Von Neumann-Morgenstern Stable Sets:

Generalizing the Solutions of Three-Person Games

One way of looking at three-person games is as games where coalitions of only three sizes are of relevance: 1-, 2-, and 3-player coalitions.

A natural first-step generalization to n-player games is to continue to consider games where coalition of only three distinct sizes – 1, k, and n – are *vital*, i.e., where

$$v(S) = \sum_{i \in S} v(i)$$

for $|S| \neq 1, k, \text{or } n$.

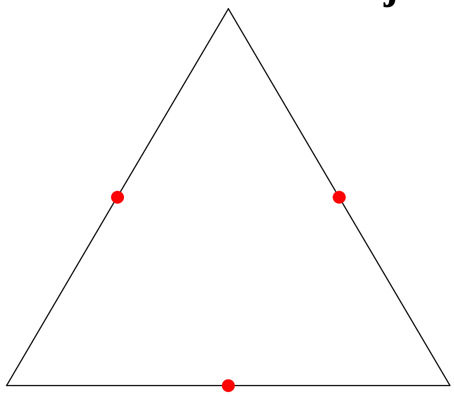
Here, we'll see how the vNM solutions of three-person games generalize to games where only the 1, n-1, and n-player coalitions are vital.

And we'll see how important the notion of "blocking" can be, as the solutions we derive will exhibit a "pairing off" of the players.

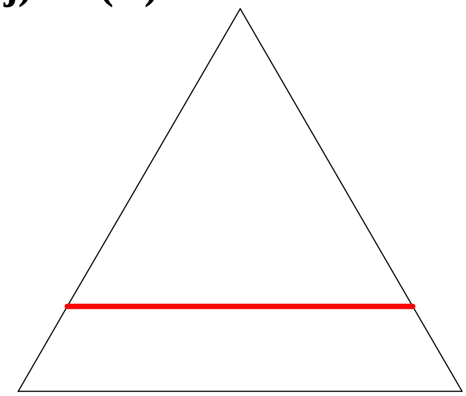
Note: All games in the following discussion are 0-normalized, i.e.,

