On Stability of the Core*

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Abstract

This paper introduces a strengthening of the notion of a stable core and characterizes it in terms of Kikiuta and Shapley's extendability condition.

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1 Introduction

A co-operative game with transferable utility consists of a set N of agents and a characteristic function v that associates with each $S \subseteq N$ a real number $v(S)$. The number $v(S)$ is usually called the value of the coalition S.

An **imputation** for a co-operative game (v, N) is a $x \in \Re^{|N|}$ such that $x_i \geq v(i)$ for all $i \in N$ and $\sum_{i \in N} x_i = v(N)$. The set of imputations is denoted $I(v, N)$. A solution concept is a rule that associates with each (v, N) a subset of $I(v, N)$.

The very first solution concept for co-operative games, proposed by von Neumann and Morgentsern (1944), was called the **stable set**. Their idea was to proposes a partial order over the elements of $I(v, N)$ and select the set of elements that were 'undominated' in that order. This set was called the stable set.

Lucas (1968) showed a co-operative game need not posess a stable set. Subsequent work by others showed that a stable set could be quite complicated. See Lucas (1992) for a survey. Attention shifted to other solution concepts. The most important of these has been the **core** introduced by Gillies (1957). Because of the similarity, in some respects, of the core to the stable set, it has been natural to ask when the core of a co-operative game is a stable set. A number of appealing sufficient conditions (see van Gellekom, Potters and Reijnierse (1999)) for the core to be stable have been identified.

The present paper identifies the key property common to the prior sufficient conditions for stability of the core. The condition suggests a strengthening of the notion of stability which is then completely characterized. We use this strengthening to suggest why no succinct characterization of stable cores is possible. One way to approach this issue is to examine the complexity of determining when the core is stable. We would expect a succinct characterization only if stability of the core could be verified efficiently.

In the next section we introduce the notation and definitions to be used in the remainder of the paper. Subsequently we state the precise result we prove and discuss its relation to prior work. The last section contains the proof. We conclude with a discussion of the question of deciding whether a core is stable.

2 Notation and Definitions

Given a vector $x \in \Re^{|N|}$, we write $\sum_{i \in S} x_i$ for any $S \subseteq N$ as $x(S)$. For any $S \subseteq N$ define $C(v, S)$ to be:

$$\{x \in \Re^{|S|} : x(S) = v(S), x(T) \geq v(T) \ \forall T \subset S\}.$$

$C(v, N)$ is called the **core** of the game (v, N).

A game (v, N) is **balanced** if and only if

$$v(N) \geq \max \sum_{T \subset N} \lambda_T v(T)$$

$$st \sum_{T \ni i} \lambda_T = 1 \ \forall i \in N$$

$$\lambda_T \geq 0 \; \forall T \subset N.$$

It follows from the duality theorem of linear programming that $C(v, N) \neq \emptyset$ if and only if (v, N) is balanced.

(v, N) is said to be **totally balanced** if (v, S) is balanced for all $S \subset N$. A totally balanced game is called **strongly totally balanced** if $C(v, S)$ has non-empty relative interior for all $S \subseteq N$ and $|S| \geq 2$.

A balanced game (v, N) is **exact** iff. for all $T \subset N$ there is an $x \in C(v, N)$ such that $x(T) = v(T)$. It is easy to see that if (v, N) is exact then $C(v, S) \neq \emptyset$ for all $S \subset N$. While an exact game is totally balanced, the converse is not true. Exact games were introduced by Schmeidler (1972).

The core, $C(v, N)$ of a balanced game is **stable** if for all $y \in I(v, N) \setminus C(v, N)$ there is an $x \in C(v, N)$ and $T \subset N$ such that $x(T) = v(T)$ and $x_i > y_i$ for all $i \in T$. In this case x is said to **dominate** y via T.

3 Prior Work

A number of classes of co-operative games are known to have stable cores. The most restrictive of these are where the characteristic function, v, is supermodular. In fact such games are exact (see Shapley(1971)).

Sharkey (1982) showed that if the core of the game is **large** then it is also stable. The core, $C(v, N)$ is called large if for every $y \in \Re^{|N|}$ such that $y(S) \geq v(S)$ for all $S \subseteq N$ there is an $x \in C(v, N)$ such that $x \leq y$. Submodular games have large cores. Also, a totally balanced game with large core is exact.

Subsequently Shapley and Kikuta (1986) introduced the notion of **extendability**, which is implied by largeness of the core, and showed that the core of a game with this property was stable. Extendability means that for each $S \subset N$ and $z \in C(v, S)$ there is an $x \in C(v, N)$ such that $x_i = z_i$ for all $i \in S$. We summarize these implications below.

