SUPPLEMENT TO "TESTING FOR TIME-VARYING JUMP ACTIVITY FOR PURE JUMP SEMIMARTINGALES'

By Viktor Todorov

Northwestern University

This supplement contains proofs of the auxiliary results (Lemmas 2-5) given in Section 6.4 of the main text.

1. Proof of Lemma 2. We start with (6.16). Recalling (6.44), we denote

$$\widetilde{X}_s = X_{(i-2)\Delta_n} + \int_{(i-2)\Delta_n}^{s} \widetilde{\sigma}_u dS_u, \quad s \in [(i-2)\Delta_n, i\Delta_n],$$

$$\widetilde{z}_i^1(u) = \left[\cos\left(u\frac{\Delta_i^n \widetilde{X} - \Delta_{i-1}^n \widetilde{X}}{(V_i^n(p))^{1/p}}\right) - \exp\left(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}\right)\right] 1_{\{\Delta_n^{-p/\beta} V_i^n(p) > \epsilon\}}.$$

Using (6.46) and applying Lemmas 2.1.5 and 2.1.7 of [1], we have

$$\Delta_n^{-1/\beta}\mathbb{E}_{i-2}^n\left|\int_{(i-2)\Delta_n}^{i\Delta_n} (\sigma_{u-} - \widetilde{\sigma}_u) dS_u\right| \leq K\Delta_n^{1/2+\iota},$$

for some sufficiently small $\iota > 0$. Then using the above bound and (6.38)-(6.39) (note $\beta' < \beta/2$ from (4.9)), we have

$$\mathbb{E}_{i-2}^n \left|z_i^1(u) - \widetilde{z}_i^1(u)\right| \leq K\Delta_n^{1/2+\iota}. \tag{A.1}$$

Next, by conditioning on the filtration generated by W and $\widetilde{W}$, we have (see e.g., Proposition 3.1 of [2])

$$\mathbb{E}_{i-2}^n\left(\cos\left(u\frac{\Delta_i^n \widetilde{X} - \Delta_{i-1}^n \widetilde{X}}{(V_i^n(p))^{1/p}}\right)\right) = \exp\left(-\frac{A_\beta u^\beta \int_{(i-2)\Delta_n}^{i\Delta_n} |\widetilde{\sigma}_s|^\beta ds}{2(V_i^n(p))^{\beta/p}}\right),$$

and from here by Taylor expansion and (6.45), we have

$$|\mathbb{E}_{i-2}^n\left(\widetilde{z}_i^1(u)\right)| \leq K\Delta_n. \tag{A.2}$$

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Combining (A.1) and (A.2), we have (6.16).

We turn next to (6.17). Using the trigonometric identity $\cos(a)\cos(b) = \frac{1}{2}\cos(a-b)+\frac{1}{2}\cos(a+b)$ for any $a, b \in \mathbb{R}$, the above result, as well as (6.11) and (6.14) of Lemma 1, we have

$$\mathbb{E}|\mathbb{E}_{i-2}^n\left(z_i^1(u)z_i^1(v)\right) - \widetilde{\overline{\Xi}}_{0,i}(p,u,v)| \le K(\alpha_n \vee k_n^{-1}). \tag{A.3}$$

Using similar steps as for the proof of the above inequality, and in addition the trigonometric identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ for $a, b \in \mathbb{R}$, $\mathbb{E}|\widehat{V}_i^n(p) - \widehat{V}_{i-1}^n(p)| \le K\Delta_n^{p/\beta}/k_n$ as well as successive conditioning:

$$\mathbb{E}|\mathbb{E}_{i-3}^n\left(z_i^1(u)z_{i-1}^1(v)\right) - \widetilde{\overline{\Xi}}_{1,i-1}(p,u,v)| \le K(\alpha_n \vee k_n^{-1}). \tag{A.4}$$

To prove (6.17), we use the decomposition

$$\begin{aligned}\Big(\sum_{i\in I_j^n} x_i\Big)^2 &= \Big(\sum_{i\in I_j^n}(x_i - \mathbb{E}_{i-2}^n(x_i))\Big)^2 + \Big(\sum_{i\in I_j^n}\mathbb{E}_{i-2}^n(x_i)\Big)^2 \\ &\quad + 2\Big(\sum_{i\in I_j^n}(x_i - \mathbb{E}_{i-2}^n(x_i))\Big)\Big(\sum_{i\in I_j^n}\mathbb{E}_{i-2}^n(x_i)\Big).\end{aligned}$$

Using the bounds in (6.16) and (A.3)-(A.4), we have

$$\mathbb{E}\Bigg|\frac{1}{m_n}\mathbb{E}_{i_j^n}^n\Big(\sum_{i\in I_j^n}(x_i - \mathbb{E}_{i-2}^n(x_i))\Big)^2 - \frac{1}{m_n}\sum_{i\in I_j^n}\overline{\Xi}_i\Bigg| \le K(\alpha_n \vee k_n^{-1}),$$

$$\frac{1}{m_n}\mathbb{E}\Big(\sum_{i\in I_j^n}\mathbb{E}_{i-2}^n(x_i)\Big)^2 \le \mathbb{E}\Big(\sum_{i\in I_j^n}(\mathbb{E}_{i-2}^n(x_i))^2\Big) \le K m_n \Delta_n^{1+\iota},$$

from which the result in (6.17) follows by an application of Cauchy-Schwarz inequality.

We turn next to (6.18). Using second-order Taylor expansion, the fact that $\mathbb{E}\left(\Delta_n^{-p/\beta}\widehat{V}_i^n(p) - |\overline{\sigma}|_i^p\right)^2 \le K/k_n$, we have

$$\sum_{j=1}^{b_n}\sum_{i\in I_j^n}(\overline{\Xi}_i - \overline{\Xi}(p,u,v)) = \sum_{j=1}^{b_n}\sum_{i\in I_j^n}\overline{\Xi}'\left(\frac{\Delta_n^{-p/\beta}\widehat{V}_i^n(p)}{|\sigma_{(i-2)\Delta_n}|^p} - 1\right) + O_p\left(\frac{n}{k_n}\right).$$

where $\overline{\Xi}' = -(\overline{\Xi}_u'(p,u,v)u + \overline{\Xi}_v'(p,u,v)v)/p$. Using the bound $\mathbb{E}|\sigma_t - \sigma_s|^2 \le K|t-s|$ for $s, t \ge 0$ (which follows from assumption SB), $\mathbb{E}|\Delta_n^{-p/\beta}\widehat{V}_i^n(p)|^2 \le$