$$v(\varnothing) = v(i) = 0 .$$

The Majority Game: v(ij) = v(N) = 1

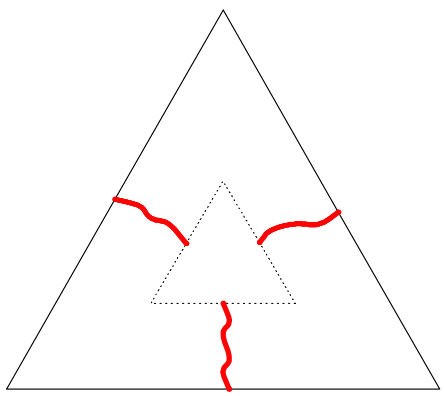


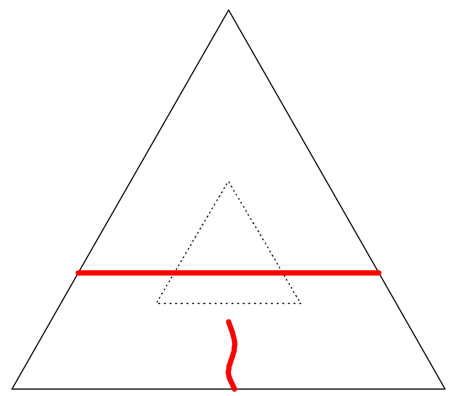
Symmetric



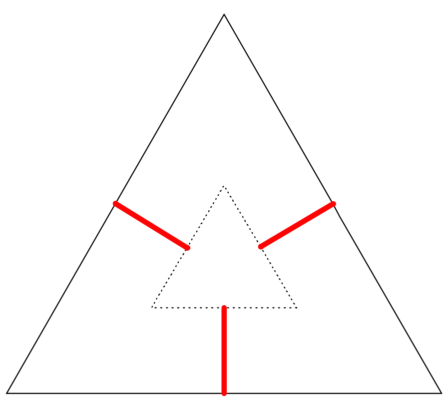
Discriminatory

More generally, games with empty core: v(ij) = 0.8, v(N) = 1

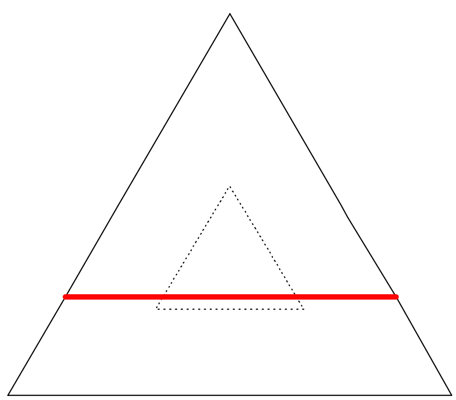




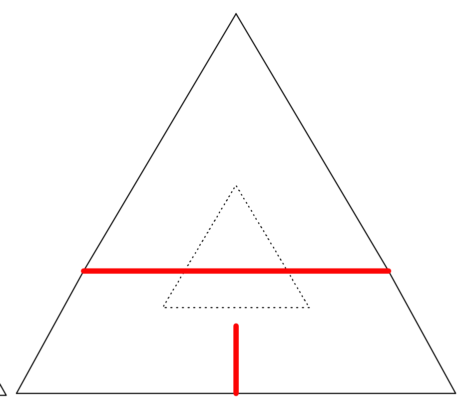
Symmetrization and semi-discriminatory symmetrizations



Lucas

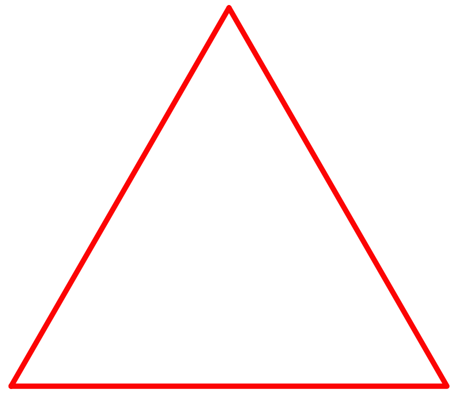


Owen (and me)

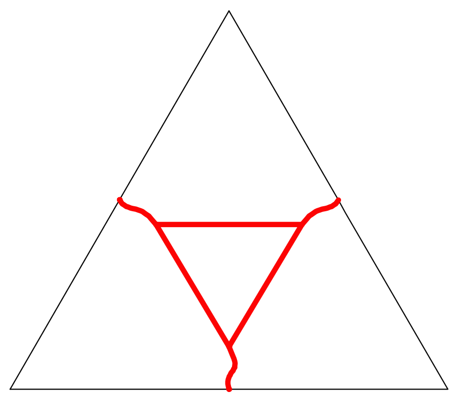


Me

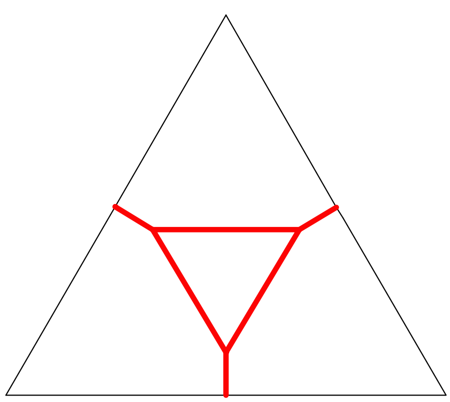
The Unanimity Game: v(ij) = 0, v(N) = 1



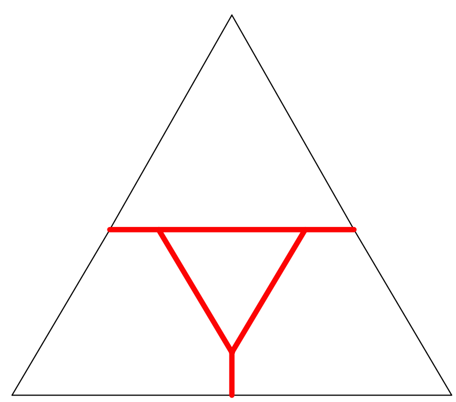
More generally, games with nonempty core: v(ij) = 0.6, v(N) = 1



Symmetrization and semi-discriminatory symmetrization



Lucas



Me

References:

Bill Lucas: The symmetric generalization.

n-Person Games with only 1, n-1, and n-Person Coalitions

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 6 (1966).

