SUPPLEMENT TO "REALIZED LAPLACE TRANSFORMS FOR PURE-JUMP SEMIMARTINGALES"

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This supplement contains some technical results and proofs of theorems in the paper.

As in Section 7 of the paper in what follows we will denote with C a constant that does not depend on T and Δ_n, and further it might change from line to line. We also use the short hand $\mathbb{E}_{i-1}^n$ for $\mathbb{E}\left(\cdot|\mathscr{F}_{(i-1)\Delta_n}\right)$.

1. Proof of preliminary results of Section 7.1 in the paper. Here we derive all the results of Section 7.1 in the paper.

1.1. *Representation of X_t.* We start with showing that X_t can be represented on an extension of the original probability space as

$$
\begin{aligned}
X_t = X_0 + \int_0^t \overline{\alpha}_s ds + \int_0^t \int_{\mathbb{R}} \sigma_{s-} x \tilde{\mu}_1(ds, dx) \\
+ \int_0^t \int_{\mathbb{R}} \sigma_{s-} x \mu_2(ds, dx) - \int_0^t \int_{\mathbb{R}} \sigma_{s-} x \mu_3(ds, dx) + Y_t,
\end{aligned}
\tag{A.1}
$$

where μ_1, μ_2 and μ_3 are homogenous Poisson measures (the three measures are not mutually independent) with compensators respectively $\nu_1(dx) = \frac{A}{|x|^{\beta+1}}dx$, $\nu_2(dx) = |\nu'(x)|dx$ and $\nu_3(dx) = 2|\nu'(x)|1\left(\nu'(x) < 0\right)dx$ and $\overline{\alpha}_s = \alpha_s - \sigma_{s-}\int_{\mathbb{R}} x\nu'(x)dx$.

The above extension of the original probability space can be done in the following way. First, we consider a very good product extension of the original probability space by introducing an auxiliary space supporting the homogenous Poisson measure μ_3, with compensator $dt \otimes \nu_3(x)dx$ where $\nu_3(x) = 2|\nu'(x)|1(\nu'(x) < 0)dx$, which is independent from the filtration of the original one. We then make a (very good) product extension of this space by using an auxiliary space endowed with a sequence of i.i.d. random variables $(U_p)_{p\geq 1}$ with uniform distribution on $[0, 1]$ with product filtration

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$(\widetilde{\mathcal{F}}_t)$ being the smallest such that U_p is $\widetilde{\mathcal{F}}_{T_p}$-measurable for every p where $(T_p)_{p\geq 1}$ is an exhausting sequence for the jump times of μ and μ_3 with corresponding sequence for the jump sizes at the times $(T_p)_{p\geq 1}$ denoted by $(S_p)_{p\geq 1}$ (note that μ and μ_3 are independent Poisson measures hence their associated Poisson processes almost surely never jump together). On this extended space we define the random measure $\underline{\mu}$ on $\mathbb{R}_+ \otimes \mathbb{R} \otimes [0,1]$ via

$$\underline{\mu}(dt, dx, du) = \sum_p 1_{\{S_p \neq 0\}} \epsilon_{\{T_p, S_p, U_p\}}(dt, dx, du),$$

for $\epsilon_x(a)$ denoting Dirac delta measure at point a. $\underline{\mu}$ is a homogenous Poisson measure (Definition II.1.20 in Jacod and Shiryaev (2003)) with compensator $dt \otimes (\frac{A}{|x|^{1+\beta}} + |\nu'(x)|)dx \otimes 1_{\{u\in[0,1]\}}du$ (Recall $(U_p)_{p\geq 1}$ are uniformly distributed independent of each other and of the σ-field $\mathcal{F}$, and further $\nu(x) + \nu_3(x) = \frac{A}{|x|^{1+\beta}} + |\nu'(x)|$). We further have for every $A \times B$ in the Borel sigma field on $\mathbb{R}_+ \otimes \mathbb{R}$

$$\underline{\mu}(A \times B \times [0,1]) = \mu(A \times B) + \mu_3(A \times B).$$

Note, $\underline{\mu}$ is randomization of the Poisson measure $\mu + \mu_3$ by the uniform distribution in the terminology of [2], Proposition 10.5. We then define μ_1 and μ_2 via thinning of $\underline{\mu}$. In particular, for each $A \times B$ in the Borel sigma field on $\mathbb{R}_+ \otimes \mathbb{R}$ we set

$$\mu_1(A \times B) = \int_A \int_B \int_{[0,1]} 1_{\{u \leq \nu_1(x)/(\nu_1(x)+\nu_2(x))\}} \underline{\mu}(ds, dx, du),$$

$$\mu_2(A \times B) = \int_A \int_B \int_{[0,1]} 1_{\{u > \nu_1(x)/(\nu_1(x)+\nu_2(x))\}} \underline{\mu}(ds, dx, du),$$

where $\nu_1(x) = \frac{A}{|x|^{\beta+1}}$ and $\nu_2(x) = |\nu'(x)|$. It is easy to see that μ_1 and μ_2 are homogenous Poisson measures with compensators $dt \otimes \nu_1(x)dx$ and $dt \otimes \nu_2(x)dx$ and further $\mu_1(A \times B) + \mu_2(A \times B) = \underline{\mu}(A \times B \times [0,1])$.

Therefore, we can write

$$Z_t = Z_t^{(1)} + Z_t^{(2)} - Z_t^{(3)}, \tag{A.2}$$

where

$$Z_t^{(1)} = \int_0^t \int_{\mathbb{R}} x\widetilde{\mu}_1(ds, dx), \;\; Z_t^{(2)} = \int_0^t \int_{\mathbb{R}} x\widetilde{\mu}_2(ds, dx), \;\; Z_t^{(3)} = \int_0^t \int_{\mathbb{R}} x\widetilde{\mu}_3(ds, dx),$$

and from here the decomposition in (A.1) holds.

In what follows, for simplicity of notation we will assume that the original probability space has been already extended in the above way so that filtration and probability will continue to be denoted with $(\mathcal{F}_t)$ and $\mathbb{P}$ respectively.

