SUPPLEMENTAL APPENDIX TO "NONPARAMETRIC IMPLIED LÉVY DENSITIES"

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This supplementary appendix contains the following items: (1) limit results for the integrated squared error of the nonparametric estimator, (2) lower bounds for the minimax risk of recovering Lévy density from noisy option data with heteroskedastic Gaussian observation errors, and (3) alternative Lévy density estimator based on the second derivatives of the characteristic function of the asset return estimated from the option data.

1. Limit Results for the Integrated Squared Error of $\widehat{h}_t$. In this section, we derive the probability limit of the part of the integrated squared error of $\widehat{h}_t$ which is due to the estimation error. More specifically, using Parseval's identity, the integrated squared error might be decomposed as

(A.1)
$$\int_{\mathbb{R}}(\widehat{h}_t(x)-h_t(x))^2dx = \frac{1}{2\pi}\int_{|u|\le u_N}|\widehat{h}^*_t(u)-h^*_t(u)|^2du+\frac{1}{2\pi}\int_{|u|>u_N}|h^*_t(u)|^2du,$$

and our focus here will be the convergence in probability of the first term on the right hand side of the above equality. The result of this section shows that the bound of Theorem 1 on the integrated squared error is sharp.

We will work in this section in a simplified setting in which x is a Lévy process and hence we will drop the subscript t in the notation of its characteristics. For stating the result, we will make use of the following notation

$$O^d(k) = \begin{cases} \int_{\mathbb{R}}(e^k - e^z)^+\nu(z)dz, & \text{if } k \le 0, \\ \int_{\mathbb{R}}(e^z - e^k)^+\nu(z)dz, & \text{if } k > 0. \end{cases} \quad \text{(A.2)}$$

With this notation, we have the following result.

Theorem 2. *Suppose that X is a Lévy process under $\mathbb{Q}$ with $\int_{\mathbb{R}}(|e^{p|z|} - 1| \vee 1)\nu(z)dz < \infty$ for some $p > 4$. In addition, let Assumption A5 hold*

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with $\eta = 1$, and assume $\epsilon_i = \sigma_\epsilon O_T(k_i) z_i$, where $\{z_i\}$ is a sequence of i.i.d. random variables with $\mathbb{E}(z_i) = 0$, $\mathbb{E}(z_i^2) = 1$ and $\mathbb{E}(z_i^4) < \infty$, and σ_ϵ is a finite positive constant.

If $\overline{\Delta} \asymp T^\alpha$, $\overline{K} \asymp T^{-\beta}$, $\underline{K} \asymp T^\gamma$, for some $\alpha > 0$, $\beta > \alpha/6$ and $\gamma > \alpha/6$, and if further $u_N = -u_{N,l} = u_{N,h}$ satisfies $u_N \to \infty$, $u_N^2 T |\ln T|^2 \to 0$ and $u_N^5 \overline{\Delta} \to 0$, then we have

(A.3)
$$\frac{1}{u_N^5 \overline{\Delta}} \left(\int_{\mathbb{R}} (\widehat{h}_t(x) - h_t(x))^2 dx - \frac{1}{2\pi} \int_{|u| > u_N} |h_t^*(u)|^2 du \right) \xrightarrow{\mathbb{P}} \frac{\sigma_\epsilon^2}{5\pi} \int_{\mathbb{R}} e^{-2k} k^6 O^d(k)^2 dk.$$

Theorem 2 shows that the bound on the integrated squared error of $\widehat{h}_t$ in Theorem 1 in the paper is sharp. Indeed, the quantity $\int_{\mathbb{R}} e^{-2k} k^6 O^d(k)^2 dk$ is strictly positive as soon as the Lévy measure ν has support outside of zero. An interesting aspect of the above result is that the probability limit of the integrated squared error of $\widehat{h}_t$ does not depend on the diffusive part of the price even though the latter is present in x. This is because for $T \downarrow 0$, the option prices with k away of zero are dominated by the jump part of the process. On the other hand, the option prices with k in vicinity of zero (more precisely a neighborhood around zero of order $O(\sqrt{T})$) are dominated by the diffusive component of x. However, $\widehat{h}_t$ puts asymptotically smaller weight on these options and hence this dependence on the diffusive component becomes of higher asymptotic order only.

Remark 1.1. *We note that in the case when $u_N \asymp \overline{\Delta}^{-\frac{1}{2r+5}}$, then we have*

$$\int_{\mathbb{R}} (\widehat{h}_t(x) - h_t(x))^2 dx - \frac{1}{2\pi} \int_{|u| > u_N} |h_t^*(u)|^2 du = O_p\left(\overline{\Delta}^{\frac{2r}{2r+5}} \right),$$

and since from Assumption A1

$$\frac{1}{2\pi} \int_{|u| > u_N} |h_t^*(u)|^2 du = O\left(\overline{\Delta}^{\frac{2r}{2r+5}} \right),$$

we get altogether

$$\int_{\mathbb{R}} (\widehat{h}_t(x) - h_t(x))^2 dx = O_p\left(\overline{\Delta}^{\frac{2r}{2r+5}} \right),$$

and hence $\widehat{h}_t$ achieves the optimal rate of recovering h_t in a minimax sense as we show in the next section (see Theorem 3 below).

2. Minimax Risk for Recovering Lévy Density from Short-Dated Options. We will now derive a lower bound for the minimax risk for recovering the Lévy density from option data. This result will show that the nonparametric estimator $\widehat{h}_t$ is rate-efficient, provided u_N is chosen optimally. The analysis in this section will be performed in a simplified setting in which the process x does not contain a diffusion and further when its jumps are of finite activity (but the jump intensity is allowed to vary over time). We will additionally simplify the analysis by assuming that the observation errors are $\mathcal{F}_t$-conditionally Gaussian and independent. We will further assume that the relative option error, that is the ratio of the error over the true (unobservable) option price is $O_p(1)$. Since the true option price decreases in value as $T \downarrow 0$, the size of the observation error shrinks at the same rate as the option price it is attached to.

