Bounds on the Inefficiency of Sequential Auctions

[Extended Abstract]

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ABSTRACT

We study the sequential second price auction of multiple units of a homogeneous commodity. It is well known that such auctions can have inefficient equilibria. For the case of two bidders, we show that the value of the allocation obtained in a the unique subgame perfect equilibrium is at least $1-e^{-1}$ of the value of the efficient allocation. Furthermore, we show that this bound is asymptotically tight.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*

General Terms

Economics, Algorithms, Theory

Keywords

Iterative Auctions, Immediate Allocation, Ascending Auctions

1. INTRODUCTION

A set of n units are to be divided between two agents via a sequential second price auction. The agents are endowed with quasi-linear utilities which satisfy diminishing marginal utilities. In each round the two agents submit their bid for the current unit. The unit is allocated to the highest bidder at the price of the lower bid. This procedure is repeated until all units are allocated.

The study of this and similar sequential auctions is not new; see for example [5], [9], [12] and [16]. The rise of eBay has renewed interest in such auctions as stylized models of the eBay market place (see for example [17]). In these papers valuations are assumed to be private but bidders have unit demands. In our case an agent is interested in consuming more than one unit. This makes the private information of agents multi-dimensional. So, to allow for tractability, we assume that valuations are common knowledge.

It is well known that the sequential second price auction has inefficient equilibria. In the spirit of [15] we attempt to bound this loss in efficiency. We show that the greatest loss in efficiency occurs when the smallest marginal valuation of agent 1, say, exceeds the largest marginal utility of agent 2. In this case, efficiency dictates that all n units should be allocated to agent 1. However, in the unique subgame perfect equilibrium of the sequential second price auction, this is not the case. In this case we prove that the value of the equilibrium allocation is always at least $1-e^{-1}$ of the value of the efficient allocation.

There are many reasons for a seller to use a sequential mechanism of the type just described. For example, bidders may be arriving over time, the seller may be unable to commit to a reserved price or the seller wishes to encourage the revelation of information ([6] provide a brief overview). Our specific motivation is the possibility of dynamically sharing spectrum in a wireless network. For example, suppose that a block of spectrum is allocated (e.g., licensed) to a primary agent, or service provider, for exclusive use. For many types of services, the spectrum may not be used continuously, so that the service provider may wish to lease portions of the spectrum to secondary users temporarily. (These types of secondary markets for spectrum allocation are currently being considered by the FCC [7, 8].) In situations where the demand of both the primary and the secondary agents fluc-

EC'07, June 11–15, 2007, San Diego, California, USA.

tuates randomly with time, a sequential auction mechanism, such as that considered here, may be attractive. Namely, the primary agent, serving as the auctioneer, would auction off pieces of available spectrum incrementally. The analysis we present can provide insight into the efficiency loss incurred by this mechanism, relative to auctioning off a fixed supply of available spectrum.

2. THE MODEL

The sequential second price auction is an extensive form game with a balanced binary game tree where each decision node designates an intermediate state of the world where a certain quantity of units are allocated to agent 1 and 2. Since the units are homogenous all the decision nodes with the same allocation can be unified and the game tree replaced with a directed graph $G = (V, E)$ where $V \subset [1, \ldots, n] \times [1, \ldots, n]$. With some abuse of notation we call a node in G a leaf if it is a union of leaves in the game tree. The root is the pair $(0, 0)$ and each decision node $v \in V$ is a pair $v = (s, t)$ satisfying $s + t = l$ where l is the number of units allocated at that point. Each node (s, t) that is not a leaf branches to nodes $(s + 1, t)$ and $(s, t + 1)$. Such decision nodes designate a state of the world where s units have been allocated to agent 1 and t to agent 2, the leaves designate the final allocations.

We assume both agents have decreasing marginal valuations, given by $u^i_1 \geq \ldots, \geq u^i_n$. Agent i's valuation for allocation (s, t) is therefore $\sum_{j=1}^{s} u^i_j$. Let H be the set of observable bidding histories. A *strategy* $\sigma_i : V \times H \rightarrow \mathbb{R}$ is a function mapping states of the world and observable histories to bids. The strategy set of an agent is the set of all such functions. The outcome path of a pair of strategies σ_1 and σ_2 is a sequence of vertices $\delta = \{(s_1, t_1), \ldots, (s_n, t_n)\}$ such that $s_{k+1} = s_k + 1$ and $t_{k+1} = t_k$ iff

$$\sigma_1((s_k, t_k), \Gamma_k) > \sigma_2((s_k, t_k), \Gamma_k)$$

and $s_{k+1} = s_k$ and $t_{k+1} = t_k + 1$ iff

$$\sigma_1((s_k, t_k), \Gamma_k) \leq \sigma_2((s_k, t_k), \Gamma_k)$$

where Γ_k is the bidding history on the path for the first k units[1].