Supermodularity $\Rightarrow$ Large Core $\Rightarrow$ Extendability $\Rightarrow$ Stability

The reverse implications are all false as discussed in papers by Biswas, Parthasarathy, Potters and Voorneveld (1999) and Biswas, Parthasarathy and Ravindran (2001).

The argument that extendability imply stability is simple. First one picks a $y \in I(v, N) \setminus C(v, N)$. Call a set T such that $y(T) < v(T)$ a **violated** set. Select a minimally violated set, T^*. Choose a $z \in C(T^*, v)$ such that $z_i > y_i$ for all $i \in T^*$. It is easy to see that such a z exists. By extendability there is an $x \in C(v, N)$ such that $x_i = z_i$ for all $i \in T^*$. In effect, each $y \in I(v, N) \setminus C(v, N)$ is not merely dominated by some $x \in C(v, N)$ via a violated set T but by a minimally violated set. This observation prompts the following definition.

The core, $C(v, N)$ of a balanced game is **strongly stable** if for all $y \in I(v, N) \setminus C(v, N)$ and for any minimally violated set T, there is an $x \in C(v, N)$ such that $x(T) = v(T)$ and $x_i > y_i$ for all $i \in T$. One example of a game with a stable core that is not strongly stable is the

5 person game in the proof of Theorem 3 of Biswas, Parthasarathy, Potters and Voorneveld (1999).

We now state our main result.

Theorem 1 *Let (v, N) be a strongly totally balanced game. Then $C(v, N)$ is strongly stable iff. (v, N) is extendable.*

The requirement that (v, N) be a strongly totally balanced game is essential. The 6 person game described below has a strongly stable core but is not extendable. It is based on example 1 of van Gellekom, Potters and Reijnierse (1999).

Let $N = \{1, 2, 3, 4, 5, 6\}$ and $v(N) = 4 - \epsilon$ where $\epsilon \in (0, 1)$. Let $K = \{\{1, 2\}, \{1, 3\}, \{4, 5\}, \{4, 6\}\}$. for any $S \in K$, $v(S) = 1$, for any $S \subset N$ that contains a member of K, $v(S) = 1$, and for any $S \subset N$ that contains two disjoint members of K, $v(S) = 2$. For all other $S \subset N$, $v(S) = 0$. It is easy to see that $C(v, N) \neq \emptyset$ and is totally balanced. To show stability of $C(v, N)$ pick an imputation y. Observe that if y satisfies all core conditions for sets of size 2 then it lies in the core. Hence, if y does not lie in the core it must violate the core constraints associated with an element of K. Suppose then, that $y_1 + y_2 < v(12) = 1$. Consider the imputation

$$x = (y_1 + \frac{1 - (y_1 + y_2)}{2}, y_2 + \frac{1 - (y_1 + y_2)}{2}, y_2 + \frac{1 - (y_1 + y_2)}{2}, 3 - \epsilon - y_2 - \frac{1 - (y_1 + y_2)}{2}, 0, 0).$$

It is straightforward to verify that $x \in C(v, N)$, $x_1 + x_2 = v(12)$, $x_1 > y_1$ and $x_2 > y_2$. To show that the game is not extendable, consider the subgame on the coalition $\{1, 4\}$. The core of this subgame is $x_1 = 0 = x_2$. However, there is no element of $C(v, N)$ for which the first two components are zero.

4 The Proof

Shapley and Kikuta (1986) prove that every extendable game has strongly stable core. We prove the converse by way of contradiction. The following notation and lemma are helpful in proving the theorem.

If v is a characteristic function and $\gamma > 0$ we denote by v^γ the characteristic function where $v^\gamma(S) = v(S)$ for all $S \subset N$ and $v^\gamma(N) = v(N) + \gamma$.

Lemma 1 *If (v, N) is not extendable there exists a $\mu > 0$ such that (v^γ, N) extendable for all $\gamma \geq \mu$ and is not extendable for all $\gamma < \mu$.*

Proof
See page 217 of van Gellekom, Potters and Reijnierse (1999). ■

Corollary 1 *Suppose v is not extendable. Consider a set $S \subset N$, and an $x \in C(v, S)$ such that no extension of x lies in $C(v, N)$. There exist a $\delta_{S,x} > 0$, such that there is an extension of x in $C(v^\gamma, N)$ for any $\gamma \geq \delta_{S,x}$ and no extension of x in $C(v^\gamma, N)$ for any $\gamma < \delta_{S,x}$.*

Lemma 2 *Let (v, N) be a balanced game. Let $S \subset N$ and $x \in \Re^S$ such that for all $i \in S$ $x_i \geq v(i)$ and $x(S) \leq v(S)$ then there exist an extension of x in $I(v, N)$.*

