Supplementary Appendix to "Time-Varying Periodicity in Intraday Volatility"*

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Abstract

This document consists of two appendices. Appendix A contains the proofs of all formal results stated in the paper while Appendix B reports additional Monte Carlo evidence concerning the finite sample performance of the testing procedure.

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Appendix A: Proofs

A.1 Auxiliary Results and Notation

Throughout the proofs we will assume without loss of generality that $\mathbb{E}(V_t) = 1$. Further, we will denote with C a positive and finite constant that does not depend on Δ_n and T, and can change from line to line. Finally, to improve the readability of the proofs we introduce the following notation:

$$V_{t,\kappa}^n = V_{t-1+(\lfloor \kappa n \rfloor -2)/n}, \quad \overline{f}_\kappa = \mathbb{E}\left(f_{t,\kappa} V_{t+\kappa}\right), \quad \kappa \in (0,1], \; t = 1, ..., T, \tag{A.1}$$

and we will use similar notation for $f_{t,\kappa}^n$, and if the latter does not depend on t (the null hypothesis for our test), we will further simplify notation to f_κ^n. We next denote with X^c and X^d the continuous and discontinuous parts of X:

$$X_t^c = X_0 + \int_0^t a_s ds + \int_0^t \widetilde{\sigma}_s dW_s, \quad X_t^d = \int_0^t \int_{\mathbb{R}} x \mu(ds, dx). \tag{A.2}$$

With this notation, we set

$$\begin{aligned} d_{t,n}(u) = \cos\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right) - \cos\left(\sqrt{2un}\sqrt{V_{t,\kappa'}^n}\Delta_{t,\kappa'}^n W\right) \\ - \frac{\pi}{2} u \mathcal{L}'(u)\left(V_{t,\kappa}^n n |\Delta_{t,\kappa}^n W||\Delta_{t,\kappa-\Delta_n}^n W| - V_{t,\kappa'}^n n |\Delta_{t,\kappa'}^n W||\Delta_{t,\kappa'-\Delta_n}^n W|\right), \end{aligned} \tag{A.3}$$

$$\breve{f}_\kappa = \frac{1}{T}\frac{\pi}{2}\sum_{t=1}^T n|\Delta_{t,\kappa}^n X^c||\Delta_{t,\kappa-\Delta_n}^n X^c|, \quad \widetilde{f}_\kappa = \frac{1}{T}\frac{\pi}{2}\sum_{t=1}^T f_{t,\kappa}^n V_{t,\kappa}^n n|\Delta_{t,\kappa}^n W||\Delta_{t,\kappa-\Delta_n}^n W|. \tag{A.4}$$

$$\widetilde{L}_\kappa'(u) = -\frac{1}{T}\sum_{t=1}^T \sin\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right)\frac{\sqrt{n}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W}{\sqrt{2u}}. \tag{A.5}$$

Lemma 1. *Suppose Assumptions 1-3 hold with* $\mathcal{K} = \{\kappa, \kappa'\}$ *and let* $\sup_{t\in\mathbb{R}_+}\mathbb{E}(|a_t|^q) + \sup_{t\in\mathbb{R}_+}\mathbb{E}(\sigma_t^q) < \infty$ *for some* $q \geq 8$. *Then, for* $p \in [1,4]$ *and* $\kappa \in (0,1]$ *and with* $\varpi > \frac{1}{2} - \frac{1}{p}$, *we have*

$$\mathbb{E}|\widehat{f}_\kappa - \breve{f}_\kappa|^p \leq C\Delta_n^{-p}[\Delta_n^{q(1/2-\varpi)+2p\varpi} \vee \Delta_n^{1+p/2+p\varpi} \vee \Delta_n^{p+1-2p/q}], \tag{A.6}$$

$$\mathbb{E}|\breve{f}_\kappa - \widetilde{f}_\kappa|^p \leq C\Delta_n^{\frac{p}{2}\wedge\frac{q-p}{q}}, \tag{A.7}$$

$$\mathbb{E}|\widetilde{f}_\kappa - \overline{f}_\kappa|^2 \leq CT^{-1}, \quad \mathbb{E}|u\widetilde{\mathcal{L}}'_\kappa(u) - u\mathcal{L}'_\kappa(u)|^2 \leq C(|u| \vee 1)T^{-1}, \tag{A.8}$$

for some positive and finite constant C *that does not depend on* u.

Proof. For the first bound in (A.6), we make use of the following algebraic inequality

$$\begin{aligned}
&||\Delta_{t,\kappa}^n X||\Delta_{t,\kappa-\Delta_n}^n X|1_{\{\mathcal{A}_{t,\kappa}^n\}} - |\Delta_{t,\kappa}^n X^c||\Delta_{t,\kappa-\Delta_n}^n X^c|| \leq Cv_n^2 1_{\{|\Delta_{t,\kappa}^n X^c|\geq v_n \,\cup\, |\Delta_{t,\kappa-\Delta_n}^n X^c|\geq v_n\}} \\
&\quad + |\Delta_{t,\kappa}^n X^c||\Delta_{t,\kappa-\Delta_n}^n X^d|1_{\{|\Delta_{t,\kappa-\Delta_n}^n X^d|\leq 2v_n\}} + |\Delta_{t,\kappa}^n X^d||\Delta_{t,\kappa-\Delta_n}^n X^c|1_{\{|\Delta_{t,\kappa}^n X^d|\leq 2v_n\}} \\
&\quad + |\Delta_{t,\kappa}^n X^c||\Delta_{t,\kappa-\Delta_n}^n X^c|1_{\{|\Delta_{t,\kappa}^n X^c|\geq\frac{v_n}{2} \,\cup\, |\Delta_{t,\kappa-\Delta_n}^n X^c|\geq\frac{v_n}{2} \,\cup\, |\Delta_{t,\kappa}^n X^d|\geq\frac{v_n}{2} \,\cup\, |\Delta_{t,\kappa-\Delta_n}^n X^d|\geq\frac{v_n}{2}\}} \\
&\quad + |\Delta_{t,\kappa}^n X^d||\Delta_{t,\kappa-\Delta_n}^n X^d|1_{\{|\Delta_{t,\kappa}^n X^d|\leq 2v_n \,\cap\, |\Delta_{t,\kappa-\Delta_n}^n X^d|\leq 2v_n\}}.
\end{aligned} \tag{A.9}$$

By Markov inequality and Burkholder-Davis-Gundy inequality and taking into account the integrability condition for a and $\widetilde{\sigma}$ as well as making use of $F(\mathbb{R}) < \infty$, we have

$$\mathbb{E}|\Delta_{t,\kappa}^n X^c|^q \leq C\Delta_n^{q/2}, \quad \mathbb{P}\left(|\Delta_{t,\kappa}^n X^c| \geq v_n\right) \leq C\Delta_n^{q(1/2-\varpi)}, \quad \mathbb{P}(\Delta_{t,\kappa}^n X^d \neq 0) \leq C\Delta_n. \tag{A.10}$$

From here, by application of Hölder's inequality, we have

$$\mathbb{E}\left(|\Delta_{t,\kappa}^n X^c|^p|\Delta_{t,\kappa-\Delta_n}^n X^d|^p 1_{\{|\Delta_{t,\kappa-\Delta_n}^n X^d|\leq 2v_n\}}\right) \leq C\Delta_n^{1+p/2+p\varpi}, \tag{A.11}$$

$$\mathbb{E}\left(|\Delta_{t,\kappa}^n X^d|^p|\Delta_{t,\kappa-\Delta_n}^n X^c|^p 1_{\{|\Delta_{t,\kappa}^n X^d|\leq 2v_n\}}\right) \leq C\Delta_n^{1+p/2+p\varpi}, \tag{A.12}$$

$$\mathbb{E}\left(|\Delta_{t,\kappa}^n X^c|^p|\Delta_{t,\kappa-\Delta_n}^n X^c|^p 1_{\{|\Delta_{t,\kappa}^n X^c|\geq\frac{v_n}{2} \,\cup\, |\Delta_{t,\kappa-\Delta_n}^n X^c|\geq\frac{v_n}{2}\}}\right) \leq C\Delta_n^{\frac{q}{2}-(q-2p)\varpi}, \tag{A.13}$$

$$\mathbb{E}\left(|\Delta^n_{t,\kappa}X^c|^p|\Delta^n_{t,\kappa-\Delta_n}X^c|^p 1_{\{|\Delta^n_{t,\kappa}X^d|\geq\frac{v_n}{2}\ \cup\ |\Delta^n_{t,\kappa-\Delta_n}X^d|\geq\frac{v_n}{2}\}}\right)\leq C\Delta_n^{p+\frac{q-2p}{q}}. \quad (A.14)$$

Next, $|\Delta^n_{t,\kappa}X^d||\Delta^n_{t,\kappa-\Delta_n}X^d|$ is nonzero only if there are jumps in both intervals and further by Markov's inequality we have $\mathbb{P}_\tau\left(\int_\tau^{\tau+\Delta_n}\int_{\mathbb{R}}\mu(ds,dx)\geq 1\right)\leq C\mathbb{E}_\tau(\int_\tau^{\tau+\Delta_n} b_s ds)$ for any τ. From here, using successive conditioning, Assumption 2, the above inequality and Hölder's inequality, we have

$$\mathbb{E}\left(|\Delta^n_{t,\kappa}X^d|^p|\Delta^n_{t,\kappa-\Delta_n}X^d|^p 1_{\{|\Delta^n_{t,\kappa}X^d|\leq 2v_n\ \cap\ |\Delta^n_{t,\kappa-\Delta_n}X^d|\leq 2v_n\}}\right)\leq C\Delta_n^{2+2p\varpi}. \quad (A.15)$$

Combining the above bounds, we get the result in (A.6). For the second bound in (A.7), we use the algebraic inequality

$$\begin{aligned}
&||\Delta^n_{t,\kappa}X^c||\Delta^n_{t,\kappa-\Delta_n}X^c| - f^n_{t,\kappa}V^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W||\\
&\leq \left|\Delta^n_{t,\kappa}X^c - \sqrt{f^n_{t,\kappa}V^n_{t,\kappa}}\Delta^n_{t,\kappa}W\right||\Delta^n_{t,\kappa-\Delta_n}X^c| \quad (A.16)\\
&\quad + \left|\Delta^n_{t,\kappa-\Delta_n}X^c - \sqrt{f^n_{t,\kappa}V^n_{t,\kappa}}\Delta^n_{t,\kappa-\Delta_n}W\right|\left|\sqrt{V^n_{t,\kappa}}\Delta^n_{t,\kappa}W\right|.
\end{aligned}$$

For $r\in[2,q]$, by application of Burkholder-Davis-Gundy inequality, inequality in means and making use of the integrability of a_t, σ_t as well as the smoothness in expectation of σ_t and f_t, we have

$$\mathbb{E}\left|\Delta^n_{t,\kappa}X^c - \sqrt{f^n_{t,\kappa}V^n_{t,\kappa}}\Delta^n_{t,\kappa}W\right|^r\leq C\Delta_n^{1+r/2}. \quad (A.17)$$

From here the result follows by an application of Hölder's inequality (raising the term $|\Delta^n_{t,\kappa-\Delta_n}X^c|^p$ or $\left|\sqrt{V^n_{t,\kappa}}\Delta^n_{t,\kappa}W\right|^p$ to power q).

