Mix, Cost, and Time Flexibility: Valuation, Relationships and Corporate Diversification

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Abstract

Flexibility measures the ability to adapt to change and often has multiple dimensions that impact value jointly yet differently. We assess this joint impact in a theoretical model of a two-product firm that makes capacity, production and pricing decisions at three points in time with an underlying continuous-time information evolution. The firm's ability to adapt is characterized by three types of flexibility. Mix flexibility is measured by the cost of switching production between the two products. Cost flexibility is measured by the fraction of product cost that is postponed until demand information is updated. Finally, time flexibility is measured by the relative timing of the production decision. We show that mix and cost flexibilities are substitutes in creating firm value but both are complementary to time flexibility. Furthermore, the marginal values of mix and time flexibility decrease in demand correlation whereas the marginal value of cost flexibility increases in demand correlation. We discuss the implications of these results for the optimal investment in different aspects of flexibility. We also relate these results to corporate strategy and show when different types of flexibility can justify a company to pursue market diversification.

1 Introduction

Flexibility measures the ability to adapt to change and often has multiple dimensions that impact value jointly yet differently. This paper considers mix, cost and time flexibility and examines their value, relationships and implications for corporate diversification. We consider a two-product firm that makes capacity, production and pricing decisions at three decision epochs with an underlying continuous-time information evolution. First, the firm chooses product-specific capacity levels based on an imperfect forecast of the future demand curves. Second, as the selling season approaches, the firm updates its forecast and at a certain point of time, called the update time, locks in output levels of its two products. The output of each product is constrained by the existing product-specific capacity but the firm has the option to convert, at a cost, one product-specific capacity to another. Finally, when the selling season comes and the demand curve uncertainty is resolved, the firm sets prices and realizes profit. We characterize the firm's optimal strategy in terms of its capacity, production and pricing decisions assuming the firm is an expected value maximizer.

Flexibility measures the adaptability to dynamically evolving conditions. In our model, it stems from three abilities, each modeled by a continuous parameter:

1. *Mix flexibility*, also known as product flexibility, is the ability to switch production among different products. In our model it is measured by the cost of switching one unit of specialized capacity to the other product. The lower this cost, the higher mix flexibility.

2. *Cost flexibility* is the ability to postpone decisions that impact cost until more information is available. In our model, it is measured by the fraction of total unit cost incurred after the update time. Cost flexibility also implies volume flexibility. When cost flexibility is high, the unit capacity cost is small relative to the marginal output cost incurred after the update time, and the firm thus chooses a relatively large capacity level that is unlikely to constrain output

volume. In addition, the bulk of unit cost is still variable at the update time so that the firm has the ability and economic incentive to adapt output volume to the updated forecast. Factors that determine cost flexibility include the capital intensity of the production process and supply lead times.

3. *Time flexibility* is the ability to delay production decision thereby benefiting from information updating. In our model, it is measured by the update time relative to the timing of capacity selection and the selling season. Factors that determine time flexibility include production lead times, machine setup and changeover times, and/or the point of product differentiation within the production process.

We show that mix and cost flexibilities are strategic substitutes in creating firm value, but are both complementary with time flexibility. Thus, for example, a firm with inherently greater mix flexibility should invest more in time flexibility and less in cost flexibility. Furthermore, the marginal values of mix and time flexibility are decreasing in demand correlation but the marginal value of cost flexibility increases in demand correlation. Therefore, as demand correlation increases, managers should invest more in cost flexibility and less in mix and time flexibilities.

Finally, we relate our results to corporate strategy, namely, to market diversification. The strategic management literature has recognized that market diversification coupled with technological or organizational flexibility may create value by reducing uncertainty in capacity planning through demand pooling. In addition to showing that the value premium for diversification increases in mix and time flexibility and decreases in cost flexibility and demand correlation, we provide insights into the trade-offs between related and unrelated diversification. These findings provide several hypotheses that can be tested empirically.

While there is an extensive body of literature on mix flexibility (Fine and Freund 1990, Van Mieghem 1998, Chod and Rudi 2005 and references therein), time and cost flexibilities have received less attention. There are several papers that examine relationships among various types of flexibility (Gupta and Goyal 1992, Parker and Wirth 1999 and references therein) but their focus is the hierarchy among these flexibilities (e.g., machine flexibility is necessary for mix flexibility). Our focus, on the other hand, is substitutability/complementarity between different types of flexibility in creating firm value.

To the best of our knowledge, the only analytical paper that addresses a similar research question is the recent work of Goyal and Netessine (2005) who examine the relationship between product (mix) and volume flexibility. They consider the optimal technology and capacity choice of a two-product price-setting firm that faces uncertain demand functions in a two-step recourse model in which the firm produces at capacity. Volume flexibility is the ability to adjust, at a quadratic cost, capacity levels up or down once uncertainty is resolved. Netessine and Goyal show that adding product flexibility to volume flexibility does not necessarily benefit the firm, even if it is costless, because of possible diseconomies of scope. In our model, adding product (mix) flexibility is beneficial unless the firm has full cost flexibility in which case product flexibility is irrelevant.

There are extensive strategic management and financial economics literatures studying motivations for companies to diversify (for a comprehensive survey of the diversification literature, see e.g., Martin and Sayrak 2003). Among the first articles explaining corporate diversification with resource flexibility is the influential qualitative article of Teece (1982) who studies a firm that chooses a product mix according to constantly changing market conditions which create opportunities in different markets at different times. Since than, the link between diversification and different types of flexibility has appeared in several other papers (Levy and Haber 1986, Haber and Levy 1988, Von Ungern-Sternberg 1989, Smith and Triantis 2001, and Matsusaka 2001). We contribute to this literature by formalizing in a rigorous mathematical model the idea that in the presence of demand uncertainty and transaction costs in the factor market, mix flexibility may justify corporate diversification. We also show how the diversification premium depends on the three dimensions of flexibility.

Finally, our work is related to the real options literature that studies the valuation of flexibility using contingent claims pricing (see e.g., Triantis and Hodder 1990, and He and Pindyck 1992). The remainder of this paper is structured as follows. Section 2 presents the model which is solved in Section 3. The flexibility premium and the interplay among different dimensions of flexibility and market parameters are examined in Section 4. Section 5 discusses the link between flexibility and market diversification. Section 6 concludes. All proofs are relegated to Appendix 2.

2 The Model

Consider a two-product firm that must make three decisions at three different points in time. Information availability and uncertainty, which are crucial to any investment strategy, are formalized by a standard probabilistic framework with a probability space $(\Omega, \mathcal{F}, P)$ and filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ as primitives. (The filtration $\mathbb{F}$ is an increasing family of sub-σ-fields that specifies how information arrives and uncertainty is resolved as time passes, with $\mathcal{F}_t$ representing the information available at time t.) Expectation conditional on $\mathcal{F}_t$ is denoted as $\mathbb{E}_t$.

At time 0, the firm must choose a vector $\mathbf{K} \in \mathbb{R}_+^2$ of two product-specific capacity levels based on the information then available and on its assessment of the uncertain future. Capacity is defined as the maximum output that can be produced given the firm's choice of capital, workforce, inventory or production levels. We assume that the capacity investment incurs a constant marginal capacity cost c_K that is adjusted for any residual value of capacity at time T. We assume, for simplicity, that all costs, capacity consumption rates and demand parameters are identical for both products.

At the update time $\tau \in [0, T]$, the firm must decide on the actual output vector $\mathbf{Q} \in \mathbb{R}_+^2$ that will be available for sales at time T. Transformation of one unit of capacity into a unit of output incurs a constant marginal output cost c_Q, which includes any production costs incurred after the update time. Although the aggregate output cannot exceed total capacity, i.e., $Q_1+Q_2 \leq K_1+K_2$, the firm has the option to convert, or "switch", capacity i to capacity $j \neq i$. Because capacity i may be less cost-efficient at producing product j, switching may increase the marginal output cost by $c_S \geq 0$, which we refer to as switching cost.

