 17 *The pair of one-dimensional equations for γ and ψ , namely (14) and (15), is equivalent to a pair of one-dimensional equations for γ and χ , namely*

$$\begin{aligned} 0 = \max_{z,g} & \left\{ y(1-y) \Phi(g, y, \gamma) - y(1-y) \chi + \frac{1}{2} (Ry + r(1-y)) \sigma^2 \gamma \right. \\ & + \frac{1}{2} \left((1-y)^2 \gamma' - \frac{R}{y} \gamma \right) \sigma^2 - \left((1-y) \gamma' - \frac{R}{y} \gamma \right) (1-y) z \sigma^2 \\ & \left. + \frac{1}{2} \left(\gamma' - \frac{R}{y} \gamma - \frac{r}{1-y} \gamma \right) (1-y)^2 z^2 \sigma^2 \right\}, \end{aligned} \quad (16)$$

and

$$\beta_R \gamma = \beta + ((1 - R) - (2 - R)y) \chi + y(1 - y) \chi'. \quad (17)$$

Proof. The only term in equation (14) that involves ψ is the second term, namely

$$y((\mu - \frac{1}{2} R \sigma^2) - \beta_R \psi).$$

Putting

$$\psi = \frac{\mu - \frac{1}{2} R \sigma^2 + (1 - y) \chi}{\beta_R}$$

in this term and rearranging yields the corresponding term in equation (16), namely

$$-y(1 - y) \chi.$$

We also need to put $s = (1 - y) z$. Similarly, we can substitute for ψ and

$$\psi' = \frac{-\chi + (1 - y) \chi'}{\beta_R}$$

in equation (15), and rearrange, to obtain equation (17). ■

6.5 A pair of one-dimensional equations for γ and χ

We are now in a position to derive the definitive form of our pair of one-dimensional equations for γ and χ . All that remains is to eliminate g and z from equations (16) and (17). Far from being more complex, as one might have expected, the resulting pair of equations is actually simpler.

The only caveat is that there is no explicit formula for the optimal g (except when $y = 0$ or $y = 1$). Instead, we work with the function ϕ given by the formula

$$\phi(y, \gamma) = \max_g \{\Phi(g, y, \gamma)\}$$

for all $y \in [0, 1]$ and $\gamma \in (0, \infty)$. This leads to the following result.

Proposition 18 *The two-dimensional equation for V , namely (Bellman 3), can be reduced to a pair of one-dimensional equations for γ and χ , namely*

$$y(1-y)\gamma' = \frac{\gamma}{\chi - \phi} \left(((1-y)R + yr)(\chi - \phi) - \frac{1}{2}(R-r)^2\sigma^2\gamma \right) \quad (18)$$

and

$$y(1-y)\chi' = \beta_R\gamma - \beta - ((1-R) - (2-R)y)\chi. \quad (19)$$

Moreover the optimal risk-sharing rules take the form

$$z = 1 + \frac{y(\chi - \phi)}{\frac{1}{2}(R-r)\sigma^2\gamma} \quad (20)$$

and

$$Z = 1 - \frac{(1-y)(\chi - \phi)}{\frac{1}{2}(R-r)\sigma^2\gamma}. \quad (21)$$

Proof: See the Appendix. ■

Remarkably, we obtain yet another characterization of optimal risk-sharing from the equations (20) and (21):

Corollary 19 *The more risk averse party takes on less than its share of the total risk: if $R > r$ then $Z < 1 < z$, and if $R < r$, then $z < 1 < Z$. ■*

Remark 20 *Economic intuition tells us that, if $R \neq r$, then we must have $\chi > \phi > 0$ for all $y \in [0, 1]$. Indeed, ϕ is the gain from sharing the termination risk, $\chi - \phi$ is the gain from sharing the investment risk, and χ is the total gain from sharing risk.*

The system (18-19) and the equations (20) and (21) are simple enough to lend themselves to numerical analysis. To proceed further we need to specify boundary conditions for γ and χ .

7 Boundary Conditions

In order to identify the boundary conditions for χ and γ , we put $y = 0$ and $y = 1$ in the system (18-19).