1.2. *Bounds for* $\xi^{(2)}_{i,u}$. We continue with deriving the bounds for $\xi^{(2)}_{i,u}$ and its subcomponents. Since

$$E^n_{i-1}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}|\tilde{\sigma}_s|^2ds+\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}|\underline{\delta}(s,x)|^2\underline{\nu}(ds,dx)\right)<\infty,$$

we have $\mathbb{E}^n_{i-1}\left(\xi^{(2)}_{i,u}(1)\right)=0$. Using Itô isometry and square integrability we further have

$$\begin{aligned}\mathbb{E}^n_{i-1}\left(\xi^{(2)}_{i,u}(1)\right)^2 \leq & C\Delta_n\Upsilon^2(\sigma_{(i-1)\Delta_n-},u)\\ &\times\mathbb{E}^n_{i-1}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\int_{(i-1)\Delta_n}^{s}\tilde{\sigma}_u dW_u+\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}\underline{\delta}(u-,x)\underline{\tilde{\mu}}(du,dx)\right)^2ds\\ \leq & C\Delta_n\Upsilon^2(\sigma_{(i-1)\Delta_n-},u)\\ &\times\mathbb{E}^n_{i-1}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\int_{(i-1)\Delta_n}^{s}\tilde{\sigma}^2_u du+\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}\underline{\delta}^2(u-,x)\underline{\nu}(du,dx)\right)ds.\end{aligned}$$

Therefore, since $|\Upsilon(x,u)|\leq C$ where the positive constant C depends only on u, and further using the integrability conditions in assumption B, we have altogether

$$\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}^n_{i-1}\left(\xi^{(2)}_{i,u}(1)\right)=0,\qquad \frac{\Delta_n^{-2}}{T}\mathbb{E}\left(\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}^n_{i-1}\left(\xi^{(2)}_{i,u}(1)\right)^2\right)\leq C. \tag{A.3}$$

For $\xi^{(2)}_{i,u}(2)$, by using Cauchy-Schwarz inequality, Itô isometry and the integrability conditions of assumption B′, we can write

$$\begin{aligned}\mathbb{E}|\xi^{(2)}_{i,u}(2)| &\leq C\int_{(i-1)\Delta_n}^{i\Delta_n}\sqrt{\mathbb{E}(\Upsilon(\sigma^*_s,u)-\Upsilon(\sigma_{(i-1)\Delta_n-},u))^2}\sqrt{s-(i-1)\Delta_n}ds\\ &\leq C\Delta_n^{3/2}\sqrt{\mathbb{E}\left(\sup_{s\in[(i-1)\Delta_n,i\Delta_n]}(\Upsilon(\sigma^*_s,u)-\Upsilon(\sigma_{(i-1)\Delta_n-},u))^2\right)},\end{aligned}$$

where the constant C does not depend on u. To continue further we make use of the following algebraic inequality

$$\begin{aligned}|\Upsilon(x,u)-\Upsilon(y,u)| \leq & Cue^{-u|y|^\beta}|y|^{2(\beta-1)}|x-y|+Cue^{-u|y|^\beta}|y|^{\beta-1}|x-y|^\beta\\ &+Cu|x-y|^{\beta-1},\quad x,y\in\mathbb{R},\ u\geq 0,\ \beta>1,\end{aligned} \tag{A.4}$$

with the constant C independent of u. We can further simplify this inequality upon noticing $e^{-u|y|^\beta}u|y|^{2(\beta-1)}+e^{-u|y|^\beta}u|y|^{\beta-1}\leq C(u\vee 1)$ where the positive constant C again does not depend on u. The bound in (A.4) then simplifies to

$$|\Upsilon(x,u)-\Upsilon(y,u)| \leq C(u\vee 1)(|x-y|^{\beta}+|x-y|^{\beta-1}), \quad x,y\in\mathbb{R},\ u\geq 0,\ \beta>1.$$

Plugging in the above inequality $x=\sigma_s^*$ and $y=\sigma_{(i-1)\Delta_n-}$ and using successive conditioning (first on the filtration $\mathscr{F}_{(i-1)\Delta_n}$) together with the Burkholder-Davis-Gundy inequality and the integrability conditions of assumption B, we get

$$\sqrt{\mathbb{E}\left(\sup_{s\in[(i-1)\Delta_n,i\Delta_n]}(\Upsilon(\sigma_s^*,u)-\Upsilon(\sigma_{(i-1)\Delta_n-},u))^2\right)} \leq C\Delta_n^{\beta/2-1/2}.$$

Therefore for any finite $\overline{u}>0$

$$(T\Delta_n^{\beta/2})^{-1}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}|\xi_{i,u}^{(2)}(2)|\right)\leq C. \tag{A.5}$$

Finally, first-order Taylor expansion implies

$$\mathbb{E}_{i-1}^n|\xi_{i,u}^{(2)}(3)| \leq Cu\mathbb{E}_{i-1}^n\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\left||\sigma_s|^{\beta}-|\widehat{\sigma}_s|^{\beta}\right|ds\right),$$

and using the integrability conditions in assumption B, we can write for any finite $\overline{u}>0$

$$(T\Delta_n)^{-1}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}|\xi_{i,u}^{(2)}(3)|\right)\leq C. \tag{A.6}$$

1.3. *Bounds for $\xi_{i,u}^{(3)}$.* We finish this section with deriving the bounds for $\xi_{i,u}^{(3)}$ and its subcomponents.

Using the basic inequality $|\sin(x)|\leq |x|^p$ for any $0<p\leq 1$, we have

$$\begin{aligned}\mathbb{E}|\xi_{i,u}^{(3)}(1)| \leq Cu^{\beta'/\beta}\Delta_n^{1-\beta'/\beta}\mathbb{E}\Bigg(&\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}\sigma_{s-}x\mu_2(ds,dx)\right| \\ &+\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}\sigma_{s-}x\mu_3(ds,dx)\right|+|\Delta_i^nY|\Bigg)^{\beta'},\end{aligned} \tag{A.7}$$

where the constant C does not depend on u. Further, using the inequality $|\sum_i |a_i||^p \leq \sum_i |a|_i^p$ for any $0 < p \leq 1$, as well as the Burkholder-Davis-Gundy inequality and Hölder inequality, we have for $j = 2, 3$

$$\begin{aligned}
&\mathbb{E}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}\sigma_{s-}x\mu_j(ds,dx)\right|^{\beta'} \\
&\leq C\left|\Delta_n\int_{|x|<\Delta_n^{1/\beta'}}|x|\nu_j(dx)\right|^{\beta'} + C\mathbb{E}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|\geq\Delta_n^{1/\beta'}}|x|^{\beta'}|\sigma_{s-}|^{\beta'}\mu_j(ds,dx)\right) \\
&\leq C\left(\Delta_n\int_{|x|<\Delta_n^{1/\beta'}}|x\nu'(x)|dx\right)^{\beta'} + C\Delta_n\int_{|x|\geq\Delta_n^{1/\beta'}}|x|^{\beta'}|\nu'(x)|dx \leq C\Delta_n|\log\Delta_n|,
\end{aligned}$$

where we also made use of the assumption that $|\nu'(x)| \leq \frac{C}{|x|^{\beta'+1}}$ for $|x|$ sufficiently small. We can get the same bound for the last term in (A.7) using assumption A. Therefore, altogether for any finite $\overline{u} > 0$

$$(T|\log(\Delta_n)|\Delta_n^{1-\beta'/\beta})^{-1}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}|\xi_{i,u}^{(3)}(1)|\right) \leq C. \tag{A.8}$$

For $\xi_{i,u}^{(3)}(2)$, using the boundedness of the function $\sin(x)$ and the square integrability of $\overline{\alpha}_s$

$$\mathbb{E}_{i-1}^n\left(\xi_{i,u}^{(3)}(2)\right) = 0, \quad \frac{(\Delta_n^{3-2/\beta})^{-1}}{T}\mathbb{E}\left(\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}_{i-1}^n\left(\xi_{i,u}^{(3)}(2)|\right)^2\right) \leq C. \tag{A.9}$$

