Appendix: Details on Estimation for "Variance Risk Premium Dynamics: The Role of Jumps"

1 Calculation of $P(IV_a(t)|\mathscr{G}_t)$

Combining central limit theorems in Barndorff-Nielsen et al. (2006) and Jacod (2008) about $TV_\delta(t)$ and $RV_\delta(t)$ respectively (in the presence of jumps) we have the following asymptotic approximation for $TV_\delta(t)$ and $RV_\delta(t)$

$$\begin{aligned} TV_\delta(t) &= (1-\pi)\theta + (IV(t) - (1-\pi)\theta) + \nu_{1t} \\ JV_\delta(t) &= (1-\pi)\int_{\mathbb{R}_0^n} h^2(\mathbf{x})G(d\mathbf{x}) + \int_{t+\pi}^{t+1}\int_{\mathbb{R}_0^n} h^2(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}) + \nu_{2t}, \end{aligned}$$

where (ν_{1t},ν_{2t}) is a martingale difference sequence with

$$\mathbb{E}\left(\nu_{1t}^2\right) = \frac{3.0613}{M}\mathbb{E}\left(\int_{t+\pi}^{t+1}\sigma^4(u)du\right), \quad \mathbb{E}\left(\nu_{1t}\nu_{2t}\right) = -\frac{1.0613}{M}\mathbb{E}\left(\int_{t+\pi}^{t+1}\sigma^4(u)du\right),$$

$$\begin{aligned} \mathbb{E}\left(\nu_{2t}^2\right) &= \frac{1.0613}{M}\mathbb{E}\left(\int_{t+\pi}^{t+1}\sigma^4(u)du\right) \\ &\quad + \frac{1}{M}\left(4\mathbb{E}(IV(t))\int_{\mathbb{R}_0^n} h^2(\mathbf{x})G(d\mathbf{x}) + 2(1-\pi)\int_{\mathbb{R}_0^n} h^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x})\right). \end{aligned}$$

In the derivation of $P(IV_a(t)|\mathscr{G}_t)$ we will assume that the approximation holds exact (see Todorov (2009b) for conditions under which this does not change the estimation results asymptotically). We can decompose the demeaned IV into two independent parts

$$\begin{aligned} IV(t) - (1-\pi)\theta &= \widetilde{IV}^c(t) + \widetilde{IV}^j(t), \\ \widetilde{IV}^c(t) &= \int_{t+\pi}^{t+1} V^c(s)ds - (1-\pi)\mathbb{E}(V^c(s)), \\ \widetilde{IV}^j(t) &= \int_{t+\pi}^{t+1} V^j(s)ds - (1-\pi)\mathbb{E}(V^j(s)), \end{aligned}$$

with $V^c(t)$ and $V^j(t)$ specified in equations (3) and (4) respectively. Both $\widetilde{IV}^c(t)$ and $\widetilde{IV}^j(t)$ have autocorrelations of ARMA(1,1) process and are independent of each other. Also, $\int_{t+\pi}^{t+1}\int_{\mathbb{R}_0^n} h^2(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})$

is an i.i.d. sequence connected with $\widetilde{IV}^j(t)$. Therefore, $(\widetilde{IV}^c(t), \widetilde{IV}^j(t), \int_{t+\pi}^{t+1}\int_{\mathbb{R}_0^n} h^2(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}))$ have the same autocorrelation structure as the following process (y_t^c, y_t^j, y_t^h)

$$
\begin{aligned}
y_t^c &= \phi_c y_{t-1}^c + e_t^c + \theta_c e_{t-1}^c, \\
y_t^j &= \phi_j y_{t-1}^j + \phi_h y_{t-1}^h + e_t^j + \theta_j e_{t-1}^j, \\
y_t^h &= e_t^h,
\end{aligned} \tag{1.1}
$$

where $(e_t) = (e_t^c, e_t^j, e_t^h)$ is a discrete time white noise process, i.e. $\mathbb{E}(e_t e_s') = \mathbf{0}$ for $t \neq s$.

Next, I derive the parameters of the above multivariate ARMA process as functions of the parameters of the underlying SV model (1)-(4). First, it is easy to see that $\phi_c = e^{-\kappa_c}$ and $\phi_j = e^{\rho_j}$.