We will now introduce the necessary notation for stating the formal result. For some constants $R > 0$ and $r \geq 0$, we define the set $\mathcal{G}_r(R)$ of risk-neutral probability measures $\mathbb{Q}$ (under which the true option prices $O_T(k)$ are computed according to (2.2)) under which x is an Itô semimartingale with characteristics triplet with respect to the identity truncation function ([3], Definition II.2.6) given by

$$\text{(A.4)} \qquad \left(-\int_0^t \int_{\mathbb{R}} (e^z - 1 - z)\nu_s(z)dzds,\ 0,\ F(t,z)\right),$$

where $F(dt, dz) = dt\nu_t(z)dz$ and we further have for $h_t(x) = x^3\nu_t(x)$:

$$\text{(A.5)} \qquad \max_{0\leq k\leq r} ||h_t^{(k)}||_{L^2(\mathbb{R})} \leq R, \quad ||h_t^{(r)}||_{L^\infty(\mathbb{R})} \leq R,$$

and in addition

$$\text{(A.6)} \qquad \mathbb{E}_t^{\mathbb{Q}}|a_s|^4 \leq R, \quad \mathbb{E}_t^{\mathbb{Q}}\left(\int_{\mathbb{R}}\left((e^{3|z|}-1)\vee 1\right)\nu_s(z)dz\right)^4 \leq R,$$

$$\text{(A.7)} \qquad \mathbb{E}_t^{\mathbb{Q}}|a_s - a_t|^p \leq R|s-t|, \quad \forall p \in [2,4],$$

$$\text{(A.8)} \quad \mathbb{E}_t^{\mathbb{Q}}\left(\int_t^s \int_{\mathbb{R}} \left(e^{z\vee 0}|z| \vee z^2\right)|\nu_s(z) - \nu_t(z)|dz\right)^p \leq R|s-t|, \ \forall p \in [2,3],$$

for some $\bar{t} > t$ and $\forall s \in [t, \bar{t}]$.

The set $\mathcal{G}_r(R)$ corresponds to risk-neutral probability laws for x with no diffusion and jumps of finite activity whose Lévy measure is allowed to vary over time.

The option observations are given by

$$\widehat{O}_T(k_i) = O_T(k_i) + (O_T(k_i) \vee T)\epsilon_i, \quad i = 1, ..., N, \tag{A.9}$$

where $\{\epsilon\}_{i\geq 1}$ is a sequence of i.i.d. $N(0,1)$ random variables defined on a product extension of $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)})_{t\geq 0}, \mathbb{P}^{(0)})$ and independent of $\mathcal{F}^{(0)}$. Note that, since x is assumed to have no diffusion and further since the jumps are of finite activity, $O_T(k)$ is of asymptotic order $O_p(T)$. Therefore, the truncation from below in the scale of the option error does not change its asymptotic order.

The grid of log-strikes is given by

$$\underline{k} \equiv k_1 < k_2 < \cdots < k_N \equiv \overline{k}, \tag{A.10}$$

with $k_i - k_{i-1} = \overline{\Delta}$ (equidistant log-strike grid) and we further denote $\underline{K} = e^{\underline{k}}$ and $\overline{K} = e^{\overline{k}}$.

In what follows, we will denote with $\mathbb{E}_{\mathcal{T}}$ expectations under which the true (unobservable) option prices $O_T(k)$ are computed according to the risk-neutral probability measure $\mathcal{T}$. We will also use the notation $a_n \gtrsim b_n$ and $a_n \lesssim b_n$ to mean the respective inequality up to a constant independent of the parameter n.

Theorem 3. *With the notation in (A.4)-(A.10), assume further that* $\overline{\Delta} \asymp T^{\alpha}$, $\overline{K} \asymp T^{-\beta}$ *and* $\underline{K} \asymp T^{\gamma}$, *for* $\alpha \in (0,1)$ *and* $\beta, \gamma > 0$ *as* $T \to 0$.
We then have almost surely

$$\inf_{\widehat{h}_t} \sup_{\mathcal{T}\in\mathcal{G}_r(R)} \mathbb{E}_{\mathcal{T}}\left(||\widehat{h}_t - h_t||^2_{L^2(\mathbb{R})} \big| \mathcal{F}^{(0)}\right) \gtrsim \overline{\Delta}^{\frac{2r}{2r+5}}, \tag{A.11}$$

where $\widehat{h}_t$ *is any estimator of* h_t *based on the option data* $\{\widehat{O}_T(k_i)\}_{i=1,...,N}$.

3. Alternative Estimator based on Second Derivatives. As mentioned in the main text, an alternative strategy for recovering the Lévy density is to use second derivatives of $\widehat{f}_T(u)$. Given the approximation in equation (4.4) in the paper, however, in this case we will also need an estimator for σ_t. In addition, when using the second derivative of $\widehat{f}_T(u)$, instead of recovering $h_t(x) = x^3\nu_t(x)$ we will estimate

$$g_t(x) = x^2\nu_t(x). \tag{A.12}$$

Since g_t dampens the Lévy measure ν_t less around zero than h_t does, this approach can potentially yield better recovery of the Lévy measure for the "small" jumps.

Formally, our estimator is defined as follows. We first set

$$\widehat{g}_t^*(u) = \frac{1}{T}\frac{\widehat{f}_T^{(1)}(u)^2}{\widehat{f}_T(u)^2} - \frac{1}{T}\frac{\widehat{f}_T^{(2)}(u)}{\widehat{f}_T(u)}, \tag{A.13}$$

and with $\widehat{\sigma}_t$ denoting an estimator of σ_t, our inference for g_t is based on

$$\widehat{g}_t(x) = \frac{1}{2\pi}\int_{u\in[u_{N,l},u_{N,h}]} e^{-iux}(\widehat{g}_t^*(u) - \widehat{\sigma}_t^2)du. \tag{A.14}$$

The asymptotic bound on the integrated squared error of g_t is given in the following theorem.