The payment of agent 1 at (s_k, t_k) is

$$p_1(k, s, t) = \sigma_2((s_k, t_k), \Gamma_k)$$

if $\sigma_1((s_k, t_k), \Gamma_k) > \sigma_2((s_k, t_k), \Gamma_k)$ and $p_1(k, s, t) = 0$ otherwise. The total payment of the path Γ is

$$P_1(\Gamma) = \sum_{k=1}^{n} p_1(k, s, t).$$

P_2 is defined similarly. The payment of agents 1 or 2 at (s, t) is the payment of a path Γ that terminates at (s, t).

2.1 Example

Suppose $n = 3$, agent 1 values each unit at \$6. Agent 2 values the first unit at \$4.1, the second at \$3.1 and the third unit at zero. Thus the agents have the following marginals: $u^1_1 = u^1_2 = u^1_3 = 6$, $u^2_1 = 4.1$, $u^2_2 = 3.1$ and $u^2_3 = 0$. Since agent 1 values each unit more than agent 2 values any unit, the efficient allocation would be to give all three units to agent 1.

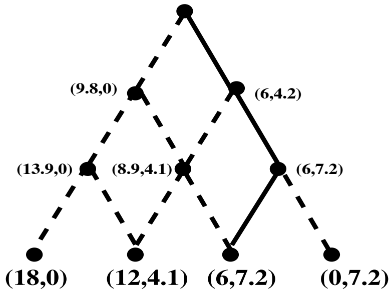

Figure 1: The equilibrium path in sequential allocation

Now let us examine how the agents will bid in equilibrium. Agent 1 should assume that agent 2 will bid \$4.1 in every round until she is allocated at least one unit and therefore her payment will be \$12.3 and her payoff \$5.7. On the other hand, if she does not bid on the first unit, she knows agent 2 will bid \$3.1 on the remaining two units and therefore her total payoff will be \$5.8. If she takes this reasoning even further she can abstain on both units. In this case agent 2's valuation drops to 0 and agent 1's gets the last unit for free with total payoff \$6. Thus, in this setting agent 1's payoff is maximized when she is allocated only one unit. We shall call this type of bidding behavior sophisticated bidding. Our objective is to model this behavior and analyze the efficiency of the equilibria it induces.

In figure 1 we see a game tree with the valuations of the allocations at the leaves and the valuation of the subtree at each node. We assume these valuations are known to the agents and determine the outcome, we can see that the equilibrium path does not terminate in an efficient allocation.

2.2 Bidding Strategies

There are two basic types of bidding strategies that will be of special interest. A bidding strategy is *myopic* if it maximizes payoff over intermediate outcomes. An extreme case of myopic bidding is truthful bidding on each round, which we call *greedy* bidding. Formally, $\sigma_1((s, t), \Gamma_k) = u^1_s$ and $\sigma_2((s, t), \Gamma_k) = u^2_t$ are the greedy strategies. As we have seen in section 2.1 greedy bidding is no longer a dominant strategy when there is more than one unit. This does not imply that this strategy cannot be rationalized in the sense that it can be associated with some sort of maximizing behavior, the latter depends on the information structure of the extensive form game. In the case where the agents are uninformed about the number of units or the valuation of the other agents it may be rational to bid truthfully in every round under the belief that that round is the terminal round. In fact, if there are no restrictions on the beliefs that can be attached with each agent, virtually any bidding strategy could be rationalized.

A bidding strategy is *sophisticated* if it maximizes payoff over final or expected final outcomes. The ability to make inferences on the final outcome requires the agent be sufficiently informed about the preferences and strategies of the other agent. We focus on sophisticated bidding in a setting of full information, but a similar analysis could be made in the case where the agent is Baysian and has information about the distribution of the marginal values.

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[1] Tie breaking can be, of course, arbitrary.