K (since $p < \beta/2$) and applying Cauchy-Schwarz inequality, we have

$$\mathbb{E}\left|\frac{\Delta_n^{-p/\beta}\widehat{V}_i^n(p)}{|\sigma_{(i-2)\Delta_n}|^p} - \frac{\Delta_n^{-p/\beta}\widehat{V}_i^n(p)}{|\sigma_{(i-k_n-3)\Delta_n}|^p}\right| + \mathbb{E}||\overline{\sigma}|_i^p - |\sigma_{(i-k_n-3)\Delta_n}|^p| \\ \leq K\sqrt{k_n\Delta_n}.$$

Using next the fact that $\mathbb{E}_{i-k_n-3}^n(\widehat{V}_i^n(p) - |\overline{\sigma}|_i^p) = 0$ as well as Burkholder-Davis-Gundy inequality, we have

$$\sum_{j=1}^{b_n}\sum_{i\in I_j^n}\overline{\Xi}'\left(\frac{\Delta_n^{-p/\beta}\widehat{V}_i^n(p)}{|\sigma_{(i-k-n-3)\Delta_n}|^p} - 1\right) = O_p\left(\sqrt{n}\right).$$

Combining the above three results, we have (6.18).

We continue with (6.19). We first decompose

$$\sum_{i\in I_j^n} x_i = \sum_{i=i_j^n+2}^{i_j^n+m_n+1}(x_i - \mathbb{E}_{i-1}^n(x_i)) + \sum_{i=i_j^n+1}^{i_j^n+m_n}\mathbb{E}_i^n(x_{i+1}),$$

and we further set $\widetilde{x}_i = x_i - \mathbb{E}_{i-1}^n(x_i) + \mathbb{E}_i^n(x_{i+1})$. Using (6.16), we have

$$\left|\mathbb{E}_{i-1}^n\left(\widetilde{x}_i\right)\right| \leq K\Delta_n^{1/2+\iota}. \tag{A.5}$$

Next using the notation

$$z_i^1(u)' = \cos\left(u\frac{\Delta_n^{-1/\beta}(\Delta_i^n S - \Delta_{i-1}^n S)}{\mu_{p,\beta}^{1/\beta}}\right) - \exp\left(-C_{p,\beta}u^\beta\right),$$

and the bounds in (6.11), (6.13), (6.14) and (6.36)-(6.39), as well as the Itô semimartingale assumption for σ, we have

$$\mathbb{E}|z_i^1(u) - z_i^1(u)'| \leq K\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2}\right).$$

From here, using also the boundedness of $\widetilde{x}_i$, we have

$$\mathbb{E}\left|\mathbb{E}_{i-2}^n(\widetilde{x}_i)^2 - \overline{\Xi}(p,u,v)\right| \leq K\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2}\right), \tag{A.6}$$

$$\mathbb{E}\left|\mathbb{E}_{i-2}^n(\widetilde{x}_i)^3 - \overline{\Upsilon}_1(p,u,v)\right| \leq K\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2}\right), \tag{A.7}$$

$$\mathbb{E}\left|\mathbb{E}_{i-3}^n(\widetilde{x}_i^2\widetilde{x}_{i-1}) - \overline{\Upsilon}_2(p,u,v)\right| \leq K\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2}\right), \tag{A.8}$$

where

$$\overline{\Upsilon}_1(p,u,v) = \mathbb{E}\left(\frac{\overline{z}_0(u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\overline{z}_0(v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}\right)^3,$$

$$\overline{\Upsilon}_2(p,u,v) = \mathbb{E}\Bigg[\left(\frac{\overline{z}_0(u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\overline{z}_0(v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}\right)^2 \times \left(\frac{\overline{z}_1(u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\overline{z}_1(v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}\right)\Bigg],$$

and for $i = 0, 1$ (recall the notation for $S_i^{(\beta)}$ in Section 4.1 for $i = 1, 2, 3$)

$$\overline{z}_i(u) = \cos\left(u\frac{S_{i+1}^{(\beta)} - S_{i+2}^{(\beta)}}{\mu_{p,\beta}^{1/\beta}}\right) - \left(\cos\left(u\frac{S_{i+2}^{(\beta)}}{\mu_{p,\beta}^{1/\beta}}\right) - \cos\left(u\frac{S_{i+1}^{(\beta)}}{\mu_{p,\beta}^{1/\beta}}\right)\right)e^{-\frac{C_{p,\beta}u^\beta}{2}} - e^{-C_{p,\beta}u^\beta}.$$

We set $\widetilde{I}_j^n = [(j-1)(m_n+1)+k_n+3, ..., j(m_n+1)+k_n]$ and we further use the shorthand $y_i = \widetilde{x}_i - \mathbb{E}_{i-1}^n(\widetilde{x}_i)$. We can then use Bukrholder-Davis-Gundy inequality for discrete martingales to get

$$\frac{1}{m_n^2}\mathbb{E}\left(\sum_{i\in\widetilde{I}_j^n} y_i\right)^4 \leq K,$$

and further using (A.5), we have

$$\frac{1}{m_n^2}\mathbb{E}\left(\sum_{i\in\widetilde{I}_j^n} \mathbb{E}_{i-1}^n(\widetilde{x}_i)\right)^4 \leq K\Delta_n^{2+\iota}m_n^2.$$

Combining these two results, and using Hölder inequality as well as $\Delta_n m_n \to 0$ (which follows from (4.9)), we then have

$$\frac{1}{m_n^2}\mathbb{E}\left|\left(\sum_{i\in\widetilde{I}_j^n}\widetilde{x}_i\right)^4 - \left(\sum_{i\in\widetilde{I}_j^n} y_i\right)^4\right| \leq K\Delta_n^{1/2+\iota}m_n^{1/2}. \tag{A.9}$$

Using the boundedness of the terms $\widetilde{x}_i$ and again $\Delta_n m_n \to 0$, we further have

$$\frac{1}{m_n^2}\mathbb{E}\left|\left(\sum_{i\in I_j^n} x_i\right)^4 - \left(\sum_{i\in\widetilde{I}_j^n}\widetilde{x}_i\right)^4\right| \leq \frac{K}{\sqrt{m_n}}. \tag{A.10}$$