7.1 Boundary conditions at $y = 0$

Recall that at $y = 0$ the function Φ is given by:

$$\Phi(g, 0, \gamma) = \beta C_R(1 - g) + \gamma \beta_r g$$

for $g \in (-\infty, 1)$ and $\gamma \in (0, \infty)$. The first-order condition for g then gives

$$\gamma = \frac{\beta}{\beta_r} C'_R(1 - g), \quad (22)$$

and the definition of ϕ gives

$$\phi = \beta C_R(1 - g) + \gamma \beta_r g = \beta (C_R(1 - g) + C'_R(1 - g) g). \quad (23)$$

Setting $y = 0$ in the system (18-19) we also obtain the equations for

$$\chi = \phi + \frac{1}{2} \sigma^2 \frac{(R - r)^2}{R} \gamma, \quad (24)$$

and

$$\gamma = \frac{\beta + (1 - R)\chi}{\beta_R}. \quad (25)$$

Solving this system of four equations for the four unknowns, γ, χ, ϕ and g , and observing that we cannot have $g \geq 1$ or $\gamma \leq 0$ we obtain the following boundary conditions:

Proposition 21 *Boundary conditions at $y = 0$:*

$$g(0) = \frac{1}{R} + \frac{(1 - R)(R - r)^2 \sigma^2}{2R^2 \beta_r} - \frac{\beta_R}{R\beta_r},$$

$$\begin{aligned}\gamma(0) &= \frac{\beta C'_R(1-g(0))}{\beta_r}, \\ \chi(0) &= \frac{\beta_R}{\beta_r} \left(\frac{\beta C'_R(1-g(0))}{1-R} \right) - \frac{\beta}{1-R}, \\ \phi(0) &= \beta C'_R(1-g(0)) \left(\frac{1-g(0)}{1-R} + g(0) \right) - \frac{\beta}{1-R}\end{aligned}$$

and

$$\beta_R > (1-R) \left(\beta_r + \frac{(R-r)^2}{2R} \sigma^2 \right).$$

Proof. Eliminating $\beta C'_R(1-g)$ from the system (22-25), solving for g , and noting that $g < 1$, we obtain the desired result. ■

7.2 Boundary conditions at $y = 1$

At $y = 1$ the function Φ takes the form:

$$\Phi(g, 1, \gamma) = -\beta g + \gamma \beta_r C_r(1+g)$$

for $g \in (-1, +\infty)$ and $\gamma \in (0, \infty)$. Proceeding as before, we obtain a system of four equations in four unknowns, γ, ϕ, χ and g , from the first-order condition for g :

$$\gamma = \frac{\beta}{\beta_r} C'_R(1-g), \tag{26}$$

the definition of ϕ ,

$$\phi = -\beta g + \gamma \beta_r C_r(1+g), \tag{27}$$

and, setting $y = 1$ in the system (18-19), from the equations for γ and χ :

$$0 = r(\chi - \phi) - \frac{1}{2}(R-r)^2 \sigma^2 \gamma, \tag{28}$$

$$0 = \beta_R \gamma - \beta + \chi \tag{29}$$

Solving this system and observing that we cannot have $g \leq -1$ or $\gamma \leq 0$ we obtain the boundary conditions at $y = 1$:

Proposition 22 *Boundary conditions at $y = 1$:*

$$(1 + g(1))^{1-r} = \frac{1}{r} - (1 - r) \left[\frac{\beta_R}{r\beta_r} + \frac{(R - r)^2 \sigma^2}{2r^2\beta_r} \right] \quad (30)$$

$$\gamma(1) = \frac{\beta(1 + g(1))}{\beta_R + \beta_r C_r(1 + g(1)) + \frac{(R-r)^2 \sigma^2}{2r}},$$

$$\chi(1) = \beta - \frac{\beta(1 + g(1))}{1 + \frac{\beta_r}{\beta_R} C_r(1 + g(1)) + \frac{(R-r)^2 \sigma^2}{2r\beta_R}},$$

$$\phi(1) = \frac{\beta_r C_r(1 + g(1))\beta(1 + g(1))}{\beta_R + \beta_r C_r(1 + g(1)) + \frac{(R-r)^2 \sigma^2}{2r}} - \beta g(1)$$

and

$$\beta_r > (1 - r) \left(\beta_R + \frac{(R - r)^2}{2r} \sigma^2 \right)$$

Proof. Eliminating χ and ϕ from the system (26-29), solving for

$$(1 + g) C_r'(1 + g) = (1 + g)^{1-r},$$

and noting that $g > -1$, we obtain the desired result. ■

8 Numerical Solutions

To solve the system (18-19) numerically, with the boundary values for γ , ϕ and χ given above, we use the boundary value problem solver for ordinary differential equations, *bvp4c* by MATLAB (see MATLAB version 7, 2005). To keep our discussion as brief as possible we only report numerical solutions for the case in which the principal has a higher coefficient of risk-aversion than the agent, i.e. for the case $R > r$. Our goal here is to illustrate how the qualitative predictions obtained from the formulae for z^* and g^* in respectively (9) and (13) match the numerical solutions, even though we are looking at numerical solutions for a very low value for β (namely $\beta = 0.05$), for which there is no a priori reason to expect that the approximations would be relevant.