For $\xi_{i,u}^{(3)}(4)$ we first can use assumption B and apply Jensen's inequality to get for $s \in [(i-1)\Delta_n, i\Delta_n]$

$$\mathbb{E}|a_s - a_{(i-1)\Delta_n}| \leq C\sqrt{s-(i-1)\Delta_n}.$$

From here, we can apply the trigonometric identities for $\cos(x) - \cos(y)$ and $\sin(x+y)$, the basic inequalities $|\sin(x)| \leq |x|^p$ and $|\sum_i |a_i||^p \leq \sum_i |a|_i^p$ for any $0 < p \leq 1$, and finally use the fact that $x - \kappa(x)$ is 0 around 0 (and hence $\int_{\mathbb{R}}(|x-\kappa(x)|^{\iota}\wedge 1)\,\nu_1(dx) < \infty$ for arbitrary small $\iota > 0$), to get

$$(T\Delta_n^{3/2-1/\beta})^{-1}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}|\xi_{i,u}^{(3)}(4)|\right) \leq C, \quad \forall\overline{u} > 0. \tag{A.10}$$

We turn now to $\xi_{i,u}^{(3)}(3)$. The proof proceeds through first splitting

$$\sigma_s - \sigma_{(i-1)\Delta_n} = \sigma_{1s} + \sigma_{2s}, \quad s \in [(i-1)\Delta_n, i\Delta_n], \tag{A.11}$$

$$
\begin{aligned}
\sigma_{1s} = & \int_{(i-1)\Delta_n}^{s} \tilde{\alpha}_u du + \int_{(i-1)\Delta_n}^{s} (\tilde{\sigma}_u - \tilde{\sigma}_{(i-1)\Delta_n}) dW_u + \int_{(i-1)\Delta_n}^{s} (\tilde{\sigma}'_u - \tilde{\sigma}'_{(i-1)\Delta_n-}) dZ_u \\
& + \int_{(i-1)\Delta_n}^{s} \int_{\mathbb{R}} (\underline{\delta}(u-, x) - \underline{\delta}((i-1)\Delta_n-, x)) \underline{\tilde{\mu}}(du, dx) \\
& + \tilde{\sigma}'_{(i-1)\Delta_n-} \int_{(i-1)\Delta_n}^{s} x \tilde{\mu}_2(ds, dx) - \tilde{\sigma}'_{(i-1)\Delta_n-} \int_{(i-1)\Delta_n}^{s} x \tilde{\mu}_3(ds, dx), \\
\sigma_{2s} = & \int_{(i-1)\Delta_n}^{s} \tilde{\sigma}_{(i-1)\Delta_n} dW_u + \int_{(i-1)\Delta_n}^{s} \tilde{\sigma}'_{(i-1)\Delta_n-} dL_u \\
& + \int_{(i-1)\Delta_n}^{s} \int_{\mathbb{R}} \underline{\delta}((i-1)\Delta_n-, x) \underline{\tilde{\mu}}(du, dx),
\end{aligned}
$$

where the terms involving the process $\tilde{\sigma}'_t$ in the above are present when only the stronger assumption B$'$ hold. We can further split

$$
\sigma_{1s} = \bar{\sigma}_{1s} + \hat{\sigma}_{1s}, \tag{A.12}
$$

where

$$
\hat{\sigma}_{1s} = \tilde{\sigma}'_{(i-1)\Delta_n-} \int_{(i-1)\Delta_n}^{s} x \mu_2(ds, dx) - \tilde{\sigma}'_{(i-1)\Delta_n-} \int_{(i-1)\Delta_n}^{s} x \mu_3(ds, dx),
$$

and we note $\hat{\sigma}_{1s}$ is different from zero only under B$'$. Then for each of the terms we can argue as follows. First we can split the range of integration:

$$
\begin{aligned}
\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_{1s-} d\widetilde{L}_s = & \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{|x|<\Delta_n^{1/\beta}} \kappa(x) \bar{\sigma}_{1s-} \tilde{\mu}_1(ds, dx) \\
& + \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{|x|<\Delta_n^{1/\beta}} \kappa(x) \hat{\sigma}_{1s-} \tilde{\mu}_1(ds, dx) \\
& + \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{|x|\geq\Delta_n^{1/\beta}} \kappa(x) \sigma_{1s-} \tilde{\mu}_1(ds, dx).
\end{aligned} \tag{A.13}
$$

Then for the first integral on the right hand side of the above decomposition we can use Burkholder-Davis-Gundy inequality and get

$$
\begin{aligned}
\mathbb{E}\left| \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{|x|<\Delta_n^{1/\beta}} \kappa(x) \bar{\sigma}_{1s-} \tilde{\mu}_1(ds, dx) \right| & \leq C \Delta_n^{1/\beta - 1/2} \sqrt{\int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}(\bar{\sigma}_{1s-}^2) ds} \\
& \leq C \Delta_n^{1/\beta + 1}.
\end{aligned}
$$

where we made use of the definition of $\nu_1(dx)$, the fact that for x sufficiently close to 0 we have $\kappa(x) = x$, as well as the second part of equation (2.7) of assumption B.

For the second integral on the right side of (A.13), we can apply Burkholder-Davis-Gundy inequality and get

$$
\begin{aligned}
&\mathbb{E}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa(x)\hat{\sigma}_{1s-}\tilde{\mu}_1(ds,dx)\right| \\
&\quad\le C\mathbb{E}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa^2(x)\hat{\sigma}_{1s-}^2\mu_1(ds,dx)\right)^{1/2} \\
&\quad\le C\mathbb{E}\left(\left(\sup_{s\in[(i-1)\Delta_n,i\Delta_n]}\hat{\sigma}_{1s}^2\right)\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa^2(x)\mu_1(ds,dx)\right)^{1/2} \\
&\quad\le C\mathbb{E}\Bigg\{|\tilde{\sigma}'_{(i-1)\Delta_n-}|\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}|x|\mu_2(ds,dx)+\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}|x|\mu_3(ds,dx)\right) \\
&\qquad\qquad\times\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa^2(x)\mu_1(ds,dx)\right)^{1/2}\Bigg\}.
\end{aligned}
$$

Now, for $k\in\mathbb{N}$ arbitrarily big (but higher than 1), we can apply Hölder inequality as well as Burkholder-Davis-Gundy successively (a total of k times), to get

(A.14)

$$
\begin{aligned}
&\mathbb{E}_{i-1}^n\left\{\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}|x|\mu_j(ds,dx)\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa^2(x)\mu_1(ds,dx)\right)^{1/2}\right\} \\
&\quad\le C\left(\mathbb{E}_{i-1}^n\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}|x|\mu_j(ds,dx)\right)^{\frac{2^k}{2^k-1}}\right)^{\frac{2^k-1}{2^k}} \\
&\qquad\times\left(\mathbb{E}_{i-1}^n\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa^2(x)\mu_1(ds,dx)\right)^{2^{k-1}}\right)^{2^{-k}} \\
&\quad\le C\Delta_n^{\frac{1}{\beta}+\frac{2^k-1}{2^k}},\quad j=2,3.
\end{aligned}
$$