Thus, we are left with analyzing $\mathbb{E}^n_{i^n_j}\left(\sum_{i\in\widetilde{I}^n_j} y_i\right)^4$. We can split $\left(\sum_{i\in\widetilde{I}^n_j} y_i\right)^4 = Y^1_j + Y^2_j + Y^3_j + Y^4_j + Y^5_j$ where for $s = 1, ..., 5$:

$$Y^s_j = \sum_{(i_1,i_2,i_3,i_4)\in A^s} y_{i_1}y_{i_2}y_{i_3}y_{i_4}, \quad i_1, i_2, i_3, i_4 \in \widetilde{I}^n_j, \tag{A.11}$$

and $A^1 = \{\tilde{i}_1 = \tilde{i}_2 = \tilde{i}_3 = \tilde{i}_4 \cup \tilde{i}_1 = \tilde{i}_2 = \tilde{i}_3 = \tilde{i}_4 + 1\}$, $A^2 = \{\tilde{i}_1 = \tilde{i}_2 = \tilde{i}_3 > \tilde{i}_4 + 1\}$, $A^3 = \{\tilde{i}_1 = \tilde{i}_2 > \tilde{i}_3 = \tilde{i}_4\}$, $A^4 = \{\tilde{i}_1 = \tilde{i}_2 > \tilde{i}_3 > \tilde{i}_4\}$ and $A^5 = \{\tilde{i}_1 > \tilde{i}_2\}$, with $(\tilde{i}_1, \tilde{i}_2, \tilde{i}_3, \tilde{i}_4)$ being the ordered rearrangment of (i_1, i_2, i_3, i_4) (with $\tilde{i}_1 \geq \tilde{i}_2 \geq \tilde{i}_3 \geq \tilde{i}_4$). Using the boundedness of y_i, we have $\mathbb{E}|Y^1_j| \leq K m_n$. Also, using law of iterated expectations and since $\mathbb{E}^n_{i-1}(y_i) = 0$, we easily have $\mathbb{E}^n_{i^n_j}(Y^5_j) = 0$. Using (A.5)-(A.7), we have

$$\mathbb{E}|\mathbb{E}^n_{i^n_j}(Y^2_j)| \leq K\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2}\right) m_n^2, \tag{A.12}$$

$$\begin{aligned} &\mathbb{E}\left|\frac{1}{m_n^2}\mathbb{E}^n_{i^n_j}(Y^3_j) - 3\overline{\Xi}^2(p,u,v)\right| \\ &\qquad \leq K\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2} \vee m_n^{-1}\right). \end{aligned} \tag{A.13}$$

We are left with Y^4_j. First, we will show

$$\mathbb{E}^n_{i^n_j}\left(\sum_{r,l\in\widetilde{I}^n_j:\ i>r>l} y_r y_l\right)^2 \leq K m_n^2, \quad i \in \widetilde{I}^n_j. \tag{A.14}$$

This bound follows from the following two estimates

$$\mathbb{E}^n_{i^n_j}\left(\sum_{l\in\widetilde{I}^n_j,\ l<r} y_l\right)^2 \leq K m_n,\ r \in \widetilde{I}^n_j,$$

$$\left|\mathbb{E}^n_{i^n_j}\left[\left(y_{r_1}\sum_{l_1\in\widetilde{I}^n_j,\ l_1<r_1} y_{l_1}\right)\left(y_{r_2}\sum_{l_2\in\widetilde{I}^n_j,\ l_2<r_2} y_{l_2}\right)\right]\right| = 0, \quad r_1 > r_2,\ r_1, r_2 \in \widetilde{I}^n_j,$$

which in turn follow from applying successive conditioning and $\mathbb{E}^n_{i-1}(y_i) = 0$.

Therefore, using successive conditioning, the bounds in (A.5), (A.6), (A.8) and (A.14), we have

$$\mathbb{E}|\mathbb{E}^n_{i^n_j}(Y^4_j)| \leq K\left[\left(\alpha_n \vee k_n^{-1/2} \vee (k_n\Delta_n)^{1/2}\right) m_n^2 \vee m_n\right]. \tag{A.15}$$

Combining the bounds in (A.9)-(A.15) we get the result in (6.19).

Finally, (6.20) follows by an application of Burkholder-Davis-Gundy and Jensen's inequality. □

2. Proof of Lemma 3. We start with the bounds involving $\widetilde{z}_i^{(2,a)}(u)$. Using Taylor series expansion, we have the following decomposition

$$\begin{aligned}\widetilde{z}_i^{(2,a)}(u) &= G_i^n(u,|\overline{\sigma}|_i^p)\left(\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}}-|\overline{\sigma}|_i^p\right)\\ &\quad + H_i^n(u,\check{x}_i)\left(\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}}-|\overline{\sigma}|_i^p\right)^2,\end{aligned} \tag{A.16}$$

where $\check{x}_i$ is between $\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}}$ and $|\overline{\sigma}|_i^p$, and for $x>0$

$$G_i^n(u,x) = -\frac{\beta}{p}e^{C_{p,\beta}u^\beta\left(1-\frac{|\sigma_{(i-2)\Delta_n}|^\beta}{x^{\beta/p}}\right)}\frac{|\sigma_{(i-2)\Delta_n}|^\beta}{x^{\beta/p+1}},$$

$$\begin{aligned}H_i^n(u,x) &= \frac{1}{2}\frac{\beta}{p}\left(\frac{\beta}{p}+1\right)e^{C_{p,\beta}u^\beta\left(1-\frac{|\sigma_{(i-2)\Delta_n}|^\beta}{x^{\beta/p}}\right)}\frac{|\sigma_{(i-2)\Delta_n}|^\beta}{x^{\beta/p+2}}\\ &\quad -\frac{1}{2}\left(\frac{\beta}{p}\right)^2 e^{C_{p,\beta}u^\beta\left(1-\frac{|\sigma_{(i-2)\Delta_n}|^\beta}{x^{\beta/p}}\right)}\frac{C_{p,\beta}u^\beta|\sigma_{(i-2)\Delta_n}|^{2\beta}}{x^{2(\beta/p+1)}}.\end{aligned}$$

We further denote with $\widetilde{G}_i^n(u,x)$ and $\widetilde{H}_i^n(u,x)$ the counterparts of $G_i^n(u,x)$ and $H_i^n(u,x)$, respectively, in which $\sigma_{(i-2)\Delta_n}$ is replaced by $\sigma_{(i-k_n-3)\Delta_n}$.