For the first of the bounds in (A.8), we make use of the decomposition

$$\widetilde{f}_\kappa-\overline{f}_\kappa=\frac{1}{T}\frac{\pi}{2}\sum_{t=1}^T f^n_{t,\kappa}V^n_{t,\kappa}\left(n|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W|-\frac{2}{\pi}\right)+\frac{1}{T}\sum_{t=1}^T(f^n_{t,\kappa}V^n_{t,\kappa}-\overline{f}_\kappa). \quad (A.18)$$

Successive application of the Burkholder-Davis-Gundy inequality, given the integrability condition for σ_t, gives

$$\mathbb{E}\left|\sum_{t=1}^T f^n_{t,\kappa}V^n_{t,\kappa}\left(n|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W|-\frac{2}{\pi}\right)\right|^p\leq CT^{p/2}. \quad (A.19)$$

Applying Lemma VIII.3.102 of [2] and Hölder's inequality and taking into account the integrability assumptions for σ_t as well as for the mixing coefficient of $\mathbf{Y}_t$, we have

$$\mathbb{E}\Big|\sum_{t=1}^{T}(f_{t,\kappa}^n V_{t,\kappa}^n - \overline{f}_\kappa)\Big|^2 \leq C\sum_{k=0}^{T}(T-k)\sqrt{\alpha_k} \leq KT. \tag{A.20}$$

Combining the above two results, we get the first bound in (A.8). The second one is proved in an analogous way. □

Lemma 2. *Suppose the setting of Lemma 1 and in addition $f_{t,\kappa} \equiv f_\kappa$ (constant time-of-day) for $t \in \mathbb{N}_+$. Let $0 < \epsilon < \inf_{\kappa\in[0,1]} f_\kappa/4$ and assume $q \geq 8$ and $\varpi \geq 2/q$. Then, we have*

$$\mathbb{E}\big|u\widehat{L}'_\kappa(u)1_{\{|\widehat{f}_\kappa|>\epsilon\}} - u\widetilde{L}'_\kappa(u)\big| \leq C(|u|\vee 1)\left[\frac{1}{\sqrt{T}}\bigvee\sqrt{\Delta_n}\bigvee\Delta_n^{(q-4)\left(\frac{1}{2}-\varpi\right)}\right], \tag{A.21}$$

where the positive and finite constant C does not depend on u.

Proof. The derivation is done on the basis of the following bound:

$$\begin{aligned}
&\Bigg|\sin\left(\sqrt{2un}\Delta_{t,\kappa}^n X/\sqrt{\widehat{f}_\kappa}\right)\frac{\sqrt{n}\Delta_{t,\kappa}^n X}{\sqrt{\widehat{f}_\kappa}}1_{\{|\Delta_{t,\kappa}^n X|\leq v_n\,\cap\,|\widehat{f}_\kappa|>\epsilon\}} \\
&\qquad\qquad - \sin\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right)\sqrt{n}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\Bigg| \\
&\leq C\big|\sqrt{n}\Delta_{t,\kappa}^n X 1_{\{|\Delta_{t,\kappa}^n X|\leq v_n\}} - \sqrt{n}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\big| + C1_{\{|\widehat{f}_\kappa|\leq\epsilon\}}\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W| \\
&+ C\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\left(\sqrt{u}\big|\sqrt{n}\Delta_{t,\kappa}^n X^c - \sqrt{n}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\big| + 1_{\{|\Delta_{t,\kappa}^n X^d|>0\}}\right) \\
&+ C|\widehat{f}_\kappa - f_\kappa^n|\left(\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W| + \sqrt{u}nV_{t,\kappa}^n|\Delta_{t,\kappa}^n W|^2\right),
\end{aligned} \tag{A.22}$$

for a positive and finite constant C that does not depend on u. Applying Cauchy-Schwarz inequality

$$\mathbb{E}\left[\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\left(\big|\sqrt{n}\Delta_{t,\kappa}^n X^c - \sqrt{n}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\big| + 1_{\{|\Delta_{t,\kappa}^n X^d|>0\}}\right)\right] \leq C\sqrt{\Delta_n}. \tag{A.23}$$

Applying Burkholder-Davis-Gundy inequality and making use of $F(\mathbb{R}) < \infty$, we get

$$\mathbb{E}\left|\sqrt{n}\Delta_{t,\kappa}^n X 1_{\{|\Delta_{t,\kappa}^n X| \leq v_n\}} - \sqrt{n}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right| \leq C(\Delta_n^{(q-1)(1/2-\varpi)} \vee \sqrt{\Delta_n}). \tag{A.24}$$

By application of Hölder's inequality as well as the results of Lemma 1 (and using $\varpi \geq \frac{2}{q}$ and $q \geq 8$), we have

$$\begin{aligned}\mathbb{E}\Bigg[&|\widehat{f}_\kappa - f_\kappa^n| \left(\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W| + nV_{t,\kappa}^n|\Delta_{t,\kappa}^n W|^2\right) \\ &+ 1_{\{|\widehat{f}_\kappa| \leq \epsilon\}}\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\Bigg] \leq C\left(\Delta_n^{(q-4)\left(\frac{1}{2}-\varpi\right)} \bigvee \sqrt{\Delta_n} \bigvee \frac{1}{\sqrt{T}}\right).\end{aligned} \tag{A.25}$$

Combining the estimates in (A.23)-(A.25) with the bound in (A.22) we get the result of the lemma. □

A.2 Proof of Theorem 1

We make the decomposition

$$\cos\left(\sqrt{2un}\Delta_{t,\kappa}^n X/\sqrt{\widehat{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\overline{f}_\kappa}\right) = \sum_{j=1}^{3} \chi_{t,n}^{(j)}(u,\kappa), \tag{A.26}$$

where

$$\chi_{t,n}^{(1)}(u,\kappa) = \cos\left(\sqrt{2un}\Delta_{t,\kappa}^n X/\sqrt{\widehat{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widehat{f}_\kappa}\right),$$

$$\chi_{t,n}^{(2)}(u,\kappa) = \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widehat{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widetilde{f}_\kappa}\right),$$

$$\chi_{t,n}^{(3)}(u,\kappa) = \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widetilde{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\overline{f}_\kappa}\right).$$

The proof will then consist of analysis of the separate terms in the decomposition. Using the inequalities $|\cos(x) - \cos(y)| \leq 2|\sin(\frac{x-y}{2})| \leq |x-y| \wedge 2$ we can bound $\chi_{t,n}^{(1)}$ as follows

$$|\chi_{t,n}^{(1)}(u,\kappa)| \leq 21_{\{\Delta_{t,\kappa}^n X^d \neq 0\ \cup\ \widehat{f}_\kappa < \epsilon\}} + C\sqrt{un}|\Delta_{t,\kappa}^n X^c - \sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W|, \tag{A.27}$$

where ϵ is some constant satisfying $0 < \epsilon < \inf_{\kappa \in [0,1]} f_\kappa/4$, and ϵ and C do not depend on u. From here, applying Burkholder-Davis-Gundy inequality, the smoothness in expectation assumption for σ_t and f_t, the integrability assumptions for a_t, b_t and σ_t and Lemma 1, we have

$$\frac{1}{T}\left|\left|\sum_{t=1}^{T}\chi_{t,n}^{(1)}(u,\kappa)\right|\right| = O_p\left(\sqrt{\Delta_n}\bigvee\frac{1}{T}\right). \tag{A.28}$$

Turning next to $\chi_{t,n}^{(2)}(u,\kappa)$, we have the following bound (note that f_κ is bounded both from below and above)

$$|\chi_{t,n}^{(2)}(u,\kappa)| \leq 2 1_{\{\widehat{f}_\kappa < \epsilon \ \cup \ \widetilde{f}_\kappa < \epsilon\}} + C\sqrt{un}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W||\widehat{f}_\kappa - \widetilde{f}_\kappa|, \tag{A.29}$$

for a positive and a finite constant C that does not depend on u. Using Assumptions 1-3 and applying Lemma VIII.3.102 of [2] and Hölder's inequality, we have

$$\mathbb{E}\left|\sum_{t=1}^{T}\sqrt{n}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W| - \sqrt{\frac{2}{\pi}}\mathbb{E}|V_t|\right|^2 \leq \frac{C}{T}. \tag{A.30}$$

Using this bound, Cauchy-Schwartz inequality and Lemma 1, we have

$$\frac{1}{T}\left|\left|\sum_{t=1}^{T}\chi_{t,n}^{(2)}(u,\kappa)\right|\right| = O_p\left(\sqrt{\Delta_n}\bigvee\Delta_n^{(q-2)\left(\frac{1}{2}-\varpi\right)}\bigvee\frac{\Delta_n^{(1-2\varpi)\wedge\varpi\wedge\frac{1}{4}}}{\sqrt{T}}\bigvee\frac{1}{T}\right), \tag{A.31}$$

for q being the constant of Lemma 1. For $\chi_{t,n}^{(3)}(u,\kappa)$, we have

$$|\chi_{t,n}^{(3)}(u,\kappa)| \leq C(|u|\vee 1)(\sqrt{n}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\vee 1)|\widetilde{f}_\kappa - \overline{f}_\kappa|, \tag{A.32}$$

for a positive and a finite constant C that does not depend on u. Then, by application of Cauchy-Schwarz inequality and Lemma 1, we have $\frac{1}{\sqrt{T}}||\sum_{t=1}^{T}\chi_{t,n}^{(3)}(u,\kappa)|| = O_p(1)$.

Finally, using Assumptions 1-3 and applying Lemma VIII.3.102 of [2] and Hölder's inequality, we have

$$\mathbb{E}\left|\left|\sum_{t=1}^{T}\left[\cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\overline{f}_\kappa}\right) - \mathbb{E}\left(e^{-uf_{t+\kappa}V_{t+\kappa}/\mathbb{E}(f_{t+\kappa}V_{t+\kappa})}\right)\right]\right|\right|^2 \leq CT, \tag{A.33}$$

for a positive and a finite constant C that does not depend on u.

Combining the above bounds, we get the consistency result of the theorem. For further use, we note also that

$$||\chi_{t,n}^{(1)}(u,\kappa)|| + ||\chi_{t,n}^{(2)}(u,\kappa)|| = o_p(1/\sqrt{T}), \tag{A.34}$$

provided $T\Delta_n \to 0$ and $\varpi \le \frac{q-3}{2q-4}$. Under this same condition we also have $\widehat{L}_\kappa^n - \mathcal{L}_\kappa = O_p(1/\sqrt{T})$.

A.3 Proof of Theorem 2

The proof consists of two lemmas.

Lemma 3. *Under the conditions of Theorem 2, we have*

$$\sqrt{T}\left|\left|\widehat{L}_\kappa^n(u) - \widehat{L}_{\kappa'}^n(u) - \frac{1}{T}\sum_{t=1}^T d_{t,n}(u)\right|\right| \xrightarrow{\mathbb{P}} 0. \tag{A.35}$$

Proof of Lemma 3. We denote $0 < \epsilon < \inf_{\kappa\in[0,1]} f_\kappa/4$ and make the following decomposition

$$\cos\left(\sqrt{2un}\Delta_{t,\kappa}^n X/\sqrt{\widehat{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right) = \sum_{j=1}^5 \chi_{t,n}^{(j)}(u,\kappa), \tag{A.36}$$

where, using the fact that $f_{t,\kappa} = f_\kappa$, we denote

$$\chi_{t,n}^{(1)}(u,\kappa) = \cos\left(\sqrt{2un}\Delta_{t,\kappa}^n X/\sqrt{\widehat{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widehat{f}_\kappa}\right),$$

$$\chi_{t,n}^{(2)}(u,\kappa) = \cos\left(\sqrt{2un}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widehat{f}_\kappa}\right) - \cos\left(\sqrt{2un}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W/\sqrt{\widetilde{f}_\kappa}\right),$$

$$\chi_{t,n}^{(3)}(u,\kappa) = \frac{1}{2}\sin\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right)\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\left(\frac{\widetilde{f}_\kappa - f_\kappa^n}{f_\kappa^n}\right),$$

$$\chi_{t,n}^{(4)}(u,\kappa) = -\frac{1}{2}\sin\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right)\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\left(\frac{\widetilde{f}_\kappa - f_\kappa^n}{f_\kappa^n}\right)1_{\{|\widetilde{f}_\kappa|\le\epsilon\}},$$

$$\chi_{t,n}^{(5)}(u,\kappa) = \frac{1}{2}g\left(\sqrt{2un}\sqrt{f_\kappa^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W; \dot{f}_\kappa\right)\left(\widetilde{f}_\kappa - f_\kappa^n\right)^2 1_{\{|\widetilde{f}_\kappa|>\epsilon\}},$$

with $\dot{f}_\kappa$ being an intermediary value between $\widetilde{f}_\kappa$ and f_κ^n, and further

$$g(a;x) = -\cos\left(\frac{a}{\sqrt{x}}\right)\frac{a^2}{4x^3} - \sin\left(\frac{a}{\sqrt{x}}\right)\frac{3a}{4x^{5/2}}. \tag{A.37}$$