Therefore, altogether we have

$$
\mathbb{E}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|<\Delta_n^{1/\beta}}\kappa(x)\hat{\sigma}_{1s-}\tilde{\mu}_1(ds,dx)\right|\le C\Delta_n^{\frac{1}{\beta}+1-\iota},\quad\forall\iota>0. \tag{A.15}
$$

For the third integral on the right side of (A.13), we can decompose it as integration with respect to μ_1 and its compensated measure, and then apply again Burkholder-Davis-Gundy inequality to get

$$\mathbb{E}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{|x|\geq\Delta_n^{1/\beta}}\kappa(x)\sigma_{1s-}\tilde{\mu}_1(ds,dx)\right| \leq C\int_{|x|\geq\Delta_n^{1/\beta}}|\kappa(x)|\nu_1(dx)\mathbb{E}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}|\sigma_{1s-}|\,ds\right)\leq C\Delta_n^{1/\beta+1}.$$

To continue further we denote

$$\widetilde{L}_s^n=\int_{(i-1)\Delta_n}^{s}d\widetilde{L}_s \text{ and } \overline{L}_s^n=\int_{(i-1)\Delta_n}^{s}d\overline{L}_s,\quad s\in[(i-1)\Delta_n,i\Delta_n].$$

We set under assumption B′

$$\begin{aligned}Y_s^{1n} &= \int_{(i-1)\Delta_n}^{s}\tilde{\sigma}_{(i-1)\Delta_n}dW_u+\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}\underline{\delta}((i-1)\Delta_n-,x)\underline{\tilde{\mu}}(du,dx)\\ &\quad+\widetilde{\sigma}'_{(i-1)\Delta_n-}\overline{L}_s^n,\\ Y_s^{2n} &= \widetilde{\sigma}'_{(i-1)\Delta_n-}\widetilde{L}_s^n,\end{aligned}$$

and under assumption B

$$Y_s^{1n}=\int_{(i-1)\Delta_n}^{s}\tilde{\sigma}_{(i-1)\Delta_n}dW_u \text{ and } Y_s^{2n}=\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}\underline{\delta}((i-1)\Delta_n-,x)\underline{\tilde{\mu}}(du,dx).$$

Note that Y_s^{1n} is a time-homogenous martingale independent from the random measure μ_1. This follows from our assumption on $\underline{\mu}$ (in the case of assumption B′) and the fact that the Brownian motion and a homogenous Poisson measure generate independent filtration. With this notation using integration by parts, we have

$$\begin{aligned}&\mathbb{E}_{i-1}^n\left(\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\widetilde{L}_{i\Delta_n}^n\right)\int_{(i-1)\Delta_n}^{i\Delta_n}Y_{s-}^{1n}d\widetilde{L}_s\right)\\ &=\mathbb{E}_{i-1}^n\left(\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\widetilde{L}_{i\Delta_n}^n\right)\left(\widetilde{L}_{i\Delta_n}^nY_{i\Delta_n}^{1n}-\int_{(i-1)\Delta_n}^{i\Delta_n}\widetilde{L}_{s-}^ndY_s^{1n}\right)\right)=0,\end{aligned}$$

where we made use of the independence of $\widetilde{L}_s^n$ and Y_s^{1n} and the symmetry of the distribution of $\widetilde{L}_s^n$. Next, under assumption B′, we have

$$\begin{aligned}&\mathbb{E}_{i-1}^n\left(\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\widetilde{L}_{i\Delta_n}^n\right)\int_{(i-1)\Delta_n}^{i\Delta_n}Y_{s-}^{2n}d\widetilde{L}_s\right)\\ &=\frac{\widetilde{\sigma}'_{(i-1)\Delta_n-}}{2}\mathbb{E}_{i-1}^n\left(\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\widetilde{L}_{i\Delta_n}^n\right)\left((\widetilde{L}_{i\Delta_n}^n)^2-\int_{(i-1)\Delta_n}^{i\Delta_n}\kappa^2(x)\widetilde{\mu}_1(ds,dx)\right)\right)\\ &=0,\end{aligned}$$

where we made use of Itô lemma for $\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\widetilde{L}_{i\Delta_n}\right)$ as well as the fact that the function κ is symmetric. Finally for the case when μ and $\underline{\mu}$ are not necessarily independent (i.e., the general case of assumption B), we have the following additional bound (which can be derived using the Burkholder-Davis-Gundy inequality twice and then the inequality $|\sum_i |a_i||^p \leq \sum_i |a_i|^p$ for any $p \in (0,1]$)

$$\begin{aligned}
\mathbb{E}\left|\int_{(i-1)\Delta_n}^{i\Delta_n} Y_{s-}^{2n} d\widetilde{L}_s\right|^{\beta+\iota} &\leq C\mathbb{E}\int_{(i-1)\Delta_n}^{i\Delta_n}\left|\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}\underline{\delta}(u-,z)\underline{\tilde{\mu}}(du,dz)\right|^{\beta+\iota} ds \\
&\leq C\mathbb{E}\int_{(i-1)\Delta_n}^{i\Delta_n}\left|\int_{(i-1)\Delta_n}^{s}\int_{\mathbb{R}}|\underline{\delta}(u-,z)|^{\beta\vee\beta''+\iota}\underline{\mu}(du,dz)\right|^{\frac{\beta+\iota}{\beta\vee\beta''+\iota}} ds \\
&\leq C\Delta_n^{1+\frac{\beta+\iota}{\beta\vee\beta''+\iota}},
\end{aligned}$$

for $\iota > 0$ sufficiently small. Thus altogether we have for ι arbitrary small and any finite $\overline{u} > 0$

$$\begin{aligned}
&(T\Delta_n)^{-1}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}\left|\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}_{i-1}^n\xi_{i,u}^{(3)}(3)\right|\right) \leq C, \quad \text{under B}', \\
&\left(T\Delta_n^{1/(\beta\vee\beta''+\iota)}\right)^{-1}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}\left|\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}_{i-1}^n\xi_{i,u}^{(3)}(3)\right|\right) \leq C, \quad \text{under B}.
\end{aligned} \tag{A.16}$$

On the other hand using the boundedness of the $\sin(x)$ function, Itô isometry (note that $\kappa(x)$ has bounded support and therefore $\int_{\mathbb{R}}\kappa^2(x)\nu_1(dx) < \infty$), and the fact that $\mathbb{E}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_s - \sigma_{(i-1)\Delta_n})^2 ds\right) \leq C\Delta_n^2$, gives

$$(T\Delta_n^{3-2/\beta})^{-1}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}|\xi_{i,u}^{(3)}(3)|^2\right) \leq C, \quad \forall\overline{u} > 0. \tag{A.17}$$