To determine the moving average coefficient θ_c and the variance of the error term in the first equation, $\mathrm{Var}(e_t^c)$, I solve the following system of equations

$$
\begin{aligned}
\mathrm{Var}(y_t^c) - \phi_c \mathrm{Cov}(y_t^c, y_{t-1}^c) &= \mathrm{Var}(e_t^c)\left(1 + \theta_c\phi_c + \theta_c^2\right), \\
\mathrm{Cov}(y_t^c, y_{t-1}^c) - \phi_c \mathrm{Var}(y_t^c) &= \mathrm{Var}(e_t^c)\theta_c.
\end{aligned}
$$

First, it is easy to derive

$$
\begin{aligned}
\mathrm{Var}(y_t^c) &= \mathrm{Var}(IV^c(t)) = 2\sigma_c^2 \frac{(e^{-(1-\pi)\kappa_c} + (1-\pi)\kappa_c - 1)}{\kappa_c^2} \\
\mathrm{Cov}(y_t^c, y_{t-1}^c) &= \mathrm{Cov}(IV^c(t), IV^c(t-1)) = \frac{\sigma_c^2(1 - e^{-(1-\pi)\kappa_c})^2 e^{-\pi\kappa_c}}{\kappa_c^2}.
\end{aligned}
$$

If we set $\rho_c = \frac{\mathrm{Cov}(IV^c(t), IV^c(t-1))}{\mathrm{Var}(IV^c(t))}$, then it is easy to verify that the invertible solution for the moving average coefficient (i.e. $|\theta_c| < 1$) is given by

$$
\theta_c = \frac{1 + \phi_c^2 - 2\phi_c\rho_c - \sqrt{(1 + \phi_c^2 - 2\phi_c\rho_c)^2 - 4(\rho_c - \phi_c)^2}}{2(\rho_c - \phi_c)},
$$

and from here the expression for $\mathrm{Var}(e_t^c)$ follows.

Because of the independence of IV^c from the jumps it is easy to see that we must have $\mathbb{E}(e_t^c e_t^j) = \mathbb{E}(e_t^c e_t^h) = 0$. For the correlation between the error terms in the last two equations of the system (1.1) I use a representation of $\widetilde{IV}^j(t)$ with respect to the compensation measure $\tilde{\mu}$ derived in Todorov (2009a) as well as the Ito isometry. After simplifying we have

$$
\begin{aligned}
\mathbb{E}(e_t^j e_t^h) &= \mathrm{Cov}\left(\widetilde{IV}^j(t), \int_{t+\pi}^{t+1}\int_{\mathbb{R}_0^n} h^2(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right) \\
&= \frac{e^{(1-\pi)\rho_j} - 1 - (1-\pi)\rho_j}{\rho_j^2} \int_{\mathbf{R}_0^n} h^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}),
\end{aligned} \tag{1.2}
$$

and for $\mathrm{Var}(e_t^h)$

$$
\mathrm{Var}(e_t^h) = \mathrm{Var}\left(\int_{t+\pi}^{t+1}\int_{\mathbb{R}_0^n} h^2(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right) = (1-\pi)\int_{\mathbb{R}_0^n} h^4(\mathbf{x})G(d\mathbf{x}).
$$

Finally, we need to determine ϕ_h, θ_j and $\mathrm{Var}(e_t^j)$. I solve the following system of equations

$$\begin{aligned}
\mathrm{Var}\left(y_t^j\right) - \phi_j \mathrm{Cov}\left(y_t^j, y_{t-1}^j\right) &= \phi_h \mathrm{Cov}\left(y_t^j, y_{t-1}^h\right) + \mathrm{Var}(e_t^j)\left(1+\theta_j\phi_j+\theta_j^2\right) \\
&\quad +\phi_h\theta_j \mathrm{Cov}\left(e_t^h, e_t^j\right), \\
\mathrm{Cov}\left(y_t^j, y_{t-1}^j\right) - \phi_j \mathrm{Var}\left(y_t^j\right) &= \phi_h \mathrm{Cov}\left(e_t^h, e_t^j\right) + \theta_j \mathrm{Var}(e_t^j), \\
\mathrm{Cov}\left(y_t^j, y_{t-1}^h\right) - \phi_j \mathrm{Cov}\left(e_t^h, e_t^j\right) &= \phi_h \mathrm{Var}\left(e_t^h\right) + \theta_j \mathrm{Cov}\left(e_t^h, e_t^j\right),
\end{aligned}$$

where I use the following expressions for $\mathrm{Var}\left(y_t^j\right)$, $\mathrm{Cov}\left(y_t^j, y_{t-1}^j\right)$ and $\mathrm{Cov}\left(y_t^j, y_{t-1}^h\right)$,which are easy to derive (see Todorov (2009a))