We can write

$$\begin{aligned}\overline{\chi}_n^{(3)}(u,\kappa) &\equiv \frac{1}{T}\sum_{t=1}^T \chi_{t,n}^{(3)}(u,\kappa) + u\mathcal{L}'(u)\frac{1}{T}\sum_{t=1}^T\left(\frac{\pi}{2}V_{t,\kappa}^n n|\Delta_{t,\kappa}^n W||\Delta_{t,\kappa-\Delta_n}^n W| - 1\right)\\ &= \big(-u\widetilde{L}_\kappa'(u) + u\mathcal{L}'(u)\big)\frac{1}{T}\sum_{t=1}^T\left(\frac{\pi}{2}V_{t,\kappa}^n n|\Delta_{t,\kappa}^n W||\Delta_{t,\kappa-\Delta_n}^n W| - 1\right).\end{aligned} \tag{A.38}$$

With this notation, we finally have

$$\begin{aligned}\widehat{L}_\kappa^n(u) - \widehat{L}_{\kappa'}^n(u) - \frac{1}{T}\sum_{t=1}^T d_{t,n}(u) = \frac{1}{T}\sum_{j=1,2,4,5}\sum_{t=1}^T\left(\chi_{t,n}^{(j)}(u,\kappa) - \chi_{t,n}^{(j)}(u,\kappa')\right)\\ + \overline{\chi}_n^{(3)}(u,\kappa) - \overline{\chi}_n^{(3)}(u,\kappa').\end{aligned} \tag{A.39}$$

The proof consists of showing the asymptotic negligibility of the terms $\frac{1}{\sqrt{T}}||\sum_{t=1}^T \chi_{t,n}^{(j)}(u,\kappa)||$ for $j = 1,2,4,5$ as well as the negligibility of $\sqrt{T}\overline{\chi}_n^{(3)}(u,\kappa)$ for arbitrary $\kappa \in (0,1]$. For $j = 1,2$, this was already established in the proof of Theorem 1 under the condition for ϖ of the theorem. For $\chi_{t,n}^{(4)}(u,\kappa)$, since $\epsilon < \inf_{\kappa\in[0,1]} f_\kappa$, we have

$$|\chi_{t,n}^{(4)}(u,\kappa)| \le C\sqrt{un}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W||\widetilde{f}_\kappa - f_\kappa^n|^2, \tag{A.40}$$

for some positive and finite C that does not depend on u. Similarly, $\chi_{t,n}^{(5)}$ can be bounded as follows

$$|\chi_{t,n}^{(5)}(u,\kappa)| \le C\left(unV_{t,\kappa}^n|\Delta_{t,\kappa}^n W|^2 + \sqrt{unV_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\right)|\widetilde{f}_\kappa - f_\kappa^n|^2, \tag{A.41}$$

with C as above. Therefore,

$$\begin{aligned} &\frac{1}{\sqrt{T}}\Big|\Big|\sum_{t=1}^{T}\chi_{t,n}^{(4)}(u,\kappa)\Big|\Big| + \frac{1}{\sqrt{T}}\Big|\Big|\sum_{t=1}^{T}\chi_{t,n}^{(5)}(u,\kappa)\Big|\Big| \\ &\qquad\qquad \leq C|\widetilde{f}_\kappa - f_\kappa^n|^2\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(nV_{t,\kappa}^n|\Delta_{t,\kappa}^n W|^2 + \sqrt{nV_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\right), \end{aligned} \tag{A.42}$$

for some positive and finite C that does not depend on u. From here, since

$$\frac{1}{T}\sum_{t=1}^{T}\left(nV_{t,\kappa}^n|\Delta_{t,\kappa}^n W|^2 + \sqrt{nV_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\right) = O_p(1),$$

and utilizing the result of Lemma 1, we have $\frac{1}{\sqrt{T}}||\sum_{t=1}^{T}\chi_{t,n}^{(4)}(u,\kappa)|| + \frac{1}{\sqrt{T}}||\sum_{t=1}^{T}\chi_{t,n}^{(5)}(u,\kappa)|| = o_p(1)$.

We are left with $\overline{\chi}_n^{(3)}(u,\kappa)$. Using Lemma 1 and Cauchy-Schwarz inequality, we have

$$\mathbb{E}||\overline{\chi}_n^{(3)}(u,\kappa)|| \leq \frac{C}{T}, \tag{A.43}$$

for some positive and finite C that does not depend on u. The asymptotic negligibility of $\sqrt{T}||\overline{\chi}_n^{(3)}(u,\kappa)||$ then readily follows. $\square$

To state the next lemma, we will need some additional notation which we now introduce. We decompose

$$d_{t,n}(u) = \xi_{t,n}^{(1)}(u) + \xi_{t,n}^{(2)}(u), \tag{A.44}$$

where

$$\begin{aligned} \xi_{t,n}^{(1)}(u) = \cos&\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right) - e^{-uV_{t,\kappa}^n} - \cos\left(\sqrt{2un}\sqrt{V_{t,\kappa'}^n}\Delta_{t,\kappa'}^n W\right) + e^{-uV_{t,\kappa'}^n} \\ &+ u\mathcal{L}'(u)V_{t,\kappa'}^n\left(n\frac{\pi}{2}|\Delta_{t,\kappa'}^n W||\Delta_{t,\kappa-\Delta_n}^n W| - 1\right) - u\mathcal{L}'(u)V_{t,\kappa}^n\left(n\frac{\pi}{2}|\Delta_{t,\kappa}^n W||\Delta_{t,\kappa-\Delta_n}^n W| - 1\right), \end{aligned}$$

$$\xi_{t,n}^{(2)}(u) = e^{-uV_{t,\kappa}^n} - e^{-uV_{t,\kappa'}^n} + u\mathcal{L}'(u)\left(V_{t,\kappa}^n - V_{t,\kappa'}^n\right).$$

Fix a positive integer l and denote for $t = 1, ..., T$:

$$\widetilde{\xi}^{(2)}_{t,n,l}(u) = \sum_{k=0}^{l-1} \left(\mathbb{E}_t(\xi^{(2)}_{t+k,n}(u)) - \mathbb{E}_{t-1}(\xi^{(2)}_{t+k,n}(u)) \right). \tag{A.45}$$

With this notation we set

$$\widetilde{d}_{t,n,l}(u) = \xi^{(1)}_{t,n}(u) + \widetilde{\xi}^{(2)}_{t,n,l}(u), \tag{A.46}$$

and denote the difference

$$R^n_{T,l}(u) = \frac{1}{T} \sum_{t=1}^{T} \left(d_{t,n}(u) - \widetilde{d}_{t,n,l}(u) \right). \tag{A.47}$$

Using the decomposition

$$\sum_{t=1}^{T} \sum_{k=0}^{l} \left(\mathbb{E}_t \xi^{(2)}_{t+k,n}(u) - \mathbb{E}_{t-1} \xi^{(2)}_{t+k,n}(u) \right) = \sum_{k=0}^{l} \left(\sum_{t=1}^{T} \mathbb{E}_t \xi^{(2)}_{t+k,n}(u) - \sum_{t=0}^{T-1} \mathbb{E}_t \xi^{(2)}_{t+k+1,n}(u) \right), \tag{A.48}$$

we have

$$R^n_{T,l}(u) = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_t \xi^{(2)}_{t+l,n}(u) - \frac{1}{T} \sum_{k=1}^{l-1} \left(\mathbb{E}_T \xi^{(2)}_{T+k,n}(u) - \mathbb{E}_0 \xi^{(2)}_{k,n}(u) \right). \tag{A.49}$$

Using the mixing condition for $\mathbf{Y}_t$, the integrability assumption for V_t, and Lemma VIII.3.102 of [2], we have $\sqrt{\mathbb{E}|\mathbb{E}_t(\xi^{(2)}_{t+k,n}(u))|^2} \leq K \alpha_k^{3/8} (\mathbb{E}|\xi^{(2)}_{t+k,n}(u)|^8)^{1/8}$. Therefore, since $\alpha_k = o(k^{-8/3})$ as $k \to \infty$, by Fatou's lemma, the limit $\widetilde{d}_{t,n,\infty} = \xi^{(1)}_{t,n} + \widetilde{\xi}^{(2)}_{t,n,\infty}$ is finite almost surely, where

$$\widetilde{\xi}^{(2)}_{t,n,\infty} := \lim_{l \to \infty} \widetilde{\xi}^{(2)}_{t,n,l} = \sum_{k=0}^{\infty} \left(\mathbb{E}_t(\xi^{(2)}_{t+k,n}) - \mathbb{E}_{t-1}(\xi^{(2)}_{t+k,n}) \right), \tag{A.50}$$

and the same holds for

$$R^n_{T,\infty} := \lim_{l \to \infty} R^n_{T,l} = \frac{1}{T} \sum_{k=1}^{\infty} \left(\mathbb{E}_0(\xi^{(2)}_{k,n}) - \mathbb{E}_T(\xi^{(2)}_{T+k,n}) \right). \tag{A.51}$$

Finally, we set

$$\xi_t^{(2)}(u) = e^{-uV_{t+\kappa}} - e^{-uV_{t+\kappa'}} + u\mathcal{L}'(u)\left(V_{t+\kappa} - V_{t+\kappa'}\right),$$

and define $\widetilde{\xi}_{t,l}^{(2)}(u)$ and $\widetilde{\xi}_{t,\infty}^{(2)}(u)$ from it exactly as we defined $\widetilde{\xi}_{t,n,l}^{(2)}(u)$ and $\widetilde{\xi}_{t,n,\infty}^{(2)}(u)$ from $\xi_{t,n}^{(2)}(u)$.