Guillermo Owen: Necessary and sufficient conditions for the existence of purely-discriminatory solutions.

n-Person Games with only 1, n-1, and n-Person Coalitions

Proc. Amer. Math. Soc. 19 (1968).

Me: The symmetric semi-discriminatory generalization.

A Generalized Discriminatory Solution for a Class of n-Person Games

Senior thesis under Al Tucker (1969), Tech. Rpt. 174, Cornell OR Department.

The Lucas stable sets

Let v be a game on N with only 1, n-1, and n-player vital coalitions.

Let $d_i = v(N) - v(N\backslash i)$. (This is the "blocking" power of player i.)

Define:

$$L_{\sigma_r} = \{x \in X : x_{\sigma(i-1)} - d_{\sigma(i-1)} = x_{\sigma(i)} - d_{\sigma(i)} \geq 0, \quad i = 2,4,\ldots,2r$$

$$x_{\sigma(j)} - d_{\sigma(j)} \leq 0, \quad j = 2r+1,\ldots,n\}$$

where σ is one of the $n! / 2^r r!(n-2r)!$ essentially different permutations of N selecting r distinct pairs of players.

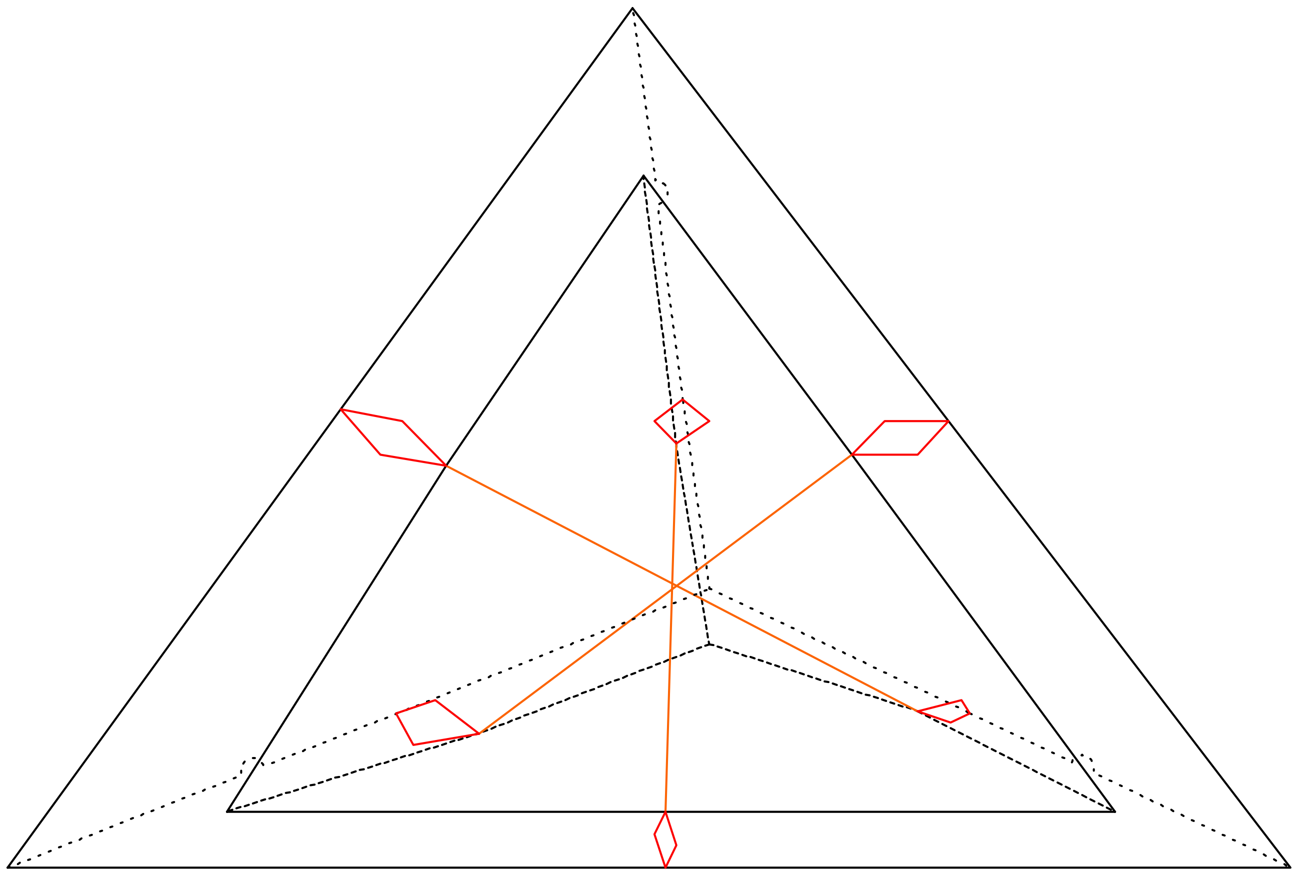