Proof

Construct $y \in \Re^N$ such that $y_i = x_i$ for all $i \in S$ and $y_i = v(i)$ for all $i \in N \setminus S$. If y is in $I(v, N)$ then y is an extension of x which proves the lemma. So assume y is not in $I(v, N)$. The only condition of $I(v, N)$ which could be violated is $y(N) = v(N)$. If $y(N) < v(N)$ then we could take any i in $N \setminus S$ and increase y_i from v_i to $v_i + v(N) - y(N)$. We complete the proof if we show that $y(N) > v(N)$ is not possible. Suppose it is then, $v(N) < y(N) \leq v(S) + \sum_{i \in N \setminus S} v(i)$, which violates the balancedness of (v, N). ■

Lemma 3 *Let (v, N) be a strongly totally balanced game. Consider a set $S \subset N$ and $x \in C(v, S)$. For every $\epsilon > 0$ there exist a $z \in I(v, N)$ such that*

1. *$z(K) \geq v(K)$ for all $K \subset S$,*
2. *$z(S) < v(S)$,*
3. *and $\sum_{j \in S} |x_j - z_j| < \epsilon$.*

Proof

Since (v, N) is strongly totally balanced there is a $y \in C(v, S)$ such that $y(S) = v(S)$ and $y(K) > v(K)$ for all $k \subset S$. Clearly, we can choose non-negative t_i for all $i \in S$ and $w_i = y_i - t_i$ such that $w(S) < v(S)$ and $w(K) \geq v(K)$ for all $K \subset S$. Since the game is totally balanced, by lemma **??** we could extend w to $w' \in I(v, N)$. Using the same lemma we extend x to $x' \in I(v, N)$. Now choose a λ such that $0 < \lambda < 1$, and set $z = \lambda w' + (1 - \lambda)x'$. Since $I(v, N)$ is a convex, and both w' and x' are in $I(v, N)$, $z \in I(v, N)$. Similarly, the first condition $z(K) \geq v(K)$ is satisfied because both w' and x' satisfy the condition. $z(S) < v(S)$ is satisfied because w' has $w'(S) < v(s)$ and x' can't compensate for it because $x'(S) = v(S)$.

For the last condition, one chooses λ sufficiently small. More formally,

$$\sum_{j \in S} |x_j - z_j| = \lambda \sum_{j \in S} |w'_j - x'_j|$$

Therefore any positive λ smaller than $\frac{\epsilon}{\sum_{j \in S} |w'_j - x'_j|}$ will complete the proof. ■

We now prove the main result. Since extendability implies stability it suffices to prove that if a game is strongly stable then it is extendable. Suppose not. Then there is a game (v, N) that is strongly totally balanced with a strongly stable core which is not extendable.

Consider any $S \subset N$ and choose an $x \in C(v, S)$ such that x does not extend to $C(v, N)$. By corollary **??** there exists a $\delta_{S,x}$, such that x does not extend to $C(v^\gamma, N)$ for any $\gamma < \delta_{S,x}$. Choose $\epsilon = \delta_{S,x}/|N|$. By Lemma 1, choose a $z \in I(v, N)$ satisfying conditions (1-3). By condition (2) of Lemma **??**, $z \notin C(v, N)$. Since S is a minimally violated constraint, it follows by strong stability, that there is a $y \in C(v, N)$ such that $y(S) = v(S)$ and $y_j > z_j$ for all $j \in S$.

Now we create $w \in \Re^N$ as follows. For all $i \in S$, $w_i = x_i$. For all $i \in N \setminus S$, $w_i = y_i + \epsilon$. We claim that w lies in $C(v^{|N \setminus S|\epsilon}, N)$. Clearly $w(N) = v(N) + |N \setminus S|\epsilon$. Take anyother T. If $T \subseteq S$ then $w(T) = x(T) \geq v(T)$ follows from $x \in C(v, S)$. If $T \subseteq N \setminus S$ then $w(T) \geq y(T) \geq v(T)$ follows from $y \in C(v, N)$. So we assume that T intersects with S as well as $N \setminus S$.

Note that $y(T) \geq v(T)$. Since $w(T \setminus S) \geq y(T \setminus S) + \epsilon$. So it is sufficient to prove that $w(T \cap S) \geq y(T \cap S) - \epsilon$, i.e., $x(T \cap S) \geq y(T \cap S) - \epsilon$. Suppose it is false, i.e., $x(T \cap S) < y(T \cap S) - \epsilon$. This is equivalent to $x(T \cap S) + z(S \setminus T) < y(T \cap S) + z(S \setminus T) - \epsilon$. This implies, $x(T \cap S) + z(S \setminus T) < v(S) - \epsilon$. This is equivalent to, $x(S \setminus T) - z(S \setminus T) > \epsilon$. This contradicts condition (3) of Lemma **??**.