Lemma 4. *Under Assumptions 1-3 with $\mathcal{K} = \{1\}$, as $n \to \infty$ and $T \to \infty$ with $T\Delta_n \to 0$, we have*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \widetilde{d}_{t,n,\infty} \xrightarrow{\mathcal{L}} N(0, K) \quad \textit{and} \quad \sqrt{T}||R_{T,\infty}|| \ \xrightarrow{\mathbb{P}} \ 0. \tag{A.52}$$

Proof of Lemma 4. By dominated convergence, we have $\mathbb{E}_{t-1}\left(\widetilde{d}_{t,n,\infty}(u)\right) = 0$ and $\mathbb{E}\left(||\widetilde{d}_{t,n,\infty}||^2\right) < \infty$, and therefore the array $\{\widetilde{d}_{t,n,\infty}\}_{t\in\mathbb{N}_+}$ is a martingale difference sequence and we can apply Theorem C of [3] to establish the CLT result. In particular, it suffices to show that the following is true:

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_{t-1}\left(||\widetilde{d}_{t,n,\infty}||^2\right) \ \xrightarrow{\mathbb{P}} \ \text{Trace}(K), \tag{A.53}$$

$$\frac{1}{T^{1+\iota/2}}\sum_{t=1}^{T} \mathbb{E}_{t-1}\left(||\widetilde{d}_{t,n,\infty}||^{2+\iota}\right) \ \xrightarrow{\mathbb{P}} \ 0, \quad \text{for some } \iota \in (0,1), \tag{A.54}$$

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_{t-1}\left(\langle\widetilde{d}_{t,n,\infty}, e_i\rangle\langle\widetilde{d}_{t,n,\infty}, e_j\rangle\right) \ \xrightarrow{\mathbb{P}} \ \langle Ke_i, e_j\rangle, \quad \forall i, j \in \mathbb{N}_+, \tag{A.55}$$

where $\{e_i\}_{i\in\mathbb{N}_+}$ is an orthonormal basis in $\mathcal{L}^2(w)$. We have

$$\mathbb{E}_{t-1}||\xi_{t,n}^{(1)}||^2 = \mathbb{E}_{t-1}||\eta_u(V_{t,\kappa}^n, V_{t,\kappa'}^n)||^2, \tag{A.56}$$

where for two positive constants C_1 and C_2, we denote $\eta_u(C_1, C_2) = \sqrt{\mathbb{E}\left(\overline{\eta}_u(C_1, C_2)^2\right)}$ with

$$\begin{aligned}\overline{\eta}_u(C_1, C_2) = \cos\left(\sqrt{2uC_1}Z_1\right) - e^{-uC_1} - \cos\left(\sqrt{2uC_2}Z_2\right) + e^{-uC_2}\\ - u\mathcal{L}'(u)\left(C_1\left(\frac{\pi}{2}|Z_1||\widetilde{Z}_1| - 1\right) - C_2\left(\frac{\pi}{2}|Z_2||\widetilde{Z}_2| - 1\right)\right),\end{aligned} \tag{A.57}$$

for some independent standard normal random variables Z_1, $\widetilde{Z}_1$, Z_2 and $\widetilde{Z}_2$. From here, we have

$$\mathbb{E}_{t-1}||\widetilde{d}_{t,n,\infty}||^2 = \mathbb{E}_{t-1}\left(||\eta_u(V^n_{t,\kappa}, V^n_{t,\kappa'})||^2 + ||\widetilde{\xi}^{(2)}_{t,n,\infty}||^2\right) + 2\mathbb{E}_{t-1}\left(\langle \xi^{(1)}_{t,n}, \widetilde{\xi}^{(2)}_{t,n,\infty}\rangle\right). \qquad (A.58)$$

Using successive conditioning, we can write

$$\begin{aligned}\mathbb{E}_{t-1}\left(\xi^{(1)}_{t,n}(u)\widetilde{\xi}^{(2)}_{t,n,\infty}(u)\right) = \mathbb{E}_{t-1}\Bigg(\xi^{(1)}_{t,n}(u)\sum_{\iota=\kappa,\kappa'}\sum_{k=0}^{\infty}\Big[&\mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor}{n}}(\xi^{(2)}_{t+k,n}(u)) \\ &- \mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor - 2}{n}}(\xi^{(2)}_{t+k,n}(u))\Big]\Bigg).\end{aligned} \qquad (A.59)$$

For $\iota = \kappa, \kappa'$ and $k \geq 0$, we have for some positive and finite C:

$$\begin{aligned}&\mathbb{E}\left|\xi^{(1)}_{t,n}(u)\left(\mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor}{n}}\xi^{(2)}_{t+k,n}(u) - \mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor-2}{n}}\xi^{(2)}_{t+k,n}(u)\right)\right| \\ &\leq \sqrt{\mathbb{E}(\xi^{(1)}_{t,n}(u))^2}\sqrt{\mathbb{E}\left(\mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor}{n}}\xi^{(2)}_{t+k,n}(u)\right)^2 - \mathbb{E}\left(\mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor-2}{n}}\xi^{(2)}_{t+k,n}(u)\right)^2} \\ &= \sqrt{\mathbb{E}(\xi^{(1)}_{t,n}(u))^2}\sqrt{\mathbb{E}\left(\mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor}{n}}\xi^{(2)}_{t+k,n}(u)\right)^2 - \mathbb{E}\left(\mathbb{E}_{t-1+\frac{\lfloor \iota n\rfloor}{n}}\xi^{(2)}_{t+k+2\Delta_n,n}(u)\right)^2} \\ &\leq C(|u|\vee 1)\sqrt{\mathbb{E}\left[\mathbb{E}_{\frac{\lfloor \iota n\rfloor}{n}}(\xi^{(2)}_{1+k,n}(u) - \xi^{(2)}_{1+k+2\Delta_n,n}(u))\mathbb{E}_{\frac{\lfloor \iota n\rfloor}{n}}(\xi^{(2)}_{1+k,n}(u) + \xi^{(2)}_{1+k+2\Delta_n,n}(u))\right]} \\ &\leq C(|u|\vee 1)\left(\mathbb{E}(\xi^{(2)}_{1+k,n}(u) - \xi^{(2)}_{1+k+2\Delta_n,n}(u))^2\right)^{1/4} \\ &\quad\times\left(\mathbb{E}\left(\left(\mathbb{E}_{\frac{\lfloor \iota n\rfloor}{n}}\xi^{(2)}_{1+k,n}(u)\right)^2 + \left(\mathbb{E}_{\frac{\lfloor \iota n\rfloor}{n}}\xi^{(2)}_{1+k+2\Delta_n,n}(u)\right)^2\right)\right)^{1/4} \\ &\leq C(|u|^{3/2}\vee 1)\Delta_n^{1/4}\alpha_k^{\frac{3}{16}}\left(\mathbb{E}|\xi^{(2)}_{1,n}(u)|^8\right)^{1/16} \leq C(|u|^2\vee 1)\Delta_n^{1/4}\alpha_k^{\frac{3}{16}},\end{aligned} \qquad (A.60)$$

where for the first inequality we have made use of Cauchy-Schwarz inequality, for the second equality we use the stationarity of V_t (and hence of its conditional expectation), for the third inequality we again made use of the stationarity of V_t as well as the integrability assumption for V_t, for the forth inequality we used Cauchy-Schwarz and Jensen's inequality, and for the remaining inequalities, we made use of the integrability and smoothness in expectation conditions for V_t, as well as Lemma VIII.3.102 of [2].

From here, since by Assumption 3, $\alpha_k = o(k^{-16/3})$ for $k \to \infty$, we have

$$\mathbb{E}|\xi_{t,n}^{(1)}(u)\widetilde{\xi}_{t,n,\infty}^{(2)}(u)| \leq C(|u|^2 \vee 1)\Delta_n^{1/4}, \tag{A.61}$$

for some positive and finite C that does not depend on u, and therefore

$$\frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}}\left(\mathbb{E}_{t-1}\left(\xi_{t,n}^{(1)}(u)\widetilde{\xi}_{t,n,\infty}^{(2)}(u)\right)\right)w(u)du = o_p(1). \tag{A.62}$$

Using the mixing condition for $\mathbf{Y}_t$, Lemma VIII.3.102 of [2], Lebesgue's dominated convergence theorem as well as Assumptions 1-2, we have

$$\mathbb{E}\left|||\eta_u(V_{t,\kappa}^n, V_{t,\kappa'}^n)||^2 - ||\eta_u(V_{t+\kappa}, V_{t+\kappa'})||^2 + ||\widetilde{\xi}_{t,n,\infty}^{(2)}||^2 - ||\widetilde{\xi}_{t,\infty}^{(2)}||^2\right| \leq \frac{C}{n}, \tag{A.63}$$

for some positive and finite C that does not depend on u. Furthermore, given the square integrability of V_t, the assumption that $\widetilde{\mathbf{Y}}_t$ is a Markov process (and hence the conditional expectation of a transformation of it is a function of the process at the time of the conditioning), and by an application of an ergodic theorem, we have

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{t-1}\left(||\eta_u(V_{t+\kappa}, V_{t+\kappa'})||^2 + ||\widetilde{\xi}_{t,\infty}^{(2)}||^2\right) \xrightarrow{\mathbb{P}} \mathbb{E}\left(||\eta_u(V_{t+\kappa}, V_{t+\kappa'})||^2 + ||\widetilde{\xi}_{t,\infty}^{(2)}||^2\right), \tag{A.64}$$

provided $\mathbb{E}(||\widetilde{\xi}_{t,\infty}^{(2)}||^2) < \infty$. The latter is guaranteed by the mixing condition for $\mathbf{Y}_t$, use of Lemma VIII.3.102 of [2] and Lebesgue's dominated convergence theorem upon making use of the inequality

$$|\widetilde{\xi}_{t,l}^{(2)}(u)|^2 \leq \sum_{k,p=0}^{l-1}|\mathbb{E}_t(\xi_{t+k}^{(2)}(u)) - \mathbb{E}_{t-1}(\xi_{t+k}^{(2)}(u))||\mathbb{E}_t(\xi_{t+p}^{(2)}(u)) - \mathbb{E}_{t-1}(\xi_{t+p}^{(2)}(u))|, \tag{A.65}$$

which in turn implies for some positive and finite C that does not depend on u

$$\mathbb{E}|\widetilde{\xi}_{t,l}^{(2)}(u)|^2 \le C \sum_{k,p=0}^{l-1} (\alpha_k^{3/8} \alpha_p^{3/8}) \le C. \tag{A.66}$$

To establish (A.53), using stationarity, we therefore need to show that

$$\mathbb{E}\left(||\eta_u(V_\kappa, V_{\kappa'})||^2 + ||\widetilde{\xi}_{1,\infty}^{(2)}||^2\right) = \mathrm{Trace}(K). \tag{A.67}$$

For this, using dominated convergence, it suffices to show

$$\mathbb{E}\left(||\eta_u(V_\kappa, V_{\kappa'})||^2\right) + \lim_{l\to\infty} \mathbb{E}\left(||\widetilde{\xi}_{1,l}^{(2)}||^2\right) = \mathrm{Trace}(K). \tag{A.68}$$

We have

$$\begin{aligned}
\mathbb{E}\left(|\widetilde{\xi}_{1,l}^{(2)}(u)|^2\right) &= \sum_{k,p=0}^{l-1} \mathbb{E}\left[\mathbb{E}_1(\xi_{1+k}^{(2)}(u))\left(\mathbb{E}_1(\xi_{1+p}^{(2)}(u)) - \mathbb{E}_0(\xi_{1+p}^{(2)}(u))\right)\right] \\
&= \sum_{k,p=0}^{l-1} \mathbb{E}\left[\xi_{1+k}^{(2)}(u)\left(\mathbb{E}_1(\xi_{1+p}^{(2)}(u)) - \mathbb{E}_0(\xi_{1+p}^{(2)}(u))\right)\right] \\
&= \sum_{k,p=0}^{l-1} \mathbb{E}[\xi_k^{(2)}(u)\mathbb{E}_0(\xi_p^{(2)}(u))] - \sum_{k,p=1}^{l} \mathbb{E}[\xi_k^{(2)}(u)\mathbb{E}_0(\xi_p^{(2)}(u))] \qquad (A.69) \\
&= \mathbb{E}[\xi_0^{(2)}(u)]^2 + 2\sum_{k=1}^{l-1} \mathbb{E}\left[\xi_0^{(2)}(u)\xi_k^{(2)}(u)\right] \\
&\quad - \sum_{k=1}^{l-1} \mathbb{E}[\xi_k^{(2)}(u)\mathbb{E}_0(\xi_l^{(2)}(u))] - \sum_{p=1}^{l} \mathbb{E}[\xi_l^{(2)}(u)\mathbb{E}_0(\xi_p^{(2)}(u))],
\end{aligned}$$

where for the first equality we made use of successive conditioning and for the second inequality we made use of the stationarity of the sequence $\{\xi_t^{(2)}(u)\xi_{t+s}^{(2)}(u)\}_{t\ge0}$ and arbitrary fixed $s \ge 0$. We now bound the last two terms in the above inequality. Using Lemma VIII.3.102 of [2] and our integrability assumption for V_t, we have

$$\sqrt{\mathbb{E}|\mathbb{E}_0(\xi_p^{(2)}(u))|^2} \le C\alpha_p^{3/8}\left(\mathbb{E}(\xi_0^{(2)}(u))^8\right)^{1/8}, \quad p \ge 0, \tag{A.70}$$

for some positive and finite C that does not depend on u. Therefore, with C as above, we have

$$\begin{aligned}\sum_{k=1}^{l-1}|\mathbb{E}[\xi_k^{(2)}(u)\mathbb{E}_0(\xi_l^{(2)}(u))]| &\leq \sum_{k=1}^{l-1}\sqrt{\mathbb{E}|\mathbb{E}_0(\xi_k^{(2)}(u))|^2\mathbb{E}|\mathbb{E}_0(\xi_l^{(2)}(u))|^2} \\ &\leq C\left(\mathbb{E}(\xi_0^{(2)}(u))^8\right)^{1/8}\alpha_l^{3/8}\sum_{k=1}^{l-1}\alpha_k^{3/8},\end{aligned} \tag{A.71}$$

$$\begin{aligned}\sum_{p=1}^{l}|\mathbb{E}[\xi_l^{(2)}(u)\mathbb{E}_0(\xi_p^{(2)}(u))]| &\leq \sum_{p=1}^{l}\sqrt{\mathbb{E}|\mathbb{E}_0(\xi_l^{(2)}(u))|^2\mathbb{E}|\mathbb{E}_0(\xi_p^{(2)}(u))|^2} \\ &\leq C\left(\mathbb{E}(\xi_0^{(2)}(u))^8\right)^{1/8}\alpha_l^{3/8}\sum_{p=1}^{l}\alpha_p^{3/8}.\end{aligned} \tag{A.72}$$