Finally, at time T, production is complete and uncertainty is resolved. The firm sets output prices $\mathbf{p} \in \mathbb{R}_+^2$ and earns a revenue $\pi(\mathbf{p},\mathbf{Q}) = \mathbf{p'q}\left(\mathbf{p},\mathbf{Q}\right)$, where $\mathbf{q}\left(\mathbf{p},\mathbf{Q}\right) \leq \mathbf{Q}$ is the optimal output vector to bring to market, and the prime denotes transpose. Given that no model dynamics occur after the start of the sales season, we compress the latter into an instantaneous sales event at time T, after which the firm is liquidated. We suppress the time value of money so that the firm terminal value, denoted as $v(T)$, is equal to the sales revenue minus the capacity investment, capacity switching and output costs:

$$v\left(T;\mathbf{K},\mathbf{Q},\mathbf{p}\right) = \pi(\mathbf{p},\mathbf{Q}) - \sum_{i=1}^{2}\left(c_K K_i + c_S \max\left(Q_i - K_i, 0\right) + c_Q Q_i\right). \tag{1}$$

The firm makes the three decisions, $(\mathbf{K},\mathbf{Q},\mathbf{p})$, with the objective to maximize its value. The firm value at time $t \in [0,T]$, denoted as $v\left(t\right)$, is the expectation of the firm terminal value conditional on information available then, i.e., $v\left(t\right) = \mathbb{E}_t v\left(T\right)$. The optimality equations simplify as follows:

$$\begin{aligned}
\mathbf{p}^*\left(\mathbf{Q}\right) &= \underset{\mathbf{p}\in\mathbb{R}_+^2}{\arg\max}\ v\left(T;\mathbf{K},\mathbf{Q},\mathbf{p}\right), && (2)\\
\mathbf{Q}^*\left(\mathbf{K}\right) &= \underset{\mathbf{Q}\in\mathbb{R}_+^2,\mathbf{1'Q}\leq\mathbf{1'K}}{\arg\max}\ v\left(\tau;\mathbf{K},\mathbf{Q},\mathbf{p}^*\left(\mathbf{Q}\right)\right), && (3)\\
\mathbf{K}^* &= \underset{\mathbf{K}\in\mathbb{R}_+^2}{\arg\max}\ v\left(0;\mathbf{K},\mathbf{Q}^*\left(\mathbf{K}\right),\mathbf{p}^*\left(\mathbf{Q}^*\left(\mathbf{K}\right)\right)\right), && (4)
\end{aligned}$$

where $\mathbf{1'} = (1,1)$. An optimal strategy $(\mathbf{K}^*,\mathbf{Q}^*,\mathbf{p}^*)$ is a solution $(\mathbf{K}^*,\mathbf{Q}^*(\mathbf{K}^*),\mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}^*)))$ to (2)–(4). Finally, we let $v^*\left(t\right) = v\left(t;\mathbf{K}^*,\mathbf{Q}^*,\mathbf{p}^*\right)$ be the firm value at time t, when the optimal strategy is followed.

For concreteness and tractability, we assume that in each of the two product-markets, the firm is a monopolist facing an iso-elastic demand curve that is subject to a multiplicative random shock $\epsilon_i(T)$, $i = 1, 2$. Thus, the inverse demand curve in market i at time T is

$$p_i = \epsilon_i(T)\, q_i^{1/b},$$

where $b \in (-\infty, -1)$ is the constant price elasticity of demand.

The random shock $\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}(t), t \geq 0\}$ is assumed to follow a two-dimensional geometric Brownian motion. Thus, $\ln \boldsymbol{\epsilon}(t)$ is a bivariate normal random vector with mean $\ln \boldsymbol{\epsilon}(0)$ and covariance matrix $t\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}_{ii} = \sigma^2$ and $\boldsymbol{\Sigma}_{ij} = \rho\sigma^2$ if $i \neq j$. The information available at time t includes the history of $\boldsymbol{\epsilon}(t)$ or, formally, the filtration $\mathbb{F}$ is generated by ϵ.

Our objective is to assess the firm's flexibility which stems from the following three abilities:

1. *Mix flexibility* φ measures the firm's ability to convert one type of capacity to another after updating the demand forecast. Since this ability depends on the switching cost c_S, we proxy mix flexibility by the continuous switching cost: $\varphi = 1/c_S$. With zero mix flexibility ($\varphi = 0$), the firm does not have the option to switch capacity. With perfect mix flexibility ($\varphi \to \infty$), both products rely on the same capacity.

2. *Cost flexibility* γ measures the ability to postpone cost-incurring decisions until the demand forecast is updated. We define cost flexibility as the fraction of the total unit cost that is incurred after the update time: $\gamma = c_Q/(c_K + c_Q)$. With zero cost flexibility ($\gamma = 0$), the firm has incentive to always utilize full capacity, i.e., the aggregate production volume is fully determined ex ante. With perfect cost flexibility ($\gamma = 1$), the aggregate production volume is unconstrained by capacity. In general, the higher the cost flexibility, the larger the optimal

capacity and the lower the impact of production volume on the average total cost. Cost flexibility thus implies *volume flexibility*, i.e., the ability to adapt production volume to the updated forecast.

3. *Time flexibility* τ measures how long the firm can wait before finalizing the output decision. Without time flexibility ($\tau = 0$), the firm has to lock in the output quantities ex ante. With perfect time flexibility ($\tau = T$), the firm can postpone the output decision until all uncertainty is resolved.

3 The Optimal strategy

We solve for the optimal strategy by backward induction starting with the pricing decision (2). It is a well-known property of the iso-elastic demand function that a monopoly always maximizes its revenue from a given output by selling all units at the market clearing price. In other words, $\mathbf{q}\left(\mathbf{p}^*\left(\mathbf{Q}\right), \mathbf{Q}\right) = \mathbf{Q}$ and

$$p_i^*\left(\mathbf{Q}\right) = \epsilon_i\left(T\right) Q_i^{1/b}, \qquad i = 1, 2. \tag{5}$$

Therefore, the firm's revenue under the optimal pricing policy is $\pi\left(\mathbf{p}^*\left(\mathbf{Q}\right), \mathbf{Q}\right) = \sum_{i=1}^{2} \epsilon_i\left(T\right) Q_i^{1+1/b}$. The optimal output vector (3) maximize the firm value at time τ, which can be written as

$$v\left(\tau; \mathbf{K}, \mathbf{Q}, \mathbf{p}^*\left(\mathbf{Q}\right)\right) = \sum_{i=1}^{2} \Big(\mathbb{E}_\tau \epsilon_i\left(T\right) Q_i^{1+1/b} - c_Q Q_i - c_S \max\left(Q_i - K_i, 0\right) - c_K K_i\Big). \tag{6}$$

Since $b < -1$, this objective function is strictly concave and the optimal output vector $\mathbf{Q}^*(\mathbf{K})$ is the unique solution to the Kuhn-Tucker optimality conditions. To characterize the optimal output vector, we partition the state space of the updated demand prospects $\boldsymbol{\epsilon}\left(\tau\right)$ into eight events $\Omega_1, ..., \Omega_8$ as illustrated in Figure 1. We also define $\Omega_{i_1 i_2 \cdots i_n} \equiv \Omega_{i_1} \cup \Omega_{i_2} \cup \cdots \cup \Omega_{i_n}$. The formal definitions

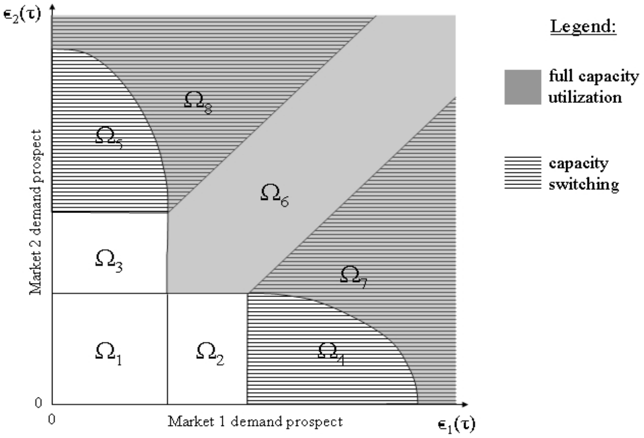

Figure 1: The state space of the demand prospects $\boldsymbol{\epsilon}(\tau)$ is partitioned into eight events that specify different optimal production vectors. If $\boldsymbol{\epsilon}(\tau) \in \Omega_{678}$, the total capacity is fully utilized. If $\boldsymbol{\epsilon}(\tau) \in \Omega_{4578}$, some capacity conversion or switching is optimal.

of $\Omega_1, ..., \Omega_8$ as well as the corresponding optimal output vectors are relegated to Appendix 1. The intuitive interpretation of these eight events is as follows:

Event $\Omega_1(\mathbf{K})$: The prospects of both markets are poor and neither capacity is fully utilized: $Q_i^* < K_i$, $i = 1, 2$.