In the full information case each agent knows the number of units being sold, the bidding history of the other agent (and, of course, his own bidding history), and the valuations of the other agent. In particular both agents know when the last unit is being sold. The last round of the auction is identical to a standard second price auction hence it is a dominant strategy for both agents to bid truthfully on the last round. Since valuations are common knowledge, both agents know beforehand what the allocation and payments on the last round and therefore their terminal payoffs. Thus, we can think of the penultimate round as an auction over the right to participate in one of two auctions in the last round. Since the payoffs of each one of these auctions is common knowledge, we can think of the last round as a second price auction with valuations equal to these payoffs. It is therefore a dominant strategy in the penultimate round to bid truthfully the payoff from participating in one of the two possibilities of the last round.

We proceed in this way inductively until we reach the root. This shows sophisticated bidding is the only strategy that survives iterative elimination of weakly dominated strategies. This does not rule out other equilibria and in fact there exist other Nash equilibria with higher payoffs for both agents (if for instance they conspire against the seller). However, these equilibria must rely on unreliable threats and commitments. We eliminate these equilibria from consideration by focusing on *subgame perfect* equilibria that survive the iterative elimination of weakly dominant strategies. This discussion is summarized in the following theorem.

Theorem 1. *In a sequential auction with fully informed agents, sophisticated bidding is the unique subgame perfect equilibrium after iterative elimination of weakly dominated strategies.*

The *equilibrium path* is the outcome produced when both agents use a sophisticated bidding strategy. The *sequential allocation* is the allocation at the terminal point of the equilibrium path.

3. A BOUND ON THE INEFFICIENCY

Our main result is the following.

Theorem 2. *For any pair of decreasing marginal valuations $u^1_1 \geq \ldots \geq u^1_n$ and $u^2_1 \geq \ldots \geq u^2_n$ let k and $n-k$ be an efficient allocation and k' and $n-k'$ the sequential allocation. Then*

$$(1-e^{-1})(\sum_{i=1}^{k} u^1_i + \sum_{i=1}^{n-k} u^2_i) \leq \sum_{i=1}^{k'} u^1_i + \sum_{i=1}^{n-k'} u^2_i.$$

The proof relies on the intuition that the greatest inefficiency in the equilibrium outcome occurs in the case when $u^1_1 = \ldots = u^1_n \geq u^2_1 \geq \ldots \geq u^2_n$. In this case efficiency dictates that all units be awarded to agent 1, however, in the sequential allocation agent 2 may receive one or more units, and in fact, there are examples were agent 2 may receive up to $n-1$ units.

3.1 Equilibrium Path Payments

We begin the proof of theorem 2 by determining the payment that agent 1 makes on each units she receives in the sequential allocation.

Lemma 3. *Suppose $u^1_1 \geq \ldots \geq u^1_n \geq u^2_1 \geq \ldots \geq u^2_n$. If (s,t) is the sequential allocation, then the payment of agent 1 for each unit on the equilibrium path is u^2_{n-s}.*

Proof: We prove the lemma by induction on the number of units allocated. It is immediate for $n = 1$ since the auction is then a standard second price auction.

For $n > 1$ we can think of the root as a decision node between two alternatives:

1. Agent 1 receives a unit and both agents participate in an auction for $n-1$ units.
2. Agent 2 receives a unit and they both participate in an auction for $n-1$ units.

If on the equilibrium path the first unit is allocated to agent 2, then agent 1 pays nothing and the lemma follows by induction since the equilibrium path is an equilibrium path on the subgame rooted at the new state of the world by the definition of subgame perfect, and the utilities on the subgame are $u^1_1 \geq \ldots \geq u^{n-1}_1 \geq u^2_2 \geq \ldots \geq u^2_n$.