8.1 Solutions for z^*

The following three sets of pictures display solutions for the three most interesting parameter constellations:

1. When both parties are fairly risk tolerant, i.e. when $2 > R > r$, the agent always insures the principal more than he would in a static risk-sharing contract.
2. When both parties are fairly risk averse, i.e. when $R > r > 2$, the agent always insures the principal less than he would in a static risk-sharing contract.
3. When the principal is fairly risk averse and the agent is fairly risk tolerant, i.e. when $R > 2 > r$, the sign of the difference between the optimal dynamic risk-sharing rule and the optimal static risk-sharing rule depends on the distribution of wealth.

Consider first Figures 1 and 2. Figure 1 plots the optimal rule z_* and the myopic rule $z_*^{(0)} = \frac{R}{R(1-y)+ry}$ for the parameter values $R = 1.9$, $r = 0.8$, $\mu = 0.05$, $\sigma = 0.15$ and $\beta = 0.05$, so that both parties are fairly risk tolerant. For these parameter values, formula (9) suggests that the principal will demand less insurance from the agent than would be optimal under the myopic rule. This is exactly what the figures show.

Similarly, Figure 2 plots the optimal rule z_* and the myopic rule $z_*^{(0)} = \frac{R}{R(1-y)+ry}$ for the parameter values $R = 8.1$, $r = 2.1$, $\mu = 0.05$, $\sigma = 0.15$ and $\beta = 0.05$, so that both parties are fairly risk averse. One would therefore expect the principal to demand substantial insurance from the agent. This is indeed the case: the myopic rule $z_*^{(0)}$, which measures the agent's risk exposure relative to his wealth, rises from 1 when $y = 0$ (at which point

the agent has all the wealth) to 4.5 when $y = 1$ (at which point the agent has none of the wealth). Moreover the optimal dynamic rule z_* , which again measures the agent's risk exposure relative to his wealth, assigns even more risk to the agent (as long as $0 < y < 1$). Both of these findings are in agreement with the qualitative predictions of formula (9).

Finally, Figure 3 plots the optimal rule z_* and the myopic rule $z_*^{(0)}$ for the parameter values $R = 13$, $r = 1.4$, $\mu = 0.05$, $\sigma = 0.15$ and $\beta = 0.05$. For these parameter values, formula (9) leads us to expect that the principal will demand less insurance than would be optimal under the myopic rule when $y > 0.948$, and more insurance than would be optimal under the myopic rule when $y < 0.948$. This is exactly what Figure 3 shows.

These figures also reveal that $s_* = z_*(1 - y)$ always lies in the interval $[0, 1]$ for the parameter values we have considered. In other words, neither party goes short the risky asset. It seems likely that this is a general result, but unfortunately we do not have a proof of this seemingly simple result.

8.2 Solutions for g_*

The parameter values we have selected to solve for g_* are motivated primarily by our desire to illustrate all the possible shapes g_* can take, and to highlight the extraordinary accuracy of some of the predictions arising from the formula for g_* in (13). As can be seen from the formula, depending on the parameter values, g_* can be everywhere positive, everywhere negative, cross the zero axis once or even twice. Also, when:

1. Both parties are fairly risk tolerant ($2 > R > r$), the formula in (13) predicts that g_* is decreasing in y .
2. Both parties are fairly risk averse ($R > r > 2$), the formula in (13) predicts that g_* is increasing in y .

3. The principal is fairly risk averse and the agent is fairly risk tolerant ($R > 2 > r$) the formula in (13) predicts that g_* is hump-shaped in y .

We illustrate each of these possibilities in figures 4, 5, 6 and 7. In figures 4 and 5 we use the same parameter values as in respectively figures 1 and 2. The last two figures, 6 and 7, are based on different parameter values than in Figure 3, mainly because the shape of g_* predicted by the formula in (13) is easier to see in the plotted solutions. Again, what is remarkable from these figures is just how accurately the formula for g_* in (13) predicts the true solution.