THEOREM 4. *Suppose Assumptions A1-A6 in the main text hold, with h replaced by g for A1, and in addition $\overline{\Delta} \asymp T^{\alpha}$, $\overline{K} \asymp T^{-\beta}$, $\underline{K} \asymp T^{\gamma}$, for some $\alpha > \frac{1}{2}$, $\beta > 0$ and $\gamma > 0$ and where $\overline{\Delta}$ is the mesh of the log-strike grid. Let $-u_{N,l} \asymp u_{N,h} \asymp u_N$ be such that:*

$$u_N \to \infty \quad \textit{and} \quad u_N^2(T+\overline{\Delta})|\ln T|^2 \to 0, \tag{A.15}$$

and

$$\widehat{\sigma}_t^2 - \sigma_t^2 = O_p(v_N), \tag{A.16}$$

for some $v_N \to 0$. Then, we have

$$\int_{\mathbb{R}} \left(\widehat{g}_t(x) - g_t(x)\right)^2 dx = O_p\left(u_N^{-2r} \bigvee u_N v_N^2 \bigvee u_N \frac{\overline{\Delta}}{\sqrt{T}} \bigvee u_N^5 \widetilde{\Gamma}_N\right), \tag{A.17}$$

where r is the constant in Assumption A1 and

$$\widetilde{\Gamma}_N = \overline{\Delta} \bigvee T(\ln^6 T)(e^{-2\underline{k}}|\underline{k}|^4 \wedge T^{-1/3}) \bigvee e^{6\underline{k}}|\underline{k}|^4 \bigvee e^{-6\overline{k}}|\overline{k}|^4. \tag{A.18}$$

The result in Theorem 4 is similar to that in Theorem 1 for $\widehat{h}_t$ with a few notable differences. First, the last three terms of $\widehat{\Gamma}_n$ are somewhat smaller than their counterparts in Γ_n. Recall that these terms are due to the approximation error resulting from the time variation in σ_t and ν_t as well as from the finite strike range of the options used in the estimation. This approximation error is slightly larger for the higher-order derivatives of $\widehat{f}_T(u)$ and hence the difference in the corresponding terms in $\widetilde{\Gamma}_n$ and Γ_n.

Second, the bound for the size of the integrated squared error of $\widehat{g}_t$ in probability in (A.17) contains the additional term $u_N\overline{\Delta}/\sqrt{T}$ which is not present in the bound for integrated squared error of $\widehat{h}_t$ we derived in Theorem 1. The reason for the presence of this additional bound here is the observation error in the options with k in the vicinity of zero. Relative to $\widehat{h}_t$, $\widehat{g}_t$ loads more heavily on these options. This is to be expected as these options contain stronger signal for the "small" jumps. At the same time, since the option error is assumed to be proportional to the option price it is attached to (see Assumption A6), this means that the option error for the options with log-strikes around zero will be bigger in absolute terms than the one for options with log-strikes away from zero.

Third, and quite naturally, given the debiasing of $\widehat{g}_t^*(u)$ by $\widehat{\sigma}_t^2$ in the construction of $\widehat{g}_t(x)$, the integrated squared error of $\widehat{g}_t$ depends on the size of the estimation error in $\widehat{\sigma}_t^2$. This is captured by the second term on the right hand side of (A.17). There are different volatility estimators that can be used. One option is given by $-2\Re(\ln \widehat{f}_T(u_N))/(Tu_N^2)$ in an analogy to the integrated volatility estimator of [4] constructed from high-frequency returns of x. Another alternative is the Black-Scholes implied volatility constructed from the options with log-strikes in the vicinity of zero. Yet another option is to use high-frequency returns in a local neighborhood of the observation time of the options and construct a spot volatility estimator, see e.g., Section 9.4 of [2].

4. Proofs of Theorems 2-4.

4.1. *Proof of Theorem 2.* In the proof, it suffices to restrict attention to the set Ω_n^c (Ω_n is defined in the proof of Theorem 1) as we have shown in the proof of Theorem 1 that $\mathbb{P}(\Omega_n) \to 0$. It is also no restriction to assume $T < \bar{t} - t$, for $\bar{t}$ being the constant in Lemmas 2-6. We will do so without further mention.

Given the Plancherel's identity, it suffices to provide a lower bound for $\int_{|u|\leq u_N} |\widehat{h}_t^*(u) - h_t^*(u)|^2 du$. We will use the notation of the proof of Theorem

1 in what follows (see the decomposition in (8.10) in particular and note that due to our Lévy assumption for x here, $\widehat{f}_{T,2}(u)$ is zero). Using Taylor series expansion, we have

$$
\begin{aligned}
\widehat{h}_t^*(u) - h_t^*(u) = {} & \frac{i}{T}\frac{\widehat{f}_{T,1}^{(3)}(u)}{f_T(u)} - \frac{i}{T}\frac{f_T^{(3)}(u)\widehat{f}_{T,1}(u)}{f_T(u)^2} - \frac{3i}{T}\frac{\widehat{f}_{T,1}^{(1)}(u)f_T^{(2)}(u)}{f_T(u)^2} \\
& - \frac{3i}{T}\frac{f_T^{(1)}(u)\widehat{f}_{T,1}^{(2)}(u)}{f_T(u)^2} + \frac{6i}{T}\frac{f_T^{(1)}(u)f_T^{(2)}(u)\widehat{f}_{T,1}(u)}{f_T(u)^3} \\
& + \frac{6i}{T}\frac{f_T^{(1)}(u)^2\widehat{f}_{T,1}^{(1)}(u)}{f_T(u)^3} - \frac{6i}{T}\frac{f_T^{(1)}(u)^3\widehat{f}_{T,1}(u)}{f_T(u)^4} + r_t(u),
\end{aligned} \tag{A.19}
$$

where $r_t(u)$ is a residual term for which using the results in (8.11), (8.14), the fact that $\sup_{|u|\le u_N}|\widehat{f}_{T,3}^{(j)}(u)| = o(1)$ and $\sup_{|u|\le u_N}|\widehat{f}_{T,4}^{(j)}(u)| = o(1)$, for $j = 0,1,2,3$ (because of (8.16)-(8.18) and the assumptions for α, β and γ of the theorem), the bounds in (8.16)-(8.18) as well as $u_N^2 T|\ln T| \to 0$ and the assumptions for α, β and γ of the theorem, we have

$$
\begin{aligned}
T|r_t(u)| \le {} & \overline{\eta}_{N,1}(u)\sum_{k=3,4}\sum_{j=0,1,2}|\widehat{f}_{T,k}^{(j)}(u)| + \overline{\eta}_{N,2}(u)\sum_{k=3,4}|\widehat{f}_{T,k}^{(3)}(u)| \\
& + \overline{\eta}_{N,3}(u)\sum_{j=0,1,2,3}|\widehat{f}_{T,1}^{(j)}(u)|^2,
\end{aligned} \tag{A.20}
$$

for some nonnegative-valued $\overline{\eta}_{N,1}(u)$, $\overline{\eta}_{N,2}(u)$ and $\overline{\eta}_{N,3}(u)$ satisfying

$$
\sup_{|u|\le u_N}\overline{\eta}_{N,1}(u) = O_p\left(\sqrt{T} \vee u_N^2 T|\ln T| \vee u_N\sqrt{T}\overline{\Delta}|\ln T|\right),
$$

and

$$
\sup_{|u|\le u_N}\overline{\eta}_{N,2}(u) = o_p(1) \quad \text{and} \quad \sup_{|u|\le u_N}\overline{\eta}_{N,3}(u) = o_p(1).
$$