If on the equilibrium path the first unit is allocated to agent 1 then, it suffices to show that she pays u^2_{n-s}. Since the equilibrium path induced on the subgame is the equilibrium path of the subgame, we can think of the first node as choice between a second price auction on the right to participate in a $n-1$ unit sequential auction and receive an extra unit, or participate in the same game without the extra unit. But this shows that the disutility of agent 2 from having the unit allocated to agent 1 is the valuation agent 2 would have for the extra unit given the allocation of the subgame, which is by definition u^2_{n-s} □

Corollary 4. *Suppose $u^1_1 \geq \ldots \geq u^1_n \geq u^2_1 \geq \ldots \geq u^2_n$. The allocation (s,t) is the sequential allocation iff*

$$s \cdot (u^1_1 - u^2_{n-s}) \geq r \cdot (u^1_1 - u^2_{n-r}) \text{ for any } 1 \leq r \leq n \quad (1)$$

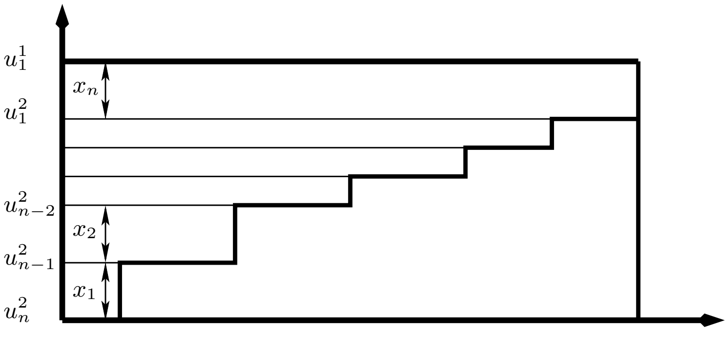

Figure 2: The extremal case, agent 1 has a constant marginal above the marginals of agent 2.

3.2 Bounds for Flat Valuations

In the case of flat valuations we assume $u^1_1 = \ldots = u^1_n \geq u^2_1 \geq \ldots \geq u^2_n$. If we let $x_1 = u^2_n$, $x_2 = u^2_{n-1} - u^2_n$, …, $x_{n-1} = u^2_1 - u^2_2$ and $x_n = u^1_n - u^2_1$ (see figure 2) then equation 1 can be written as

$$s \cdot \sum_{k=s}^{n} x_k \geq r \cdot \sum_{k=r}^{n} x_k$$

for any r.

The difference in value between the efficient allocation and the sequential allocation (s, t) is

$$n \cdot u_1^1 - (s \cdot u_1^1 + \sum_{k=1}^{n-s} u_k^2) = \sum_{k=s+1}^{n} (k - s)x_k$$

Under this same assumption the value of the efficient allocation can be written as $n \sum_{k=1}^{n} x_k$. In the lemma below we bound from above the expression

$$\frac{\sum_{k=s+1}^{n} (k - s)x_k}{n \sum_{k=1}^{n} x_k}$$

LEMMA 5. *Let $x_1^j, \ldots, x_n^j$ the solution to the following maximization problem*

$$\begin{aligned} \max \quad & \sum_{k=j+1}^{n} (k - j)x_k \\ \text{s.t.} \quad & j \cdot \sum_{k=j}^{n} x_k \geq r \cdot \sum_{k=r}^{n} x_k \\ & \text{for } r > j \\ & x_1, \ldots, x_n \geq 0 \end{aligned}$$

then

$$\left[\sum_{k=j+1}^{n} (k-1)x_k^j\right] / \left[n \cdot \sum_{k=1}^{n} x_k^j\right] = \frac{j}{n} \sum_{k=j}^{n-1} \frac{1}{k+1}$$

for $j = 1, \ldots, n$.

Proof: We begin with the case $j = 1$. In this case we are maximizing the linear functional

$$\phi(x_1, \ldots, x_n) = \sum_{k=2}^{n} (k - 1)x_k$$

on the polytope defined by the inequalities:

$$\begin{aligned} 0 &\leq x_1, \ldots, x_n \\ x_2 + \ldots + x_n &\leq x_1 \\ 2 \cdot (x_3 + \ldots + x_n) &\leq x_1 + x_2 \\ &\vdots \\ (n-1) \cdot x_n &\leq x_1 + \ldots + x_{n-1} \end{aligned}$$

W.l.g we may assume that $x_1 = 1$, hence with some rearranging we get the set of inequalities

$$\begin{aligned} x_2, \ldots, x_n &\geq 0 \\ x_2 + \ldots + x_n &\leq 1 \\ -x_2 + 2 \cdot (x_3 + \ldots + x_n) &\leq 1 \\ &\vdots \\ -x_2 - \ldots - x_{n-1} + (n-1) \cdot x_n &\leq 1 \end{aligned}$$

and in matrix form

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ -1 & 2 & 2 & \ldots & 2 & 1 \\ -1 & -1 & 3 & \ldots & 3 & 1 \\ & & & \vdots & & \\ -1 & -1 & -1 & \ldots & n-1 & 1 \end{pmatrix}$$