9 Conclusion

In this paper we have analyzed an optimal risk-sharing problem between two agents investing in a common constant-returns-to-scale risky asset, where the two agents have different constant coefficients or relative risk aversion and start with a different finite wealth endowment. We have taken out many interesting features from the model to keep the analysis tractable. In particular, we have only allowed for consumption at the end and we have only considered an extreme bargaining situation, where one of the parties can make take-it-or-leave-it offers. Within this model, we have, however, been able to push the characterization of optimal risk-sharing quite far, obtaining formulae for optimal risk-sharing rules that are approximately optimal. These rules capture in a transparent way the main tradeoffs the contracting parties face in the recursive formulation for the risk-sharing problem (in particular the tradeoff between getting more insurance coverage today versus preserving future insurance options). Moreover, these rules are explicit and easy to apply.

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10 Appendix

1. **1.** Proof of proposition 5:

Proof. Substituting the expressions

$$V_T = U' \left(W + \frac{1}{\beta} (P + G) \right) \left(1 + \frac{1}{\beta} (P_T + G_T) \right),$$

$$v_w = u' \left(w + \frac{1}{\beta} A \right) \left(1 + \frac{1}{\beta} A \right),$$

$$V_{TT} = U' \left(W + \frac{1}{\beta} (P + G) \right) \left[\frac{1}{\beta} G_{TT} - \frac{R}{W} \left(1 + \frac{1}{\beta} (P_T + G_T) \right)^2 \right],$$

$$V_{TW} = U' \left(W + \frac{1}{\beta} (P + G) \right) \left[\frac{1}{\beta} G_{TW} - \frac{R}{W} \left(1 + \frac{1}{\beta} (P_T + G_T) \right) \left(1 + \frac{1}{\beta} (P_W + G_W) \right) \right],$$

and

$$v_{ww} = u' \left(w + \frac{1}{\beta} A \right) - \frac{r}{w} \left(1 + \frac{1}{\beta} A \right)^2$$

into the general formula for s_* :

$$s_* = \frac{-\frac{V_{TW}}{V_T}}{-\frac{V_{TT}}{V_T} - \frac{v_{ww}}{v_w}}$$

we obtain the expanded expression for s_* :

$$s_* = \frac{\frac{R}{W} + \frac{1}{\beta} (P_W + G_W) \left(\frac{R}{W} - \frac{G_{TW}}{P_W + G_W} \right)}{\left(\frac{R}{W} + \frac{r}{w} \right) \left(1 + \frac{1}{\beta} \frac{(P_T + G_T) \left(\frac{R}{W} - \frac{G_{TT}}{P_T + G_T} \right)}{\frac{R}{W} + \frac{r}{w}} \right) + \frac{1}{\beta} \frac{A_w \frac{r}{w}}{\frac{R}{W} + \frac{r}{w}}},$$

or inverting,

$$s_* = \left(\frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}} \right) \left[1 + \frac{1}{\beta} (P_W + G_W) \left(1 - \frac{W}{R} \frac{G_{TW}}{P_W + G_W} \right) \right] \times \left[1 - \frac{1}{\beta} \frac{(P_T + G_T) \left(\frac{R}{W} - \frac{G_{TT}}{P_T + G_T} \right)}{\frac{R}{W} + \frac{r}{w}} - \frac{1}{\beta} \frac{A_w \frac{r}{w}}{\frac{R}{W} + \frac{r}{w}} \right],$$

or, ignoring terms of order two or higher in $\frac{1}{\beta}$, we obtain:

$$s^{(1)} = \left(\frac{\frac{R}{W}}{\frac{R}{W} + \frac{r}{w}} \right) \left[1 + \frac{1}{\beta} (P_W + G_W) \left(1 - \frac{W}{R} \frac{G_{TW}}{P_W + G_W} \right) - \frac{1}{\beta} \frac{(P_T + G_T) \left(\frac{R}{W} - \frac{G_{TT}}{P_T + G_T} \right)}{\frac{R}{W} + \frac{r}{w}} - \frac{1}{\beta} \frac{A_w \frac{r}{w}}{\frac{R}{W} + \frac{r}{w}} \right].$$

■

Proof. Finally, substituting for:

- $A_w = (\mu - \frac{1}{2}r\sigma^2)$
- $P_W = (\mu - \frac{1}{2}R\sigma^2)$ and $G_W = \frac{1}{2}(R - r)^2 \sigma^2 \left[\frac{\frac{R}{W^2}}{(\frac{R}{W} + \frac{r}{w})^2} \right]$,
- $P_T = (\mu - \frac{1}{2}R\sigma^2)$ and $G_T = \frac{1}{2}(R - r)^2 \sigma^2 \left[\frac{\frac{R}{W^2} - \frac{r}{w^2}}{(\frac{R}{W} + \frac{r}{w})^2} \right]$,
- $G_{TW} = \frac{1}{2}(R - r)^2 \sigma^2 \left[\frac{-\frac{2Rr}{W^3w} - \frac{2rR}{w^2W^2}}{(\frac{R}{W} + \frac{r}{w})^3} \right]$, and
- $G_{TT} = \frac{1}{2}(R - r)^2 \sigma^2 \left[\frac{-\frac{4Rr}{W^3w} - \frac{4rR}{w^2W^2} - 2r^2(\frac{1}{W^4} - \frac{1}{w^2W^2})}{(\frac{R}{W} + \frac{r}{w})^3} \right]$,

and rearranging we obtain the final formula:

$$s^{(1)} = \frac{Rr(R - r)^3 \sigma^2 W^2 w^2 ((R - 2)w + (r - 2)W)}{2(Rw + rW)^5}.$$

■

2. Proof of Lemma 13:

Proof. Differentiating the formula obtained for V_T in Lemma 10 with respect to T , we obtain

$$\begin{aligned} V_{TT} &= \gamma' \frac{\partial y}{\partial T} C'_R + \gamma C''_R \\ &= \gamma' \frac{1}{W + w} C'_R + \gamma \frac{W}{y(W + w)} C''_R \\ &= \left(\gamma' - \frac{R}{y} \gamma \right) \frac{C'_R}{W + w}, \end{aligned}$$

as required. Similarly, differentiating the same formula with respect to W , we obtain

$$\begin{aligned}
V_{TW} &= \gamma' \frac{\partial y}{\partial W} C'_R + \gamma C''_R \\
&= \gamma' \frac{1-y}{W+w} C'_R + \gamma \frac{W}{y(W+w)} C''_R \\
&= \left((1-y) \gamma' - \frac{R}{y} \gamma \right) \frac{C'_R}{W+w}.
\end{aligned}$$

The required formula for V_{WT} follows on noting that $V_{WT} = V_{TW}$. Finally, differentiating the formula obtained for V_W in Lemma 10 with respect to W , we obtain

$$\begin{aligned}
V_{WW} &= ((1 + (1 - R) \psi) + y(1 - R) \psi' - \gamma + (1 - y) \gamma') \frac{\partial y}{\partial W} C'_R \\
&\quad + (y(1 + (1 - R) \psi) + (1 - y) \gamma) C''_R \\
&= ((1 + (1 - R) \psi) + y(1 - R) \psi' - \gamma + (1 - y) \gamma') \frac{1-y}{W+w} C'_R \\
&\quad + (y(1 + (1 - R) \psi) + (1 - y) \gamma) \frac{W}{y(W+w)} C''_R \\
&= \left((1 - y) \left((1 + (1 - R) \psi) + y(1 - R) \psi' - \gamma + (1 - y) \gamma' \right) \right. \\
&\quad \left. - \frac{R}{y} \left(y(1 + (1 - R) \psi) + (1 - y) \gamma \right) \right) \frac{C'_R}{W+w}.
\end{aligned}$$

The required formula for V_{WW} then follows on noting that $y \psi' = \gamma - (1 + (1 - R) \psi)$ and collecting terms in γ' , γ and $\gamma - (1 + (1 - R) \psi)$. ■

3. Proof of Lemma 14:

Proof. We have

$$\begin{aligned}
\frac{U(W - b(W + w)) - V}{(W + w) C'_R(W)} &= \frac{C_R(W - b(W + w)) - C_R(\rho W)}{(W + w) C'_R(W)} \\
&= \frac{W}{W + w} \frac{C_R(W - b(W + w)) - C_R(\rho W)}{W C'_R(W)} \\
&= y \left(C_R \left(1 - \frac{b}{y} \right) - C_R(\rho) \right) \\
&= y (C_R(1 - (1 - y)g) - \psi).
\end{aligned}$$

Similarly, recalling that $v_w = \gamma_A C'_r(w)$, we obtain:

$$\begin{aligned}
\frac{V_T}{v_w} \frac{u(w + b(W + w)) - v}{(W + w) C'_R(W)} &= \frac{\gamma C'_R(W)}{\gamma_A} \frac{C_r(w + b(W + w)) - C_r(\rho_A w)}{(W + w) C'_R(W)} \\
&= \frac{w}{W + w} \frac{\gamma}{\gamma_A} \frac{C_r(w + b(W + w)) - C_r(\rho_A w)}{w C'_r(w)} \\
&= (1 - y) \frac{\gamma}{\gamma_A} \left(C_r \left(1 + \frac{b}{1 - y} \right) - C_r(\rho_A) \right) \\
&= (1 - y) \frac{\gamma}{\gamma_A} (C_r(1 + yg) - \psi_A).
\end{aligned}$$