Similar transformations yield

$$(T\Delta_n^{2-2/\beta})^{-1}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}\left(\sup_{0\leq u\leq\overline{u}}|\xi_{i,u}^{(3)}(5)|\right) \leq C, \quad \forall\overline{u} > 0. \tag{A.18}$$

Combining the results in (A.8), (A.9), (A.10), (A.16), (A.17) and (A.18), we get

(A.19)
$$\mathbb{E}\left|\sum_{i=1}^{[T/\Delta_n]} \xi_{i,u}^{(3)}\right| \leq CT\left(|\log\Delta_n|\Delta_n^{1-\beta'/\beta} \vee \Delta_n^{(2-2/\beta)}\right), \quad \text{under B}',$$
$$\mathbb{E}\left|\sum_{i=1}^{[T/\Delta_n]} \xi_{i,u}^{(3)}\right| \leq CT\left(|\log\Delta_n|\Delta_n^{1-\beta'/\beta} \vee \Delta_n^{(2-2/\beta)\wedge\frac{1}{\beta\vee\beta''+\iota}}\right), \quad \text{under B.}$$

□

2. Proof of Theorem 2. We follow the same steps as those for the proof of Theorem 1. First, we can make a decomposition similar to that of $V_T(X,\Delta_n,\beta,u) - \int_0^T e^{-u|\sigma_t|^\beta}dt$, mainly

$$\widetilde{V}_T(X,\Delta_n,\beta,u) - \int_0^T e^{-u|\sigma_t|^\beta}dt = \sum_{i=2}^{[T/\Delta_n]}(\widetilde{\xi}_{i,u}^{(1)} + \xi_{i-1,u}^{(2)} + \widetilde{\xi}_{i,u}^{(3)}) + \int_{([T\Delta_n]-2)\Delta_n}^T e^{-u|\sigma_s|^\beta}ds,$$

$$\widetilde{\xi}_{i,u}^{(1)} = \Delta_n\left(\cos[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}(\Delta_i^n L - \Delta_{i-1}^n L)] - e^{-u|\sigma_{(i-2)\Delta_n-}|^\beta}\right),$$
$$\widetilde{\xi}_{i,u}^{(3)} = \Delta_n\left(\cos[u^{1/\beta}\Delta_n^{-1/\beta}(\Delta_i^n X - \Delta_{i-1}^n X)] - \cos[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}(\Delta_i^n L - \Delta_{i-1}^n L)]\right).$$

Then for $\widetilde{\xi}_{i,u}^{(1)}$ we can further decompose for any $t>0$

$$\begin{aligned}
\sum_{i=2}^{[t/\Delta_n]}\widetilde{\xi}_{i,u}^{(1)} &= \sum_{i=2}^{[t/\Delta_n]-1}\gamma_{i,u} + \Delta_n e^{-u|\sigma_0|^\beta/2}\left(\cos[u^{1/\beta}\sigma_0\Delta_n^{-1/\beta}\Delta_1^n L] - e^{-u|\sigma_0|^\beta/2}\right)\\
&+ \Delta_n\left(\cos[u^{1/\beta}\sigma_{([t/\Delta_n]-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{[t/\Delta_n]}^n L] - e^{-u|\sigma_{([t/\Delta_n]-2)\Delta_n-}|^\beta/2}\right)\\
&\quad\times\cos[u^{1/\beta}\sigma_{([t/\Delta_n]-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{[t/\Delta_n]-1}^n L]\\
&+ \Delta_n\sin[u^{1/\beta}\sigma_{([t/\Delta_n]-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{[t/\Delta_n]}^n L]\sin[u^{1/\beta}\sigma_{([t/\Delta_n]-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{[t/\Delta_n]-1}^n L],
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{i,u} &= \Delta_n\left(\cos[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L] - e^{-u|\sigma_{(i-2)\Delta_n-}|^\beta/2}\right)\cos[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{i-1}^n L]\\
&\quad+ \Delta_n e^{-u|\sigma_{(i-1)\Delta_n-}|^\beta/2}\left(\cos[u^{1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L] - e^{-u|\sigma_{(i-1)\Delta_n-}|^\beta/2}\right)\\
&\quad+ \Delta_n\sin[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L]\sin[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{i-1}^n L].
\end{aligned}$$

We have

(A.20)
$$\mathbb{E}_{i-1}^n(\gamma_{i,u}) = 0, \quad \mathbb{E}_{i-1}^n(\widetilde{\xi}_{i,u}^{(1)})^4 \leq C\Delta_n^4,$$

(A.21)
$$
\begin{aligned}
\mathbb{E}_{i-1}^n(\gamma_{i,u})^2 &= \Delta_n^2 F_\beta((u/2)^{1/\beta}|\sigma_{(i-2)\Delta_n-}|)\cos^2[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{i-1}^n L]\\
&+ \Delta_n^2 e^{-u|\sigma_{(i-1)\Delta_n-}|^\beta} F_\beta((u/2)^{1/\beta}|\sigma_{(i-1)\Delta_n-}|)\\
&+ \Delta_n^2\left(\frac{1-e^{-u2^{\beta-1}|\sigma_{(i-2)\Delta_n-}|^\beta}}{2}\right)\sin^2[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{i-1}^n L]\\
&+ 2\Delta_n^2 e^{-u|\sigma_{(i-1)\Delta_n-}|^\beta/2}\cos[u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}\Delta_{i-1}^n L]\\
&\times\left\{\frac{e^{-u|\sigma_{(i-1)\Delta_n-}+\sigma_{(i-2)\Delta_n-}|^\beta/2}}{2}+\frac{e^{-u|\sigma_{(i-1)\Delta_n-}-\sigma_{(i-2)\Delta_n-}|^\beta/2}}{2}\right.\\
&\qquad\left.-\,e^{-u|\sigma_{(i-1)\Delta_n-}|^\beta/2-u|\sigma_{(i-2)\Delta_n-}|^\beta/2}\right\}.
\end{aligned}
$$

Using the fact that the process σ is càdlàg, we finally have

$$
\begin{aligned}
\sum_{i=2}^{[t/\Delta_n]-1}\mathbb{E}_{i-1}^n(\gamma_{i,u})^2 &= \Delta_n^2\widetilde{F}_\beta(u^{1/\beta}|\sigma_{(i-2)\Delta_n-}|)+o_p(1)\\
&= \Delta_n\int_0^t \widetilde{F}_\beta(u^{1/\beta}|\sigma_s|)ds+o_p(1),\quad \forall t>0.
\end{aligned}
\tag{A.22}
$$

Combining (A.20)-(A.22) and using the same steps as in the proof of Theorem 1 (in particular showing the analogue of the result in (7.16) in the text), we have the stable convergence of $\frac{1}{\sqrt{\Delta_n}}\sum_{i=2}^{[t/\Delta_n]}\widetilde{\xi}_{i,u}^{(1)}$, as a process in t for the Skorkhod topology when $\Delta_n\to 0$, to a $\int_0^t\widetilde{F}_\beta(u^{1/\beta}|\sigma_s|)d\widetilde{W}_s'$ where $\widetilde{W}_t'$ is a Brownian motion defined on extension of the original probability space and independent of the σ-filed $\mathcal{F}$.