By adding the first row to each of the other rows and dividing by the leftmost non zero coefficient we obtain

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & 1 & \ldots & 1 & \frac{2}{3} \\ 0 & 0 & 1 & \ldots & 1 & \frac{1}{2} \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 & \frac{2}{n} \end{pmatrix}$$

By extracting from each row the row immediately below it we get

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \ldots & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & \ldots & 0 & 0 & \frac{1}{10} \\ & & & \vdots & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 & \frac{2}{n(n-1)} \\ 0 & 0 & 0 & \ldots & & 1 & \frac{2}{n} \end{pmatrix}$$

This show that the set of equations are independant and the polytope is not degenerate. Moreover $x_1 = 1$, $x_2 = \frac{1}{3}$, $x_k = \frac{2}{k(k+1}$ for $k = 3, \ldots, x_{n-1}$ and $x_n = \frac{2}{n}$ are the coordinates of the only vertex of the polytope with all coordinates positive. It is not difficult to see that any vertex with non positive coordinates does not maximize ϕ. This implies

$$\phi_{\max} = \frac{1}{3} + \sum_{k=3}^{n-1} \frac{2(k-1)}{k(k+1)} + \frac{2(n-1)}{n} = 2 \cdot \sum_{k=1}^{n-1} \frac{1}{k+1} \tag{2}$$

on the other hand

$$n \cdot \left(1 + \frac{1}{3} + \sum_{k=3}^{n-1} \frac{2}{k(k+1)} + \frac{2}{n}\right) = 2 \cdot n \tag{3}$$

Substituting 2 and 3 gives

$$\left[\sum_{k=2}^{n} (k-1)x_k^1\right] / \left[n \cdot \sum_{k=1}^{n} x_k^1\right] = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{(k+1)}$$

The proof for $j > 1$ is similar, in these cases we are maximizing the linear functional

$$\phi(x_1, \ldots, x_n) = \sum_{k=j+1}^{n} (k - j)x_k$$

on the polytope defined by the inequalities:

$$\begin{aligned} 0 &\leq x_1, \ldots, x_n \\ x_{j+1} + \ldots + x_n &\leq j \cdot (x_1 + \ldots + x_j) \\ 2 \cdot (x_{j+2} + \ldots + x_n) &\leq j \cdot (x_1 + \ldots + x_{j+1}) \\ &\vdots \\ (n-j) \cdot x_n &\leq j \cdot (x_1 + \ldots + x_{n-1}) \end{aligned}$$

We may think of $x_1 + \ldots + x_j$ as one variable which w.l.g equals 1 and look at a system with $n - j - 1$ variables,

$$\begin{aligned} 0 &\leq x_1, \ldots, x_{n-j} \\ x_{j+1} + \ldots + x_n &\leq j \\ 2 \cdot (x_{j+2} + \ldots + x_n) &\leq j + j \cdot x_{j+1} \\ &\vdots \\ (n-j) \cdot x_n &\leq j + j \cdot x_{j+1} + \ldots + j \cdot x_{n-1} \end{aligned}$$

As before, with some rearranging we obtain the matrix form

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & j \\ -j & 2 & 2 & \dots & 2 & j \\ -j & -j & 3 & \dots & 3 & j \\ & & & \vdots & & \\ -j & -j & -j & \dots & n-j & j \end{pmatrix}$$

By multiplying the first row by j and adding it to each of the other rows and dividing by the leftmost non zero coefficient we obtain

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & j \\ 0 & 1 & 1 & \dots & 1 & \frac{j(j+1)}{j+2} \\ 0 & 0 & 1 & \dots & 1 & \frac{j(j+1)}{j+3} \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & \frac{j(j+1)}{n} \end{pmatrix}$$

By extracting from each row the row immediately below it we get

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \frac{j}{j+2} \\ 0 & 1 & 0 & \dots & 0 & 0 & \frac{j(j+1)}{(j+2)(j+3)} \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & \frac{j(j+1)}{n(n-1)} \\ 0 & 0 & 0 & \dots & & 1 & \frac{j(j+1)}{n} \end{pmatrix}$$