To bound this residual term $r_t(u)$, we will then make use of the following inequality for $p = 0,1,2,3$:

$$
\begin{aligned}
\mathbb{E}\left(\sum_{j=2}^N \cos(uk_{j-1})e^{-k_{j-1}}k_{j-1}^p\epsilon_j\overline{\Delta}\right)^4 \le {} & C\left(\sum_{j=2}^N e^{-2k_{j-1}}k_{j-1}^{2p}O_T^2(k_{j-1})\overline{\Delta}^2\right)^2 \\
& + C\sum_{j=2}^N e^{-4k_{j-1}}k_{j-1}^{4p}O_T^4(k_{j-1})\overline{\Delta}^4,
\end{aligned} \tag{A.21}
$$

which follows from the i.i.d. assumption for the sequence $\{z_i\}_i$, successive application of Burkholder-Davis-Gundy inequality as well as $\mathbb{E}(z_i^4) < \infty$. From here, making use of the integrability condition for the Lévy measure in the theorem as well as Lemmas 2 and 5, we have

$$\sup_{u\in\mathbb{R}} \mathbb{E}\left(\sum_{j=2}^{N} \cos(uk_{j-1})e^{-k_{j-1}}\epsilon_j \overline{\Delta}\right)^4 = O\left(T^3\overline{\Delta}^2 \vee T^{5/2}\overline{\Delta}^3\right), \tag{A.22}$$

$$\sup_{u\in\mathbb{R}} \mathbb{E}\left(\sum_{j=2}^{N} \cos(uk_{j-1})e^{-k_{j-1}}k_{j-1}^p\epsilon_j \overline{\Delta}\right)^4 = O\left(T^4\overline{\Delta}^2\right), \quad p=1,2,3, \tag{A.23}$$

and the same two bounds hold obviously with the cosine replaced by sine. Making use of these bounds as well as the bounds in (8.11)-(8.18) in the proof of Theorem 1 and the rate conditions $u_N^5\overline{\Delta} \to 0$ in the theorem, we have

$$\int_{|u|\leq u_N} |r_t(u)|^2 du = o_p\left(u_N^5\overline{\Delta}\right), \tag{A.24}$$

and

$$\int_{|u|\leq u_N} |\widehat{h}_t^*(u) - h_t^*(u) - r_t(u)|^2 du = O_p\left(u_N^5\overline{\Delta}\right). \tag{A.25}$$

From here, applying Cauchy-Schwarz inequality, we have

$$\int_{|u|\leq u_N} |\widehat{h}_t^*(u) - h_t^*(u)|^2 du = \int_{|u|\leq u_N} |\widehat{h}_t^*(u) - h_t^*(u) - r_t(u)|^2 du + o_p\left(u_N^5\overline{\Delta}\right). \tag{A.26}$$

Using the definition of $f_T(u)$ in (8.7), we have for $|u| \leq u_N$ that $|f_T^{(1)}(u)| \leq C\sqrt{T}$, $|f_T^{(2)}(u)| + |f_T^{(3)}(u)| \leq CT$, because $u_N^2 T \to 0$. Making use of this fact, Lemma 8 as well as $u_N^2 T \to 0$ again, we have

$$\int_{|u|\leq u_N} |\widehat{h}_t^*(u) - h_t^*(u)|^2 du = \int_{|u|\leq u_N} |\widetilde{h}_t^*(u)|^2 du + o_p\left(u_N^5\overline{\Delta}\right), \tag{A.27}$$

where

$$\widetilde{h}_t^*(u) = \frac{u^2}{T}\sum_{j=2}^{N} e^{(iu-1)k_{j-1}}k_{j-1}^3\epsilon_j\Delta_j. \tag{A.28}$$

Furthermore, we can write

$$\int_{|u|\le u_N} |\widetilde{h}_t^*(u)|^2 du = 2\int_0^{u_N} \left[\Re(\widetilde{h}_t^*(u))^2 + \Im(\widetilde{h}_t^*(u))^2\right] du, \tag{A.29}$$

and given the assumption for ϵ_i in the theorem,

$$\begin{aligned} &\mathbb{E}\left(\Re(\widetilde{h}_t^*(u))^2\right) + \mathbb{E}\left(\Im(\widetilde{h}_t^*(u))^2\right) \\ &\qquad = \frac{u^4}{T^2}\sigma_\epsilon^2 \sum_{j=2}^N e^{-2k_{j-1}} k_{j-1}^6 O_T^2(k_{j-1})\Delta_j^2. \end{aligned} \tag{A.30}$$

By application of Burkholder-Davis-Gundy inequality (due to the fact that $\{z_i^2-1\}_i$ is i.i.d. sequence with mean zero and finite second moment) and then Lemma 2, we have

$$\begin{aligned} &\mathbb{E}\left(\int_{|u|\le u_N} (|\widetilde{h}_t^*(u)|^2 - \mathbb{E}|\widetilde{h}_t^*(u)|^2) du\right)^2 \\ &\le C\frac{u_N^{10}}{T^4}\sum_{j=2}^N e^{-4k_{j-1}} k_{j-1}^{12} O_T^4(k_{j-1})\Delta_j^4 \le C u_N^{10}\overline{\Delta}^3(|\underline{k}| + \overline{k}). \end{aligned} \tag{A.31}$$