Using the Itô semimartingale assumption for σ (from Assumption SB), we have:

$$\mathbb{E}_{i-k_n-3}^n|G_i^n(u,|\overline{\sigma}|_i^p)-\widetilde{G}_i^n(u,|\sigma_{(i-k_n-3)\Delta_n}|^p)|^2 \leq K(k_n\Delta_n), \tag{A.17}$$

$$\mathbb{E}_{i-k_n-3}^n|H_i^n(u,|\overline{\sigma}|_i^p)-\widetilde{H}_i^n(u,|\sigma_{(i-k_n-3)\Delta_n}|^p)|^2 \leq K(k_n\Delta_n). \tag{A.18}$$

Further, using the bounds (6.11) and (6.13), we have

$$\mathbb{E}_{i-k_n-3}^n(\Delta_n^{-p/\beta}V_i^n(p)-\mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p)^4 \leq K(\alpha_n^4\vee k_n^{-2}). \tag{A.19}$$

From here, using Cauchy-Schwarz inequality, we have

$$\left|\mathbb{E}_{i-k_n-3}^n\left(G_i^n(u,|\overline{\sigma}|_i^p)\left(\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}}-|\overline{\sigma}|_i^p\right)\right)\right| \leq K(\alpha_n\vee\sqrt{\Delta_n}). \tag{A.20}$$

Using $G_i^n(u, |\sigma_{(i-2)\Delta_n}|^p) = G_i^n(v, |\sigma_{(i-2)\Delta_n}|^p)$ and Taylor series expansion (and the boundedness of the first derivative of $H_i^n(u,x)$ in its second argument), we also have

$$\begin{aligned}&G_i^n(u, |\overline{\sigma}|_i^p) - G_i^n(v, |\overline{\sigma}|_i^p)\\&\quad = 2(\widetilde{H}_i^n(u, |\sigma_{(i-k_n-3)\Delta_n}|^p) - \widetilde{H}_i^n(v, |\sigma_{(i-k_n-3)\Delta_n}|^p))(|\overline{\sigma}|_i^p - |\sigma_{(i-2)\Delta_n}|^p) + R_i^n,\end{aligned}$$

where R_i^n is residual term satisfying

$$|R_i^n| \leq K||\overline{\sigma}|_i^p - |\sigma_{(i-2)\Delta_n}|^p|^2 + K||\sigma_{(i-2)\Delta_n}|^p - |\sigma_{(i-k_n-3)\Delta_n}|^p|^2.$$

From here, using successive conditioning, (6.11), (6.12) and (6.13), as well as the Itô semimartingale assumption for σ, we have for some sufficiently small $\iota > 0$

$$\begin{aligned}&\left|\mathbb{E}_{i-k_n-3}^n\left(\left(G_i^n(u, |\overline{\sigma}|_i^p) - G_i^n(v, |\overline{\sigma}|_i^p)\right)\left(\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}} - |\overline{\sigma}|_i^p\right)\right)\right|\\&\qquad\qquad \leq K(\Delta_n^{1/2+\iota} \vee \alpha_n\sqrt{k_n\Delta_n} \vee k_n\Delta_n).\end{aligned} \tag{A.21}$$

Next, given the boundedness of the first derivative of $H_i^n(u,x)$ in its second argument, we have (recall the definition of $\check{x}_i$ in (A.16)):

$$|H_i^n(u, \check{x}_i) - H_i^n(u, |\overline{\sigma}|_i^p)| \leq K\left|\Delta_n^{-p/\beta}V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p\right|,$$

and therefore using (A.19), we have

$$\begin{aligned}&\mathbb{E}_{i-k_n-3}^n\left|(H_i^n(u, \check{x}_i) - H_i^n(u, |\overline{\sigma}|_i^p))\left|\Delta_n^{-p/\beta}V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p\right|^2\right|\\&\quad \leq K(\alpha_n^3 \vee k_n^{-3/2}).\end{aligned} \tag{A.22}$$

Further, using (A.18) and (A.19), we have by Cauchy-Schwarz inequality

$$\begin{aligned}&\mathbb{E}_{i-k_n-3}^n\left|(H_i^n(u, |\overline{\sigma}|_i^p) - \widetilde{H}_i^n(u, |\sigma_{(i-k_n-3)\Delta_n}|^p))\left|\Delta_n^{-p/\beta}V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p\right|^2\right|\\&\quad \leq K\sqrt{k_n\Delta_n}(\alpha_n^2 \vee k_n^{-1}).\end{aligned} \tag{A.23}$$

We then note that

$$\widetilde{H}_i^n(u, |\sigma_{(i-k_n-3)\Delta_n}|^p) \equiv \frac{k_n\mathcal{B}^n(u)}{\Sigma_{p,\beta}}\frac{1}{|\sigma_{(i-k_n-3)\Delta_n}|^{2p}},$$

and therefore we need to look at

$$\frac{k_n}{|\sigma_{(i-k_n-3)\Delta_n}|^{2p}}\left(\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}} - |\overline{\sigma}|_i^p\right)^2 - \Sigma_{p,\beta}.$$

Using the bounds in (6.11) and (6.13) as well as Cauchy-Schwarz inequality,

$$\begin{aligned}\Big|\mathbb{E}^n_{i-k_n-3}\left(\Delta_n^{-p/\beta}V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p\right)^2 \\ - \Delta_n^{-2p/\beta}\mathbb{E}^n_{i-k_n-3}\left(V_i^n(p) - \widetilde{V}_i^n(p)\right)^2\Big| \leq K\left(\alpha_n^2 \vee \frac{\alpha_n}{\sqrt{k_n}}\right).\end{aligned} \tag{A.24}$$

Further, using the Itô semimartingale assumption for σ (from Assumption SB) and successive conditioning, we have

$$\mathbb{E}^n_{i-k_n-3}\left|\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\xi_j(|\sigma_{(j-2)\Delta_n}|^p - |\sigma_{(i-k_n-3)\Delta_n}|^p)\right|^2 \leq K\Delta_n,$$

where we denote

$$\xi_i = \frac{\Delta_n^{-p/\beta}(|\Delta_i^n X - \Delta_{i-1}^n X|^p - \mathbb{E}^n_{i-2}|\Delta_i^n X - \Delta_{i-1}^n X|^p)}{|\sigma_{(i-2)\Delta_n}|^p}, \tag{A.25}$$

and from here by using Cauchy-Schwarz inequality and (6.13),

$$\mathbb{E}^n_{i-k_n-3}\left|\frac{\Delta_n^{-2p/\beta}(V_i^n(p) - \widetilde{V}_i^n(p))^2}{|\sigma_{(i-k_n-3)\Delta_n}|^{2p}} - \left(\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\xi_j\right)^2\right| \leq K\frac{\sqrt{\Delta_n}}{\sqrt{k_n}}. \tag{A.26}$$

With this notation, we can next split $\left(\frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\xi_j\right)^2$ into

$$\frac{1}{k_n^2}\sum_{r,l:|r-l|\leq 1}\xi_r\xi_l + \frac{2}{k_n^2}\sum_r \xi_r \sum_{r>l+1}\xi_l,$$

where in the above summations the indexes r and l take values in the set $[i-k_n-1, ..., i-2]$. Using successive conditioning we trivially have

$$\mathbb{E}^n_{i-k_n-3}\left(\sum_r\left(\xi_r\sum_{r>l+1}\xi_l\right)\right) = 0. \tag{A.27}$$

Next, using successive conditioning and similar steps as in the proof of (6.11), together with $k_n \asymp n^{\varpi}$ with $\varpi > 1/3$, we have for some sufficiently small $\iota > 0$:

$$\mathbb{E}^n_{i-k_n-3}\left|\frac{1}{k_n^2}\sum_{r,l:|r-l|\le 1}\xi_r\xi_l - \frac{\mu_{p,\beta}^{2p/\beta}}{k_n}\Sigma_{p,\beta}\right| \le K\Delta_n^{1/2+\iota}. \tag{A.28}$$

Combining the bounds in (A.20)-(A.24) and (A.26)-(A.28), we have (6.21) and (6.22).