As before any vertex with non positive coordinates does not maximize ϕ. This implies

$$\begin{aligned} \phi_{\max} = \; & \tfrac{j}{j+2} + j(j+1) \cdot \textstyle\sum_{k=j+2}^{n-1} \frac{k-j}{k(k+1)} + j(j+1) \cdot \frac{n-1}{n} \\ & = j(j+1) \cdot \textstyle\sum_{k=j}^{n-1} \frac{1}{(k+1)} \end{aligned} \quad (4)$$

As for the denominator,

$$n \cdot \left(1 + \frac{j}{j+2} + j(j+1) \cdot \sum_{k=j+2}^{n-1} \frac{1}{k(k+1)} + \frac{j(j+1)}{n} \right) = (j+1) \cdot n \quad (5)$$

Substituting 4 and 5 gives

$$\left[\sum_{k=j+1}^{n} (k-j) x_k^j \right] / \left[n \cdot \sum_{k=1}^{n} x_k^j \right] = \frac{j}{n} \sum_{k=j}^{n-1} \frac{1}{(k+1)} \quad \square$$

3.3 Proof of Theorem 2

Corollary 4 proves Theorem 2 in the case when one agent has a constant marginal valuation that exceeds the marginal valuation of the second agent for all units. To complete the proof we need to show that this is the extremal case. We do so in two steps. First we show that in the case where the marginals of one agent are above the other, the inefficiency can only increase if we 'flatten' his marginal valuations. Then we show that for any pair of marginals for agents 1 and 2 there exists a pair of marginals where agent 1's marginals are above those of agent 2 with a higher inefficiency in the outcome.

Step 1: Let $u_1^1 \geq \ldots \geq u_n^1 \geq u_1^2 \geq \ldots \geq u_n^2$, Lemma 3 shows that if (s,t) is a sequential allocation then

$$\sum_{j=1}^{s} (u_j^1 - u_{n-s}^2) \geq \sum_{j=1}^{r} (u_j^1 - u_{n-r}^2)$$

for any $r \neq s$. If we replace the utilities of the first agent with the utilities $\bar{u}_1^1 = \ldots = \bar{u}_n^1 = u_n^1$ then

$$s \cdot (\bar{u}_1^1 - u_{n-s}^2) \geq r \cdot (\bar{u}_1^1 - u_{n-r}^2)$$

for any $r > s$. This implies that for utilities $\bar{u}_1^1 = \ldots = \bar{u}_n^1 \geq u_1^2 \geq \ldots \geq u_n^2$ the sequential equilibrium $(\bar{s}, \bar{t})$ satisfies $\bar{s} < s$. In other words 'flattening' the marginal utilities of agent 1 can only increase inefficiency.

Step 2: For any pair of decreasing marginal valuations $u_1^1 \geq \ldots \geq u_n^1$ and $u_1^2 \geq \ldots u_n^2$ let k and $n-k$ be the efficient (VCG) allocation for agents 1 and 2. After allocating part of the units we reach a decision node where either agent 1 or agent 2 obtain their efficient allocation k or $n-k$ respectfully. Since one agent meets his efficient allocation it follows that in the subgame rooted at that point, the valuations of this agent are below the valuations of the other agent. Up to this node, there is no loss in efficiency, any inefficiency in the final allocation procures in the subtree rooted at this node, hence, efficiency loss of the full game tree cannot be larger than the efficiency loss on a subtree (see figure 3). This shows that for any pair of valuations $u_1^1, \ldots, u_n^1$ and $u_1^2, \ldots, u_n^2$ there exists a pair $\bar{u}_1^1, \ldots, \bar{u}_n^1$ and $\bar{u}_1^2, \ldots, \bar{u}_n^2$ satisfying $\bar{u}_1^1 \geq \ldots \geq \bar{u}_n^1 \geq \bar{u}_1^2 \geq \ldots \geq \bar{u}_n^2$ with at least as much inefficiency.