Therefore,

$$\int_{|u|\le u_N} (|\widetilde{h}_t^*(u)|^2 - \mathbb{E}|\widetilde{h}_t^*(u)|^2) du = O_p(u_N^5 \overline{\Delta}^{3/2} |\ln T|), \tag{A.32}$$

and hence this term is $o_p(u_N^5\overline{\Delta})$. We are thus left with analyzing the term $\int_{|u|\le u_N} \mathbb{E}(|\widetilde{h}_t^*(u)|^2)du$ which we can write as

$$\int_{|u|\le u_N} \mathbb{E}(|\widetilde{h}_t^*(u)|^2)du = \frac{2u_N^5}{5T^2}\sigma_\epsilon^2 \sum_{j=2}^N e^{-2k_{j-1}} k_{j-1}^6 O_T^2(k_{j-1})\Delta_j^2. \tag{A.33}$$

Using Lemmas 2 and 7 and taking into account $\overline{\Delta}(|\underline{k}| + \overline{k}) \to 0$, we further have

$$\int_{|u|\le u_N} \mathbb{E}(|\widetilde{h}_t^*(u)|^2)du = \frac{2u_N^5}{5T^2}\sigma_\epsilon^2\overline{\Delta}\int_{\mathbb{R}} e^{-2k}k^6 O_T^2(k)dk + O\left(u_N^5\overline{\Delta}^2\right). \tag{A.34}$$

Therefore, we are left with analyzing $\int_{\mathbb{R}} e^{-2k}k^6 O_T^2(k)dk$ for which we will first derive an expansion of $O_T(k)$ for $T \downarrow 0$.

Since jumps are of finite activity and the process x is Lévy, we have $\mathbb{P}(\mu([t,t+T],\mathbb{R})>1)\leq CT^2$, where recall μ denotes the counting measure for the jumps in x. Using this fact, the inequalities in the proof of Lemma 2, as well as Hölder's inequality and the integrability assumption in the theorem for ν, we have for some $0<\iota<\frac{p-4}{2}$ (where p is the constant in the statement of the theorem)

$$\mathbb{E}_t^{\mathbb{Q}}[(e^k-e^{x_{t+T}})^+1(\mu([t,t+T],\mathbb{R})>1)]\leq C\frac{e^{2k}T^{\frac{2+2\iota}{2+\iota}}}{(e^{-k}-1)},\quad \text{for } k<0, \tag{A.35}$$

$$\mathbb{E}_t^{\mathbb{Q}}[(e^{x_{t+T}}-e^k)^+1(\mu([t,t+T],\mathbb{R})>1)]\leq C\frac{T^{\frac{2+2\iota}{2+\iota}}}{(e^{k}-1)},\quad \text{for } k>0, \tag{A.36}$$

where the constant $C>0$ does not depend on k. Similarly, by an application of Burkholder-Davis-Gundy inequality twice, we have for $\iota\in(0,2)$:

(A.37)
$$\mathbb{E}_t^{\mathbb{Q}}\left|\int_t^{t+T}\int_{\mathbb{R}}\eta(s,z)\mu(ds,dz)\right|^{2+\iota}\leq C\mathbb{E}_t^{\mathbb{Q}}\left|\int_t^{t+T}\int_{\mathbb{R}}\eta(s,z)ds\nu(z)dz\right|^{2+\iota}+$$
$$C\mathbb{E}_t^{\mathbb{Q}}\left|\int_t^{t+T}\int_{\mathbb{R}}\eta^2(s,z)ds\nu(z)dz\right|^{1+\iota/2}+C\mathbb{E}_t^{\mathbb{Q}}\left(\int_t^{t+T}\int_{\mathbb{R}}|\eta(s,z)|^{2+\iota}ds\nu(z)dz\right),$$

for arbitrary predictable in s function $\eta(s,z)$, and therefore by application of Hölder's inequality for some sufficiently small $\iota>0$, we have

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}&\left[\int_{\mathbb{R}}(e^k-e^{z+x^c_{t+T}})^+\mu(ds,dz)1(\mu([t,t+T],\mathbb{R})>1)\right]\\&\leq Ce^{2k}T^{\frac{2+2\iota}{2+\iota}}/(e^{-k}-1),\quad\text{for } k<0,\end{aligned}\tag{A.38}$$

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}&\left[\int_{\mathbb{R}}(e^{z+x^c_{t+T}}-e^k)^+\mu(ds,dz)1(\mu([t,t+T],\mathbb{R})>1)\right]\\&\leq CT^{\frac{2+2\iota}{2+\iota}}/(e^{k}-1),\quad\text{for } k>0,\end{aligned}\tag{A.39}$$

where recall x^c_s denotes the continuous component of x and where we made use of the independence of the processes x^c and x^d due to the Lévy assumption for the process x of the current theorem. In addition, using direct computations, it is easy to show

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{Q}}(e^k-e^{x^c_{t+T}})^+\leq Ce^{-p\frac{k^2}{T}},\ \text{if } k\leq-\sqrt{T},\\&\mathbb{E}_t^{\mathbb{Q}}(e^{x^c_{t+T}}-e^k)^+\leq Ce^{-p\frac{k^2}{T}},\ \text{if } k\geq\sqrt{T},\end{aligned}\tag{A.40}$$

for some constants $C > 0$ and $p > 0$ that depend on the values of a, σ and $\bar{t}$. From here, if we denote

$$\widetilde{O}^d(k) = \begin{cases} \mathbb{E}_t^{\mathbb{Q}}\left(\int_{\mathbb{R}}(e^k - e^{z+x^c_{t+T}})^+\nu(z)dz\right), & \text{if } k \leq 0, \\ \mathbb{E}_t^{\mathbb{Q}}\left(\int_{\mathbb{R}}(e^{z+x^c_{t+T}} - e^k)^+\nu(z)dz\right), & \text{if } k > 0, \end{cases} \tag{A.41}$$

we have

$$\int_{\mathbb{R}} e^{-2k}k^6 O_T^2(k)dk - T^2\int_{\mathbb{R}} e^{-2k}k^6\widetilde{O}^d(k)^2dk \leq CT^{\iota}, \tag{A.42}$$

for some $\iota > 0$ sufficiently small. We further have

(A.43)
$$|\widetilde{O}^d(k) - O^d(k)| \leq \mathbb{E}_t^{\mathbb{Q}}\left[|e^{x^c_{t+T}} - 1|\int_{z\vee(z+x^c_{t+T})>k} e^z\nu(z)dz\right], \quad \text{for } k > 0,$$

with an analogous inequality holding for the case $k \leq 0$ as well. Therefore, we have

$$\int_{\mathbb{R}} e^{-2k}k^6\widetilde{O}^d(k)^2dk - \int_{\mathbb{R}} e^{-2k}k^6O^d(k)^2dk = O(\sqrt{T}). \tag{A.44}$$

From here, the claim of the theorem follows.