We turn next to (6.23). Recalling (A.16) and using $G_i^n(u, |\sigma_{(i-2)\Delta_n}|^p) = G_i^n(v, |\sigma_{(i-2)\Delta_n}|^p)$, we have

$$\begin{aligned}
&|\widetilde{z}_i^{(2,a)}(u) - \widetilde{z}_i^{(2,a)}(v)|^2 \le K[(H_i^n(u,\check{x}_i))^2 + (H_i^n(v,\check{x}_i))^2](\Delta_n^{-p/\beta}V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p)^4\\
&\quad + K[(G_i^n(u,|\sigma_{(i-2)\Delta_n}|^p) - G_i^n(u,|\overline{\sigma}|_i^p))^2 + (G_i^n(v,|\sigma_{(i-2)\Delta_n}|^p) - G_i^n(v,|\overline{\sigma}|_i^p))^2]\\
&\quad\quad \times \left(\Delta_n^{-p/\beta}V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p\right)^2.
\end{aligned}$$

Using (6.11) and the fact that σ is an Itô semimartingale, we have

$$\begin{aligned}
&\mathbb{E}\left|(G_i^n(u,|\sigma_{(i-2)\Delta_n}|^p) - G_i^n(u,|\overline{\sigma}|_i^p))^2\left(\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p\right)^2\right|\\
&\quad \le K\alpha_n^2(k_n\Delta_n).
\end{aligned} \tag{A.29}$$

We next bound $(G_i^n(u,|\sigma_{(i-2)\Delta_n}|^p) - G_i^n(u,|\overline{\sigma}|_i^p))^2\Delta_n^{-2p/\beta}(V_i^n(p) - \widetilde{V}_i^n(p))^2$. First, we set $\widetilde{\xi}_i = \xi_i|\sigma_{(i-2)\Delta_n}|^p$ (recall the definition of ξ_i in (A.25)). With this notation we have $\Delta_n^{-p/\beta}(V_i^n(p) - \widetilde{V}_i^n(p)) = \frac{1}{k_n}\sum_{j=i-k_n-1}^{i-2}\widetilde{\xi}_j$.

Second, we denote for $u, x \in \mathbb{R}_+$ and $y \in \mathbb{R}$:

$$G(u,x,y) = -\frac{\beta}{p}e^{C_{p,\beta}u^\beta\left(1-\frac{|y|^\beta}{x^{\beta/p}}\right)}\frac{|y|^\beta}{x^{\beta/p+1}},$$

and recalling the notation $G_i^n(u,x)$ and $\widetilde{G}_i^n(u,x)$ after (A.16), we have $G_i^n(u,x) = G(u,x,\sigma_{(i-2)\Delta_n})$ and $\widetilde{G}_i^n(u,x) = G(u,x,\sigma_{(i-k_n-3)\Delta_n})$.

Then, for $l = i-k_n, ..., i-2$, we split $G_i^n(u,|\sigma_{(i-2)\Delta_n}|^p) - \widetilde{G}_i^n(u,|\sigma_{(i-k_n-3)\Delta_n}|^p) = G_{i,l}^{n,1} + G_{i,l}^{n,2} + G_{i,l}^{n,3}$, where

$$\begin{cases}
G_{i,l}^{n,1} = G(u,|\sigma_{(i-2)\Delta_n}|^p,\sigma_{(i-2)\Delta_n}) - G(u,|\sigma_{l\Delta_n}|^p,\sigma_{l\Delta_n}),\\
G_{i,l}^{n,2} = G(u,|\sigma_{l\Delta_n}|^p,\sigma_{l\Delta_n}) - G(u,|\sigma_{(l-3)\Delta_n}|^p,\sigma_{(l-3)\Delta_n}),\\
G_{i,l}^{n,3} = G(u,|\sigma_{(l-3)\Delta_n}|^p,\sigma_{(l-3)\Delta_n}) - G(u,|\sigma_{(i-k_n-3)\Delta_n}|^p,\sigma_{(i-k_n-3)\Delta_n}),
\end{cases}$$

and similarly we split $G_i^n(u, |\overline{\sigma}|_i^p) - \widetilde{G}_i^n(u, |\sigma_{(i-k_n-3)\Delta_n}|^p) = \overline{G}_{i,l}^{n,1} + \overline{G}_{i,l}^{n,2} + \overline{G}_{i,l}^{n,3}$ for $l = i-k_n, ..., i-2$, where

$$\begin{cases} \overline{G}_{i,l}^{n,1} = G(u, |\overline{\sigma}|_i^p, \sigma_{(i-2)\Delta_n}) - G(u, |\overline{\sigma}|_{i,l+2}^p, \sigma_{l\Delta_n}), \\ \overline{G}_{i,l}^{n,2} = G(u, |\overline{\sigma}|_{i,l+2}^p, \sigma_{l\Delta_n}) - G(u, |\overline{\sigma}|_{i,l-1}^p, \sigma_{(l-3)\Delta_n}), \\ \overline{G}_{i,l}^{n,3} = G(u, |\overline{\sigma}|_{i,l-1}^p, \sigma_{(l-3)\Delta_n}) - G(u, |\sigma_{(i-k_n-3)\Delta_n}|^p, \sigma_{(i-k_n-3)\Delta_n}), \end{cases}$$

with

$$|\overline{\sigma}|_{i,l}^p = \frac{1}{k_n} \sum_{j=i-k_n-1}^{i-2} |\sigma_{(j\wedge l-2)\Delta_n}|^p, \quad l = i-k_n-1, ..., i-2.$$

Using the fact that σ is an Itô semimartingale, we have

$$\begin{cases} \mathbb{E}_l^n(|G_{i,l}^{n,1}|^2 + |\overline{G}_{i,l}^{n,1}|^2) \leq K(k_n\Delta_n), \\ \mathbb{E}_{l-3}^n(|G_{i,l}^{n,2}|^4 + |\overline{G}_{i,l}^{n,2}|^4) \leq K\Delta_n, \\ \mathbb{E}_{i-k_n-3}^n(|G_{i,l}^{n,3}|^2 + |\overline{G}_{i,l}^{n,3}|^2) \leq K(k_n\Delta_n). \end{cases}$$