Next, $\frac{1}{\sqrt{\Delta_n}}\sum_{i=2}^{[T/\Delta_n]}\xi_{i,u}^{(2)}$ is asymptotically negligible from the proof of Theorem 1. We are left with $\frac{1}{\sqrt{\Delta_n}}\sum_{i=2}^{[T/\Delta_n]}\widetilde{\xi}_{i,u}^{(3)}$. We make the decomposition

$$
\widetilde{\xi}_{i,u}^{(3)}=\sum_{j=1}^4\widetilde{\xi}_{i,u}^{(3)}(j),
$$

$$
\begin{aligned}
\widetilde{\xi}_{i,u}^{(3)}(1) &= -2\Delta_n\sin\left(\frac{u^{1/\beta}\Delta_n^{-1/\beta}(\Delta_i^nX-\Delta_{i-1}^nX+\Delta_i^n\widetilde{X}-\Delta_{i-1}^n\widetilde{X})}{2}\right)\\
&\times\sin\left(\frac{u^{1/\beta}\Delta_n^{-1/\beta}(\Delta_i^nX-\Delta_{i-1}^nX-\Delta_i^n\widetilde{X}+\Delta_{i-1}^n\widetilde{X})}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\xi}_{i,u}^{(3)}(2) &= -u^{1/\beta}\Delta_n^{1-1/\beta}\sin\left(u^{1/\beta}\sigma_{(i-2)\Delta_n-}\Delta_n^{-1/\beta}(\Delta_i^n\widetilde{L}-\Delta_{i-1}^n\widetilde{L})\right)\\
&\times\left(\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_{s-}-\sigma_{(i-2)\Delta_n-})d\widetilde{L}_s-\int_{(i-2)\Delta_n}^{(i-1)\Delta_n}(\sigma_{s-}-\sigma_{(i-2)\Delta_n-})d\widetilde{L}_s\right),
\end{aligned}
$$

$$\begin{aligned}\widetilde{\xi}^{(3)}_{i,u}(3) &= \Delta_n \cos(u^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-2)\Delta_n-}(\Delta_i^n\widetilde{L}-\Delta_{i-1}^n\widetilde{L})) + \Delta_n\cos(u^{1/\beta}\Delta_n^{-1/\beta}(\Delta_i^n\widetilde{X}-\Delta_{i-1}^n\widetilde{X}))\\ &\quad - \Delta_n\cos(u^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-2)\Delta_n-}(\Delta_i^nL-\Delta_{i-1}^nL))\\ &\quad - \Delta_n\cos\left(u^{1/\beta}\Delta_n^{-1/\beta}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\sigma_{s-}d\widetilde{L}_s - \int_{(i-2)\Delta_n}^{(i-1)\Delta_n}\sigma_{s-}d\widetilde{L}_s\right)\right),\\ \widetilde{\xi}^{(3)}_{i,u}(4) &= -u^{1/\beta}\Delta_n^{1-1/\beta}\left(\sin(\widetilde{\chi}) - \sin(u^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-2)\Delta_n-}(\Delta_i^n\widetilde{L}-\Delta_{i-1}^n\widetilde{L}))\right)\\ &\qquad\times\left(\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_{s-}-\sigma_{(i-2)\Delta_n-})d\widetilde{L}_s - \int_{(i-2)\Delta_n}^{(i-1)\Delta_n}(\sigma_{s-}-\sigma_{(i-2)\Delta_n-})d\widetilde{L}_s\right),\end{aligned}$$

where $\widetilde{X}_t = \int_0^t\overline{\alpha}_sds+\int_0^t\sigma_sdL_s$ and $\widetilde{\chi}$ is between $u^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-2)\Delta_n-}(\Delta_i^n\widetilde{L}-\Delta_{i-1}^n\widetilde{L})$ and $u^{1/\beta}\Delta_n^{-1/\beta}\left(\int_{(i-1)\Delta_n}^{i\Delta_n}\sigma_{s-}d\widetilde{L}_s - \int_{(i-2)\Delta_n}^{(i-1)\Delta_n}\sigma_{s-}d\widetilde{L}_s\right)$.

Using similar techniques as for the bounds of $\xi^{(3)}_{i,u}(j)$ we can show that

$$\begin{aligned}&(T|\log(\Delta_n)|\Delta_n^{1-\beta'/\beta})^{-1}\sum_{i=2}^{[T/\Delta_n]}\mathbb{E}|\widetilde{\xi}^{(3)}_{i,u}(1)| \le C,\\ &(T\Delta_n^{1/(\beta\vee\beta''+\iota)})^{-1}\mathbb{E}\left|\sum_{i=2}^{[T/\Delta_n]}\mathbb{E}_{i-1}^n\widetilde{\xi}^{(3)}_{i,u}(2)\right| \le C, \quad (T\Delta_n^{3-2/\beta})^{-1}\sum_{i=2}^{[T/\Delta_n]}\mathbb{E}(\widetilde{\xi}^{(3)}_{i,u}(2))^2 \le C,\\ &(T\Delta_n^{3/2-1/\beta})^{-1}\sum_{i=2}^{[T/\Delta_n]}\mathbb{E}|\widetilde{\xi}^{(3)}_{i,u}(3)| \le C, \quad (T\Delta_n^{\beta/2-\iota})^{-1}\sum_{i=2}^{[T/\Delta_n]}\mathbb{E}|\widetilde{\xi}^{(3)}_{i,u}(4))| \le C.\end{aligned}$$

Therefore, $\frac{1}{\sqrt{\Delta_n}}\sum_{i=2}^{[T/\Delta_n]}\widetilde{\xi}^{(3)}_{i,u}$ is asymptotically negligible under the conditions of the Theorem when $\Delta_n \to 0$ and T is fixed. □

3. Proof of Theorem 4. Part (a). Given Theorem 3, we need to prove that the difference $\sqrt{T}\left(\widehat{\mathcal{L}}_{\widehat{\beta}}(u) - \widehat{\mathcal{L}}_\beta(u)\right)$ is asymptotically negligible. We have

$$\begin{aligned}\left|\widehat{\mathcal{L}}_{\widehat{\beta}}(u) - \widehat{\mathcal{L}}_\beta(u)\right| \le& \frac{1}{(\beta^*)^2}|\log(2u/\Delta_n)|(\Delta_n/(2u))^{-1/\beta^*+1/\beta}(\widehat{\beta}-\beta)\\ &\times\frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta_n]}|\sin\left((2u)^{1/\beta^*}\Delta_n^{-1/\beta^*}\Delta_i^nX\right)(2u)^{1/\beta}\Delta_n^{-1/\beta}\Delta_i^nX|,\end{aligned}$$

where β^* is between β and $\widehat{\beta}$. Then, using the integrability of the absolute values of the increments of X and also the fact that $\beta > 1$, and upon applying Markov's inequality, we get

$$\mathbb{P}\left(\frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta_n]}|\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta^*}\Delta_i^nX\right)(2u)^{1/\beta}\Delta_n^{-1/\beta}\Delta_i^nX| > |\log(\Delta_n)|\right) \to 0.$$

Since $\widehat{\beta} - \beta = o_p(1)$, we have $\mathbb{P}\left(\left|\frac{1}{\widehat{\beta}} - \frac{1}{\beta}\right| > \alpha/2\right) \to 0$. Taking into account the assumed rate of convergence of $\widehat{\beta}$ the result follows.