From here, taking into account the rate of decay of α_k, we have

$$\lim_{l\to\infty}\mathbb{E}\left(||\widetilde{\xi}_{1,l}^{(2)}||^2\right) = \mathrm{Trace}(\widetilde{K}) \equiv \int_0^\infty \widetilde{k}(u,u)w(u)du, \tag{A.73}$$

where the operator $\widetilde{K}$ has kernel $\widetilde{k}(z,u) = \sum_{j=-\infty}^{\infty}\mathbb{E}[\widetilde{d}_1(z)\widetilde{d}_j(u)]$ and we denote

$$\widetilde{d}_t(u) = \left(e^{-uV_{t-1+\kappa}} - e^{-uV_{t-1+\kappa'}}\right) - u\mathcal{L}'(u)(V_{t-1+\kappa} - V_{t-1+\kappa'}).$$

From here the result in (A.68) and hence (A.53) readily follows. The convergence in (A.55) is shown analogously.

We are left with establishing (A.54). First, using the integrability condition for V_t as well as the mixing assumption for $\mathbf{Y}_t$ and applying Lemma VIII.3.102 of [2], we have

$$\mathbb{E}|\mathbb{E}_0(\xi_{k,n}^{(2)}(u))|^{2+\iota} \leq C\alpha_k^{1-\frac{2+\iota}{8}}\left(\mathbb{E}|\xi_{k,n}^{(2)}(u)|^8\right)^{\frac{2+\iota}{8}}, \tag{A.74}$$

for some constant C that does not depend on u and any $\iota \in (0,6)$. From here, by inequality in means, the exponential decay of the weight function w in the tails, since $\alpha_k = o(k^{-8/3})$, and by the monotone convergence theorem, we have for C as above

$$\mathbb{E}||\widetilde{d}_{t,n,\infty}||^{2+\iota} \leq C, \quad \text{for some } \iota \in (0,1), \tag{A.75}$$

and from here the result in (A.54) follows trivially.

We continue with the bound for $R^n_{T,\infty}(u)$. Using monotone convergence, the bound in (A.70) above as well as the rate of decay condition for the mixing coefficient α_k, we have

$$\mathbb{E}||R^n_{T,\infty}|| \leq \frac{C}{T}, \tag{A.76}$$

for some positive and finite C, and therefore $\sqrt{T}||R^n_{T,\infty}|| = o_p(1)$. $\square$

Combining Lemmas 3 and 4, the result of the theorem follows.

A.4 Proof of Corollary 1

Let $Y = N(0, K)$, with the operator K given in equation (13). By the spectral theorem for compact self-adjoint operators (see [4]) it follows that there exists a complete set of eigenfunctions (ϵ_i) in $\mathcal{L}^2(w)$ and associated (real) eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq 0$ such that

$$K\epsilon_i = \lambda_i \epsilon_i. \tag{A.77}$$

Moreover, the eigenfunctions form an orthonormal basis for $\mathcal{L}^2(w)$. By Parseval's identity, it then follows that

$$||Y||^2 = \sum_{i=1}^{\infty} \lambda_i \left(\frac{\langle Y, \epsilon_i \rangle}{\sqrt{\lambda_i}} \right)^2, \tag{A.78}$$

Theorem 2 implies that $\langle Y, \epsilon_i \rangle$ is normally distributed with mean zero and variance $\langle K\epsilon_i, \epsilon_i \rangle$. The result then follows upon showing that $Cov(\langle Y, \epsilon_i \rangle, \langle Y, \epsilon_j \rangle) = \lambda_i \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta. Thus, for any $i, j \in \mathbb{N}_+$, we have

$$\begin{aligned}
Cov(\langle Y, \epsilon_i\rangle, \langle Y, \epsilon_j\rangle) &= \mathbb{E}\left[\int_{\mathbb{R}^+} Y(u)\epsilon_i(u)w(u)du \int_{\mathbb{R}_+} Y(u)\epsilon_j(u)w(u)du\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y(u)Y(t)\epsilon_i(u)w(u)\epsilon_j(t)w(t)dudt\right] \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathbb{E}\left[Y(u)Y(t)\right]\epsilon_i(u)w(u)\epsilon_j(t)w(t)dudt \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} k(u,t)\epsilon_i(u)w(u)\epsilon_j(t)w(t)dudt \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} k(u,t)\epsilon_i(u)w(u)\epsilon_j(t)w(t)dudt \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \lambda_i\epsilon_i(t)\epsilon_j(t)w(t)dt = \lambda_j\delta_{i,j},
\end{aligned}$$

where the last equality follows by the definition of the eigenvalues of K.

A.5 Proof of Theorem 3

We will first proof the following Lemma about the error in the kernel K_T.

Lemma 5. *Suppose Assumptions 1-4 hold with $\mathcal{K} = \{\kappa, \kappa'\}$, and with $f_{t,\kappa} \equiv f_\kappa$ (constant time-of-day) for $t \in \mathbb{N}_+$. Then, for $\varpi \in \left[\frac{2}{q}, \frac{q-5}{2q-8}\right]$, for q the constant of Lemma 1, and as $n \to \infty$ and $T \to \infty$, we have*

$$||K_T - K||_{HS} = O_p\left(B_T^{-6} \bigvee \frac{B_T}{T} \bigvee B_T^2\Delta_n\right). \tag{A.79}$$

Proof of Lemma 5. We have

$$||K_T - K||_{HS}^2 = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |k_T(u,s) - k(u,s)|^2 w(u)w(s)duds. \tag{A.80}$$

Our goal will be to decompose suitably $k_T(u,s) - k(u,s)$ and bound the second moments in this decomposition. We first we introduce some auxiliary notation. We set

$$\gamma_0(u,s) = \mathbb{E}(d_0(u)d_0(s)),\ \gamma_k(u,s) = \mathbb{E}[d_1(u)d_{k+1}(s) + d_{k+1}(u)d_1(s)],\ k \geq 1,$$

$$\widehat{\gamma}_0^n(u,s) = \frac{1}{T}\sum_{t=1}^{T}\widehat{d}_{t,n}(u)\widehat{d}_{t,n}(s),\ \widehat{\gamma}_k^n(u,s) = \frac{1}{T-k}\sum_{t=1}^{T}[\widehat{d}_{t,n}(u)(\widehat{d}_{t-k,n}(s)+\widehat{d}_{t+k,n}(s))],\ k\geq 1,$$

and the corresponding quantities in which $\widehat{d}_{t,n}(u)$ are replaced with $d_{t,n}(u)$ are denoted with $\gamma_k^n(u,s)$. With this notation, we can decompose

$$\begin{aligned} k_T(u,s) - k(u,s) = \sum_{j=1}^{B_T}\gamma_j(u,s)\left(\frac{T-j}{T}h\left(\frac{j}{B_T}\right)-1\right) - \sum_{j=B_T+1}^{\infty}\gamma_j(u,s) \\ + \sum_{j=1}^{B_T}\frac{T-j}{T}h\left(\frac{j}{B_T}\right)\left(\widehat{\gamma}_j^n(u,s)-\gamma_j(u,s)\right). \end{aligned} \tag{A.81}$$

By conditioning on the sigma algebra of the original probability space, we have

$$\mathbb{E}[d_1(z)d_j(u)] = \mathbb{E}[\widetilde{d}_1(z)\widetilde{d}_j(u)], \quad \text{for } j>1, \tag{A.82}$$

where $\widetilde{d}_t(u)$ is defined in the proof of Theorem 2 and using again the notation of that proof, we can write

$$\mathbb{E}[d_0(z)d_0(u)] = \mathbb{E}(\overline{\eta}_{z,u}(V_\kappa, V_{\kappa'})). \tag{A.83}$$

From here, using Lemma VIII.3.102 of [2], Hölder's inequality and the fact that $\mathbb{E}|V_t|^8<\infty$, we have

$$|\mathbb{E}(d_1(z)d_{j+1}(u))| \leq C\alpha_j^{3/4}\left(\mathbb{E}|d_1(z)|^8\mathbb{E}|d_1(u)|^8\right)^{1/8}, \quad j\geq 0, \tag{A.84}$$

for positive and finite C that does not depend on u, z and j. Similarly, for $k\in[0,j]$ by considering separately the cases $j-k<k$ and $j-k\geq k$, we have

$$|\mathbb{E}\left(d_{t,n}(u)d_{t-j,n}(s)d_{t-k}(u)d_{t-j-k,n}(s)\right)| \leq C\sqrt{\alpha_{(j-k)\vee j}}\left(\mathbb{E}|d_{0,n}(u)|^8\mathbb{E}|d_{0,n}(s)|^8\right)^{1/4}, \tag{A.85}$$

for positive and finite C that does not depend on u, s, j and k. Finally, for $k\geq j+1$ and $j\geq 0$

$$\begin{aligned} &|\mathbb{E}\left((d_{t,n}(u)d_{t-j,n}(s)-\gamma_j(u,s))d_{t-k}(u)d_{t-j-k,n}(s)\right)| \\ &\qquad \leq C\sqrt{\alpha_{(k-j)\vee j}}\left(\mathbb{E}|d_{0,n}(u)|^8\mathbb{E}|d_{0,n}(s)|^8\right)^{1/4}, \end{aligned} \tag{A.86}$$

where C is a positive and finite constant that does not depend on u, s, j and k. Using the first of the above bounds as well as the continuous differentiability of h in a neighborhood of zero, we get

$$\left| \sum_{|j|>B_T} \mathbb{E}[d_1(u) d_{j+1}(s)] \right|^2 \leq C B_T^{-6} \left(\mathbb{E}|d_1(u)|^8 \mathbb{E}|d_1(s)|^8 \right)^{1/4}, \tag{A.87}$$

$$\left| \sum_{j=1}^{B_T} \gamma_j(u,s) \left(\frac{T-j}{T} h\left(\frac{j}{B_T} \right) - 1 \right) \right|^2 \leq C B_T^{-6} \left(\mathbb{E}|d_1(u)|^8 \mathbb{E}|d_1(s)|^8 \right)^{1/4}. \tag{A.88}$$

For $j = 0, 1, ..., T-1$, we have

$$\begin{aligned} \mathbb{E}|\gamma_j^n(u,s) - \gamma_j(u,s)|^2 \leq \frac{C}{T-j} & \left(\sum_{k=0}^{j} \sqrt{\alpha_{(j-k)\vee k}} + \sum_{k=j+1}^{T-j-1} \sqrt{\alpha_{(k-j)\vee j}} + j \alpha_j^{3/2} \right) \\ & \times \left(\mathbb{E}|d_{0,n}(u)|^8 \mathbb{E}|d_{0,n}(s)|^8 \right)^{1/4}, \end{aligned} \tag{A.89}$$

where in the above three bounds C is a constant that does not depend on j, T, u and s. Taking into account Assumption 3, with C as above, we have

$$\mathbb{E}|\gamma_j^n(u,s) - \gamma_j(u,s)|^2 \leq C \frac{j}{T} \sqrt{\alpha_{\lfloor j/2 \rfloor}} \left(\mathbb{E}|d_{0,n}(u)|^8 \mathbb{E}|d_{0,n}(s)|^8 \right)^{1/4}. \tag{A.90}$$

Hence, using the boundedness of h and Assumption 3,

$$\left| \sum_{j=0}^{B_T} \frac{T-j}{T} h\left(\frac{j}{B_T} \right) \left(\gamma_j^n(u,s) - \gamma_j(u,s) \right) \right|^2 \leq C \frac{B_T}{T} \left(\mathbb{E}|d_{0,n}(u)|^8 \mathbb{E}|d_{0,n}(s)|^8 \right)^{1/4}, \tag{A.91}$$

for some positive and finite C that does not depend on u and s.

