Econometric Analysis of Jump-Driven Stochastic Volatility Models

Technical Appendix

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1 Preliminary Estimates for Integrated Variance

I start with a Lemma that will be used for the derivation of the moments of the return process below.

Lemma 1 (Moments of the Integrated Variance)

For the integrated variance $IV_a(t) = \int_t^{t+a}\sigma^2(s)ds$, *where* $\sigma^2(t) = \int_{-\infty}^t\int_{\mathbb{R}_0^n} f(t-s)k(\mathbf{x})\mu(ds, d\mathbf{x})$ *assume that the following holds* $\int_0^\infty f(s)ds < \infty$, $\int_0^\infty f^2(s)ds < \infty$, $\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) < \infty$ *and* $\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) < \infty$. *Then for any* $s \le t$ *we have*

$$\mathbb{E}_s(IV_a(t)) = \int_{-\infty}^s\int_{\mathbb{R}_0^n} H^a(t,u)k(\mathbf{x})\tilde{\mu}(du, d\mathbf{x}) + a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}), \tag{1}$$

$$\mathbb{E}(IV_a(t)) = a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}), \tag{2}$$

$$Var(IV_a(t)) = \int_{-\infty}^a (H^a(0,u))^2du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}). \tag{3}$$

Proof: I start with deriving the first two moments of the spot variance $\sigma^2(t)$. Under the conditions in the Lemma we have

$$\mathbb{E}(\sigma^2(t)) = \mathbb{E}\Big(\int_{-\infty}^t\int_{R_0^n} f(t-s)k(\mathbf{x})\mu(ds, d\mathbf{x})\Big) = \int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}). \tag{4}$$

For the mean of the squared spot variance under the conditions in the Lemma we have

$$\begin{aligned}\mathbb{E}(\sigma^4(t)) &= \mathbb{E}\Big(\int_{-\infty}^t\int_{R_0^n} f(t-s)k(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}) + \int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}))\Big)^2 \\ &= \int_0^\infty f^2(s)ds\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + \Big(\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}))\Big)^2.\end{aligned} \tag{5}$$

Then for the mean of the integrated variance we have

$$\mathbb{E}(IV_a(t)) = \mathbb{E}\Big(\int_t^{t+a} \sigma^2(s)ds,\Big) = a\mathbb{E}(\sigma^2(s)). \tag{6}$$

and (2) follows from (4).

Next I derive the result in (3) for the variance of the integrated variance. First note that for $s \le t$

$$\mathbb{E}\Big(\int_s^t \sigma^2(u)du\Big)^2 \le (t-s)\mathbb{E}\Big(\int_s^t \sigma^4(u)du\Big),$$

and therefore the finiteness of the second moment of the integrated variance follows from the conditions in the Lemma. Then using the expression for the integrated variance we have

$$\begin{aligned}
\mathbb{E}(IV_a^2(t)) &= \mathbb{E}\Big(\int_t^{t+a} \sigma^2(s)ds\Big)^2 \\
&= \mathbb{E}\Big(\int_{-\infty}^a \int_{\mathbb{R}_0^n} H^a(0,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + \int_{-\infty}^a H^a(0,u)du \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&= \int_{-\infty}^a (H^a(0,u))^2 du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + \Big(\int_{-\infty}^a H^a(0,u)du \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&= \int_{-\infty}^a (H^a(0,u))^2 du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + \Big(a\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2,
\end{aligned} \tag{7}$$

and from here using (2), the result for the variance of the integrated variance in (3) follows.