Event $\Omega_2(\mathbf{K})$: The prospects of market 1 are relatively good while those of market 2 are poor. Capacity 1 is fully utilized but capacity 2 is not and no capacity switching occurs: $Q_1^* = K_1$ and $Q_2^* < K_2$. (The event Ω_3 is symmetric: $Q_1^* < K_1$ and $Q_2^* = K_2$.)

Event $\Omega_4(\mathbf{K})$: The prospects of market 1 are very good so that not only full capacity 1 but also some of capacity 2 is used to make product 1: $Q_1^* > K_1$. The prospects of market 2, however, are poor so that only a fraction of the remaining capacity of resource 2 is used for product 2: $Q_2^* < K_2 - (Q_1^* - K_1)$. (The event Ω_5 is symmetric: $Q_2^* > K_2$ and $Q_1^* < K_1 - (Q_2^* - K_2)$.)

Event $\Omega_6(\mathbf{K})$: The prospects of both markets are good enough to justify full utilization of both resources and not sufficiently different to justify any capacity conversion: $Q_i^* = K_i$, $i = 1, 2$.

Event $\Omega_7(\mathbf{K})$: The overall market prospects are very good so that both resources are fully utilized. Furthermore, the prospects of market 1 are significantly better than those of market 2 warranting some capacity 2 conversion to product 1: $Q_1^* > K_1$ and $Q_2^* = K_2 - (Q_1^* - K_1)$. (The event Ω_8 is symmetric: $Q_2^* > K_2$ and $Q_1^* = K_1 - (Q_2^* - K_2)$.)

The capacity investment problem (4) is concave and so the optimal capacity vector is characterized by the first order condition. Furthermore, if the switching cost c_S is positive, the optimal capacity is unique. (If capacity can be switched between the products costlessly, only the *total* capacity matters. In that case, there is a continuum of optimal capacity vectors characterized by the first order conditions, all of which represent the same total capacity.)

Proposition 1 *If* $c_S > 0$, *there exists a unique optimal capacity vector* $\mathbf{K}^*$ *which satisfies* $K_1 = K_2$ *and*

$$\begin{aligned} & \Pr(\Omega_{78}(\mathbf{K}))\,\mathbb{E}\left(\sum_{i=1}^{2}\frac{\partial Q_i^*}{\partial K_1}\frac{\partial(p_i^* Q_i^*)}{\partial Q_i^*} - c_Q - c_S\frac{\partial\left|Q_1^* - K_1\right|}{\partial K_1}\middle|\,\Omega_{78}(\mathbf{K})\right) \\ + \quad & \Pr(\Omega_{26}(\mathbf{K}))\,\mathbb{E}\left(\frac{\partial(p_1^* K_1)}{\partial K_1} - c_Q\middle|\,\Omega_{26}(\mathbf{K})\right) + \Pr(\Omega_4(\mathbf{K}))\,c_S = c_K, \end{aligned} \tag{7}$$

where $\mathbf{p}^* = \mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}))$ *is given by (5) and* $\mathbf{Q}^* = \mathbf{Q}^*(\mathbf{K})$ *is given in Appendix 1.*

Proof: All proofs are relegated to Appendix 2.□

Optimality equation (7) sets marginal capacity value equal to its marginal cost c_K. A marginal increase in capacity 1 affects firm value only when resource 1 is fully utilized, which is in events $\Omega_2, \Omega_4, \Omega_6, \Omega_7$ and Ω_8. There are three cases to distinguish: (i) If all capacity is used and capacity switching takes place ($\boldsymbol{\epsilon}(\tau) \in \Omega_{78}$), an additional unit of capacity 1 will increase the total revenue, incur marginal output cost c_Q, and either increase or decreases the total cost of switching depending on which capacity is being converted. (ii) If capacity 1 is fully utilized and no switching occurs

($\boldsymbol{\epsilon}(\tau) \in \Omega_{26}$), an additional unit of this capacity will increase the revenue from product 1 after incurring marginal output cost c_Q. (iii) Finally, if capacity 2 is not fully utilized but some of it is converted to capacity 1 ($\boldsymbol{\epsilon}(\tau) \in \Omega_4$), an additional unit of capacity 1 will reduce the amount of capacity 2 that is being converted, saving unit switching cost c_S.

To evaluate the benefits of flexibility, we also consider the no-flexibility case as a reference case. A firm that has no flexibility must choose its output vector $\tilde{\mathbf{Q}}$ ex ante, together with the capacity vector $\tilde{\mathbf{K}}$. (Obviously, both are equal then.) Let $\tilde{\mathbf{p}}$ be the price vector set by the non-flexible firm. Flexibility typically comes at a cost which is reflected by lower capacity and production costs: $\tilde{c}_K \leq c_K$ and $\tilde{c}_Q \leq c_Q$. Except for possibly lower capacity and output costs, the non-flexible firm is a limiting case of the flexible firm with no time flexibility ($\tau = 0$) or, alternatively, with no mix and cost flexibilities ($\varphi = 0$ and $\gamma = 0$). Let $\tilde{v}\left(t; \tilde{\mathbf{K}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{p}}\right)$, or $\tilde{v}(t)$ for short, denote the value of the non-flexible firm at time t, and let $\tilde{v}^*(t) \equiv \tilde{v}\left(t; \tilde{\mathbf{K}}^*, \tilde{\mathbf{Q}}^*, \tilde{\mathbf{p}}^*\right)$ be its value if the optimal strategy is followed. The optimal capacity vector and the corresponding value of a non-flexible firm can be both obtained in closed form:

Corollary 1 *The optimal capacity vector and value of the non-flexible firm are, respectively,*

$$\tilde{K}_1^* = \tilde{K}_2^* = \left(\frac{1 + 1/b}{\tilde{c}_Q + \tilde{c}_K} \mathbb{E}_0 \epsilon_i(T)\right)^{-b} \qquad \textit{and} \qquad \tilde{v}^*(0) = \frac{\tilde{c}_Q + \tilde{c}_K}{|1 + b|} \left(\tilde{K}_1^* + \tilde{K}_2^*\right).$$

In the next section, we define the value of flexibility and discuss how its depends on the three dimensions of flexibility.

4 The Value of flexibility and its drivers

We define the value premium for flexibility as the relative difference between the value of a flexible and a non-flexible firm:

$$\Delta_F \equiv \frac{v^*(0) - \tilde{v}^*(0)}{\tilde{v}^*(0)}.$$

The value premium for flexibility is difficult to study analytically and we must resort to a numerical analysis except for the boundary case of zero switching and marginal output costs, which we examine next.

If $c_Q = c_S = 0$, the firm has zero cost flexibility and perfect mix flexibility. The value of flexibility stems from risk pooling and the revenue maximizing option, and can be expressed in closed form:

Lemma 1 *If $c_Q = c_S = 0$, the value premium for flexibility simplifies into*

$$\Delta_F = \left(\frac{\tilde{c}_K}{c_K}\right)^{-b-1}\left(\frac{\|\boldsymbol{\epsilon}(T)\|_\tau}{\mathbb{E}_0\epsilon_i(T)}\right)^{-b} - 1, \text{ where } \|\boldsymbol{\epsilon}(T)\|_\tau = \mathbb{E}_0\left\{\left(\frac{\mathbb{E}_\tau^{-b}\epsilon_1(T) + \mathbb{E}_\tau^{-b}\epsilon_2(T)}{2}\right)^{-1/b}\right\}.$$

It follows from the Minkowski inequality that $\|\boldsymbol{\epsilon}(T)\|_\tau \geq \mathbb{E}_0\epsilon_i(T)$. The inequality holds as equality only if the firm has zero time flexibility, the demand curves are deterministic, or the demand shocks are perfectly positively correlated. Except for these limiting cases, flexibility has benefits. Whether these benefits justify the higher cost of flexible technology depends on model parameters. In particular,

$$\Delta_F > 0 \qquad \Leftrightarrow \qquad \frac{c_K}{\tilde{c}_K} < \left(\frac{\|\boldsymbol{\epsilon}(T)\|_\tau}{\mathbb{E}_0\epsilon_i(T)}\right)^{\frac{b}{1+b}} \equiv \delta.$$

The next three lemmas characterize the effects of time flexibility, demand variability and demand correlation on the optimal capacity, firm value, flexibility premium and the maximum sustainable cost of flexibility δ. The longer the firm can wait before exercising the option to switch capacity, the higher the value of this option and the higher the optimal capacity investment:

Lemma 2 *If* $c_Q = c_S = 0$, *the optimal capacity, firm value, flexibility premium and the maximal sustainable cost of flexibility increase in time flexibility* τ: $\frac{\partial}{\partial \tau}\left(K_1^* + K_2^*\right) \geq 0$, $\frac{\partial}{\partial \tau} v^*(0) \geq 0$, $\frac{\partial}{\partial \tau}\Delta_F \geq 0$, *and* $\frac{\partial}{\partial \tau}\delta \geq 0$.