Part (b). In the case when $\widehat{\beta}$ uses an initial part of the sample (with fixed span) that is used in the construction of $V_T(X, \Delta_n, \beta, u)$, we can replace the latter with the same statistic but using only that part of the sample that is not used in the calculation of $\widehat{\beta}$. Since the time span of the sample used in the calculation of $\widehat{\beta}$ is fixed, this will have no asymptotic effect. Therefore, it is sufficient to consider only the case when $\widehat{\beta}$ uses only information before the beginning of the sample and we do so in the proof of the theorem.

First, since $\widehat{\beta} \xrightarrow{\mathbb{P}} \beta$, it is no limitation to restrict attention on the set for which $|1/\widehat{\beta} - 1/\beta| < \epsilon/2$ for some arbitrary small $\epsilon > 0$. Then, using the proof of Theorem 3 and notation of that proof, we can write for arbitrary small $\iota > 0$

$$\sqrt{T}\widehat{\mathcal{L}}_\beta(u) - \frac{1}{\sqrt{T}} \sum_{i=1}^{[T/\Delta_n]} \cos\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right) = o_p\left(\sqrt{T}\Delta_n^{(1-\beta'/\beta-\iota)\wedge(2-2/\beta)\wedge 1/2}\right).$$

Similar, using successive conditioning on the set of data used in the estimation of $\widehat{\beta}$, we can write

$$\sqrt{T}\widehat{\mathcal{L}}_{\widehat{\beta}}(u) - \frac{1}{\sqrt{T}} \sum_{i=1}^{[T/\Delta_n]} \cos\left((2u)^{1/\widehat{\beta}}\Delta_n^{-1/\widehat{\beta}}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right) = o_p\left(\sqrt{T}\Delta_n^{(1-\beta'/\beta-\iota-\epsilon)\wedge(2-2/\beta-\epsilon)\wedge 1/2}\right),$$

for ϵ defined above. Thus, we need to find the asymptotic limit of

$$\begin{aligned}
&\frac{\Delta_n}{\sqrt{T}} \sum_{i=1}^{[T/\Delta_n]} \left(\cos\left((2u)^{1/\widehat{\beta}}\Delta_n^{-1/\widehat{\beta}}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right) - \cos\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\right) \\
&\quad = -(\Delta_n/(2u))^{1/\beta-1/\beta^*} \frac{\Delta_n \log(2u/\Delta_n)}{\sqrt{T}(\beta^*)^2}\left(\widehat{\beta} - \beta\right) \\
&\quad\quad \times \sum_{i=1}^{[T/\Delta_n]} \left\{\sin\left((2u)^{1/\beta^*}\Delta_n^{-1/\beta^*}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\sigma_{(i-1)\Delta_n-}(2u)^{1/\beta}\Delta_n^{-1/\beta}\Delta_i^n L\right\},
\end{aligned}$$

where we used a first-order Taylor expansion around the true value β and we further denoted with β^* some value between $\widehat{\beta}$ and β. The proof consists of the following steps.

Step 1. We show

$$\frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta_n]}\left\{\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)(2u)^{1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L\right\}$$
$$\xrightarrow{\mathbb{P}}\ \mathbb{E}(G_\beta(u^{1/\beta}|\sigma_t|)).$$

First upon differentiating in u both sides of the identity $\mathbb{E}(\cos((2u)^{1/\beta}L)) = e^{-u}$, and using the self-similarity of the stable process, we easily have

$$\mathbb{E}_{i-1}^n\left(\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)(2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)$$
$$= G_\beta(u^{1/\beta}|\sigma_{(i-1)\Delta_n-}|).$$

From here using the fact that the function $G_\beta(x)$ is differentiable (in x), assumption B, the ergodicity of σ_t combined with a law of large numbers, we get

$$\frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta_n]}\mathbb{E}_{i-1}^n\left(\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)(2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)$$
$$\xrightarrow{\mathbb{P}}\ \mathbb{E}(G_\beta(u^{1/\beta}|\sigma_t|)).$$

The result then follows using Theorem VIII.2.29 of [1] and the fact that for some $1 < p < \beta$ we have

$$\mathbb{E}\left|\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L\right|^p \le C.$$

Step 2. We have

(A.23)
$$\frac{\Delta_n^{1+\alpha-\epsilon/2}|\log(\Delta_n/2u)|}{\sqrt{T}}\sum_{i=1}^{[T/\Delta_n]}\left\{\left(\sin\left((2u)^{1/\beta^*}\Delta_n^{-1/\beta^*}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\right.\right.$$
$$\left.\left.-\sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\right)\sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L\right\}$$
$$\xrightarrow{\mathbb{P}}\ 0.$$

First, we can write for some $1 < p < \beta$

$$\begin{aligned}
&\left|\sin\left((2u)^{1/\beta^*}\Delta_n^{-1/\beta^*}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right) - \sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\right| \\
&\quad \le 2\left|\sin\left(0.5((\Delta_n/(2u))^{-1/\beta^*} - (\Delta_n/(2u))^{-1/\beta})\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\right| \\
&\quad \le C|(\Delta_n/(2u))^{-1/\beta^*+1/\beta} - 1|^{p-1}|\Delta_n^{-1/\beta}\Delta_i^n L|^{p-1},
\end{aligned}$$

where we have made use of $|\cos(x)| \leq 1$ and the property $|\sin(x)| \leq |\sin(x)|^{p-1} \leq |x|^{p-1}$ since $0 < p-1 < 1$. Then we have

(A.24)

$$\frac{\Delta_n}{T}\left|\sum_{i=1}^{[T/\Delta_n]}\left\{\left(\sin\left((2u)^{1/\beta^*}\Delta_n^{-1/\beta^*}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right) - \sin\left((2u)^{1/\beta}\Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L\right)\right)\right.\right.$$
$$\left.\left.\times\, \sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L\right\}\right|$$
$$\leq C|(\Delta_n/(2u))^{-1/\beta^*+1/\beta} - 1|^{p-1} \times \frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta_n]}|\sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L|^p.$$

The second term above converges in L^1 norm. Now, using Taylor expansion we have $|(\Delta_n/(2u))^{-1/\beta^*+1/\beta} - 1| \leq C|\log(\Delta_n/2u)|\Delta_n^{-\epsilon/2}|\beta^* - \beta|$. Then we can choose $p = \beta - \iota$ for $\iota > 0$ sufficiently small in (A.24) and taking into account $T\Delta_n \to 0$ as well as $\alpha > 1/(2\beta)$ we have the result in (A.23) provided $\epsilon > 0$ is chosen sufficiently small.