Finally I prove the claim in (1) for the conditional expectation of the integrated variance.

$$\begin{aligned}
\mathbb{E}_s(IV_a(t)) &= \mathbb{E}_s\Big(\int_t^{t+a} \sigma^2(s)ds\Big) \\
&= \mathbb{E}_s\Big(\int_{-\infty}^{t+a} \int_{\mathbb{R}_0^n} H^a(t,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + \int_{-\infty}^{t+a} H^a(t,u)du \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big) \\
&= \int_{-\infty}^s \int_{\mathbb{R}_0^n} H^a(t,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + a\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}).
\end{aligned} \tag{8}$$

□

2 Moments of Jump-Diffusion JDSV Model

Theorem 1 (Moments of the Jump-Diffusion JDSV model)

For the Jump-Diffusion JDSV model assume that conditions **H1-H4** *in the paper are satisfied. Then we have*

$$Var(r^a(t)) = a\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) + a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{9}$$

$$\mathbb{E}(r^a(t) - \mathbb{E}(r^a(t)))^3 = 3\int_0^a H^a(0,u)du \int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}) + a\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}), \tag{10}$$

$$\mathbb{E}(r^a(t)-\mathbb{E}(r^a(t)))^4 = 3\int_{-\infty}^{a}(H^a(0,u))^2du\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x}) + 3\Big(a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2$$
$$+a\int_{\mathbb{R}_0^n}g^4(\mathbf{x})G(d\mathbf{x}) + 6a^2\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})$$
$$+6\int_0^a H^a(0,u)du\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + 3a^2\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\Big)^2. \tag{11}$$

and for $h = a, 2a, 3a....$ we also have

$$\begin{aligned}&Cov((r_a(0)-\mathbb{E}(r_a(0)))^2,(r_a(h)-\mathbb{E}(r_a(h)))^2) = \\ &\int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + \int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x}).\end{aligned} \tag{12}$$

Proof: In the proof of the theorem about I use the following notation

$$X(t) = \int_0^t \sigma(s-)dW(s), \tag{13}$$

and

$$Y(t) = \int_0^t\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}). \tag{14}$$

Therefore over the period $(t, t+a]$, the return process is split into continuous and discontinuous martingale parts, denoted respectively as

$$X^a(t) = X(t+a) - X(t), \tag{15}$$

$$Y^a(t) = Y(t+a) - Y(t). \tag{16}$$

Therefore

$$r_a(t) = a\alpha + X^a(t) + Y^a(t). \tag{17}$$

Given the integrability conditions **H1**-**H3** in the paper it is not hard to derive the joint characteristic function of arbitrary number of returns.

This characteristic function could be used to prove all the theorems about moments of the return process. Computations are somewhat involved and therefore in the proofs I decided not to use the characteristic function.

Since the integrals $X^a(t)$ and $Y^a(t)$ are with respect to martingale measures we have

$$\mathbb{E}(X^a(0)) = 0, \tag{18}$$

and

$$\mathbb{E}(Y^a(0)) = \int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}) = 0. \tag{19}$$

To derive the variance of the returns, I derive the variance of its continuous and discontinuous parts. For the continuous part, using the integrability of the integrated variance established in Lemma 1 and Ito isometry we have

$$\mathrm{Var}(X^a(0)) = \mathbb{E}(X^a(0))^2 = \mathbb{E}\Big(\int_0^a \sigma^2(s-)ds\Big) = a\mathbb{E}(\sigma^2(s)), \tag{20}$$

For the variance of the jump component of the returns, I use condition **H1** in the paper and Ito isometry to get

$$\mathrm{Var}(Y^a(0)) = \mathbb{E}(Y^a(0))^2 = a \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}). \tag{21}$$

Using

$$\mathrm{Var}(r^a(0)) = \mathrm{Var}(X^a(0)) + \mathrm{Var}(Y^a(0)), \tag{22}$$

the result in (9) follows.

Now I derive the third central moment in (10). I make use of the fact that the filtration generated by the random measure μ is orthogonal to $W(t)$ and therefore conditional on it $X^a(t)$ is normal. This implies

$$\begin{aligned}\mathbb{E}(r^a(0) - \mathbb{E}(r^a(0)))^3 &= \mathbb{E}(X^a(0))^3 + 3\mathbb{E}(X^a(0)^2 Y^a(0)) + 3\mathbb{E}(X^a(0) Y^a(0)^2) + \mathbb{E}(Y^a(0))^3 \\ &= 3\mathbb{E}\left(\int_0^a \sigma^2(s) ds \int_0^a \int_{\mathbb{R}_0^n} g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x})\right) + a \int_{\mathbb{R}_0^n} g^3(\mathbf{x}) G(d\mathbf{x}),\end{aligned} \tag{23}$$

where I made use of

$$\mathbb{E}(X^a(0))^3 = 0 \qquad \text{and} \qquad \mathbb{E}(X^a(0)(Y^a(0)^2)) = 0, \tag{24}$$

and provided conditions **H1** and **H4** in the paper are satisfied

$$\mathbb{E}(Y^a(0))^3 = a \int_{\mathbb{R}_0^n} g^3(\mathbf{x}) G(d\mathbf{x}). \tag{25}$$