Similar to financial options, price volatility increases the value of the option to switch and, hence, the value of a flexible firm. It also increases the optimal capacity level:

Lemma 3 *If* $c_Q = c_S = 0$, *the optimal capacity, firm value, flexibility premium and the maximal sustainable cost of flexibility increase in demand volatility* σ: $\frac{\partial}{\partial \sigma}\left(K_1^* + K_2^*\right) \geq 0$, $\frac{\partial}{\partial \sigma} v^*(0) \geq 0$, $\frac{\partial}{\partial \sigma}\Delta_F \geq 0$, *and* $\frac{\partial}{\partial \sigma}\delta \geq 0$.

As expected, higher demand correlation reduces the value of the switching option. It also leads to a lower capacity investment:

Lemma 4 *If* $c_Q = c_S = 0$, *the optimal capacity, firm value, flexibility premium and the maximal sustainable cost of flexibility decrease in demand correlation* ρ: $\frac{\partial}{\partial \rho}\left(K_1^* + K_2^*\right) \leq 0$, $\frac{\partial}{\partial \rho} v^*(0) \leq 0$, $\frac{\partial}{\partial \rho}\Delta_F \leq 0$ *and* $\frac{\partial}{\partial \rho}\delta \leq 0$.

While it is intuitive that higher market volatility and lower market correlation both increase the value of flexibility, it is less obvious that they also result in a higher optimal capacity level $(K_1^* + K_2^*)$. In the newsvendor model of Eppen (1979), the optimal flexible capacity increases in demand volatility and correlation if, and only if, the capacity exceeds the expected demand. In that model, the effect of demand volatility and correlation on the optimal capacity depends on whether it increases or decreases the probability that all capacity will be used. With zero marginal output cost ($c_Q = 0$), that probability is always one. But with price-setting, higher market volatility and lower market correlation increase the expected output prices and, hence, the marginal value of capacity. For the remainder, we return to the general case with nonnegative marginal output and switching costs ($c_Q \geq 0$ and $c_S \geq 0$).

4.1 Three dimensions of flexibility: Complements or substitutes?

Both mix and cost flexibilities mitigate the expected cost of mismatch between capacity and demand which results from the capacity investment being made under demand uncertainty. Mix flexibility reduces this mismatch cost by providing the opportunity to reallocate capacity based on the additional information revealed by the update time. Cost flexibility reduces the mismatch cost by enabling to postpone some of the irreversible expenditures until the update time when more information is available. Mix and cost flexibilities thus provide two distinct ways to reduce the initial capacity commitment. As a result, one would expect mix and cost flexibilities to be strategic substitutes. At the same time, both mix and cost flexibilities become more valuable as the update time moves closer to the selling season making more information available. One would therefore expect both mix and cost flexibilities to be complementary with time flexibility. We can formalize these notions as follows:

Conjecture 1 *Mix and time flexibilities are strategic complements:* $\frac{\partial^2 \Delta_F}{\partial \varphi \partial \tau} \geq 0$ *for any cost flexibility.*

Conjecture 2 *Time and cost flexibilities are strategic complements:* $\frac{\partial^2 \Delta_F}{\partial \tau \partial \gamma} \geq 0$ *for any mix flexibility.*

Conjecture 3 *Cost and mix flexibilities are strategic substitutes:* $\frac{\partial^2 \Delta_F}{\partial \varphi \partial \gamma} \leq 0$ *for any time flexibility.*

Our numerical investigation supports all three conjectures, as illustrated by Figure 2 for a typical set of parameter values. (All numerical results are based on simulation using 50,000 demand

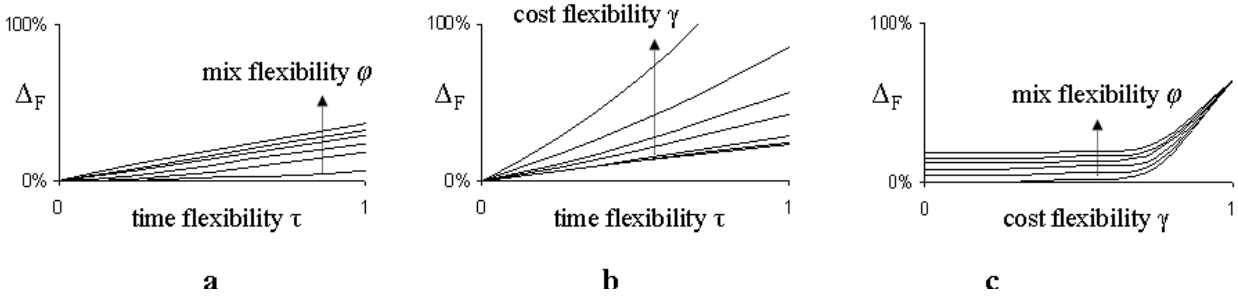

Figure 2: The effects of mix, cost and time flexibilities on the flexibility premium Δ_F for independent demands. ($T = 1$, $b = -2$, $c_K + c_Q = 0.2$, $\epsilon(0) = 1$, $\sigma = 1$ and $\rho = 0$.)

scenarios. In the boundary cases that were also solved analytically, the simulation errors were below 1%.)

Figure 2a shows the value premium for flexibility Δ_F for cost flexibility $\gamma = 0.5$, time flexibility $\tau \in [0, 1]$ and mix flexibility $\varphi \in \{0, ..., \infty\}$ (i.e., switching cost $c_S \in \{0, ..., \infty\}$). We observe that the marginal value of time flexibility $\partial\Delta_F/\partial\tau$ increases in mix flexibility φ, which confirms Conjecture 1 that mix and time flexibilities are strategic complements. Note that even in the absence of mix flexibility ($\varphi = 0$), time flexibility is valuable ($\partial\Delta_F/\partial\tau > 0$) because the firm has some cost flexibility ($\gamma > 0$). However, without time flexibility ($\tau = 0$), mix flexibility has no value ($\partial\Delta_F/\partial\varphi = 0$). This complementarity result has two managerial implications:

1. The closer to the selling season a firm chooses its output mix, or the more demand information it has at that time, the more it should invest in product-flexible technology, workforce cross-training and other enablers of mix flexibility.

2. The easier (cheaper) it is for a firm to convert one type of capacity to another, the more the firm should invest in obtaining accurate and timely market information and/or in postponing the point of product differentiation.

Figure 2b illustrates the complementarity of time and cost flexibilities for time flexibility $\tau \in [0,1]$, cost flexibility $\gamma \in [0,1]$ and mix flexibility $\varphi = 20$ (i.e., switching cost $c_S = 0.05$). We notice that the marginal value of time flexibility $\partial \Delta_F / \partial \tau$ increases in cost flexibility γ, confirming Conjecture 2. We also note that cost flexibility has no value ($\partial \Delta_F / \partial \gamma = 0$) without time flexibility ($\tau = 0$), while time flexibility has value ($\partial \Delta_F / \partial \tau > 0$) even without cost flexibility ($\gamma = 0$) due to the existing mix flexibility ($\varphi > 0$). The managerial implication is again twofold:

1. The more information a firm can gain by waiting, the more it should strive to postpone purchasing, hiring or production decisions.
2. And vice versa, the more of its quantity commitments a firm can postpone, the more it should invest in gathering information that will be available prior to making the postponed commitments.