Then

$$K = \bigcup_{r=0}^{[n/2]} \bigcup_{\sigma_r} L_{\sigma_r}$$

is a stable set of v.

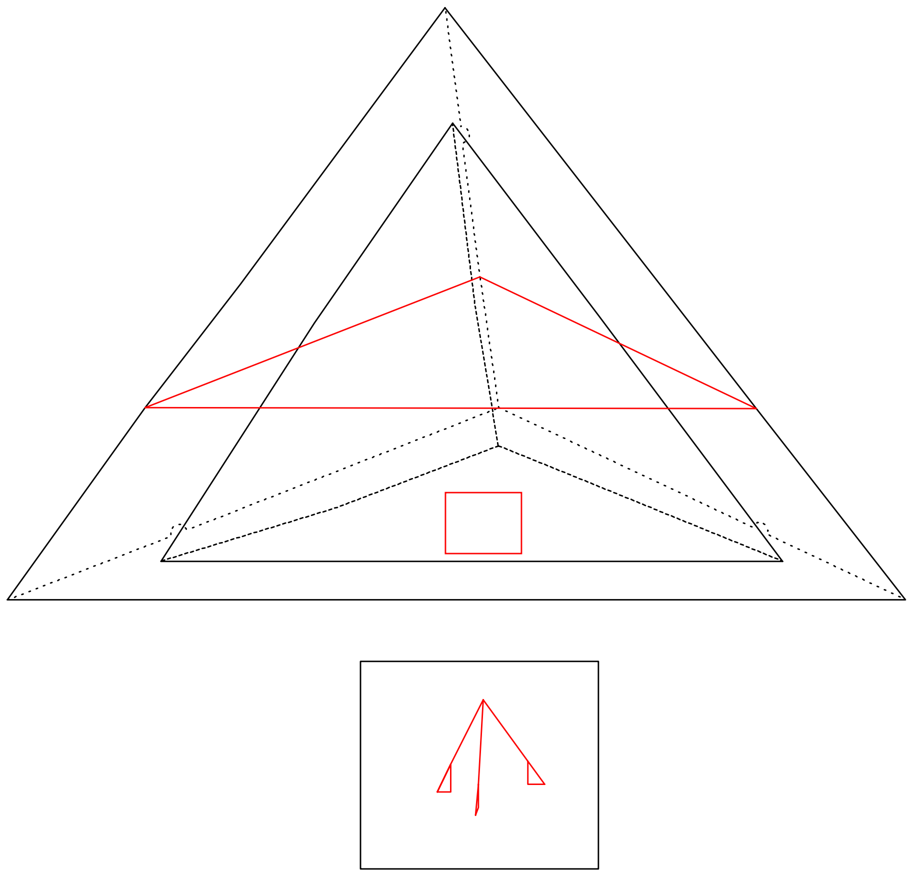
(Mine are messier, with a discriminatory component, and two L-type components: one constrains the non-paired players' gains relative to the paired players', and both constrain the sum of the paired players' gains relative to the discriminated player's gain. In compensation, when the core is empty, I get an infinite family of solutions. ☺)

The Lucas “Six Flags” solution



v(ijk) = 0.8, v(N) = 1

The Me "Little Lucas" solution



Why the "little Lucas" solution? All cross-sections of the lower portion of the stable set are "Lucas" solutions to appropriately-defined 3-person games!

Generally, the "Me" solutions contain, at their "bottoms," sets which have as cross-sections the Lucas solutions to [1, n-2, n-1]-games which result from removing player n from the game (but assuming he's available to any coalition).