This completes the proof of the lemma. ■

4. Proof of Lemma 15:

Proof. The objective in the Bellman equation for V is

$$\begin{aligned}
&\beta (U(W - b(W + w)) - V) + \beta \frac{V_T}{v_w} (u(w + b(W + w)) - v) + V_W \mu(W + w) \\
&\quad + \frac{1}{2} \left(V_{WW} - 2V_{WT} s + \left(V_{TT} + \frac{v_{ww}}{v_w} V_T \right) s^2 \right) \sigma^2 (W + w)^2.
\end{aligned}$$

Making the substitution $V = C_R(W) + W C'_R(W) \psi$ in this expression, using Lemma 10, Corollary 11, Lemma 13 and Lemma 14 to simplify the individual

terms that result, and dividing through by $(W + w) C'_R(W)$, we obtain

$$\begin{aligned} & \beta y (C_R(1 + (1 - y)g) - \psi) + \beta(1 - y) \frac{\gamma}{\gamma_A} (C_r(1 + yg) - \psi_A) \\ & \quad + \mu((1 - y)\gamma + y(1 + (1 - R)\psi)) \\ & \quad + \frac{1}{2} \left((1 - y)^2 \gamma' - \frac{R}{y} \gamma + Ry(\gamma - 1 - (1 - R)\psi) \right) \sigma^2 \\ & \quad - \left((1 - y)\gamma' - \frac{R}{y} \gamma \right) s \sigma^2 + \frac{1}{2} \left(\gamma' - \frac{R}{y} \gamma - \frac{r}{1 - y} \gamma \right) s^2 \sigma^2. \end{aligned}$$

This new expression can be divided into three parts, namely

$$\beta y C_R(1 - (1 - y)g) + \beta(1 - y) \frac{\gamma}{\gamma_A} C_r(1 + yg) \quad (31)$$

and

$$\begin{aligned} & -\beta y \psi - \beta(1 - y) \frac{\gamma}{\gamma_A} \psi_A + \mu((1 - y)\gamma + y(1 + (1 - R)\psi)) \\ & \quad + \frac{1}{2} Ry(\gamma - 1 - (1 - R)\psi) \sigma^2 \end{aligned} \quad (32)$$

and

$$\begin{aligned} & \frac{1}{2} \left((1 - y)^2 \gamma' - \frac{R}{y} \gamma \right) \sigma^2 \\ & - \left((1 - y)\gamma' - \frac{R}{y} \gamma \right) s \sigma^2 + \frac{1}{2} \left(\gamma' - \frac{R}{y} \gamma - \frac{r}{1 - y} \gamma \right) s^2 \sigma^2. \end{aligned} \quad (33)$$

Now, noting that v can be expanded in the form

$$v(w) = C_r(\rho_A w) = C_r(\rho_A) + \rho_A C'_r(\rho_A) C_r(w) = \psi_A + \gamma_A C_r(w),$$

it is easy to see that

$$\psi_A = \frac{\mu - \frac{1}{2} r \sigma^2}{\beta_r}, \quad \gamma_A = \frac{\beta}{\beta_r}.$$

Substituting for ψ_A in expression (31), we obtain

$$\begin{aligned}
& \beta y C_R(1 - (1 - y)g) + \gamma \beta_r (1 - y) C_r(1 + yg) \\
= & y(1 - y) \left(\beta \frac{C_R(1 - (1 - y)g)}{1 - y} + \gamma \beta_r \frac{C_r(1 + yg)}{y} \right) \\
= & y(1 - y) \Phi(g, y, \gamma).
\end{aligned} \tag{34}$$

Substituting for ψ_A and γ_A in expression (32), and collecting terms in ψ and γ , we obtain

$$-\beta_R y \psi + \frac{1}{2} (Ry + r(1 - y)) \sigma^2 \gamma + (\mu - \frac{1}{2} R \sigma^2) y. \tag{35}$$

Finally, adding expressions (34), (35) and (33), we obtain the required result.