Finally, **Figure 2c** shows how the flexibility premium Δ_F depends on cost flexibility $\gamma \in [0,1]$ and mix flexibility $\varphi \in \{0, ..., \infty\}$ for time flexibility $\tau = 0.5$. The marginal value of cost flexibility $\partial \Delta_F / \partial \gamma$ decreases in mix flexibility φ indicating that mix and cost flexibilities are strategic substitutes as conjectured. The figure also indicates that mix flexibility has no value ($\partial \Delta_F / \partial \varphi = 0$) under perfect cost flexibility ($\gamma = 1$), whereas cost flexibility is valuable ($\partial \Delta_F / \partial \gamma > 0$) even under perfect mix flexibility ($\varphi \to \infty$). In other words, cost flexibility can deliver all benefits of mix flexibility but not vice versa. The following two managerial implications ensue.

1. The higher the relative cost of assets (capacity), the more the firm should invest in its ability to switch between the production of different products. Thus, mix flexibility is particularly important in expensive, highly utilized capital equipment.
2. The higher the mix flexibility of a resource, the less can be gained from postponing its acquisition. In other words, it is more valuable to postpone the acquisition of product-specific

resources than that of a product-flexible resource.

4.2 Three dimensions of flexibility and demand correlation

Demand correlation is an important driver of the value of flexibility. Positive demand correlation reduces the efficacy of risk pooling and, thus, diminishes the value of mix flexibility. This insight, which we stated in Lemma 4 for the boundary case of $c_Q = c_S = 0$ and which can be verified numerically for any value of c_Q and c_S, is intuitive and well-known (see e.g., Fine and Freund 1990).

It is less obvious how demand correlation affects the *marginal* value of different types of flexibility, which is important because it determines the optimal investment in flexibility. Given that risk pooling is the main value driver of mix flexibility we expect the marginal value of mix flexibility to decrease in demand correlation. Similarly for the marginal value of time flexibility (which stems from increasing the benefits of risk pooling as well as the benefits of cost flexibility). Since cost and mix flexibilities are strategic substitutes, less risk pooling increases the marginal value of cost flexibility so we expect an increase in demand correlation to increase the marginal value of cost flexibility as well. We formalize our intuition in the following three conjectures:

Conjecture 4 *The value premium for flexibility is submodular in mix flexibility and demand correlation:* $\frac{\partial^2 \Delta_F}{\partial \varphi \partial \rho} \leq 0$ *for any cost and time flexibility.*

Conjecture 5 *The value premium for flexibility is supermodular in cost flexibility and demand correlation:* $\frac{\partial^2 \Delta_F}{\partial \gamma \partial \rho} \geq 0$ *for any mix and time flexibility.*

Conjecture 6 *The value premium for flexibility is submodular in time flexibility and demand correlation:* $\frac{\partial^2 \Delta_F}{\partial \tau \partial \rho} \leq 0$ *for any mix and cost flexibility.*

Our numerical investigation confirms the three conjectures, as illustrated in Figure 3. (This

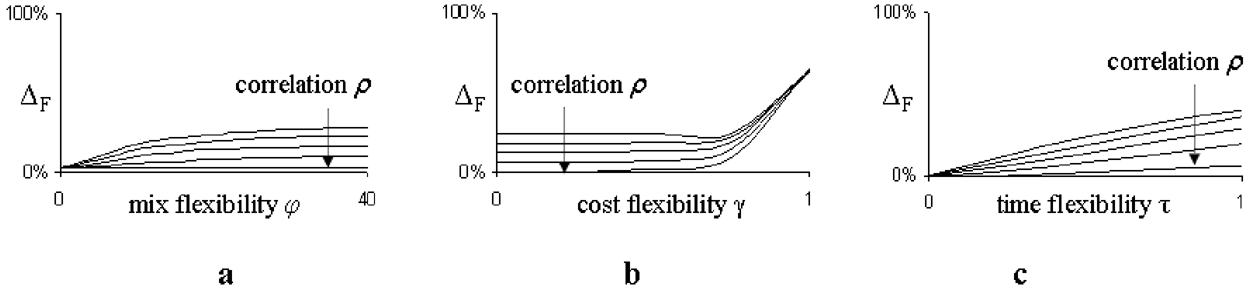

Figure 3: The effects of mix, cost and time flexibilities on the flexibility premium Δ_F under different demand correlations.

figure is based on the same parameter values as Figure 2 except that the demand correlation coefficient ρ is varied between -1 and 1.) We note that the value premium for flexibility Δ_F is strictly decreasing in demand correlation except for the following three cases in which demand correlation has no effect on this premium: (i) the firm has no mix flexibility (switching cost is infinite); (ii) the firm has perfect cost flexibility (no capacity commitment has to be made under demand uncertainty); and (iii) firm's time flexibility is zero (output decision is made ex ante).

Figure 3a plots the flexibility premium Δ_F as a function of mix flexibility φ (switching cost c_S) for given cost flexibility $\gamma = 0.5$ and time flexibility $\tau = 0.5$. It confirms our conjecture that the marginal value of mix flexibility $\partial\Delta_F/\partial\varphi$ decreases in demand correlation. The higher the demand correlation, the less capacity is likely to be converted and, therefore, the less value results from reducing the switching cost. As a result, the higher the demand correlation, the lower the optimal investment in mix flexibility (switching cost reduction).

Figure 3b graphs the flexibility premium Δ_F as a function of cost flexibility γ for given mix flexibility $\varphi = 20$ (switching cost $c_S = 0.05$) and time flexibility $\tau = 0.5$. As conjectured, the marginal value of cost flexibility $\partial\Delta_F/\partial\gamma$ increases in demand correlation. As demand correlation increases, mix flexibility becomes less effective in reducing the mismatch between capacity and demand, which makes cost flexibility relatively more important. Therefore, the higher the demand correlation, the more a firm should invest in cost flexibility. At the same time, as cost flexibility increases, capacity becomes less constraining and its conversion as well as demand correlation become less important.

Finally, **Figure 3c** shows how the flexibility premium Δ_F depends on time flexibility τ for given mix flexibility $\varphi = 20$ (switching cost $c_S = 0.05$) and cost flexibility $\gamma = 0.5$. As expected, the marginal value of time flexibility $\partial\Delta_F/\partial\tau$ decreases in demand correlation. More time flexibility means that more information is available before capacity conversion has to be made. As demand correlation increases, additional demand information is less likely to result in capacity conversion and, hence, is less valuable. Therefore, when demand correlation is high, the firm should invest less in time flexibility (forecasting, shorter lead times, etc.) than when demand correlation is low.

The next section links the value of mix flexibility to the value of market diversification with flexible assets.

5 Flexibility and Market Diversification

Economic theory recognizes that diversification can create shareholder value in the presence of market imperfections such as transaction costs in the factor market. In particular, if capacity investment is irreversible, market diversification coupled with technological or organizational flexibility may create value by reducing the aggregate uncertainty in capacity planning through statistical aggregation or "pooling" of demands. Although this argument has been formulated in the economic literature (e.g., Teece 1982, Smith and Triantis 2001), the value of diversification with flexible assets has not been studied in a rigorous mathematical model.

We quantify the benefits of diversification by comparing the value of a two-product firm to the sum of the values of two separate single-product firms under the assumption of sufficiently high transaction costs in the factor market that make firms' capacity investments irreversible. Thus, once acquired by either type of firm, capacity cannot be traded before the selling season is over. The optimal value of a two-product, or "diversified" firm is $v^*(0)$. The sum of the values of two single-product firms that cannot trade capacity is equal to $v^*(0)$ with $\varphi = 0$. We define the relative diversification premium as

$$\Delta_D \equiv \frac{v^*(0)|_{\varphi>0} - v^*(0)|_{\varphi=0}}{v^*(0)|_{\varphi=0}}. \tag{8}$$

Notice that in the boundary case of negligible output and switching costs characterized in Lemma 1, the value of diversification equals the value of flexibility. Given that the diversification premium decreases in cost flexibility and increases in mix flexibility, this value of flexibility is also an upper bound for the diversification premium in general:

$$\Delta_D \leq \left(\frac{\tilde{c}_K}{c_K}\right)^{-b-1} \left(\frac{\|\boldsymbol{\epsilon}(T)\|_\tau}{\mathbb{E}_0 \epsilon_i(T)}\right)^{-b} - 1, \tag{9}$$

where equality holds in the boundary case when $c_Q = c_S = 0$ and where c_K and $\tilde{c}_K$ are the unit capacity costs of the diversified and single-product firms, respectively. It directly follows from Lemmas 2-4 that in the boundary case the diversification premium Δ_D increases in time flexibility τ and demand volatility σ and decreases in demand correlation ρ.