Further I simplify the first term in equation (23) by using the representation of the integrated variance as an integral with respect to the Poisson measure

$$\begin{aligned}&\mathbb{E}\left(\int_0^a \sigma^2(s) ds \int_0^a \int_{\mathbb{R}_0^n} g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x})\right) \\ &= \mathbb{E}\left[\left(\int_{-\infty}^a \int_{\mathbb{R}_0^n} H^a(0,u) k(\mathbf{x}) \tilde{\mu}(du, d\mathbf{x}) + \int_{-\infty}^a H^a(0,u) du \int_{\mathbb{R}_0^n} k(\mathbf{x}) G(d\mathbf{x})\right) \int_0^a \int_{\mathbb{R}_0^n} g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x})\right] \\ &= \int_0^a H^a(0,u) du \int_{\mathbb{R}_0^n} k(\mathbf{x}) g(\mathbf{x}) G(d\mathbf{x}),\end{aligned} \tag{26}$$

and the result in (10) follows.

I proceed with the expression for the forth central moment in (11). Similar argument as the one used for the second and third central moments leads to

$$\begin{aligned}\mathbb{E}(r^a(0) - \mathbb{E}(r^a(0)))^4 &= \mathbb{E}(X^a(0))^4 + \mathbb{E}(Y^a(0))^4 + 6\mathbb{E}(X^a(0))^2 (Y^a(0))^2 \\ &= 3\mathbb{E}\left(\int_0^a \sigma^2(s-) ds\right)^2 + a^2 \left(\int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x})\right)^2 \\ &\quad + a \int_{\mathbb{R}_0^n} g^4(\mathbf{x}) G(d\mathbf{x}) \\ &\quad + 6\mathbb{E}\left[\left(\int_0^a \sigma^2(s-) ds\right)\left(\int_0^a \int_{\mathbb{R}_0^n} g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x})\right)^2\right],\end{aligned} \tag{27}$$

where I made use of

$$\mathbb{E}(X^a(0))^4 = 3\mathbb{E}\Big(\int_0^a \sigma^2(s-)ds\Big)^2, \tag{28}$$

and (provided condition (11) is satisfied)

$$\mathbb{E}(Y^a(0))^4 = a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2. \tag{29}$$

The first term in equation (27) is just the expected value of the squared integrated variance and consequently the expression in (3) could be used to simplify. Now I simplify the last term in (27).

By Ito's lemma we have

$$\begin{aligned} Y^2(t) - Y^2(s) &= \int_s^t 2Y(u-)dY(u) + \sum_{s<u\leq t}(Y^2(u) - Y^2(u-) - 2Y(u-)\Delta Y(u)) \\ &= \int_s^t 2Y(u-)dY(u) + \sum_{s<u\leq t}(\Delta Y(u))^2, \end{aligned} \tag{30}$$

from where using the integrability condition **H1** in the paper we have

$$Y^2(t) - Y^2(s) = \int_s^t \int_{\mathbb{R}_0^n}(2Y(u-)g(\mathbf{x}) + g^2(\mathbf{x}))\tilde{\mu}(du, d\mathbf{x}) + (t-s)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{31}$$

therefore provided the conditions **H1**-**H4** in the paper are satisfied, using the square-integrability of the integrated variance established in Lemma 1 the last term in (27) could be further written as