■

5. Proof of Proposition 20:

Proof. Optimizing with respect to g in equation (16), we obtain an equation of the form

$$0 = \max_z \{ (a_0 + a_1 \gamma') - 2(b_0 + b_1 \gamma')z + (c_0 + c_1 \gamma')z^2 \}.$$

Provided that $c_0 + c_1 \gamma' < 0$, which in the present case amounts to the requirement that

$$\frac{\gamma'}{\gamma} < \frac{R}{y} + \frac{r}{1 - y},$$

the optimal z is given by the formula

$$z = \frac{b_0 + b_1 \gamma'}{c_0 + c_1 \gamma'}.$$

Substituting back into the original equation, we obtain a quadratic equation for γ' , namely

$$0 = (a_0 + a_1 \gamma') (c_0 + c_1 \gamma') - (b_0 + b_1 \gamma')^2.$$

Ordinarily, this would lead to two solutions for γ' . However, in the present case, we have

$$a_1 c_1 = b_1^2.$$

This quadratic equation therefore reduces to a linear equation, namely

$$0 = (a_0 c_0 - b_0^2) + (a_0 c_1 + a_1 c_0 - 2 b_0 b_1) \gamma'.$$

Provided that

$$a_0 c_1 + a_1 c_0 - 2 b_0 b_1 \neq 0,$$

which in the present case amounts to the requirement that

$$-\frac{1}{2} \sigma^2 y (1 - y) (\chi - \phi) \neq 0,$$

this equation can be solved to obtain

$$\gamma' = -\frac{a_0 c_0 - b_0^2}{a_0 c_1 + a_1 c_0 - 2 b_0 b_1}.$$

Substituting for a_0 , a_1 , b_0 , b_1 , c_0 and c_1 , and rearranging, we arrive at equation (18). Similarly, substituting for b_0 , b_1 , c_0 and c_1 in the expression for z , and rearranging, we obtain equation (20). Equation (19) is identical to equation (17). Finally, equation (21) follows at once from equation (20) on noting that $ZW + zw = W + w$, or $Zy + z(1 - y) = 1$. ■

Remark 23 *Rearranging equation (18), we obtain*

$$\frac{\gamma'}{\gamma} = \frac{R}{y} + \frac{r}{1 - y} - \frac{\frac{1}{2}(R - r)^2 \sigma^2 \gamma}{y(1 - y)(\chi - \phi)}.$$

Since $\gamma > 0$ and $\chi > \phi$, it follows that

$$\frac{\gamma'}{\gamma} < \frac{R}{y} + \frac{r}{1 - y}.$$

In other words, the first of the two conditions that arose in the course of the proof of Proposition 18 is satisfied. On the other hand, since $\chi > \phi$, it follows that

$$-\frac{1}{2}\sigma^2 y(1-y)(\chi - \phi) < 0.$$

In other words, the second of the two conditions is also satisfied.