The key drivers of the diversification premium are demand volatility, demand correlation, time and cost flexibility (which are both assumed to be the same for the diversified firm and the single-product firms) and mix flexibility of the diversified firm. Note that even though we are assuming the same time and cost flexibility for the diversified firm as for the two single-product firms, both of these parameters affect the diversification premium due to their complementarity/substitutability with mix flexibility. These effects are illustrated in Figure 4, which is based on the same parameter values as Figures 2 and 3.

The value of diversification stems from the diversified firm's ability to switch capacity between the products based on the additional information revealed up to time τ. As a consequence, the diversification premium increases in time flexibility and the diversified firm's mix flexibility and decreases in demand correlation. At the same time, the diversification premium decreases in the firms' cost flexibility. This is because cost and mix flexibilities are substitutes, i.e., as cost flexibility increases, additional mix flexibility is less valuable. In the extreme case of perfect cost flexibility ($\gamma = 1$), the firms do not have to make any capacity investment ex ante and, therefore, the ability to switch capacity within a diversified firm is worthless. In conclusion, diversification creates more value if demand correlation is not high, the diversified firm can switch capacity relatively close to the selling season (when accurate demand information is available) and at a relatively low cost, and the irreversible capacity investment represents a considerable part of the firm's total cost.

Figure 4 further indicates the following (admittedly intuitive) complementarity and substitutability results. More time flexibility magnifies both the positive effect of mix flexibility ($\partial^2 \Delta_D / \partial\varphi\partial\tau \geq 0$) and the negative effect of cost flexibility ($\partial^2 \Delta_D / \partial\gamma\partial\tau \leq 0$). Higher cost flexibility reduces the positive effect of mix flexibility ($\partial^2 \Delta_D / \partial\varphi\partial\gamma \leq 0$). Higher demand correlation reduces the positive effects of mix flexibility ($\partial^2 \Delta_D / \partial\varphi\partial\rho \leq 0$) and time flexibility ($\partial^2 \Delta_D / \partial\tau\partial\rho \leq 0$) and the negative effect of cost flexibility ($\partial^2 \Delta_D / \partial\gamma\partial\rho \geq 0$). Thus, the lower the correlation among the different business segments of a diversified firm, the more this firm should invest in technological and organizational flexibility that allows it employing its resources across different business segments based on existing market conditions.

One aspect of diversification that has attracted significant attention in the empirical finance and strategy literatures is the relative benefit of diversification into related versus unrelated industries. Related diversification is defined as one "involving businesses that share related production or marketing technologies" (Lubatkin and O'Neill 1987). While resource sharing arguments favor related diversification, risk and internal capital market considerations support unrelated diversification.

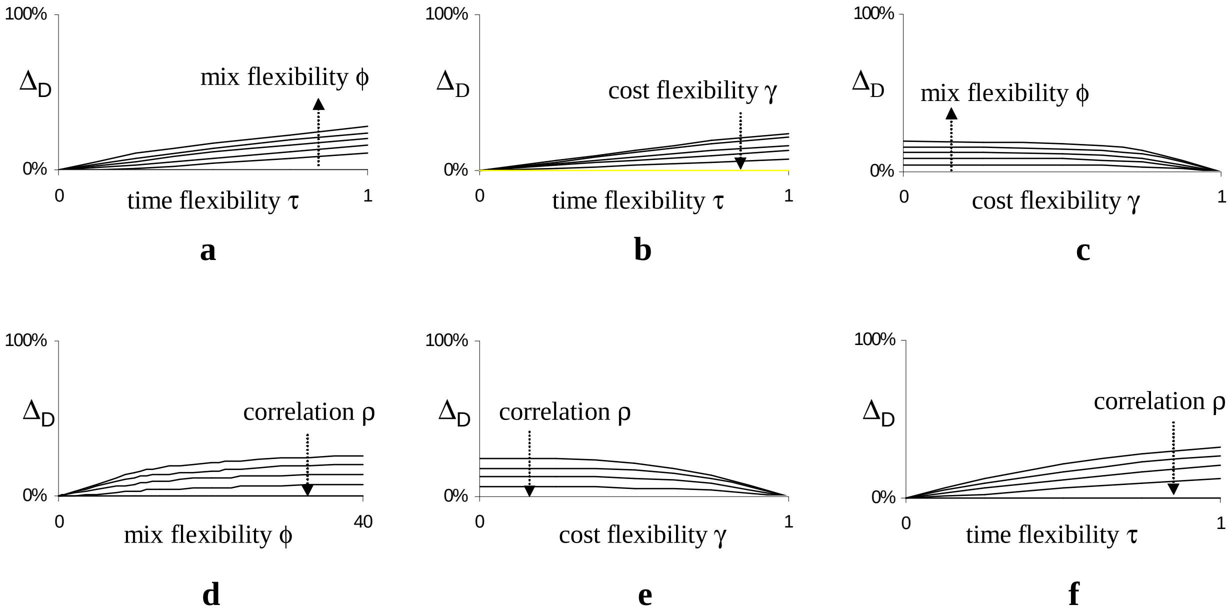

Figure 4: The effects of mix flexibility φ, cost flexibility γ, time flexibility τ and demand correlation ρ on the diversification premium Δ_D.

The extensive empirical evidence on the effect of diversification relatedness is also very fragmented (see e.g., Palich et al. 2000). This ambiguity stems partially from the fact that the notion of relatedness in the existing literature involves two aspects with very different implications for the value of diversification: relatedness of markets and relatedness of technology. To shed more light on this issue, we propose two distinct measures for both types of relatedness, and examine their relationship. Because more similar markets for the two products are likely to be more (positively) correlated, market correlation ρ is a reasonable proxy for (at least some parts of) *market relatedness* of diversification.

Because the switching cost is likely to be lower for technologically similar products, a natural proxy for *technological relatedness* of diversification is mix flexibility $\varphi = 1/c_S$. An alternative proxy for technological relatedness would be time flexibility τ because it is typically easier to postpone product differentiation until a later stage of the production process for technologically close products. No matter which of the two possible surrogates for technological relatedness we consider, Figure 4 shows that the diversification premium Δ_D increases in technological relatedness and market unrelatedness and, furthermore, technological relatedness and market unrelatedness are complementary in increasing this premium. These findings are supported by the empirical evidence of Amit and Livnat (1989) who demonstrate that efficient diversifiers operate in related business segments that have differential responses to business cycle changes and thereby enjoy the benefits from (technologically) related diversification as well as from the portfolio effect (market unrelatedness).

6 Summary

This paper considers a two-product firm that chooses capacities, output levels, and prices at three different points in time while continuously updating its demand forecast. We identify three aspects of the firm's ability to respond to the dynamically evolving demand conditions, which we refer to as mix, cost and time flexibility. We show that while mix and cost flexibilities are substitutes, they are both complementary with time flexibility. This has important implications for the optimal level of each of these flexibilities. For example, a firm that has inherently more mix flexibility should invest more in time flexibility and less in cost flexibility. Furthermore, we show that as demand correlation increases, managers should invest more in cost flexibility and less in mix and time flexibilities. Finally, we link the value of flexibility to the value of market diversification and discuss the main drivers of the diversification premium such as the three types of flexibility and demand correlation.

Our stylized model has several limitations. The symmetry assumption undervalues the revenue maximization option embedded in flexibility. The assumption that the firm is a monopoly in product markets may need to be relaxed because it is has been shown that competition may significantly impact the value of flexibility. For example, Anand and Girotra (2004) show that in the presence of Cournot-like competition, a firm's inability to adjust output has a "commitment" value which can sometimes dominate the risk pooling benefits of flexibility. While competitive considerations are beyond the scope of our paper, they would be an interesting extension.