Then, using the above result, successive conditioning as well as the Cauchy-Schwarz inequality, we have:

$$\text{(A.30)} \quad \mathbb{E}\left|(G_i^n(u, |\sigma_{(i-2)\Delta_n}|^p) - G_i^n(u, |\overline{\sigma}|_i^p))^2 \left(\frac{1}{k_n^2} \sum_{r,l:|r-l|\leq 1} \widetilde{\xi}_r \widetilde{\xi}_l\right)\right| \leq K\frac{\sqrt{\Delta_n}}{k_n}.$$

We turn next to $(G_i^n(u, |\sigma_{(i-2)\Delta_n}|^p) - G_i^n(u, |\overline{\sigma}|_i^p))^2 \left(\frac{1}{k_n^2}\sum_{r,l:|r-l|>1} \widetilde{\xi}_r\widetilde{\xi}_l\right)$. We can apply Hölder inequality and use the fact that σ is an Itô semimartingale, to get

$$\mathbb{E}|T_1T_2| \leq K(\mathbb{E}|T_1|^{4/3})^{3/4}(\mathbb{E}|T_2|^4)^{1/4} \leq K(k_n\Delta_n)^{3/4}(\mathbb{E}|T_2|^4)^{1/4},$$

where we set $T_1 = (G_i^n(u, |\sigma_{(i-2)\Delta_n}|^p) - G_i^n(u, |\overline{\sigma}|_i^p))^2$ and $T_2 = \frac{2}{k_n^2}\sum_r \psi_r$ with $\psi_r = \widetilde{\xi}_r \sum_{r>l+1} \widetilde{\xi}_l$ with r and l taking values in $[i-k_n-1, ..., i-2]$. Using successive conditioning we have

$$\mathbb{E}\left(\psi_r|\mathcal{F}_{r-2}^n\right) = 0, \quad \mathbb{E}\left(\psi_r\right)^4 \leq Kk_n^2, \; r \in [i-k_n-1, ..., i-2].$$

Then we can apply the above bounds and Burkholder-Davis-Gundy inequality for discrete martingales

$$\mathbb{E}\left(\sum_r \psi_r\right)^4 \leq Kk_n \sum_r \mathbb{E}(\psi_r)^4 \leq Kk_n^4,$$

where in the above summations r takes values in $[i-k_n-1, ..., i-2]$. Therefore $\mathbb{E}|T_2|^4 \leq \frac{K}{k_n^4}$ and from here

$$\mathbb{E}|T_1T_2| \leq K\frac{(k_n\Delta_n)^{3/4}}{k_n}. \tag{A.31}$$

Combining the bounds in (A.19) and (A.29)-(A.31), we get the result in (6.23).

Next, the result in (6.24) follows from the first-order Taylor expansion

$$\widetilde{z}_i^{(2,a)}(u) = G_i^n(u, \check{x}_i)\left(\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}} - |\overline{\sigma}|_i^p\right),$$

where $\check{x}_i$ is between $\frac{\Delta_n^{-p/\beta}V_i^n(p)}{\mu_{p,\beta}^{p/\beta}}$ and $|\overline{\sigma}|_i^p$, the boundedness of the function $G_i(u,x)$, and the bound in (A.19).

We continue with (6.25). Using Taylor expansion we have

$$\widetilde{z}_i^{(2,b)}(u) = -\frac{\beta}{p}\left(\frac{|\overline{\sigma}|_i^p}{|\sigma_{(i-2)\Delta_n}|^p} - 1\right) + r_i(u),$$

for some $r_i(u)$ satisfying

$$|r_i(u)| \leq K\left||\overline{\sigma}|_i^p - |\sigma_{(i-2)\Delta_n}|^p\right|^2.$$

From here the result in (6.25) follows directly from the assumption for σ being Itô semimartingale in Assumption SB.

Finally, (6.26) follows from the boundedness of $\widetilde{z}_i^{(2,a)}$ and $\widetilde{z}_i^{(2,b)}$ as well as the bound in (6.14). □

3. Proof of Lemma 4. We make the following decomposition for any $u \in \mathbb{R}_+$:

$$\widehat{\mathcal{L}}^n(p,u) - \mathcal{L}(p,u) = \frac{1}{n-k_n-2}\sum_{j=1}^{3} R_j^n(u),$$

where $R_j^n(u) = \sum_{i=k_n+3}^n r_i^j(u)$ for $j = 1,2,3$ and

$$r_i^1(u) = \cos\left(u\frac{\sigma_{(i-2)\Delta_n}(\Delta_i^nS - \Delta_{i-1}^nS)}{(V_i^n(p))^{1/p}}\right) - \exp\left(-\frac{A_\beta u^\beta|\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}\right),$$

$$r_i^2(u) = \cos\left(u\frac{\Delta_i^nX - \Delta_{i-1}^nX}{(V_i^n(p))^{1/p}}\right) - \cos\left(u\frac{\sigma_{(i-2)\Delta_n}(\Delta_i^nS - \Delta_{i-1}^nS)}{(V_i^n(p))^{1/p}}\right),$$

$$r_i^3(u) = \exp\left(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}\right) - \exp\left(-C_{p,\beta}u^\beta\right).$$

First, using (6.14) and $\mathbb{E}_{i-2}^n(r_i^1(u)) = 0$, we easily have

$$R_1^n(u) = O_p\left(\Delta_n^{-1/2} \vee nk_n^{-\beta/(2p)+\iota}\right), \quad \forall \iota > 0. \tag{A.32}$$

Similarly, using the bounds in (6.36)-(6.39) and (6.14), we have

$$R_2^n(u) = O_p\left(\Delta_n^{-1/2-\iota} \vee nk_n^{-\beta/(2p)+\iota}\right), \quad \forall \iota > 0. \tag{A.33}$$

We are left with $R_3^n(u)$. Using Taylor expansion we can write

$$r_i^3(u) = e^{-C_{p,\beta}u^\beta} C_{p,\beta}u^\beta \frac{\beta}{p} \frac{\Delta_n^{-p/\beta}(V_i^n(p) - \widetilde{V}_i^n(p))}{\mu_{p,\beta}^{p/\beta}|\sigma_{(i-k_n-3)\Delta_n}|^p} + \widetilde{r}_i^3(u),$$

where using the bounds of Lemma 1 and the Itô semimartingale assumption for the process σ from Assumption SB we have

$$\mathbb{E}|\widetilde{r}_i^3(u)| \leq K\left(\sqrt{\Delta_n} \vee \alpha_n \vee k_n^{-1}\right).$$