4.2. *Proof of Theorem 3.* Since $T \downarrow 0$, it is no restriction to assume $T < \bar{t} - t$ and we will do so in the proof without further mention. We will also define with K some positive constant which can change from line to line and depends on R in (A.5)-(A.8).

The idea of the proof is to perturb locally the time-t Lévy measure and then derive the order of magnitude of the Kullback-Leibler divergence of the resulting two probability distributions of the observed option prices. For this, we use similar set of functions to those used in the proof in Section 7.1 of [1] which are given as follows. We let $\psi^{(j)} \in C^\infty(\mathbb{R})$ be a function with support in $[0, 1]$ and further satisfying $||\psi^{(j)}||_{L^2} = 1$, $\int_{\mathbb{R}}\psi^{(j)}(x)dx = 0$ and $\int_{\mathbb{R}}\psi^{(j)}(x)e^{-2^{-j}x}dx = 0$. We then introduce the functions

$$\psi_{jk}(x) = 2^{j/2}\psi^{(j)}(2^jx - k), \quad j \geq 0, k = 0, ..., 2^j - 1. \tag{A.45}$$

Finally, let $\varrho = (\varrho_k) \in \{-1, +1\}^{2^j}$. Then, the local perturbation of the time-t Lévy measure ν_t^0 we consider (which we take to be such that $\inf_{x\in[1,2]}\nu_t^0(x) > 0$), is given by

$$\nu_t^{\varrho}(x) = \nu_t^0(x) + \zeta 2^{-j(r+1/2)}e^{-x}\sum_{k=0}^{2^j-1}\varrho_k\psi_{jk}(x-1), \quad x \in \mathbb{R}, \tag{A.46}$$

for some $\zeta > 0$. For sufficiently small ζ, the risk-neutral probability measure $\mathcal{T}_\varrho$ corresponding to ν_t^ϱ will satisfy (A.4)-(A.8), and hence will belong to $\mathcal{G}_r(R)$, provided that $\mathcal{T}_0$, corresponding to ν_t^0, is picked in the interior of $\mathcal{G}_r(R)$ which we assume to be the case henceforth.

Then, by an application of Assouad's lemma (Lemma 2.12 and Example 2.2 in [5]), we have

$$\inf_{\widehat{h}_t} \sup_{\mathcal{T}\in\mathcal{G}_r(R)} \mathbb{E}_\mathcal{T}\left(||\widehat{h}_t - h_t||^2_{L^2(\mathbb{R})}\middle|\mathcal{F}^{(0)}\right) \gtrsim 2^j ||h_t^\varrho - h_t^{\varrho'}||^2_{L^2(\mathbb{R})} \sim 2^{-2jr}, \tag{A.47}$$

where $h_t^\varrho(x) = x^3\nu_t^\varrho(x)$ and $h_t^{\varrho'}(x) = x^3\nu_t^{\varrho'}(x)$, and provided that the Kullback-Leibler (KL) divergence between $\mathcal{T}_\varrho$ and $\mathcal{T}_{\varrho'}$ remains uniformly bounded by a constant, where ϱ and ϱ' are equal for every $k = 0, ..., 2^j - 1$ except one k_0. The rest of the proof is devoted to determining the order of magnitude of the KL divergence in our setting which in turn will allow us to pick the minimal rate at which $2^j \to \infty$ and such that the KL divergence stays uniformly bounded.

The KL divergence between the $\mathcal{F}^{(0)}$-conditional probability measures for the observed noisy option prices, corresponding to $\mathcal{T}$ with $h_t^{(1)}$ and $h_t^{(2)}$ is given by

$$\begin{aligned} KL(h_t^{(1)}, h_t^{(2)}) = {} & \mathbb{E}_{\mathcal{T}_1}\left(\sum_{i=1}^N \frac{(\widehat{O}_T(k_i) - O_{T,2}(k_i))^2}{2(O_{T,2}(k_i)\vee T)^2}\middle|\mathcal{F}^{(0)}\right) \\ & - \mathbb{E}_{\mathcal{T}_1}\left(\sum_{i=1}^N \frac{(\widehat{O}_T(k_i) - O_{T,1}(k_i))^2}{2(O_{T,1}(k_i)\vee T)^2}\middle|\mathcal{F}^{(0)}\right) \\ & + \mathbb{E}_{\mathcal{T}_1}\left(\sum_{i=1}^N \log\left(\frac{O_{T,2}(k_i)\vee T}{O_{T,1}(k_i)\vee T}\right)\middle|\mathcal{F}^{(0)}\right), \end{aligned} \tag{A.48}$$

where $\mathcal{T}_1$ corresponds to true option prices being generated from the risk-neutral probability with $x^3\nu_t(x)$ equal to $h_t^{(1)}(x)$. We have

$$\mathbb{E}_{\mathcal{T}_1}\left((\widehat{O}_T(k_i) - O_{T,2}(k_i))^2\middle|\mathcal{F}^{(0)}\right) = (O_{T,1}(k_i) - O_{T,2}(k_i))^2 + (O_{T,1}(k_i)\vee T)^2,$$

and

$$\mathbb{E}_{\mathcal{T}_1}\left((\widehat{O}_T(k_i) - O_{T,1}(k_i))^2\middle|\mathcal{F}^{(0)}\right) = (O_{T,1}(k_i)\vee T)^2.$$

Using these results, we can further simplify the expression for $KL(h_t^{(1)}, h_t^{(2)})$

to:

(A.49)
$$KL(h_t^{(1)},h_t^{(2)}) = \sum_{i=1}^{N} \frac{(O_{T,1}(k_i)-O_{T,2}(k_i))^2}{2(O_{T,2}(k_i)\vee T)^2} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(\frac{O_{T,1}(k_i)\vee T}{O_{T,2}(k_i)\vee T}\right)^2 - 1 - \log\left(\frac{O_{T,1}(k_i)\vee T}{O_{T,2}(k_i)\vee T}\right)^2\right),$$

where the option prices corresponding to $h_t^{(j)}$ are denoted with $O_{T,j}(k)$ for $j=1,2$.