We proceed with the difference $\widehat{\gamma}_j^n(u,s) - \gamma_j^n(u,s)$. If we denote $0 < \epsilon < \inf_{\kappa \in [0,1]} f_\kappa / 4$ and $\overline{\epsilon} > 2 \sup_{\kappa \in [0,1]} f_\kappa$, then it suffices to analyze this difference on $\Omega_n = \left\{ \omega : \widehat{f}_\kappa, \widehat{f}_{\kappa'} \in [\epsilon, \overline{\epsilon}] \right\}$, since $\mathbb{P}(\Omega_n) \to 1$ from the results of Lemma 1. We will do so henceforth without further mention. Making use of $\sup_{u\in\mathbb{R}_+}|u\mathcal{L}'(u)|\leq e^{-1}$, we have for some positive and finite C that does not depend on u:

$$
\begin{aligned}
&\left|(|u\widehat{L}'_\kappa(u)|\wedge e^{-0.5})n|\Delta^n_{t,\kappa}X||\Delta^n_{t,\kappa-\Delta_n}X|1_{\{\mathcal{A}^n_{t,\kappa}\}}/\widehat{f}_\kappa+u\mathcal{L}'(u)nV^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W|\right|\\
&\quad\leq CnV^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W||\widehat{f}_\kappa-f^n_\kappa|\\
&\quad+Cn||\Delta^n_{t,\kappa}X||\Delta^n_{t,\kappa-\Delta_n}X|1_{\{\mathcal{A}^n_{t,\kappa}\}}-f^n_\kappa V^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W||\\
&\quad+CnV^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W|||u\widehat{L}'_\kappa(u)|\wedge e^{-0.5}+u\mathcal{L}'(u)|.
\end{aligned}
\tag{A.92}
$$

Therefore, with C as above, we have

$$
|\widehat{d}_{t,n}(u)-d_{t,n}(u)|\leq C(|u|\vee 1)(\zeta^{(1)}_{t,\kappa}\zeta_\kappa(u)+\zeta^{(1)}_{t,\kappa'}\zeta_{\kappa'}(u)+\zeta^{(2)}_t),\ \ |d_{t,n}(u)|\leq C(\zeta^{(1)}_{t,\kappa}+\zeta^{(1)}_{t,\kappa'}),\tag{A.93}
$$

where we denote

$$
\begin{aligned}
\zeta^{(1)}_{t,\kappa}&=nV^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W|+\sqrt{n}\sqrt{V^n_{t,\kappa}}|\Delta^n_{t,\kappa}W|,\\
\zeta^{(2)}_t&=n||\Delta^n_{t,\kappa}X||\Delta^n_{t,\kappa-\Delta_n}X|1_{\{\mathcal{A}^n_{t,\kappa}\}}-f^n_\kappa V^n_{t,\kappa}|\Delta^n_{t,\kappa}W||\Delta^n_{t,\kappa-\Delta_n}W||,\\
\zeta_\kappa(u)&=|\widehat{f}_\kappa-f^n_\kappa|+||u\widehat{L}'_\kappa(u)|\wedge e^{-0.5}+u\mathcal{L}'(u)|,
\end{aligned}
$$

and we note that by application of Lemmas 1 and 2 and for $q\geq 8$ (q is the constant of Lemma 1), we have

$$
||\zeta_\kappa(u)||=O_p\left(\frac{1}{\sqrt{T}}\bigvee\sqrt{\Delta_n}\bigvee\Delta_n^{(q-4)\left(\frac{1}{2}-\varpi\right)}\right).\tag{A.94}
$$

We can bound

$$\sum_{j=0}^{B_T} |\widehat{\gamma}_j^n(u,s) - \gamma_j^n(u,s)| \leq \frac{1}{T-B_T} \sum_{t=1}^{T} |\widehat{d}_{t,n}(u) - d_{t,n}(u)| \sum_{j=-B_T}^{B_T} |d_{t-j,n}(s)|$$
$$+ \frac{1}{T-B_T} \sum_{t=1}^{T} |d_{t,n}(u)| \sum_{j=-B_T}^{B_T} |\widehat{d}_{t-j,n}(s) - d_{t-j,n}(s)|$$
$$+ \frac{1}{T-B_T} \sum_{t=1}^{T} |\widehat{d}_{t,n}(u) - d_{t,n}(u)| \sum_{j=-B_T}^{B_T} |\widehat{d}_{t-j,n}(s) - d_{t-j,n}(s)|. \tag{A.95}$$

Using inequality in means as well as the fact that $\mathbb{E}|V_t|^4 < \infty$, we have

$$\mathbb{E}\left(\frac{1}{T-B_T}\sum_{t=1}^{T} \zeta_{t,\alpha}^{(1)}\zeta_{t-j,\beta}^{(1)}\right)^2 \leq C, \quad j = -B_T, ..., 0, ..., B_T, \tag{A.96}$$

where $\alpha, \beta = \kappa, \kappa'$. To proceed further, we bound the k-th moments of $\zeta_t^{(2)}$. Using successive conditioning, Hölder's inequality as well as the fact that $E|b_t|^4 < \infty$, we have

$$\begin{aligned} &n^k \mathbb{E}[|\Delta_{t,\kappa}^n X||\Delta_{t,\kappa-\Delta_n} X| 1_{\{\mathcal{A}_{t,\kappa}^n\}} 1_{\{\Delta_{t,\kappa}^n X^d \neq 0,\ \Delta_{t,\kappa-\Delta_n}^n X^d \neq 0\}}]^k \\ &\quad \leq \Delta_n^{k(2\varpi-1)} \mathbb{P}\left(\Delta_{t,\kappa}^n X^d \neq 0 \text{ and } \Delta_{t,\kappa-\Delta_n}^n X^d \neq 0\right) \leq C\Delta_n^{1+\frac{3}{4}+k(2\varpi-1)}. \end{aligned} \tag{A.97}$$

Applying successive conditioning, the smoothness in expectation condition for σ_t, Hölder's inequality as well as the integrability conditions for a_t, b_t and σ_t, we have

$$n^k \mathbb{E}[|\Delta_{t,\kappa}^n X||\Delta_{t,\kappa-\Delta_n} X| 1_{\{\mathcal{A}_{t,\kappa}^n\}} 1_{\{\Delta_{t,\kappa}^n X^d = 0,\ \Delta_{t,\kappa-\Delta_n}^n X^d \neq 0\}}]^k \leq C\Delta_n^{1+\frac{k}{2}(2\varpi-1)}, \ \ k \in [1,2], \tag{A.98}$$

$$n^k \mathbb{E}[|\Delta_{t,\kappa}^n X||\Delta_{t,\kappa-\Delta_n} X| 1_{\{\mathcal{A}_{t,\kappa}^n\}} 1_{\{\Delta_{t,\kappa}^n X^d \neq 0,\ \Delta_{t,\kappa-\Delta_n}^n X^d = 0\}}]^k \leq C\Delta_n^{1+\frac{k}{2}(2\varpi-1)}, \ \ k \in [1,2]. \tag{A.99}$$

Using these bounds, Hölder's inequality, the smoothness in expectation condition for σ_t as well as the integrability conditions for a_t and σ_t, we have

$$\mathbb{E}|\zeta_t^{(2)}|^k \leq C\big[\Delta_n^{\frac{7}{4}+k(2\varpi-1)} \vee \Delta_n^{1+\frac{k}{2}(2\varpi-1)} \vee \Delta_n^{(q-2k)\left(\frac{1}{2}-\varpi\right)} \vee \Delta_n^{\frac{k}{2} \wedge \frac{q-k}{q}}\big], \quad k \in [1,2], \tag{A.100}$$

where q is the constant of Lemma 1. Using $\mathbb{E}|V_t|^8 < \infty$ and applying Hölder's inequality, we have

$$\mathbb{E}|\zeta_t^{(2)}\zeta_{s,\iota}^{(1)}| \leq C\sqrt{\Delta_n}, \quad \mathbb{E}|\zeta_t^{(2)}\zeta_s^{(2)}| \leq C\sqrt{\Delta_n}, \quad \iota = \kappa, \kappa', \quad \forall s, t \geq 0, \tag{A.101}$$

provided $(q-4)\left(\frac{1}{2} - \varpi\right) \geq \frac{1}{2}$. Therefore,

$$\mathbb{E}\left(\frac{1}{T-B_T}\sum_{t=1}^{T}\left[\zeta_t^{(2)}\sum_{j=-B_T}^{B_T}\zeta_{t-j,\iota}^{(1)} + \zeta_{t,\iota}^{(1)}\sum_{j=-B_T}^{B_T}\zeta_{t-j}^{(2)}\right]\right) \leq CB_T\sqrt{\Delta_n}, \tag{A.102}$$

$$\mathbb{E}\left(\frac{1}{T-B_T}\sum_{t=1}^{T}\left[\zeta_t^{(2)}\sum_{j=-B_T}^{B_T}\zeta_{t-j}^{(2)}\right]\right) \leq CB_T\sqrt{\Delta_n}. \tag{A.103}$$

where $\iota = \kappa, \kappa'$. Combining these results, we get altogether

$$\left|\left|\sum_{j=0}^{B_T}\frac{T-j}{T}h\left(\frac{j}{B_T}\right)\left(\widehat{\gamma}_j^n(u,s) - \gamma_j^n(u,s)\right)\right|\right|^2 = O_p(B_T^2\Delta_n). \tag{A.104}$$

Combining the bounds in (A.87)-(A.88), (A.91) and (A.104), we have altogether the result of the lemma. □

We can decompose,

$$Z(\kappa,\kappa') - \widehat{Z}_T(\kappa,\kappa') = \sum_{i=p_T+1}^{\infty}\lambda_i\chi_i^2 + \sum_{i=1}^{p_T}\left(\lambda_i - \widehat{\lambda}_{i,T}\right)\chi_i^2. \tag{A.105}$$

By assumption we have that $p_T \to \infty$ as $T \to \infty$. Hence, Parseval's identity implies that $\sum_{i=p_T+1}^{\infty}\lambda_i\chi_i^2 = o_p(1)$. Furthermore, by Theorem 4.4 in [1] it follows that

$$\sup_{j\geq 1}|\widehat{\lambda}_{j,T} - \lambda_j| \leq ||K_T - K||_{HS}. \tag{A.106}$$

Therefore, we have

$$\begin{aligned}|Z(\kappa,\kappa') - \widehat{Z}_T(\kappa,\kappa')| &\leq \sum_{i=1}^{p_T}\left|\lambda_i - \widehat{\lambda}_{i,T}\right|\chi_i^2 + o_p(1) \\ &\leq \sup_{j\geq 1}|\widehat{\lambda}_{j,T} - \lambda_j|\sum_{i=1}^{p_T}\chi_i^2 + o_p(1) = o_p(1),\end{aligned} \tag{A.107}$$

where for the last bound, we made use of the result of Lemma 5 and the rate condition for p_T in the theorem.

