Appendix 1

The partitioning of the state space of $\boldsymbol{\epsilon}(\tau)$:

$$\Omega_1(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 < \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ x_2 < \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)} \right\},$$

$$\Omega_2(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)} < x_1 < \frac{(c_Q+c_S) K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ x_2 < \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)} \right\},$$

$$\Omega_3(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 < \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)} < x_2 < \frac{(c_Q+c_S) K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)} \right\},$$

$$\Omega_4(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 > \frac{(c_Q+c_S) K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ \left(\frac{x_1}{c_Q+c_S}\right)^{-b} + \left(\frac{x_2}{c_Q}\right)^{-b} < \frac{K_1+K_2}{((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T))^{-b}} \right\},$$

$$\Omega_5(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_2 > \frac{(c_Q+c_S) K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)},\ \left(\frac{x_1}{c_Q}\right)^{-b} + \left(\frac{x_2}{c_Q+c_S}\right)^{-b} < \frac{K_1+K_2}{((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T))^{-b}} \right\},$$

$$\Omega_6(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 > \frac{c_Q K_1^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)},\ x_2 > \frac{c_Q K_2^{-1/b}}{(1+1/b)\,\mathbb{E}_0\epsilon_2(\tau,T)}, \right.$$
$$\left. \left| x_1 K_1^{1/b} - x_2 K_2^{1/b} \right| < \frac{c_S}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)} \right\},$$

$$\Omega_7(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \left(\frac{x_1}{c_Q+c_S}\right)^{-b} + \left(\frac{x_2}{c_Q}\right)^{-b} > \frac{K_1+K_2}{((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T))^{-b}}, \right.$$
$$\left. x_1 K_1^{1/b} - x_2 K_2^{1/b} > \frac{c_S}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)} \right\},$$

$$\Omega_8(\mathbf{K}) \equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \left(\frac{x_1}{c_Q}\right)^{-b} + \left(\frac{x_2}{c_Q+c_S}\right)^{-b} > \frac{K_1+K_2}{((1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T))^{-b}}, \right.$$
$$\left. x_2 K_2^{1/b} - x_1 K_1^{1/b} > \frac{c_S}{(1+1/b)\,\mathbb{E}_0\epsilon_1(\tau,T)} \right\}.$$

The corresponding optimal output vector $\mathbf{Q}^*(\mathbf{K}, \boldsymbol{\epsilon}(\tau))$:

$$\text{If } \boldsymbol{\epsilon}(\tau) \in \Omega_1(\mathbf{K}),\ Q_i^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_i(T)}{c_Q}\right)^{-b}, i = 1,2.$$

$$\text{If } \boldsymbol{\epsilon}(\tau) \in \Omega_2(\mathbf{K}),\ Q_1^* = K_1, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_2(T)}{c_Q}\right)^{-b}.$$

$$\text{If } \boldsymbol{\epsilon}(\tau) \in \Omega_3(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_1(T)}{c_Q}\right)^{-b}, Q_2^* = K_2.$$

$$\text{If } \boldsymbol{\epsilon}(\tau) \in \Omega_4(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_1(T)}{c_Q+c_S}\right)^{-b}, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_2(T)}{c_Q}\right)^{-b}.$$

$$\text{If } \boldsymbol{\epsilon}(\tau) \in \Omega_5(\mathbf{K}),\ Q_1^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_1(T)}{c_Q}\right)^{-b}, Q_2^* = \left(\frac{(1+1/b)\,\mathbb{E}_\tau\epsilon_2(T)}{c_Q+c_S}\right)^{-b}.$$

$$\text{If } \boldsymbol{\epsilon}(\tau) \in \Omega_6(\mathbf{K}),\ \mathbf{Q}^* = \mathbf{K}.$$

If $\boldsymbol{\epsilon}(\tau) \in \Omega_7(\mathbf{K})$, $\mathbf{Q}^*$ is the unique solution to $Q_1 + Q_2 = K_1 + K_2$ and

$$\mathbb{E}_\tau\epsilon_1(T)\,Q_1^{1/b} - \mathbb{E}_\tau\epsilon_2(T)\,Q_2^{1/b} = \frac{c_S}{(1+1/b)}.$$

If $\boldsymbol{\epsilon}(\tau) \in \Omega_8(\mathbf{K})$, $\mathbf{Q}^*$ is the unique solution to $Q_1 + Q_2 = K_1 + K_2$ and

$$\mathbb{E}_\tau\epsilon_2(T)\,Q_2^{1/b} - \mathbb{E}_\tau\epsilon_1(T)\,Q_1^{1/b} = \frac{c_S}{(1+1/b)}.$$