Using second-order Taylor expansion and the fact that $O_T(k) \leq K \times T$ for $\mathbb{Q} \in \mathcal{G}_r(R)$ and T small enough (which follows by taking into account that there is no diffusion and then applying Lemma 1), we have

$$KL(h_t^{(1)},h_t^{(2)}) \lesssim \frac{1}{T^2}\sum_{i=1}^{N}(O_{T,1}(k_i)-O_{T,2}(k_i))^2. \quad \text{(A.50)}$$

Using Lemma 3 (note that in the special case here $\overline{O}_T(k) = \widetilde{O}_T(k)$ for $\mathbb{Q} \in \mathcal{G}_r(R)$), we have

$$KL(h_t^{(1)},h_t^{(2)}) \lesssim \frac{1}{T^2}\sum_{i=1}^{N}(\widetilde{O}_{T,1}(k_i)-\widetilde{O}_{T,2}(k_i))^2 + \frac{|\ln T|}{\sqrt{T}}\sum_{i=1}^{N}|\widetilde{O}_{T,1}(k_i)-\widetilde{O}_{T,2}(k_i)| + TN|\ln T|^2. \quad \text{(A.51)}$$

Next, by using the notation μ^x and δ^x of Section 8.1, we set $\mathcal{A} = \{s \in [t,t+T] : \Delta\widetilde{x}_s \neq 0\}$. Then, $\mathcal{F}_t$-conditionally, $\mu^x(\mathcal{A})$ is the count of jumps in the interval $[t,t+T]$ of a compound Poisson process with compensator $ds\nu_t(x)dx$ (this follows using Grigelionis representation, see e.g., Theorem 2.1.2 of [2]), and therefore we have

$$\mathbb{Q}_t\left(\mu^x(\mathcal{A}) > 1\right) \leq KT^2. \quad \text{(A.52)}$$

Using the inequality $(e^{\widetilde{x}_{t+T}} - e^k)^+ \leq e^{a_tT}\left(|e^{\widetilde{x}^d_{t+T}}-1| + |e^{-a_tT}-1|\right)$ for $k>0$ and analogous one for $(e^k - e^{\widetilde{x}_{t+T}})^+$ for $k<0$ (recall here $\widetilde{x}^c_s = a_t(s-t)$ for $s \geq t$), as well as $\mathbb{E}^{\mathbb{Q}}_t|e^{\widetilde{x}^d_{t+T}}-1|^3 + \mathbb{E}^{\mathbb{Q}}_t|e^{-\widetilde{x}^d_{t+T}}-1|^3 \leq KT$ which follows from Lemma 1, we have

$$\mathbb{E}^{\mathbb{Q}}_t[(e^k - e^{\widetilde{x}_{t+T}})^+]^3 \leq KT, \quad k<0, \quad \text{(A.53)}$$

$$\mathbb{E}_t^{\mathbb{Q}}[(e^{\widetilde{x}_{t+T}} - e^k)^+]^3 \leq KT, \quad k > 0. \tag{A.54}$$

From here, by applying Hölder's inequality (with powers 3 and 3/2 for the first and second term, respectively), we have for $\mathbb{Q} \in \mathcal{G}_r(R)$:

$$\mathbb{E}_t^{\mathbb{Q}}[(e^k - e^{\widetilde{x}_{t+T}})^+ 1(\mu^x(\mathcal{A}) > 1)] \leq KT^{5/3}, \quad k < 0, \tag{A.55}$$

$$\mathbb{E}_t^{\mathbb{Q}}[(e^{\widetilde{x}_{t+T}} - e^k)^+ 1(\mu^x(\mathcal{A}) > 1)] \leq KT^{5/3}, \quad k > 0. \tag{A.56}$$

We similarly have

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+T}\int_E (e^k - e^{\delta^x(t,z)+a_t T})^+ \mu^x(ds,dz) 1(\mu^x(\mathcal{A}) > 1)\right] \\ \leq KT^{5/3}, \quad k < 0, \end{aligned} \tag{A.57}$$

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+T}\int_E (e^{\delta^x(t,z)+a_t T} - e^k)^+ \mu^x(ds,dz) 1(\mu^x(\mathcal{A}) > 1)\right] \\ \leq KT^{5/3}, \quad k > 0. \end{aligned} \tag{A.58}$$

In addition

$$\begin{aligned} \left|\int_{\mathbb{R}} (e^k - e^{z+a_t T})^+ \nu_t(z) dz - \int_{\mathbb{R}} (e^k - e^z)^+ \nu_t(z) dz\right| \\ \leq |e^{a_t T} - 1| \int_{\mathbb{R}} e^z \nu_t(z) dz \leq KT, \quad k < 0, \end{aligned} \tag{A.59}$$

and a similar inequality holds for the corresponding difference in the case $k > 0$.

From here, by looking separately on the events in which there is at most one jump in x in the interval $[t, t+T]$ and its complement, we have for $\mathbb{Q} \in \mathcal{G}_r(R)$:

$$\left|\widetilde{O}_T(k) - T\int_{\mathbb{R}} \left(e^k - e^x\right)^+ \nu_t(x) dx\right| \leq KT^{5/3}, \quad k < 0, \tag{A.60}$$

$$\left|\widetilde{O}_T(k) - T\int_{\mathbb{R}} \left(e^x - e^k\right)^+ \nu_t(x) dx\right| \leq KT^{5/3}, \quad k > 0. \tag{A.61}$$