$$\begin{aligned} &\mathbb{E}\ \Big[\Big(\int_0^a \sigma^2(s-)ds\Big)\Big(\int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\Big)^2\Big] \\ = &\ \mathbb{E}\Big[\Big(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(0,u)k(\mathbf{x})\tilde{\mu}(du, d\mathbf{x}) + \int_{-\infty}^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)\Big(\int_0^a\int_{\mathbb{R}_0^n}(2Y(s-)g(\mathbf{x}) \\ &+ g^2(\mathbf{x}))\tilde{\mu}(ds, d\mathbf{x}) + a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)\Big] \\ = &\ a\int_{-\infty}^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) + \int_0^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})g^2(\mathbf{x})G(d\mathbf{x}), \end{aligned} \tag{32}$$

plugging this expression back in (27), together with the expression for the expected value of the square of the integrated variance we get (11).

I finish the proof of the Theorem with showing the result in equation (12) for the covariance of the squared demeaned returns.

I begin with

$$\begin{aligned} \mathbb{E}[(X^a(0) + Y^a(0))^2(X^a(h) + Y^a(h))^2] &= \mathbb{E}[((X^a(0))^2 + (Y^a(0))^2)((X^a(h))^2 + (Y^a(h))^2)] \\ &= \mathbb{E}((X^a(0))^2(X^a(h))^2) + \mathbb{E}((X^a(0))^2(Y^a(h))^2) \\ &\quad + \mathbb{E}((Y^a(0))^2(X^a(h))^2) + \mathbb{E}((Y^a(0))^2(Y^a(h))^2), \end{aligned} \tag{33}$$

and treat each of the four terms separately.

1. Expression for $\mathbb{E}((Y^a(0))^2(Y^a(h))^2)$.

Using the time-homogeneity property of the Poisson random measure

$$\mathbb{E}((Y^a(0))^2(Y^a(h))^2) = \mathbb{E}(Y^a(0))^2\mathbb{E}(Y^a(h))^2 = a^2\bigg(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\bigg)^2. \tag{34}$$

2. Expression for $\mathbb{E}((X^a(0))^2(X^a(h))^2)$.

By conditioning on the filtration generated by the spot variance, making use of the fact that the Brownian motion $W(t)$ is independent from the spot variance process and using the fact that $\sigma^2(s)$ has no fixed time of discontinuity we have

$$\begin{aligned}
\mathbb{E}[(X^a(0))^2(X^a(h))^2] &= \mathbb{E}\bigg[\bigg(\int_0^a \sigma(s-)dW(s)\bigg)^2\bigg(\int_h^{h+a} \sigma(s-)dW(s)\bigg)^2\bigg] \\
&= \mathbb{E}\bigg[\int_0^a \sigma^2(s-)ds\int_h^{h+a} \sigma^2(s-)ds\bigg] \\
&= \mathbb{E}(IV_a(0)IV_a(h)).
\end{aligned} \tag{35}$$

Therefore using the result in (1) for the conditional expectation of the integrated variance

$$\begin{aligned}
&\mathbb{E}[(X^a(0))^2(X^a(h))^2] = \mathbb{E}\bigg[\int_0^a \sigma^2(s-)ds\mathbb{E}_a\bigg(\int_h^{h+a} \sigma^2(s-)ds\bigg)\bigg] \\
&= \mathbb{E}\bigg[\int_0^a \sigma^2(s-)ds\bigg(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(h,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)\bigg] \\
&= \mathbb{E}\bigg[\int_0^a \sigma^2(s-)ds\bigg(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(h,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x})\bigg)\bigg] + a^2\bigg(\int_0^\infty f(s)ds\bigg)^2\bigg(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2,
\end{aligned} \tag{36}$$

where $\mathbb{E}_a(\cdot)$ stands for $\mathbb{E}(\cdot|\mathcal{F}_a)$.