Further, using Burkholder-Davis-Gundy inequality for discrete martingales

$$\frac{1}{n-k_n-2}\sum_{i=k_n+3}^{n}(r_i^3(u) - \widetilde{r}_i^3(u)) = O_p\left(\sqrt{\Delta_n}\right).$$

Thus, altogether

$$R_3^n(u) = O_p\left(n\alpha_n \vee nk_n^{-1}\right). \tag{A.34}$$

Combining (A.32), (A.33) and (A.34), we have

$$\widehat{\mathcal{L}}^n(p,u) - \mathcal{L}(p,u) = O_p\left(\alpha_n \vee k_n^{-1}\right). \tag{A.35}$$

Now we turn attention to the difference

$$\frac{\widehat{\mathcal{L}}^n(p,u) - \mathcal{L}(p,u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\widehat{\mathcal{L}}^n(p,v) - \mathcal{L}(p,v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}.$$

For the part of the difference involving $R_3^n(u)$ and $R_3^n(v)$ there is cancelation of a bias term which allows to improve on the bound implied by (A.35). Henceforth we denote

$$r_i^3 = \frac{r_i^3(u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{r_i^3(v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))}.$$

Using third-order Taylor expansion, we have

$$r_i^3 = \frac{C_{p,\beta}}{2}(v^\beta - u^\beta)\left(\frac{\beta}{p}\right)^2 \frac{|\Delta_n^{-p/\beta} V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\sigma_{(i-2)\Delta_n}|^p|^2}{\mu_{p,\beta}^{2p/\beta}|\sigma_{(i-2)\Delta_n}|^{2p}} + \widetilde{r}_i^3,$$

where

$$|\widetilde{r}_i^3| \leq K|\Delta_n^{-p/\beta} V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\sigma_{(i-2)\Delta_n}|^p|^3.$$

We can split

$$\begin{aligned}\Delta_n^{-p/\beta} V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\sigma_{(i-2)\Delta_n}|^p &= \Delta_n^{-p/\beta}(V_i^n(p) - \widetilde{V}_i^n(p)) \\ &+ (\Delta_n^{-p/\beta}\widetilde{V}_i^n(p) - \mu_{p,\beta}^{p/\beta}|\overline{\sigma}|_i^p) + \mu_{p,\beta}^{p/\beta}(|\overline{\sigma}|_i^p - |\sigma_{(i-2)\Delta_n}|^p).\end{aligned}$$

Using then Cauchy-Schwarz inequality and the bounds of Lemma 1, as well as the Itô semimartingale assumption for the process σ, we have

$$\begin{aligned}&\mathbb{E}|(\Delta_n^{-p/\beta} V_i^n(p) - \mu_{p,\beta}^{p/\beta}|\sigma_{(i-2)\Delta_n}|^p)^2 - \Delta_n^{-2p/\beta}(V_i^n(p) - \widetilde{V}_i^n(p))^2| \\ &\quad\leq K\left(\alpha_n^2 \bigvee \frac{\alpha_n}{\sqrt{k_n}} \bigvee \frac{k_n}{n} \bigvee \sqrt{\Delta_n}\right).\end{aligned} \tag{A.36}$$

Next, using successive conditioning, we easily have

$$\Delta_n^{-2p/\beta}\mathbb{E}_{i-k_n-3}^n(V_i^n(p) - \widetilde{V}_i^n(p))^2 = \frac{1}{k_n^2}\mathbb{E}_{i-k_n-3}^n\left(\sum_{r,l:|r-l|\leq 1}\widetilde{\xi}_r\widetilde{\xi}_l\right), \tag{A.37}$$

where recall $\widetilde{\xi}_r = \Delta_n^{-p/\beta}(|\Delta_r^n X - \Delta_{r-1}^n X|^p - \mathbb{E}_{r-2}^n(|\Delta_r^n X - \Delta_{r-1}^n X|^p))$. Using the same steps as in the proof of (A.28) in Lemma 3 as well as the Itô semimartingale assumption for the process σ, we have (and the bound can be further improved but suffices for the purposes here):

$$\left|\frac{1}{k_n}\mathbb{E}_{i-k_n-3}^n\left(\frac{\sum_{r,l:|r-l|\leq 1}\widetilde{\xi}_r\widetilde{\xi}_l}{|\sigma_{(i-k_n-3)\Delta_n}|^{2p}}\right) - \mu_{p,\beta}^{2p/\beta}\Sigma_{p,\beta}\right| \leq K\Delta_n^{1/6}. \tag{A.38}$$

Combining the results in (A.36)-(A.38) and since $\varpi > 1/3$, we then have altogether

$$|\mathbb{E}_{i-k_n-3}^n(r_i^3 - \mathcal{B}_n)| \leq K\left(\alpha_n^2 \bigvee \frac{\alpha_n}{\sqrt{k_n}} \bigvee \frac{k_n}{n} \bigvee \sqrt{\Delta_n}\right). \tag{A.39}$$

Next, using second-order Taylor expansion for r_i^3, we easily have

$$\mathbb{E}_{i-k_n-3}^n(r_i^3)^2 \leq K(\alpha_n^4 \vee k_n^{-2} \vee k_n\Delta_n). \tag{A.40}$$

Combining the above two results, we have

$$\frac{1}{n-k_n-2}\sum_{i=k_n+3}^{n} r_i^3 - \mathcal{B}_n = O_p\left(\alpha_n^2 \bigvee \frac{\alpha_n}{\sqrt{k_n}} \bigvee \frac{k_n}{n} \bigvee \sqrt{\Delta_n}\right). \tag{A.41}$$

Hence, using the above bound and the ones in (A.32) and (A.33), as well as the fact that $p < \beta/3$, $\varpi \in (1/3, 1/2)$ and $\frac{p}{\beta'} \wedge 1 - \frac{p}{\beta} > \frac{1}{4}$ (these conditions follow from the assumptions of Theorem 2), we have for $\forall \iota > 0$

$$\frac{\widehat{\mathcal{L}}^n(p,u) - \mathcal{L}(p,u)}{\mathcal{L}(p,u)\log(\mathcal{L}(p,u))} - \frac{\widehat{\mathcal{L}}^n(p,v) - \mathcal{L}(p,v)}{\mathcal{L}(p,v)\log(\mathcal{L}(p,v))} - \mathcal{B}_n = O_p\left(\Delta_n^{1/2-\iota} \vee \frac{\alpha_n}{\sqrt{k_n}}\right).$$