Appendix 2

Proof of Proposition 1: For simplicity, we use $\mathbf{p}^*$ and $\mathbf{Q}^*$ as shorthand for $\mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}))$ and $\mathbf{Q}^*(\mathbf{K})$, respectively, where $\mathbf{p}^*$ is given by (5) and $\mathbf{Q}^*$ is characterized in Appendix 1. Given the optimal pricing and output decisions, the firm value at time τ is

$$v(\tau;\mathbf{K}) = \sum_{i=1}^{2}\left(\mathbb{E}_\tau\epsilon_i(T)\,(Q_i^*)^{1+1/b} - c_Q Q_i^* - c_S\max(Q_i^* - K_i, 0) - c_K K_i\right). \tag{9}$$

The Hessian matrix of (9) with respect to $\mathbf{K}$ is

$$H_{\mathbf{K}} v(\tau; \mathbf{K}) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_{145}, \\ \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_1(T) K_1^{1/b-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_2, \\ \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_2(T) K_2^{1/b-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_3, \\ \begin{pmatrix} \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_1(T) K_1^{1/b-1} & 0 \\ 0 & \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_2(T) K_2^{1/b-1} \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_6, \\ \frac{1+b}{b^2} \frac{\mathbb{E}_\tau \epsilon_1(T) \mathbb{E}_\tau \epsilon_2(T) \left(Q_1^*\right)^{1/b-1} \left(Q_2^*\right)^{1/b-1}}{\mathbb{E}_\tau \epsilon_1(T) \left(Q_1^*\right)^{1/b-1} + \mathbb{E}_\tau \epsilon_2(T) \left(Q_2^*\right)^{1/b-1}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \boldsymbol{\epsilon}(\tau) \in \Omega_{78}. \end{cases}$$

Thus, $H_{\mathbf{K}} v(\tau; \mathbf{K})$ is negative definite if $\boldsymbol{\epsilon}(\tau) \in \Omega_6$ and negative semidefinite otherwise. Therefore, $v(\tau; \mathbf{K})$ is concave in $\mathbf{K}$ for any $\boldsymbol{\epsilon}(\tau)$ and the concavity is strict if $\boldsymbol{\epsilon}(\tau) \in \Omega_6$. This means that $v(0; \mathbf{K}) = \mathbb{E}_0 v(\tau; \mathbf{K})$ is concave in $\mathbf{K}$ and the first-order optimality condition $\nabla_{\mathbf{K}} v(0; \mathbf{K}) = \mathbf{0}$ is sufficient. Furthermore, if $c_S > 0$, then $\Pr(\Omega_6(\mathbf{K})) > 0$ and the concavity is strict, implying that $\mathbf{K}^*$ is unique. The uniqueness of $\mathbf{K}^*$ together with the symmetry of all parameters implies that $K_1^* = K_2^*$. Taking the derivative of $v(0; \mathbf{K})$ with respect to K_1 yields

$$\frac{\partial v(0; \mathbf{K})}{\partial K_1} = \frac{\partial}{\partial K_1} \mathbb{E}_0 v(T; \mathbf{K}) = \frac{\partial}{\partial K_1} \sum_{i=1}^{8} \Pr(\Omega_i(\mathbf{K})) \mathbb{E}_0 \left(v(T; \mathbf{K}) \middle| \Omega_i(\mathbf{K}) \right), \tag{10}$$

where the firm terminal value, given the optimal pricing and output decisions, is

$$v(T; \mathbf{K}) = \sum_{i=1}^{2} \left(Q_i^* p_i^* - c_P Q_i^* - c_S \max(Q_i^* - K_i, 0) - c_K K_i \right).$$

Note that $v(T;\mathbf{K})$ is continuous in $\boldsymbol{\epsilon}(\tau)$ and, therefore, the terms from differentiating the boundaries of $\Omega_1, ..., \Omega_8$ with respect to K_1 in (10) cancel out. This leaves us with

$$\frac{\partial v(0;\mathbf{K})}{\partial K_1} = \sum_{i=1}^{8} \Pr(\Omega_i(\mathbf{K}))\, \mathbb{E}_0\left(\frac{\partial v(T;\mathbf{K})}{\partial K_1} \middle| \Omega_i(\mathbf{K}) \right).$$

Differentiating $v(T;\mathbf{K})$ with respect to K_1 and setting $\partial v(0;\mathbf{K})/\partial K_1 = 0$ results in (7).□

Proof of Corollary 1: The result follows from Proposition 1 with $\tau = 0$, $c_K = \tilde{c}_K$ and $c_Q = \tilde{c}_Q$.□

Proof of Lemma 1: It follows from Proposition 1 that if $c_S = c_Q = 0$, the optimal total capacity and firm value are, respectively,

$$K_1^* + K_2^* = \left[\frac{1+1/b}{c_K}\mathbb{E}_0\left(\left(\mathbb{E}_\tau^{-b}\epsilon_1(T) + \mathbb{E}_\tau^{-b}\epsilon_2(T)\right)^{-1/b}\right)\right]^{-b} \text{ and } v^*(0) = \frac{c_K}{|1+b|}(K_1^* + K_2^*).$$