Therefore we have the claim that w lies in $C(v^{|N \setminus S|\epsilon}, N)$. Since $|N \setminus S|\epsilon < \delta_{S,x}$, we get a contradiction to corollary **??**. This completes the proof of Theorem **??**. ∎

5 Stability and Decidability

There are two problems which naturally arise. First, characterize the set of games which have a stable set. Second, characterize the set of balanced games whose core is stable. One implication of solving these two problems is that we get a finite time algorithms for checking whether a stable set exist or the core is stable resp. It is unlikely that these algorithms will be efficient (polynomial running time in the number of agents).

In this section we solve the second problem. We give an algorithmic characterization of balanced games whose core is stable. Our characterization is based on a propositional logic system over linear constraints in vector space $\Re^N$.

Let V be a set of N variables over reals. A linear constraint is one of the following two forms. A linear expression strictly greater than 0. Or a linear expression at least 0. Note that if L is a linear constraint then so is the **Not** of L, where **Not** has the obvious meaning. Further note that if L is a linear constraint of one form then **Not** of L is the linear constraint of the other. Since we are dealing with digital computers, we assume that all the co-efficients of a linear expression are rational numbers. In fact, by scaling without loss of generality we assume that all the co-efficients of a linear expression are integer.

We further have two more operators, **And** and **Or**. These operators have their obvious meaning. A *proposition* is an expression formed by applying the three operators, **Not**, **And**, and **Or** over linear constraints. A proposition is *feasible* if there is a point in $\Re^N$ satisfying the proposition, it is a *tautology* if all the points in $\Re^N$ satisfy the proposition, and it is a *contradiction* is no point of $\Re^N$ satisfy the proposition. Not that if we have an algorithm to decide whether a proposition is feasible or not then we also have an algorithm to test whether a proposition is a tautology, contradiction or none.(Contradiction means that the proposition is not feasible and tautology means its **Not** is not feasible.)

Theorem 2 *Testing whether a proposition, with all co-efficient integers, is feasible or not is decidable.*

Proof

By De Morgan's law, we could move all the **Not** to the constraint level, which can be absorbed there by negating the constraints themselves. So we assume that a proposition has only **And** and **Or**. We convert the proposition in the disjunctive normal form (DNF) (**Or**s of **And**s). Note that each clause of a DNF is an **And** of linear constraints. This is called a linear program. We can test whether a linear program is feasible or not. So in effect we may be given several linear programs. If any one of the linear programs is feasible then the whole proposition is feasible. If none of the linear programs is feasible then the proposition is not feasible.

■

Theorem 3 *Testing whether a given balanced game, (v, N), has stable core or not is decidable.*

Proof

First note that the core of a game (v, N), $C(v, N)$ is simply a linear program or a proposition. We denote this by P_0. Let P_I be a proposition encapulating all the vectors in $I(v, N)$. For every set S, we find a proposition P_S encapsulating the set of vectors in $I(v, N) \setminus C(v, N)$ but could be dominated by a vector in $C(v, N)$ via S. We create the **Or**s of P_0, **Not** P_I, P_S, over all $S \subset N$. By theorem **??** we test whether this is a tautology or not. If this is a tautology then (v, N) has stable core otherwise there is a point which lies in $I(v, N)$ but not in $C(v, N)$ which can't be dominated by any point in $C(v, N)$ via any set. Therefore (v, N) is not stable in that case.

We complete the proof if we show that P_S is a proposition. Take any vector $x \in I(v, N) \setminus C(v, N)$. We want to see if there is a vector $y \in C(v, N)$ which dominate x via S. We write the condition on x and y as the following linear program.

$$\forall\ i \in N \quad x_i \geq v(i)$$

$$x(N) = v(N)$$

$$\forall\ T \subset N \ \ y(T) \geq v(T)$$

$$y(N) = v(N)$$

$$y(S) = v(S)$$

$$\forall\ j \in N \quad y_j > x_j$$

Using Fourier-Motzkin method we eliminate all the y's from the above linear program. This operation basically projects the linear program on x variables only. We encapuslate the resulting linear program by proposition P_S.

■

6 Conclusion

Our main result suggests that a weaker sufficient condition for stability of the core cannot rely on proving domination via minimally violated sets. The condition must be more 'combinatorial' in nature as it must provide sufficient information to pin down the right violated set from which a domination argument must proceed. One might conjecture that a possible candidate is exactness of the core. The following 3-person game shows this to be false. Let $N = \{1, 2, 3\}$, $v(N) = 3$, $v(S) = 1.5$ for all $S \subset N$ such that $|S| = 2$ and $v(i) = 1$ for all $i \in N$. Since the core is the set of all imputations it is clearly stable. Notice that there is no point in the core such that $x_1 = 1$.

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