Furthermore, for $h_t^{(1)} = h_t^{\varrho}$ and $h_t^{(2)} = h_t^{\varrho'}$, and using the above bounds as well as $\int_{\mathbb{R}} \psi^{(j)}(x) dx = 0$ and $\int_{\mathbb{R}} \psi^{(j)}(x) e^{-2^{-j}x} dx = 0$, we have

$$|\widetilde{O}_{T,1}(x) - \widetilde{O}_{T,2}(x)| \leq \begin{cases} KT^{5/3}, & \text{for } x \leq 1 + \frac{k_0}{2^j} \text{ or } x \geq 1 + \frac{k_0+1}{2^j}, \\ KT^{5/3} + KT2^{-jr-2j}, & \text{for } x \in \left[1 + \frac{k_0}{2^j}, 1 + \frac{k_0+1}{2^j}\right]. \end{cases} \tag{A.62}$$

Therefore, altogether we get

$$KL(h_t^{\varrho}, h_t^{\varrho'}) \lesssim \frac{2^{-j(2r+5)}}{\overline{\Delta}} + |\ln T|\frac{\sqrt{T}}{\overline{\Delta}}2^{-jr-3j} + TN|\ln T|^2. \tag{A.63}$$

From here, by taking into account the restrictions for α, β and γ in the theorem, we have boundedness for $KL(h_t^{\varrho}, h_t^{\varrho'})$ if we set $2^{-j(2r+5)} \sim \overline{\Delta}$, and hence the result of the theorem follows.

4.3. *Proof of Theorem 4.* The proof follows the same steps as the proof of Theorem 1 and as in that proof we set for simplicity $u_N = -u_{N,l} = u_{N,h}$. By Plancherel's identity we can decompose

(A.64)

$$\int_{\mathbb{R}} (\widehat{g}_t(x) - g_t(x))^2 \, dx = \underbrace{\frac{1}{2\pi}\int_{|u|\le u_N} \left|\widehat{g}_t^*(u) - g_t^*(u) + \widehat{\sigma}_t^{\,2} - \sigma_t^2\right|^2 du}_{IV} + \underbrace{\frac{1}{2\pi}\int_{|u|>u_N} |g_t^*(u)|^2 du}_{IB}.$$

By Assumption A1 with h replaced by g, for the bias due to the truncation of the higher frequencies, we have $IB = O(u_N^{-2r})$. Using triangular inequality we have,

$$IV \le \frac{1}{\pi}\int_{|u|\le u_N} |\widehat{g}_t^*(u) - g_t^*(u)|^2 \, du + 2\frac{u_N}{\pi}\left(\widehat{\sigma}_t^{\,2} - \sigma_t^2\right)^2. \tag{A.65}$$

Therefore, we are left with the analysis of $\int_{|u|\le u_N} |\widehat{g}_t^*(u) - g_t^*(u)|^2 \, du$. For it, using the notation of the proof of Theorem 1, it suffices to restrict attention to the set Ω_n^c on which we have $\inf_{|u|\le u_N} |\widehat{f}_T(u)| \ge \epsilon/2$ for some fixed $\epsilon > 0$. We will further restrict attention to a subset of Ω_n^c on which $\sup_{|u|\le u_N}\sum_{k=1}^4\sum_{j=0,1,2} |\widehat{f}_{T,k}^{(j)}(u)| < \epsilon$ (recall the notation $\widehat{f}_{T,k}^{(j)}(u)$ from the proof of Theorem 1). The probability of this subset of Ω_n^c is going to one due to the bounds for $\widehat{f}_{T,k}^{(j)}(u)$ derived in the proof of Theorem 1 as well as the condition $u_N^2(T+\overline{\Delta})|\ln T|^2 \to 0$ of the theorem. Finally, we will assume that $T < \overline{t} - t$, for $\overline{t}$ being the constant in Lemmas 2-6.

Given the above and upon using Taylor expansion, we have

(A.66)

$$\begin{aligned}
T|\widehat{g}_t^*(u) - g_t^*(u)| \le\ & C_t|f_T^{(1)}(u)||\widehat{f}_T^{(1)}(u) - f_T^{(1)}(u)| + C_t|\widehat{f}_T^{(1)}(u) - f_T^{(1)}(u)|^2 \\
& + C_t|\widehat{f}_T^{(2)}(u) - f_T^{(2)}(u)| + C_t|f_T^{(1)}(u)|^2|\widehat{f}_T(u) - f_T(u)|^2 \\
& + C_t(|f_T^{(1)}(u)^2 f_T(u)| + |f_T^{(2)}(u)|)|\widehat{f}_T(u) - f_T(u)|, \ |u| \le u_N,
\end{aligned}$$

for some $\mathcal{F}_t$-adapted and finite valued positive C_t (which depends on ϵ). Taking into account the bounds for $f_T^{(j)}(u)$, with $j = 0, 1, 2$, from the proof of Theorem 1 as well as the fact that we are constraining attention to the set on which $\sup_{|u|\le u_N}\sum_{j=0,1,2}|\widehat{f}_T^{(j)}(u)| < \epsilon$, this bound simplifies further to

(A.67)
$$\begin{aligned}T|\widehat{g}_t^*(u) - g_t^*(u)| \le C_t\sqrt{T}|\widehat{f}_T^{(1)}(u) - f_T^{(1)}(u)| + C_t|\widehat{f}_T^{(1)}(u) - f_T^{(1)}(u)|^2 \\ + C_t|\widehat{f}_T^{(2)}(u) - f_T^{(2)}(u)| + C_tT|\widehat{f}_T(u) - f_T(u)|,\ |u| \le u_N.\end{aligned}$$

Further, using notation as in the proof of Theorem 1, we have

$$\begin{aligned}T|\widehat{g}_t^*(u) - g_t^*(u)| \le C_tT\sum_{k=1}^4|\widehat{f}_{T,k}(u)| + C_t\sum_{k=1}^4|\widehat{f}_{T,k}^{(2)}(u)| \\ + C_t\left(\sqrt{T} + \sum_{k=1}^4\sup_{|u|\le u_N}|\widehat{f}_{T,k}^{(1)}(u)|\right)\sum_{k=1}^4|\widehat{f}_{T,k}^{(1)}(u)|,\ |u| \le u_N.\end{aligned} \tag{A.68}$$

From here, the result of the theorem follows by application of the bounds on $\widehat{f}_{T,k}^{(j)}(u)$ for $j = 0, 1, 2$, and $k = 1, 2, 3, 4$ derived in the proof of Theorem 1 and taking into account the restriction $\alpha > \frac{1}{2}$ and (A.15).

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