This together with Corollary 1 gives the desired result.□

Proof of Lemma 2: To simplify the notation, we normalize $T = 1$ and $\boldsymbol{\epsilon}(0) = \mathbf{1}$. To prove the desired results, it is sufficient to show that $\frac{\partial}{\partial \tau}\|\boldsymbol{\epsilon}(1)\|_\tau \geq 0$. Recall that $\|\boldsymbol{\epsilon}(1)\|_\tau = \mathbb{E}_0\left\{\left(\frac{\mathbb{E}_\tau^{-b}\epsilon_1(1)+\mathbb{E}_\tau^{-b}\epsilon_2(1)}{2}\right)^{-1/b}\right\}$ and $\ln\boldsymbol{\epsilon}(t) \sim N(\ln\boldsymbol{\epsilon}(0), t\boldsymbol{\Sigma})$. Using the fact that $\mathbb{E}_\tau\epsilon_i(1) = \epsilon_i(\tau)\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)$, we can write

$$\begin{aligned}
\|\boldsymbol{\epsilon}(1)\|_\tau &= \mathbb{E}_0\left[\left(\frac{\left(\epsilon_1(\tau)\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\right)^{-b} + \left(\epsilon_2(\tau)\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\right)^{-b}}{2}\right)^{-1/b}\right] \\
&= 2^{1/b}\exp\left(\frac{1}{2}\sigma^2(1-\tau)\right)\mathbb{E}_0\left[\left(\epsilon_1^{-b}(\tau) + \epsilon_2^{-b}(\tau)\right)^{-1/b}\right].
\end{aligned}$$

The normal vector $\ln\boldsymbol{\epsilon}(\tau)$ can be rewritten in terms of two independent standard normal random variables as $\ln\boldsymbol{\epsilon}(\tau) = \sqrt{\tau\boldsymbol{\Sigma}}\mathbf{Z}$, where $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ and $\mathbf{I}$ is a 2×2 identity matrix. Since $\tau\boldsymbol{\Sigma}$ is positive definite, $\sqrt{\tau\boldsymbol{\Sigma}}$ exists and can be obtained using eigenvector decomposition, $\sqrt{\tau\boldsymbol{\Sigma}} =$

$\sqrt{\tau}\sigma \begin{pmatrix} \sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2} \\ -\sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2} \end{pmatrix}$. Using this transformation, we obtain

$$\|\epsilon(1)\|_\tau = 2^{1/b} \exp\left(\frac{1}{2}\sigma^2(1-\tau)\right) \times$$
$$\mathbb{E}_0\left[\exp\left(\sqrt{\tau}\sigma\sqrt{(1+\rho)/2}Z_2\right)\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b}\right].$$

Since Z_1 and Z_2 are independent, we can further simplify

$$\|\epsilon(1)\|_\tau = 2^{1/b} \exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right) \times$$
$$\mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b}\right]. \tag{11}$$

Next, we take the derivative of (11) with respect to τ. After some algebra, we obtain

$$\frac{\partial}{\partial\tau}\|\epsilon(1)\|_\tau = \frac{1}{4}\sigma^2(\rho-1)\|\epsilon(1)\|_\tau + 2^{1/b}\exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right)\frac{\sigma\sqrt{(1-\rho)/2}}{\sqrt{\tau}} \times$$
$$\mathbb{E}_0\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)\right)^{-1/b-1}\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right)Z_1\right]. \tag{12}$$

To evaluate (12), we make use of the fact that for a differentiable function g and a standard normal random variable Z_1, $\mathbb{E}(g(Z_1)Z_1) = \mathbb{E}g'(Z_1)$ (Rubinstein 1976). Applying this result and some algebra to (12), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \tau}\left\|\epsilon\left(1\right)\right\|_{\tau} &= -\sigma^{2}\left(1-\rho\right)\left(1+b\right)2^{1/b}\exp\left(\frac{1}{2}\sigma^{2}-\frac{1}{4}\tau\sigma^{2}+\frac{1}{4}\tau\sigma^{2}\rho\right)\times \\
&\mathbb{E}_{0}\left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{\left(1-\rho\right)/2}Z_{1}\right)+\exp\left(b\sqrt{\tau}\sigma\sqrt{\left(1-\rho\right)/2}Z_{1}\right)\right)^{-1/b-2}\right] \\
&= -\sigma^{2}\left(1-\rho\right)\left(1+b\right)2^{1/b}\mathbb{E}_{0}\left[\left(\prod_{i=1,2}\mathbb{E}_{\tau}^{-b}\epsilon_{i}\left(1\right)\right)\left(\sum_{i=1,2}\mathbb{E}_{\tau}^{-b}\epsilon_{i}\left(1\right)\right)^{-1/b-2}\right] \\
&\geq 0.\square
\end{aligned}$$

Proof of Lemma 3: The proof is similar to the proof of Lemma 2 and is omitted.□

Proof of Lemma 4: The proof is similar to the proof of Lemma 2 and is omitted.□