A.6 Proof of Corollary 2

The result under the null hypothesis follows from Corollary 1 and Theorem 3 and application of portmanteau theorem.

Under the alternative hypothesis, one can easily show using the integrability conditions of the theorem and using some of the bounds in the proof of Lemma 5 that we have $|\widehat{Z}_T(\kappa,\kappa')| = O_p(B_T)$. Furthermore, from the proof of Theorem 1, under the conditions of the theorem, we have $T||\widehat{L}_\kappa - \widehat{L}'_\kappa||^2 = O_p(T)$. These two results yield the asymptotic power of one by taking into account that $B_T/T \to 0$.

A.7 Proof of Theorem 4

We start with introducing some auxiliary notation and establishing some preliminary results. We introduce the set

$$\Upsilon_n = \left\{\omega : ||\widehat{\theta} - \theta_0|| \leq C\alpha_n/n\right\}, \tag{A.108}$$

for some finite constant $C > 0$ and some deterministic sequence $\alpha_n \to \infty$ when $n \to \infty$ such that $\alpha_n/\log n \to 0$. Then, since $\widehat{\theta} - \theta_0 = O_p(1/n)$, we have $\mathbb{P}(\Upsilon_n) \to 1$.

We next denote

$$\begin{aligned}\widehat{\epsilon}^n_{t,\kappa} = g\left(Z_{\frac{(t-1)n+\lfloor\kappa n\rfloor}{n}}, \theta_0\right) - g\left(Z_{\frac{(t-1)n+\lfloor\kappa n\rfloor-1}{n}}, \theta_0\right) \\ - g\left(Z_{\frac{(t-1)n+\lfloor\kappa n\rfloor}{n}}, \widehat{\theta}\right) + g\left(Z_{\frac{(t-1)n+\lfloor\kappa n\rfloor-1}{n}}, \widehat{\theta}\right),\end{aligned} \tag{A.109}$$

and with this notation we can write

$$\Delta^n_{t,\kappa}\widehat{X} = \Delta^n_{t,\kappa}X + \widehat{\epsilon}^n_{t,\kappa}. \tag{A.110}$$

We finally set

$$\widehat{f}_{\kappa}^{rc} = \frac{n}{T}\frac{\pi}{2}\sum_{t=1}^{T}|\Delta_{t,\kappa}^{n}X||\Delta_{t,\kappa-\Delta}X|1_{\{\mathcal{A}_{t,\kappa}^{r,n}\}}. \tag{A.111}$$

Using the definition of the set Υ_n as well as the Lipschitz condition for the function g, we have

$$|\widehat{f}_{\kappa}^{r} - \widehat{f}_{\kappa}^{rc}|1_{\{\Upsilon_n\}} \leq C||\widehat{\theta}-\theta_0||\frac{n}{T}\sum_{t=1}^{T}|\Delta_{t,\kappa}^{n}X||\Delta_{t,\kappa-\Delta}^{n}X| + C\left(\frac{\alpha_n}{\sqrt{n}}\right)^2. \tag{A.112}$$

Next, due to $\alpha_n/\Delta_n^{\varpi-1}\to 0$, we have $|\widehat{\epsilon}_{t,\kappa}^{n}| \leq v_n$ for n sufficiently large. Therefore, for n sufficiently large, the following bounds holds

$$|\widehat{f}_{\kappa}^{rc} - \breve{f}_{\kappa}|1_{\{\Upsilon_n\}} \leq |\widehat{f}_{\kappa}^{(1)} - \breve{f}_{\kappa}| + |\widehat{f}_{\kappa}^{(2)} - \breve{f}_{\kappa}|, \tag{A.113}$$

where $\widehat{f}_{\kappa}^{(1)}$ and $\widehat{f}_{\kappa}^{(2)}$ are defined as $\widehat{f}_{\kappa}$ but with v_n in the set $\mathcal{A}_{t,\kappa}^{n}$ replaced by $v_n/2$ and $2v_n$, respectively.

Part (a). Similar to the proof of Theorem 1, we decompose

$$\cos\left(\sqrt{2un}\Delta_{t,\kappa}^{n}\widehat{X}/\sqrt{\widehat{f}_{\kappa}^{r}}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^{n}V_{t,\kappa}^{n}}\Delta_{t,\kappa}^{n}W/\sqrt{\overline{f}_{\kappa}}\right) = \sum_{j=1}^{3}\widehat{\chi}_{t,n}^{(j)}(u,\kappa), \tag{A.114}$$

where we denote

$$\widehat{\chi}_{t,n}^{(1)}(u,\kappa) = \cos\left(\sqrt{2un}\Delta_{t,\kappa}^{n}\widehat{X}/\sqrt{\widehat{f}_{\kappa}^{r}}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^{n}V_{t,\kappa}^{n}}\Delta_{t,\kappa}^{n}W/\sqrt{\widehat{f}_{\kappa}^{r}}\right),$$

$$\widehat{\chi}_{t,n}^{(2)}(u,\kappa) = \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^{n}V_{t,\kappa}^{n}}\Delta_{t,\kappa}^{n}W/\sqrt{\widehat{f}_{\kappa}^{r}}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^{n}V_{t,\kappa}^{n}}\Delta_{t,\kappa}^{n}W/\sqrt{\widetilde{f}_{\kappa}}\right),$$

$$\widehat{\chi}_{t,n}^{(3)}(u,\kappa) = \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^{n}V_{t,\kappa}^{n}}\Delta_{t,\kappa}^{n}W/\sqrt{\widetilde{f}_{\kappa}}\right) - \cos\left(\sqrt{2un}\sqrt{f_{t,\kappa}^{n}V_{t,\kappa}^{n}}\Delta_{t,\kappa}^{n}W/\sqrt{\overline{f}_{\kappa}}\right).$$

We start with $\widehat{\chi}_{t,n}^{(1)}(u,\kappa)$. Using the algebraic inequality $|\cos(x)-\cos(y)|\leq |x-y|\wedge 2$ for $x,\, y\in\mathbb{R}$, we have

$$
\begin{aligned}
|\widehat{\chi}_{t,n}^{(1)}(u,\kappa)| \leq\, & 2 1_{\{\Delta_{t,\kappa}^n X^d\neq 0\, \cup\, \widehat{f}_\kappa^r<\epsilon\, \cup\, \Upsilon_n^c\}} + C\sqrt{un}|\Delta_{t,\kappa}^n X^c - \sqrt{f_{t,\kappa}^n V_{t,\kappa}^n}\Delta_{t,\kappa}^n W| \\
& + C\sqrt{un}|\widehat{\epsilon}_{t,\kappa}^n|,
\end{aligned}
\tag{A.115}
$$

where $0<\epsilon<\inf_{\kappa\in[0,1]} f_\kappa/4$, and ϵ and C do not depend on u and n. Form here, by performing similar steps as for the analysis of the term $\chi_{t,n}^{(1)}(u,\kappa)$ in the proof of Theorem 1 as well as by making use of Lemma 1 and the bounds in (A.112) and (A.113) and the fact that $\alpha_n/\log n\to 0$, we have

$$
\frac{1}{T}\left|\left|\sum_{t=1}^T \widehat{\chi}_{t,n}^{(1)}(u,\kappa)\right|\right| = O_p\left(\sqrt{\Delta_n}\bigvee\frac{1}{T}\right). \tag{A.116}
$$

We continue next with $\widehat{\chi}_{t,n}^{(2)}(u,\kappa)$. For n sufficiently large, using the algebraic inequality $|\cos(x)-\cos(y)|\leq|x-y|\wedge 2$ for $x,\, y\in\mathbb{R}$ and Taylor expansion as well as (A.113), we have

$$
\begin{aligned}
&|\widehat{\chi}_{t,n}^{(2)}(u,\kappa)| \leq 2 1_{\{\widehat{f}_\kappa^r<\epsilon\, \cup\, \widetilde{f}_\kappa<\epsilon\, \cup\, \Upsilon_n^c\}} \\
&\quad + C 1_{\{\Upsilon_n\}}\sqrt{un}\sqrt{V_{t,\kappa}^n}|\Delta_{t,\kappa}^n W|\left[|\widehat{f}_\kappa^r-\widehat{f}_\kappa^{rc}| + |\widehat{f}_\kappa^{(1)}-\breve{f}_\kappa| + |\widehat{f}_\kappa^{(2)}-\breve{f}_\kappa| + |\breve{f}_\kappa-\widetilde{f}_\kappa|\right].
\end{aligned}
\tag{A.117}
$$

Using the integrability conditions of Assumption 1, inequality in means as well as Cauchy-Schwartz inequality, we have

$$
\mathbb{E}\left|\frac{n}{T}\sum_{t=1}^T |\Delta_{t,\kappa}^n X||\Delta_{t,\kappa-\Delta}^n X|\right|^p \leq C\Delta_n^{1-p},\quad p\in[1,2]. \tag{A.118}
$$

From here, using the same steps as in the analysis of the term $\chi_{t,n}^{(2)}(u,\kappa)$ in the proof of Theorem 1 as well as the bound for $\widehat{f}_\kappa^r-\widehat{f}_\kappa^{rc}$ in (A.112) and upon applying Hölder inequality and taking into account the integrability conditions in Assumption 1 and that $\alpha_n/\log n\to$

0, we have altogether

$$\frac{1}{T}\left|\left|\sum_{t=1}^{T}\widehat{\chi}_{t,n}^{(2)}(u,\kappa)\right|\right| = O_p\left(\sqrt{\Delta_n}\bigvee\Delta_n^{(q-2)\left(\frac{1}{2}-\varpi\right)}\bigvee\frac{\Delta_n^{(1-2\varpi)\wedge\varpi\wedge\frac{1}{4}}}{\sqrt{T}}\bigvee\frac{1}{T}\right), \tag{A.119}$$

for q being the constant in Lemma 1. Next, the term $\widehat{\chi}_{t,n}^{(3)}(u,\kappa)$ equals the term $\chi_{t,n}^{(3)}(u,\kappa)$ in the proof of Theorem 1, and hence using this proof we have

$$\frac{1}{T}\left|\left|\sum_{t=1}^{T}\widehat{\chi}_{t,n}^{(3)}(u,\kappa)\right|\right| = O_p\left(\frac{1}{\sqrt{T}}\right). \tag{A.120}$$

From here, the proof of the result in part(a) of the theorem follows from the proof of Theorem 1.

Part (b). From the above bounds for $\widehat{\chi}_{t,n}^{(1)}(u,\kappa)$ and $\widehat{\chi}_{t,n}^{(2)}(u,\kappa)$, we have

$$\frac{1}{T}\left|\left|\sum_{t=1}^{T}\widehat{\chi}_{t,n}^{(1)}(u,\kappa)\right|\right| + \frac{1}{T}\left|\left|\sum_{t=1}^{T}\widehat{\chi}_{t,n}^{(2)}(u,\kappa)\right|\right| = o_p(1/\sqrt{T}), \tag{A.121}$$

provided $T\Delta_n \to 0$ and $\varpi < \frac{q-3}{2q-4}$. Under this same condition, we also have $\widehat{L}^{r,n} - \mathcal{L}_\kappa = O_p(1/\sqrt{T})$. From here the proof of the result in part (b) of the theorem follows from the proof of Theorem 2 upon using the same decomposition of $\cos\left(\sqrt{2un}\Delta_{t,\kappa}^n\widehat{X}/\sqrt{\widehat{f}_\kappa^r}\right) - \cos\left(\sqrt{2un}\sqrt{V_{t,\kappa}^n}\Delta_{t,\kappa}^n W\right)$ as in that proof, with $\chi_{t,n}^{(1)}(u,\kappa)$ and $\chi_{t,n}^{(2)}(u,\kappa)$ replaced with $\widehat{\chi}_{t,n}^{(1)}(u,\kappa)$ and $\widehat{\chi}_{t,n}^{(2)}(u,\kappa)$ defined above.