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Appendix 1

The partitioning of the state space of $\boldsymbol{\epsilon}\left(\tau\right)$:

$$\Omega_1\left(\mathbf{K}\right) \equiv \left\{\mathbf{x}\in\mathbb{R}_+^2 : x_1 < \frac{c_Q K_1^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)},\ x_2 < \frac{c_Q K_2^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_2\left(\tau,T\right)}\right\},$$

$$\Omega_2\left(\mathbf{K}\right) \equiv \left\{\mathbf{x}\in\mathbb{R}_+^2 : \frac{c_Q K_1^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)} < x_1 < \frac{\left(c_Q+c_S\right) K_1^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)},\ x_2 < \frac{c_Q K_2^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_2\left(\tau,T\right)}\right\},$$

$$\Omega_3\left(\mathbf{K}\right) \equiv \left\{\mathbf{x}\in\mathbb{R}_+^2 : x_1 < \frac{c_Q K_1^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)},\ \frac{c_Q K_2^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_2\left(\tau,T\right)} < x_2 < \frac{\left(c_Q+c_S\right) K_2^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_2\left(\tau,T\right)}\right\},$$

$$\Omega_4\left(\mathbf{K}\right) \equiv \left\{\mathbf{x}\in\mathbb{R}_+^2 : x_1 > \frac{\left(c_Q+c_S\right) K_1^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)},\ \left(\frac{x_1}{c_Q+c_S}\right)^{-b} + \left(\frac{x_2}{c_Q}\right)^{-b} < \frac{K_1+K_2}{\left(\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)\right)^{-b}}\right\},$$

$$\Omega_5\left(\mathbf{K}\right) \equiv \left\{\mathbf{x}\in\mathbb{R}_+^2 : x_2 > \frac{\left(c_Q+c_S\right) K_2^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_2\left(\tau,T\right)},\ \left(\frac{x_1}{c_Q}\right)^{-b} + \left(\frac{x_2}{c_Q+c_S}\right)^{-b} < \frac{K_1+K_2}{\left(\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)\right)^{-b}}\right\},$$

$$\begin{aligned}\Omega_6\left(\mathbf{K}\right) \equiv & \left\{\mathbf{x}\in\mathbb{R}_+^2 : x_1 > \frac{c_Q K_1^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)},\ x_2 > \frac{c_Q K_2^{-1/b}}{\left(1+1/b\right)\mathbb{E}_0\epsilon_2\left(\tau,T\right)},\right. \\ & \left. \left|x_1 K_1^{1/b} - x_2 K_2^{1/b}\right| < \frac{c_S}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)}\right\},\end{aligned}$$

$$\begin{aligned}\Omega_7\left(\mathbf{K}\right) \equiv & \left\{\mathbf{x}\in\mathbb{R}_+^2 : \left(\frac{x_1}{c_Q+c_S}\right)^{-b} + \left(\frac{x_2}{c_Q}\right)^{-b} > \frac{K_1+K_2}{\left(\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)\right)^{-b}},\right. \\ & \left. x_1 K_1^{1/b} - x_2 K_2^{1/b} > \frac{c_S}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)}\right\},\end{aligned}$$

$$\begin{aligned}\Omega_8\left(\mathbf{K}\right) \equiv & \left\{\mathbf{x}\in\mathbb{R}_+^2 : \left(\frac{x_1}{c_Q}\right)^{-b} + \left(\frac{x_2}{c_Q+c_S}\right)^{-b} > \frac{K_1+K_2}{\left(\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)\right)^{-b}},\right. \\ & \left. x_2 K_2^{1/b} - x_1 K_1^{1/b} > \frac{c_S}{\left(1+1/b\right)\mathbb{E}_0\epsilon_1\left(\tau,T\right)}\right\}.\end{aligned}$$

The corresponding optimal output vector $\mathbf{Q}^*(\mathbf{K}, \boldsymbol{\epsilon}(\tau))$:

$$\begin{aligned}
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_1(\mathbf{K}),\ Q_i^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_i(T)}{c_Q}\right)^{-b}, i=1,2. \\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_2(\mathbf{K}),\ Q_1^* = K_1, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_2(T)}{c_Q}\right)^{-b}. \\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_3(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_1(T)}{c_Q}\right)^{-b}, Q_2^* = K_2.
\end{aligned}$$

$$\begin{aligned}
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_4(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_1(T)}{c_Q+c_S}\right)^{-b}, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_2(T)}{c_Q}\right)^{-b}. \\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_5(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_1(T)}{c_Q}\right)^{-b}, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau \epsilon_2(T)}{c_Q+c_S}\right)^{-b}. \\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_6(\mathbf{K}),\ \mathbf{Q}^* = \mathbf{K}. \\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_7(\mathbf{K}),\ \mathbf{Q}^* \text{ is the unique solution to } Q_1 + Q_2 = K_1 + K_2 \text{ and} \\
&\quad \mathbb{E}_\tau \epsilon_1(T)\, Q_1^{1/b} - \mathbb{E}_\tau \epsilon_2(T)\, Q_2^{1/b} = \frac{c_S}{(1+1/b)}. \\
\text{If } \boldsymbol{\epsilon}(\tau) &\in \Omega_8(\mathbf{K}),\ \mathbf{Q}^* \text{ is the unique solution to } Q_1 + Q_2 = K_1 + K_2 \text{ and} \\
&\quad \mathbb{E}_\tau \epsilon_2(T)\, Q_2^{1/b} - \mathbb{E}_\tau \epsilon_1(T)\, Q_1^{1/b} = \frac{c_S}{(1+1/b)}.
\end{aligned}$$

Appendix 2

Proof of Proposition 1: For simplicity, we use $\mathbf{p}^*$ and $\mathbf{Q}^*$ as shorthand for $\mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}))$ and $\mathbf{Q}^*(\mathbf{K})$, respectively, where $\mathbf{p}^*$ is given by (5) and $\mathbf{Q}^*$ is characterized in Appendix 1. Given the optimal pricing and output decisions, the firm value at time τ is

$$v(\tau; \mathbf{K}) = \sum_{i=1}^{2} \left(\mathbb{E}_\tau \epsilon_i(T)\,(Q_i^*)^{1+1/b} - c_Q Q_i^* - c_S \max(Q_i^* - K_i, 0) - c_K K_i \right). \tag{10}$$

The Hessian matrix of (10) with respect to $\mathbf{K}$ is

$$H_{\mathbf{K}} v\left(\tau ; \mathbf{K}\right)=\begin{cases}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}\left(\tau\right) \in \Omega_{145}, \\ \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_1\left(T\right) K_1^{1/b-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}\left(\tau\right) \in \Omega_2, \\ \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_2\left(T\right) K_2^{1/b-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}\left(\tau\right) \in \Omega_3, \\ \begin{pmatrix} \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_1\left(T\right) K_1^{1/b-1} & 0 \\ 0 & \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_2\left(T\right) K_2^{1/b-1} \end{pmatrix} & \text{if } \boldsymbol{\epsilon}\left(\tau\right) \in \Omega_6, \\ \frac{1+b}{b^2} \frac{\mathbb{E}_\tau \epsilon_1(T) \mathbb{E}_\tau \epsilon_2(T)\left(Q_1^*\right)^{1/b-1}\left(Q_2^*\right)^{1/b-1}}{\mathbb{E}_\tau \epsilon_1(T)\left(Q_1^*\right)^{1/b-1}+\mathbb{E}_\tau \epsilon_2(T)\left(Q_2^*\right)^{1/b-1}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}\left(\tau\right) \in \Omega_{78}. \end{cases}$$