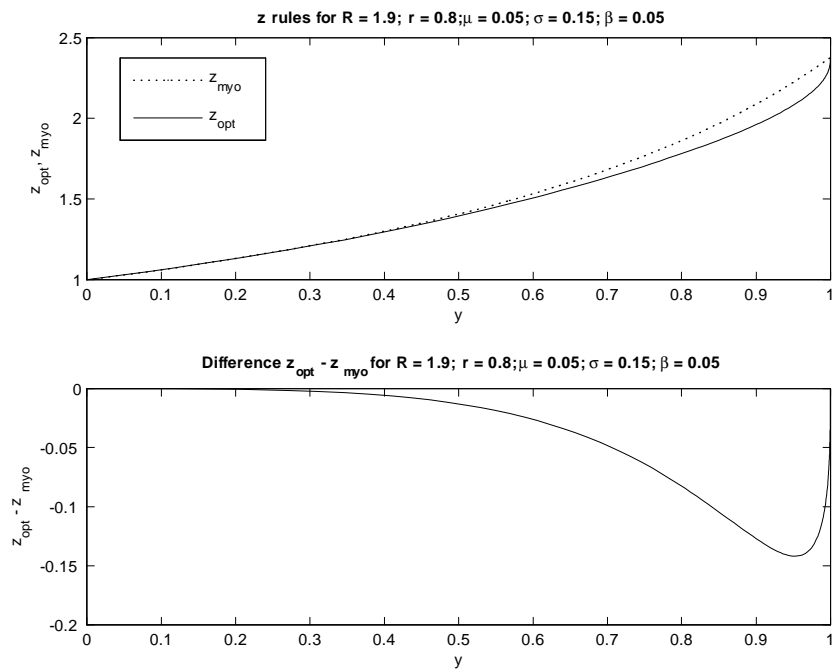


Figure 1: Optimal and myopic sharing rules when $2 > R > r$

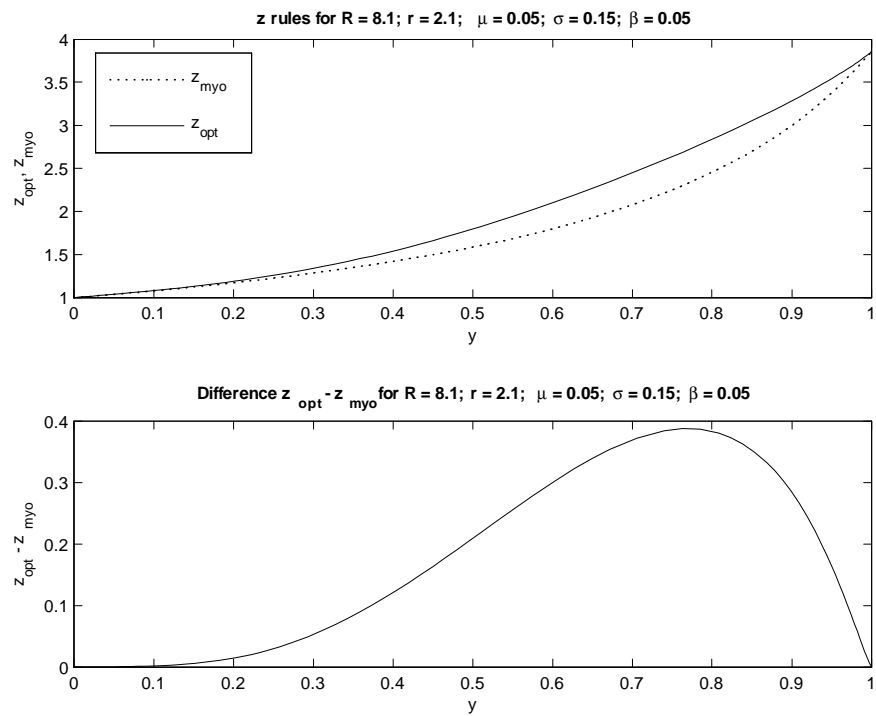


Figure 2: Optimal and myopic sharing rules when $R > r > 2$

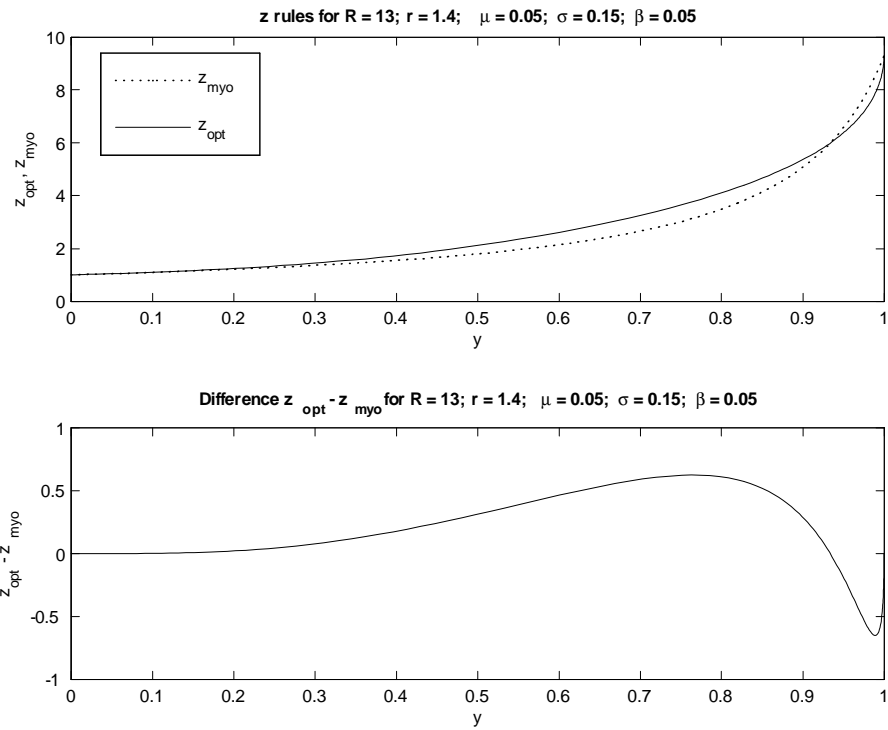


Figure 3: Optimal and myopic sharing rules when $R > 2 > r$