Appendix B: Additional Monte Carlo Results

B.1 Sensitivity to the Choice of u_{max}

In Table 1 and Table 2 we provide Monte Carlo evidence regarding the sensitivity of the testing procedure to the different choices of u_{max} under the null and alternative hypotheses, respectively. Recall from Section 4 in the main text that we calculate the test statistic as $\int_0^{u_{max}} \left(\widehat{L}^n_\kappa(u) - \widehat{L}^n_{\kappa'}(u)\right)^2 w(u)du$.

The middle panel of Table 1 and Table 2 corresponds to the original choice of u_{max} reported in Table 1 in the main text. As seen from the tables, there is very little sensitivity of the performance of the test to the choice of u_{max}.

B.2 Sensitivity to the Choice of B_T

In Table 3 and Table 4 we provide Monte Carlo evidence regarding the sensitivity of the testing procedure to the different choices of B_T under the null and alternative hypotheses, respectively. Recall from the main text that B_T is the cutoff parameter determining the number of lags used in the computation of the critical values of the test statistic. The middle panel of Table 3 and Table 4 corresponds to the original choice of B_T. As seen from the reported results, the performance of the test under the null hypothesis is not very sensitive to the choice of B_T. This is consistent with the fact that $d_t(u)$ defined in Theorem 2 has little time series persistence. Similarly, the power of the test does not change much across the different values of B_T.

	p_T					
T	1	2	3	4	5	6
			u_{max}: $\frac{1}{n}\sum_{i=1}^{n}\widehat{L}_{i\Delta_n}(u_{max}) = 0.025$			
250	0.064	0.055	0.051	0.050	0.049	0.049
500	0.063	0.049	0.047	0.047	0.047	0.047
1000	0.069	0.053	0.052	0.052	0.051	0.051
1500	0.069	0.056	0.052	0.051	0.051	0.051
2000	0.065	0.049	0.046	0.045	0.045	0.044
2500	0.056	0.044	0.042	0.042	0.042	0.042
			u_{max}: $\frac{1}{n}\sum_{i=1}^{n}\widehat{L}_{i\Delta_n}(u_{max}) = 0.05$			
250	0.054	0.046	0.045	0.044	0.044	0.044
500	0.067	0.057	0.055	0.055	0.055	0.055
1000	0.055	0.050	0.049	0.048	0.047	0.047
1500	0.059	0.053	0.048	0.047	0.047	0.047
2000	0.055	0.048	0.047	0.047	0.047	0.047
2500	0.069	0.062	0.061	0.060	0.060	0.060
			u_{max}: $\frac{1}{n}\sum_{i=1}^{n}\widehat{L}_{i\Delta_n}(u_{max}) = 0.1$			
250	0.065	0.055	0.054	0.052	0.052	0.052
500	0.045	0.038	0.036	0.036	0.036	0.036
1000	0.063	0.057	0.055	0.053	0.053	0.053
1500	0.064	0.060	0.058	0.058	0.058	0.058
2000	0.065	0.055	0.054	0.053	0.053	0.053
2500	0.061	0.055	0.053	0.052	0.052	0.052

Table 1: Monte Carlo Results under the Null Hypothesis, $f_{t,\kappa} \equiv f_\kappa$, for different choices of u_{max}. The table reports empirical rejection rates of the test of nominal size 0.05 using $1,000$ simulations. $\mathcal{K}_n$ and $\mathcal{K}'_n$ correspond to 8:40-9:10 and 12:30-13:00, respectively. The values of v_n and B_T are set as in Section 4 in the main text.

	8:40 - 9:10 vs 12:30- 13:00						8:40 - 9:10 vs 14:30 - 15:00					
	p_T						p_T					
T	1	2	3	4	5	6	1	2	3	4	5	6
u_{max}: $\frac{1}{n}\sum_{i=1}^{n}\widehat{L}_{i\Delta_n}(u_{max}) = 0.025$												
250	0.081	0.058	0.052	0.046	0.045	0.045	0.319	0.270	0.250	0.238	0.233	0.232
500	0.126	0.095	0.077	0.070	0.065	0.063	0.598	0.566	0.540	0.528	0.520	0.519
1000	0.214	0.154	0.130	0.116	0.112	0.109	0.930	0.918	0.907	0.905	0.900	0.900
1500	0.221	0.178	0.160	0.148	0.144	0.144	0.989	0.985	0.984	0.983	0.982	0.982
2000	0.340	0.265	0.238	0.220	0.213	0.209	0.998	0.997	0.997	0.997	0.997	0.997
2500	0.406	0.312	0.284	0.270	0.263	0.258	1.000	1.000	1.000	1.000	1.000	1.000
u_{max}: $\frac{1}{n}\sum_{i=1}^{n}\widehat{L}_{i\Delta_n}(u_{max}) = 0.05$												
250	0.093	0.075	0.064	0.062	0.061	0.061	0.322	0.292	0.278	0.269	0.267	0.266
500	0.139	0.100	0.086	0.076	0.075	0.074	0.623	0.588	0.571	0.562	0.555	0.554
1000	0.168	0.126	0.106	0.099	0.096	0.094	0.918	0.903	0.901	0.893	0.890	0.887
1500	0.229	0.173	0.149	0.140	0.135	0.133	0.993	0.990	0.989	0.989	0.989	0.988
2000	0.339	0.285	0.266	0.251	0.240	0.232	1.000	0.998	0.998	0.998	0.998	0.998
2500	0.410	0.347	0.308	0.292	0.286	0.279	0.999	0.999	0.999	0.999	0.999	0.999
u_{max}: $\frac{1}{n}\sum_{i=1}^{n}\widehat{L}_{i\Delta_n}(u_{max}) = 0.1$												
250	0.102	0.083	0.075	0.071	0.069	0.067	0.359	0.328	0.318	0.314	0.313	0.310
500	0.141	0.127	0.115	0.111	0.111	0.110	0.660	0.643	0.634	0.626	0.622	0.622
1000	0.214	0.182	0.171	0.171	0.167	0.165	0.951	0.939	0.935	0.934	0.934	0.934
1500	0.288	0.245	0.233	0.230	0.227	0.227	0.993	0.993	0.992	0.992	0.992	0.992
2000	0.418	0.364	0.351	0.342	0.341	0.337	0.997	0.997	0.997	0.997	0.997	0.997
2500	0.447	0.407	0.397	0.390	0.382	0.379	1.000	1.000	1.000	1.000	1.000	1.000

Table 2: Monte Carlo Results under the Alternative Hypothesis, $f_{t,\kappa} \neq f_\kappa$, for different choices of u_{max}. The table reports empirical rejection rates for the test at nominal size 0.05 using 1000 simulations.

T	p_T					
	1	2	3	4	5	6
			$B_T = \lfloor T^{1/5} \rfloor /2$			
250	0.061	0.055	0.053	0.051	0.050	0.049
500	0.060	0.053	0.050	0.050	0.050	0.050
1000	0.065	0.059	0.057	0.057	0.056	0.055
1500	0.055	0.049	0.046	0.046	0.045	0.045
2000	0.059	0.048	0.046	0.044	0.044	0.042
2500	0.069	0.057	0.054	0.054	0.054	0.054
			$B_T = \lfloor T^{1/5} \rfloor$			
250	0.054	0.046	0.045	0.044	0.044	0.044
500	0.067	0.057	0.055	0.055	0.055	0.055
1000	0.055	0.050	0.049	0.048	0.047	0.047
1500	0.059	0.053	0.048	0.047	0.047	0.047
2000	0.055	0.048	0.047	0.047	0.047	0.047
2500	0.069	0.062	0.061	0.060	0.060	0.060
			$B_T = 2\lfloor T^{1/5} \rfloor$			
250	0.057	0.051	0.049	0.048	0.047	0.047
500	0.059	0.046	0.044	0.044	0.044	0.044
1000	0.066	0.055	0.054	0.054	0.054	0.053
1500	0.053	0.046	0.045	0.045	0.045	0.045
2000	0.056	0.048	0.048	0.046	0.046	0.045
2500	0.068	0.057	0.055	0.055	0.054	0.054

Table 3: Monte Carlo Results under the Null Hypothesis, $f_{t,\kappa} \equiv f_\kappa$, for different choices of B_T. The table reports empirical rejection rates of the test of nominal size 0.05 using $1,000$ simulations. $\mathcal{K}_n$ and $\mathcal{K}_n'$ correspond to 8:40-9:10 and 12:30-13:00, respectively. The values of u_{max} and v_n are set as in Section 4 in the main text.

	8:40 - 9:10 vs 12:30- 13:00						8:40 - 9:10 vs 14:30 - 15:00					
	p_T						p_T					
T	1	2	3	4	5	6	1	2	3	4	5	6
						$B_T = \lfloor T^{1/5} \rfloor /2$						
250	0.080	0.066	0.059	0.057	0.056	0.054	0.348	0.319	0.310	0.297	0.290	0.285
500	0.124	0.095	0.076	0.071	0.067	0.067	0.690	0.663	0.647	0.635	0.627	0.623
1000	0.188	0.142	0.126	0.120	0.113	0.113	0.929	0.919	0.910	0.907	0.903	0.902
1500	0.242	0.195	0.173	0.167	0.163	0.161	0.983	0.982	0.979	0.978	0.977	0.977
2000	0.331	0.260	0.234	0.221	0.217	0.216	0.998	0.998	0.997	0.997	0.997	0.997
2500	0.337	0.283	0.268	0.258	0.247	0.243	0.999	0.998	0.998	0.998	0.998	0.998
						$B_T = \lfloor T^{1/5} \rfloor$						
250	0.093	0.075	0.064	0.062	0.061	0.061	0.322	0.292	0.278	0.269	0.267	0.266
500	0.139	0.100	0.086	0.076	0.075	0.074	0.623	0.588	0.571	0.562	0.555	0.554
1000	0.168	0.126	0.106	0.099	0.096	0.094	0.918	0.903	0.901	0.893	0.890	0.887
1500	0.229	0.173	0.149	0.140	0.135	0.133	0.993	0.990	0.989	0.989	0.989	0.988
2000	0.339	0.285	0.266	0.251	0.240	0.232	1.000	0.998	0.998	0.998	0.998	0.998
2500	0.410	0.347	0.308	0.292	0.286	0.279	0.999	0.999	0.999	0.999	0.999	0.999
						$B_T = 2\lfloor T^{1/5} \rfloor$						
250	0.080	0.053	0.047	0.046	0.044	0.043	0.311	0.264	0.238	0.231	0.229	0.226
500	0.119	0.076	0.064	0.060	0.059	0.059	0.668	0.631	0.610	0.601	0.595	0.593
1000	0.199	0.142	0.121	0.115	0.110	0.105	0.925	0.909	0.902	0.897	0.896	0.894
1500	0.242	0.195	0.170	0.161	0.157	0.153	0.982	0.982	0.979	0.978	0.978	0.977
2000	0.340	0.260	0.226	0.214	0.208	0.207	0.998	0.998	0.997	0.997	0.997	0.997
2500	0.351	0.293	0.258	0.248	0.242	0.239	0.999	0.998	0.998	0.998	0.998	0.998

Table 4: Monte Carlo Results under the Alternative Hypothesis, $f_{t,\kappa} \neq f_\kappa$, for different choices of B_T. The table reports empirical rejection rates for the test at nominal size 0.05 using 1000 simulations.

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