Thus, $H_{\mathbf{K}} v\left(\tau ; \mathbf{K}\right)$ is negative definite if $\boldsymbol{\epsilon}\left(\tau\right) \in \Omega_6$ and negative semidefinite otherwise. Therefore, $v\left(\tau ; \mathbf{K}\right)$ is concave in $\mathbf{K}$ for any $\boldsymbol{\epsilon}\left(\tau\right)$ and the concavity is strict if $\boldsymbol{\epsilon}\left(\tau\right) \in \Omega_6$. This means that $v\left(0 ; \mathbf{K}\right)=\mathbb{E}_0 v\left(\tau ; \mathbf{K}\right)$ is concave in $\mathbf{K}$ and the first-order optimality condition $\nabla_{\mathbf{K}} v\left(0 ; \mathbf{K}\right)=\mathbf{0}$ is sufficient. Furthermore, if $c_S>0$, then $\operatorname{Pr}\left(\Omega_6\left(\mathbf{K}\right)\right)>0$ and the concavity is strict, implying that $\mathbf{K}^*$ is unique. The uniqueness of $\mathbf{K}^*$ together with the symmetry of all parameters implies that $K_1^*=K_2^*$. Taking the derivatrive of $v\left(0 ; \mathbf{K}\right)$ with respect to K_1 yields

$$\frac{\partial v\left(0 ; \mathbf{K}\right)}{\partial K_1}=\frac{\partial}{\partial K_1} \mathbb{E}_0 v\left(T ; \mathbf{K}\right)=\frac{\partial}{\partial K_1} \sum_{i=1}^8 \operatorname{Pr}\left(\Omega_i\left(\mathbf{K}\right)\right) \mathbb{E}_0\left(\left.v\left(T ; \mathbf{K}\right)\right| \Omega_i\left(\mathbf{K}\right)\right), \tag{11}$$

where the firm terminal value, given the optimal pricing and output decisions, is

$$v\left(T ; \mathbf{K}\right)=\sum_{i=1}^2\left(Q_i^* p_i^*-c_P Q_i^*-c_S \max \left(Q_i^*-K_i, 0\right)-c_K K_i\right).$$

Note that $v\left(T;\mathbf{K}\right)$ is continuous in $\boldsymbol{\epsilon}\left(\tau\right)$ and, therefore, the terms from differentiating the boundaries of $\Omega_1,...,\Omega_8$ with respect to K_1 in (11) cancel out. This leaves us with

$$\frac{\partial v\left(0;\mathbf{K}\right)}{\partial K_1}=\sum_{i=1}^{8}\Pr\left(\Omega_i\left(\mathbf{K}\right)\right)\mathbb{E}_0\left(\left.\frac{\partial v\left(T;\mathbf{K}\right)}{\partial K_1}\right|\Omega_i\left(\mathbf{K}\right)\right).$$

Differentiating $v\left(T;\mathbf{K}\right)$ with respect to K_1 and setting $\partial v\left(0;\mathbf{K}\right)/\partial K_1=0$ results in (7).□

Proof of Corollary 1: The result follows from Proposition 1 with $\tau=0$, $c_K=\tilde{c}_K$ and $c_Q=\tilde{c}_Q$.□

Proof of Lemma 1: It follows from Proposition 1 that if $c_S=c_Q=0$, the optimal total capacity and firm value are, respectively,

$$K_1^*+K_2^*=\left[\frac{1+1/b}{c_K}\mathbb{E}_0\left(\left(\mathbb{E}_\tau^{-b}\epsilon_1\left(T\right)+\mathbb{E}_\tau^{-b}\epsilon_2\left(T\right)\right)^{-1/b}\right)\right]^{-b}\text{ and }v^*\left(0\right)=\frac{c_K}{\left|1+b\right|}\left(K_1^*+K_2^*\right).$$

This together with Corollary 1 gives the desired result.□

Proof of Lemma 2: To simplify the notation, we normalize $T=1$ and $\boldsymbol{\epsilon}\left(0\right)=\mathbf{1}$. To prove the desired results, it is sufficient to show that $\frac{\partial}{\partial\tau}\left\|\boldsymbol{\epsilon}\left(1\right)\right\|_\tau\geq 0$. Recall that $\left\|\boldsymbol{\epsilon}\left(1\right)\right\|_\tau=\mathbb{E}_0\left\{\left(\frac{\mathbb{E}_\tau^{-b}\epsilon_1(1)+\mathbb{E}_\tau^{-b}\epsilon_2(1)}{2}\right)^{-1/b}\right\}$ and $\ln\boldsymbol{\epsilon}\left(t\right)\sim N\left(\ln\boldsymbol{\epsilon}\left(0\right),t\boldsymbol{\Sigma}\right)$. Using the fact that $\mathbb{E}_\tau\epsilon_i\left(1\right)=\epsilon_i\left(\tau\right)\exp\left(\frac{1}{2}\sigma^2\left(1-\tau\right)\right)$, we can write

$$\begin{aligned}\left\|\boldsymbol{\epsilon}\left(1\right)\right\|_\tau &= \mathbb{E}_0\left[\left(\frac{\left(\epsilon_1\left(\tau\right)\exp\left(\frac{1}{2}\sigma^2\left(1-\tau\right)\right)\right)^{-b}+\left(\epsilon_2\left(\tau\right)\exp\left(\frac{1}{2}\sigma^2\left(1-\tau\right)\right)\right)^{-b}}{2}\right)^{-1/b}\right]\\ &= 2^{1/b}\exp\left(\frac{1}{2}\sigma^2\left(1-\tau\right)\right)\mathbb{E}_0\left[\left(\epsilon_1^{-b}\left(\tau\right)+\epsilon_2^{-b}\left(\tau\right)\right)^{-1/b}\right].\end{aligned}$$

The normal vector $\ln\boldsymbol{\epsilon}\left(\tau\right)$ can be rewritten in terms of two independent standard normal random variables as $\ln\boldsymbol{\epsilon}\left(\tau\right)=\sqrt{\tau\boldsymbol{\Sigma}}\mathbf{Z}$, where $\mathbf{Z}\sim N\left(\mathbf{0},\mathbf{I}\right)$ and $\mathbf{I}$ is a 2×2 identity matrix. Since $\tau\boldsymbol{\Sigma}$ is positive definite, $\sqrt{\tau\boldsymbol{\Sigma}}$ exists and can be obtained using eigenvector decomposition, $\sqrt{\tau\boldsymbol{\Sigma}}=$

$\sqrt{\tau}\sigma \begin{pmatrix} \sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2} \\ -\sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2} \end{pmatrix}$. Using this transformation, we obtain

$$\begin{aligned} &\left\|\epsilon(1)\right\|_\tau = 2^{1/b} \exp\left(\frac{1}{2}\sigma^2(1-\tau)\right) \times \\ &\mathbb{E}_0\left[\exp\left(\sqrt{\tau}\sigma\sqrt{(1+\rho)/2}Z_2\right)\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b}\right]. \end{aligned}$$

Since Z_1 and Z_2 are independent, we can further simplify

$$\begin{aligned} \left\|\epsilon(1)\right\|_\tau &= 2^{1/b} \exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right) \times \\ &\mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b}\right]. \end{aligned} \tag{12}$$

Next, we take the derivative of (12) with respect to τ. After some algebra, we obtain

$$\begin{aligned} &\frac{\partial}{\partial\tau}\left\|\epsilon(1)\right\|_\tau = \frac{1}{4}\sigma^2(\rho-1)\left\|\epsilon(1)\right\|_\tau + 2^{1/b}\exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right)\frac{\sigma\sqrt{(1-\rho)/2}}{\sqrt{\tau}} \times \\ &\mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b-1}\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)Z_1\right]. \end{aligned} \tag{13}$$

To evaluate (13), we make use of the fact that for a differentiable function g and a standard normal random variable Z_1, $\mathbb{E}(g(Z_1)Z_1) = \mathbb{E}g'(Z_1)$ (Rubinstein 1976). Applying this result and some algebra to (13), we obtain

$$\begin{aligned}\frac{\partial}{\partial\tau}\left\|\epsilon\left(1\right)\right\|_{\tau} &= -\sigma^{2}\left(1-\rho\right)\left(1+b\right)2^{1/b}\exp\left(\frac{1}{2}\sigma^{2}-\frac{1}{4}\tau\sigma^{2}+\frac{1}{4}\tau\sigma^{2}\rho\right)\times\\ &\quad\mathbb{E}_{0}\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{\left(1-\rho\right)/2}Z_{1}\right)+\exp\left(b\sqrt{\tau}\sigma\sqrt{\left(1-\rho\right)/2}Z_{1}\right)\right)^{-1/b-2}\right]\\ &= -\sigma^{2}\left(1-\rho\right)\left(1+b\right)2^{1/b}\mathbb{E}_{0}\left[\left(\prod_{i=1,2}\mathbb{E}_{\tau}^{-b}\epsilon_{i}\left(1\right)\right)\left(\sum_{i=1,2}\mathbb{E}_{\tau}^{-b}\epsilon_{i}\left(1\right)\right)^{-1/b-2}\right]\\ &\geq 0.\square\end{aligned}$$

Proof of Lemma 3: The proof is similar to the proof of Lemma 2 and is omitted.□

Proof of Lemma 4: The proof is similar to the proof of Lemma 2 and is omitted.□