$$\begin{aligned}
\mathrm{Var}\left(y_t^j\right) &= \mathrm{Var}\left(\widetilde{IV}^j(t)\right) = \frac{1-e^{(1-\pi)\rho_j}+(1-\pi)\rho_j}{\rho_j^3}\int_{\mathbb{R}_0^2} k^2(\mathbf{x})G(d\mathbf{x}), \\
\mathrm{Cov}\left(y_t^j, y_{t-1}^j\right) &= \mathrm{Cov}\left(\widetilde{IV}^j(t), \widetilde{IV}^j(t-1)\right) = -\frac{e^{\pi\rho_j}(1-e^{(1-\pi)\rho_j})^2}{2\rho_j^3}\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}), \\
\mathrm{Cov}\left(y_t^j, y_{t-1}^h\right) &= \mathrm{Cov}\left(\widetilde{IV}^j(t), \int_{t-1}^t\int_{\mathbb{R}_0^n} h^2(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right) \\
&= e^{\pi\rho_j}\left(\frac{1-e^{(1-\pi)\rho_j}}{\rho_j}\right)^2\int_{\mathbb{R}_0^n} h^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}).
\end{aligned}$$

Finally, to calculate $P(IV_a(t)|\mathscr{G}_t)$ I use the following state-space representation for TV, which is easy to verify

$$\begin{pmatrix} TV_\delta(t) \\ JV_\delta(t) \end{pmatrix} = (1-\pi)\begin{pmatrix} \theta \\ \int_{\mathbb{R}_0^n} h^2(\mathbf{x})G(d\mathbf{x}) \end{pmatrix} + \mathbf{H}'\xi_t + \nu_t \tag{1.3}$$

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1}, \tag{1.4}$$

where

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ \theta_c & 0 \\ 1 & 0 \\ \theta_j & 0 \\ 0 & 1 \\ \frac{\phi_h\theta_j}{\phi_j+\theta_j} & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \phi_c & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_j & 0 & \frac{\phi_h\phi_j}{\phi_j+\theta_j} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{v}_t = \begin{pmatrix} e_t^c \\ 0 \\ e_t^j \\ 0 \\ e_t^h \\ 0 \end{pmatrix}, \quad \nu_t = \begin{pmatrix} \nu_{1t} \\ \nu_{2t} \end{pmatrix}.$$

From here generating the linear projection $P(IV_a(t)|\mathscr{G}_t)$, given a set of parameters, follows easily (see e.g. Hamilton (1994), Chapter 13).

2 Calculation of Point Estimates and Standard Errors

All estimates in the paper come from GMM estimation which is conducted using the Chernozhukov and Hong (2003) MCMC approach which I briefly outline here.

First the generic GMM objective function is given by

$$J(\theta) = T\widehat{m}_T(\theta)'\widehat{W}m_T(\theta), \tag{2.5}$$

where the moment vector $\widehat{m}(\theta)$ has been specified in the paper for each estimation. The weighting function $\widehat{W}$ is a consistent estimate of the optimal one (i.e. the inverse of the asymptotic variance-covariance of the moment vector). Due to the separability of parameters and data in the moment vectors used in the paper, $\widehat{W}$ is constructed on a single step directly from the data. In the construction of $\widehat{W}$, I use a HAC estimator with a Parzen kernel with lag length of 80.

Next, following Chernozhukov and Hong (2003), I treat the Laplace transform of $J(\theta)$, $L(\theta) := \exp(-0.5J(\theta))$ (the multiplication by $1/2$ of $J(\theta)$ is in order to make the information equality for the efficient GMM hold), as unnormalized likelihood function and apply standard MCMC estimation. The prior and the updating of the posterior is done as follows.

1. For all parameters I use flat (uninformative) priors and further I impose all inequality restrictions on the parameters as priors.

2. On each step in the MCMC, I use random walk Metropolis-Hasting algorithm with a normal proposal, updating one parameter at a time.

The output from the MCMC estimation (after the burn-in) is the MCMC chain $\{\theta_i\}_{i=1}^R$. Then, the GMM estimator is computed simply as

$$\widehat{\theta} = \text{mode}\{\theta_i\}_{i=1}^R. \tag{2.6}$$

The estimate for the asymptotic variance of $\widehat{\theta}$ is calculated analytically (via numerically computed derivatives) and is given by $\nabla_\theta m_T(\widehat{\theta})'\widehat{W}\nabla_\theta m_T(\widehat{\theta})$. (Alternatively, one can use the variance of the MCMC chain, since the GMM estimators used in the paper are efficient, i.e. information equality holds for them).

References

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