From here, using first-order Taylor series expansion, the bound in (A.35) as well as $\varpi \in (1/3, 1/2)$ and $\frac{p}{\beta'} \wedge 1 - \frac{p}{\beta} > \frac{1}{4}$, we have

$$\frac{\log(\widetilde{\mathcal{L}}^n(p,u))}{\log(\mathcal{L}(p,u))} - \frac{\log(\widetilde{\mathcal{L}}^n(p,v))}{\log(\mathcal{L}(p,v))} - \mathcal{B}_n = O_p\left(\Delta_n^{1/2-\iota} \vee \frac{\alpha_n}{\sqrt{k_n}}\right).$$

Similar arguments lead to

$$\begin{aligned}\log(-\log(\widetilde{\mathcal{L}}^n(p,u))) - \log(-\log(\widetilde{\mathcal{L}}^n(p,v)))& \\ - \beta\log(u/v) - \mathcal{B}_n &= O_p\left(\Delta_n^{1/2-\iota} \vee \frac{\alpha_n}{\sqrt{k_n}}\right),\end{aligned}$$

from which the result to be proved follows. □

4. Proof of Lemma 5. First, when M is a discontinuous martingale, we can show (6.28) similar to the proof of the corresponding result of Theorem 1 in [3]. So, we are left with showing (6.28) when M is a continuous martingale. We denote

$$\widehat{X}_s = X_{(i-2)\Delta_n} + \int_{(i-2)\Delta_n}^{s} \sigma_{(i-2)\Delta_n} dS_u, \quad s \in [(i-2)\Delta_n, i\Delta_n],$$

and

$$\widehat{z}_i^1(u) = \left[\cos\left(u\frac{\Delta_i^n\widehat{X} - \Delta_{i-1}^n\widehat{X}}{(V_i^n(p))^{1/p}}\right) - \exp\left(-\frac{A_\beta u^\beta |\sigma_{(i-2)\Delta_n}|^\beta}{\Delta_n^{-1}(V_i^n(p))^{\beta/p}}\right)\right] 1_{\{\Delta_n^{-p/\beta}V_i^n(p)>\epsilon\}},$$

and we further set $\widehat{x}_i = \widehat{z}_i^1(u)/(\mathcal{L}(p,u)\log(\mathcal{L}(p,u))) - \widehat{z}_i^1(v)/(\mathcal{L}(p,v)\log(\mathcal{L}(p,v)))$.

Using (6.16) and since $\mathbb{E}_{i-2}^n(\widehat{z}_i^1(u)) = 0$, we have for some sufficiently small $\iota > 0$

$$|\mathbb{E}_{i-2}^n(z_i^1(u) - \widehat{z}_i^1(u))| \le K\Delta_n^{1/2+\iota}. \tag{A.42}$$

With the notation of x_i given in the statement of Lemma 2, and using the above result and the bounds in (6.36)-(6.39), the fact that $\beta' < \beta/2$ and since $m_n/\sqrt{n} \to \infty$, we also have for some sufficiently small $\iota > 0$

$$\mathbb{E}\bigg(\frac{1}{m_n}\sum_{i\in I_j^n}(x_i - \widehat{x}_i)\bigg)^2 \le K\Delta_n^{1+\iota}. \tag{A.43}$$

Then with the notation $y_j^n = \frac{1}{m_n}\sum_{i\in I_j^n} x_i$ and $\widehat{y}_j^n = \frac{1}{m_n}\sum_{i\in I_j^n}\widehat{x}_i$, we have for sufficiently small $\iota > 0$

$$\mathbb{E}|(y_j^n)^2 - (\widehat{y}_j^n)^2| \le K\frac{\Delta_n^{1/2+\iota}}{\sqrt{m_n}}, \tag{A.44}$$

where we made use of Cauchy-Schwarz inequality and $\mathbb{E}(\widehat{y}_j^n)^2 \le K/m_n$. From here, using the boundedness of the martingale M, we have for sufficiently small $\iota > 0$

(A.45)

$$\begin{aligned}\frac{m_n}{\sqrt{b_n}}\sum_{j=1}^{\lfloor tb_n\rfloor}&\mathbb{E}^n_{i_j^n}[((y_j^n)^2 - (\widehat{y}_j^n)^2)(M_{k_n+1+j(m_n+1)} - M_{k_n+1+(j-1)(m_n+1)})]\\ &= o_p(\Delta_n^{\iota}).\end{aligned}$$

So we are left with analyzing $\mathbb{E}^n_{i_j^n}[(m_n(\widehat{y}_j^n)^2 - \frac{1}{m_n}\sum_{i\in I_j^n}\overline{\Xi}_i)(M_{k_n+1+j(m_n+1)} - M_{k_n+1+(j-1)(m_n+1)})]$. Using a martingale representation theorem, we have

$$\mathbb{E}^n_{i-2}(\widehat{x}_i(M_{i\Delta_n} - M_{(i-2)\Delta_n})) = 0. \tag{A.46}$$

Using a martingale representation theorem and (6.11)-(6.14) as well as $p < \beta/4$, we have

$$|\mathbb{E}^n_{i-2}((\widehat{z}_i^1(u)\widehat{z}_i^1(v) - \overline{\Xi}_{0,i}(p,u,v))(M_{i\Delta_n} - M_{(i-2)\Delta_n}))| \le K(\alpha_n \vee k_n^{-2}). \tag{A.47}$$

Similarly, a martingale representation theorem plus (6.11)-(6.14) as well as $p < \beta/4$, yields

$$\begin{aligned}&|\mathbb{E}^n_{i-3}((\widehat{z}_i^1(u)\widehat{z}_{i-1}^1(v) - \overline{\Xi}_{1,i-1}(p,u,v))(M_{i\Delta_n} - M_{(i-3)\Delta_n}))|\\ &\quad\le K(\alpha_n \vee k_n^{-1}).\end{aligned} \tag{A.48}$$

Combining the above three bounds and using successive conditioning, we get altogether

(A.49)

$$\begin{aligned}&\Big|\mathbb{E}^n_{i_j^n}[(m_n(\widehat{y}_j^n)^2 - \frac{1}{m_n}\sum_{i\in I_j^n}\overline{\Xi}_i)(M_{k_n+1+j(m_n+1)} - M_{k_n+1+(j-1)(m_n+1)})]\Big|\\ &\quad\le K(\alpha_n \vee k_n^{-1}).\end{aligned}$$

From here, since $\alpha_n\sqrt{b_n} \to 0$ and $\sqrt{b_n}/k_n \to 0$, we have (6.28) for the case when M is a continuous martingale as well. □

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Department of Finance
Northwestern University
Evanston, IL 60208-2001
E-mail: v-todorov@northwestern.edu