With the help of the representation of the integrated variance as an integral with respect to the random measure μ we finally have

$$\begin{aligned}
\mathbb{E}[(X^a(0))^2(X^a(h))^2] &= \int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) \\
&\quad + a^2\bigg(\int_0^\infty f(s)ds\bigg)^2\bigg(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2.
\end{aligned} \tag{37}$$

From here

$$\mathrm{Cov}((X^a(0))^2,(X^a(h))^2) = \int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}). \tag{38}$$

3. Expression for $\mathbb{E}((X^a(0))^2(Y^a(h))^2)$.

Using the fact that the Brownian motion $W(t)$ is independent from the filtration generated by the random measure μ and the properties of the Poisson random measure we have

$$\begin{aligned}
\mathbb{E}[(X^a(0))^2(Y^a(h))^2] &= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds(Y^a(h))^2\Big] \\
&= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds\Big(\int_h^{h+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)^2\Big] \\
&= a^2\mathbb{E}(\sigma^2(s))\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\
&= a^2\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned} \tag{39}$$

4. Expression for $\mathbb{E}((Y^a(0))^2(X^a(h))^2)$.

Using the expression in (1) for the conditional expected value of the integrated variance, conditions **H1-H4** in the paper, as well as the square-integrability of $Y(s)$ we have

$$\begin{aligned}
\mathbb{E}[(Y^a(0))^2(X^a(h))^2] &= \mathbb{E}\Big[(Y^a(0))^2\int_h^{h+a}\sigma^2(s-)ds\Big] \\
&= \mathbb{E}\Big[(Y^a(0))^2\mathbb{E}_a\int_h^{h+a}\sigma^2(s-)ds\Big] \\
&= \mathbb{E}\Big[\int_0^a\int_{\mathbb{R}_0^n}(2Y(s-)g(\mathbf{x})+g^2(\mathbf{x}))\tilde{\mu}(ds,d\mathbf{x})\int_{-\infty}^a\int_{\mathbb{R}_0^n}H^a(h,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x})\Big] \\
&\quad +a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\mathbb{E}\Big[\int_0^a\int_{\mathbb{R}_0^n}(2Y(s-)g(\mathbf{x})+g^2(\mathbf{x}))\mu(ds,d\mathbf{x})\Big] \\
&= \int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&\quad +a^2\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x}).
\end{aligned} \tag{40}$$

Combining the expressions for $\mathbb{E}((Y^a(0))^2(Y^a(h))^2)$, $\mathbb{E}((X^a(0))^2(X^a(h))^2)$, $\mathbb{E}((X^a(0))^2(Y^a(h))^2)$, $\mathbb{E}((Y^a(0))^2(X^a(h))^2)$ into equation (33) we have

$$\begin{aligned}
\mathbb{E}[(X^a(0)+Y^a(0))^2(X^a(h)+Y^a(h))^2] &= \int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&\quad +\int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x}) \\
&\quad +a^2\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})+\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2.
\end{aligned} \tag{41}$$

Using the fact that

$$\mathbb{E}(r_a(0)-\mathbb{E}(r_a(0))^2 = \mathbb{E}[(X^a(0)+Y^a(0))^2] = a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})+a\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x}), \tag{42}$$

we finally have

$$
\begin{aligned}
&\operatorname{Cov}((r_a(0)-\mathbb{E}(r_a(0)))^2,(r_a(h)-\mathbb{E}(r_a(h)))^2)=\operatorname{Cov}[(X^a(0)+Y^a(0))^2,(X^a(h)+Y^a(h))^2] \\
&=\mathbb{E}[((X^a(0))^2+(Y^a(0))^2)(X^a(h))^2+(Y^a(h))^2)]-\Big(\mathbb{E}(r_a(0)-\mathbb{E}(r_a(0)))^2\Big)^2.
\end{aligned}
\tag{43}
$$

and hence the result in (12) follows. □

3 Moments of the Pure-Jump JDSV Model

Theorem 2 (Moments of the Pure-Jump JDSV Model)
For the pure-jump JDSV model assume conditions **S1-S5** *in the paper hold. Then we have*

$$
Var(r_a(t))=a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x}),
\tag{44}
$$

$$
\begin{aligned}
&E(r^a(t)-\mathbb{E}(r_a(t)))^4=a\int_0^\infty f^2(s)ds\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^4(\mathbf{x})G(d\mathbf{x}) \\
&+a\Big(\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2\int_{\mathbb{R}_0^n}g^4(\mathbf{x})G(d\mathbf{x}) \\
&+6\int_0^a\int_0^s f(s-u)duds\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&+3a^2\Big(\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&+6\int_0^a\int_{-\infty}^u H^u(0,s)f(u-s)dsdu\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x})\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\Big)^2,
\end{aligned}
\tag{45}
$$

and for $h=a,2a,...$

$$
\begin{aligned}
Cov(r_a^2(t),r_a^2(t+h))=&\int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x})\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
+&\int_0^\infty f(s)ds\int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
\tag{46}
$$