We have seen that for any $j \leq n$ the inefficiency is bounded by $\frac{j}{n} \sum_{k=j}^{n-1} \frac{1}{(k+1)} = -\frac{j}{n} \log \frac{j}{n}$ which can be approximated by $-x \log x$. The later obtains a maximum when $x = e^{-1}$ hence $j^* \approx [e^{-1}]$ and is bounded from above by e^{-1} $\square$

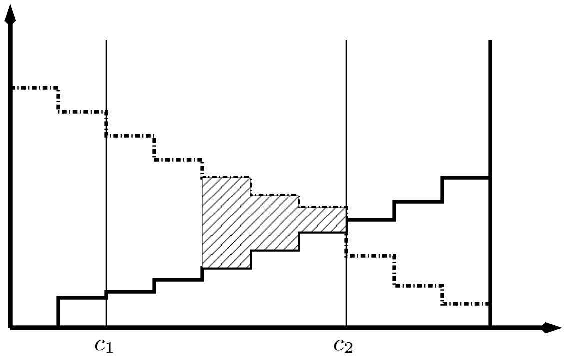

Figure 3: The general case, c_2 is the efficient allocation for agent 2 and c_1 is the allocation to agent 1 at this node. The subgraph on the segment $[c_1, c_2]$ is the valuations on the subtree (valuation decreases from left to right for agent 1 and from right to left for agent 2).

3.4 Sharpness of the Bound

We show that the bound of Theorem 2 is tight. First consider the sequential game with $n = 2$. If $u_1^1 = u_2^1 = 1$ and $u_1^2 = \frac{1}{2} + \varepsilon_1$, $u_2^2 = 0$, then the allocation of the goods after the sequential auction ends is (1,1). Instead of getting two goods with payoff $1 - 2\varepsilon_1$, user 1 is better off by just getting one good with payoff 1. The efficiency loss of this auction with the marginal valuations as above is $1 - \frac{1+\frac{1}{2}}{1+1} = \frac{1}{4}$, and this is the worst case among all possible profiles of marginal valuations for two users and two goods. Similarly, we can construct the marginal valuations of each user to get the

n	Marginals	Inefficiency
2	$1, 1 \;\; ; \;\; 1/2 + \varepsilon_1, 0$	1/4
3	$1, 1, 1 \;\; ; \;\; 2/3 + \varepsilon_1, 1/2 + \varepsilon_2, 0$	5/18
4	$1, 1, 1, 1 \; ; \; 1/2 + \varepsilon_1 ; 1/3 + \varepsilon_2, 0, 0$	7/24

Table 1: Maximal inefficiency for $n = 2, 3, 4$.

worst efficiency loss for given $n > 2$.

Consider the following profile of marginal valuations for two agents and n goods: n: $u_1^1 = \ldots = u_n^1 = 1$ for user 1 and $u_1^2 = 1 - \frac{j}{n} + \varepsilon_1, u_2^2 = 1 - \frac{j}{n-1} + \varepsilon_2, u_3^2 = 1 - \frac{j}{n-2} + \varepsilon_3 \ldots, u_{n-j}^2 = 1 - \frac{j}{j+1} + \varepsilon_{n-j}, u_{n-j+1}^2 = 0, \ldots, u_n^2 = 0$ for user 2, where $j \in [1, n]$ and integer. The sequential auction ends with the allocation $(j, n-j)$. This can be shown using Lemma 3. If user 1 gets j goods, the payoff of user 1 after the auction is $j \cdot 1$. However, if user 1 gets $j+1$ goods, then the payoff becomes $(j+1) \cdot (1 - u_{n-j}^2) = j - (j+1) \cdot \varepsilon_{n-j}$, which is smaller than j. In a similar way, the payoff of user 1 would be smaller than j if user 1 gets $r \neq j$ goods. Therefore, the sequential auction outcome with sophisticated bidding results in the allocation $(j, n-j)$.

Efficiency of the sequential auction in this case is given by

$$\frac{j + \sum_{k=1}^{j} u_k^2}{n} = \frac{j}{n} + \sum_{k=1}^{n-j} \frac{1}{n} \cdot \left(1 - \frac{j}{n-k+1}\right)$$

and the efficiency loss is $\frac{j}{n} \sum_{k=j}^{n-1} \frac{1}{k+1}$ as shown in section 3.2. The maximum efficiency loss should be for

$$j^* = \arg \max_{1 \leq j \leq n} \left[\frac{j}{n} \sum_{k=j}^{n-1} \frac{1}{k+1} \right]$$

Table 1 shows the marginal valuations of user 2 which give the maximum efficiency loss for given n and corresponding j^*. It is not too difficult to show that these marginal valuations of user 2 with j^* give the worst-case efficiency for given n. As we see in the table, the efficiency loss becomes larger as the total number of goods n increases □

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