Step 3. The result of the theorem follows by taking into account that

$$(\Delta_n/(2u))^{1/\beta-1/\beta^*} = 1 + (\beta^* - \beta)\frac{(\Delta_n/(2u))^{1/\beta-1/\beta^{**}}}{(\beta^{**})^2}\log(2u/\Delta_n),$$

where β^{**} is between β^* and β, and the fact that $\widehat{\beta} - \beta = o_p(\Delta_n^\alpha)$ for some $\alpha > 0$.

Part (c). Given the result of Step 1 above, the only thing that remains to be proved is

$$\frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta_n]}\Big(\Delta_n^{-1/\beta}\Delta_i^n X\sin((2u)^{1/\beta}\Delta_n^{-1/\beta}\Delta_i^n X)$$
$$- \sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L\sin((2u)^{1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_n^{-1/\beta}\Delta_i^n L)\Big) \xrightarrow{\mathbb{P}} 0.$$

For this we need only use the following algebraic inequality for $\forall x, y \in \mathbb{R}$

$$|x\sin(x) - y\sin(y)| \leq |x - y| + |y\sin((x-y)/2)|,$$

with $x = \Delta_n^{-1/\beta}\Delta_i^n X$ and $y = \Delta_n^{-1/\beta}\sigma_{(i-1)\Delta_n-}\Delta_i^n L$, Hölder inequality, the fact that $\mathbb{E}|L|^p < \infty$ for $p < \beta$, and the following basic inequalities

$$\Delta_n^{-1/\beta}\mathbb{E}\left|\Delta_i^n X - \int_{(i-1)\Delta_n}^{i\Delta_n}\int_{\mathbb{R}}\sigma_{s-}x\mu_1(ds,dx)\right| \leq C\Delta_n^{1-1/\beta},$$

$$\Delta_n^{-1/\beta}\mathbb{E}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_{s-} - \sigma_{(i-1)\Delta_n-})x\tilde{\mu}_1(ds,dx)\right| \leq C\Delta_n^{1+\beta/2-1/\beta-\epsilon},$$

where $\epsilon > 0$ is arbitrary small. □

3.1. *Proof of Theorem 5.* We first show that $\widehat{\theta}$ is consistent estimator of θ_0. We have

$$\sup_{\theta}\left|\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)^2\kappa(u)du-\int_{\mathbb{R}_+}\left(\mathcal{L}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)^2\kappa(u)du\right| \xrightarrow{\mathbb{P}} 0. \tag{A.25}$$

Denote $u_{\max}=\sup\{u:\kappa(u)>0\}$ which is finite due to the assumption of the theorem. Then, using Theorem 3, we have uniformly in θ

$$\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)^2 1_{\{u\le u_{\max}\}}(\widehat{\kappa}(u)-\kappa(u))du \xrightarrow{\mathbb{P}} 0. \tag{A.26}$$

Next, using the boundedness of $\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)$ and the property of the kernel $\widehat{\kappa}$, we have

$$\begin{aligned}\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)^2 1_{\{u>u_{\max}\}}\widehat{\kappa}(u)du &\le \int_{u>u_{\max}}\widehat{\kappa}(u)du\\ &\le C\sup_{u>u_{\max}}u^{2+\iota}\widehat{\kappa}(u) \xrightarrow{\mathbb{P}} 0.\end{aligned} \tag{A.27}$$

This obviously holds uniformly in θ. Combining the results together with the fact that $\int_{\mathbb{R}_+}\left(\mathcal{L}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)^2\kappa(du)$ is uniquely minimized at $\theta=\theta_0$, we have that $\widehat{\theta} \xrightarrow{\mathbb{P}} \theta_0$.

Given the established consistency, with probability approaching 1, $\widehat{\theta}$ solves

$$\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta)\right)\nabla_\theta\mathcal{L}_\beta(u;\theta)\widehat{\kappa}(u)du=0.$$

Therefore, with probability approaching 1, we have

$$\sqrt{T}\left(\widehat{\theta}-\theta_0\right)=\widehat{H}^{-1}\widehat{E},$$

where

$$\widehat{H}=-\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\widetilde{\theta})\right)\nabla_{\theta\theta'}\mathcal{L}_\beta(u;\widetilde{\theta})\widehat{\kappa}(u)du+\int_{\mathbb{R}_+}\nabla_\theta\mathcal{L}_\beta(u;\theta)\nabla_\theta\mathcal{L}_\beta(u;\theta)'\widehat{\kappa}(u)du,$$

$$\widehat{E}=\sqrt{T}\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta_0)\right)\nabla_\theta\mathcal{L}_\beta(u;\theta_0)\widehat{\kappa}(u)du,$$

with $\widetilde{\theta}$ being a number between $\widehat{\theta}$ and θ_0. Then, using exactly the same decomposition as in (A.25)-(A.27) and Theorem 3, we can show (since $\widehat{\theta}$ is consistent for θ_0, with probability approaching we have 1 $\widehat{\theta}\in\Theta^l$)

$$\widehat{H} \xrightarrow{\mathbb{P}} \int_{\mathbb{R}_+}\nabla_\theta\mathcal{L}_\beta(u;\theta)\nabla_\theta\mathcal{L}_\beta(u;\theta)'\kappa(u)du. \tag{A.28}$$

Hence we are left with showing

$$\widehat{E} \xrightarrow{\mathcal{L}} \Xi^{1/2}E'. \tag{A.29}$$

Using Theorem 3, we have

$$\sqrt{T}\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta_0)\right)\nabla_\theta\mathcal{L}_\beta(u;\theta_0)\kappa(u)du \xrightarrow{\mathcal{L}} \Xi^{1/2}E'.$$

Finally, we can write

$$\begin{aligned}&\sqrt{T}\left|\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta_0)\right)\nabla_\theta\mathcal{L}_\beta(u;\theta_0)(\widehat{\kappa}(u)-\kappa(u))du\right|\\&\leq C\sup_{u>u_{\max}}u^{2+\iota}|\widehat{\kappa}(u)-\kappa(u)|\sqrt{T}\int_{\mathbb{R}_+}\left(\widehat{\mathcal{L}}_\beta(u)-\mathcal{L}_\beta(u;\theta_0)\right)\frac{\nabla_\theta\mathcal{L}_\beta(u;\theta_0)}{u^{2+\iota}}du.\end{aligned}$$

The first term on the right hand side of the above is asymptotically negligible from the assumption of the theorem, while for the second one we can use the decomposition and the corresponding bounds in Section 1 of this supplement to show that it converges either in L^1 or in L^2 to a constant. □

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