Proof: I introduce similar notation to the one used in the proofs of the theorems for the jump-diffusion JDSV model.

$$
Y(t)=\int_0^t\int_{\mathbb{R}_0^n}\sigma(s-)g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}),
\tag{47}
$$

and for the period $(t,t+a]$

$$
Y^a(t)=Y(t+a)-Y(t),
$$

therefore the return over the period $(t,t+a]$ is

$$
r_a(t)=a\alpha+Y^a(t).
$$

Under assumptions **S1** and **S2** in the paper, $Y(t)$ in (47) is a square-integrable martingale. Before proceeding with the derivations of the moments in the theorem I establish that under the conditions **S1**-**S4** in the paper we have $\mathbb{E}(\sup_{s\leq t} Y^4(t)) < \infty$ for every t. For this I use the Burkholder-Davis-Gundy inequality (see Protter (2004)), from which it is sufficient to establish the finiteness of $\mathbb{E}([Y,Y]_{[0,t]})^2$. The quadratic variation of the jump martingale $Y(t)$ is given by

$$[Y,Y]_{[0,t]} = \int_0^t \int_{\mathbb{R}_0^n} \sigma^2(s-) g^2(\mathbf{x}) \mu(ds, d\mathbf{x}),$$

therefore using conditions **S1**-**S3** in the paper, combined with the result for the finiteness of the second moment of $\int_0^t \sigma^2(s)ds$ shown in Lemma 1 we have

$$\begin{aligned}
\mathbb{E}([Y,Y]_{[0,t]})^2 &= \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}_0^n} \sigma^2(s-) g^2(\mathbf{x}) \mu(ds, d\mathbf{x}) \bigg)^2 \\
&= \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}_0^n} \sigma^2(s-) g^2(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x}) + \int_0^t \sigma^2(s) ds \int_{\mathbf{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}) \bigg)^2 \\
&\leq 2\mathbb{E}\bigg(\int_0^t \sigma^4(s-) ds \bigg) \int_{\mathbb{R}_0^n} g^4(\mathbf{x}) G(d\mathbf{x}) + 2\mathbb{E}\bigg(\bigg(\int_0^t \sigma^2(s) ds \bigg) \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}) \bigg)^2 < \infty.
\end{aligned}$$

This establishes the claim.

I proceed with deriving the moments of the return process. First we trivially have

$$\mathbb{E}(r_a(0)) = a\alpha + \mathbb{E}(Y(a)) = a\alpha.$$

For the variance of the returns I make use of the fact that $Y(t)$ is a square-integrable martingale.

$$\begin{aligned}
\mathrm{Var}(r_a(0)) &= E\bigg(\int_0^a \int_{\mathbb{R}_0^n} \sigma(s-) g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x}) \bigg)^2 \\
&= E\bigg(\int_0^a \sigma^2(s) ds \bigg) \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}) \\
&= aE(\sigma^2(s)) \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}),
\end{aligned}$$

and the result in (44) for the variance of the returns follows from the result in (4).

I prove now the result for the forth central moment of the returns in equation (45).

$$E(r_a(0) - \mathbb{E}(r_a(0)))^4 = E(Y(a))^4.$$

By Ito's lemma

$$Y^4(t) = Y^4(s) + \int_s^t 4Y^3(u-) dY(u) + \sum_{s<u\leq t} \Big((\Delta Y(u))^4 + 4Y(u-)(\Delta Y(u))^3 + 6Y^2(u-)(\Delta Y(u))^2 \Big),$$

and using conditions **S1-S5** in the paper we can further write

$$
\begin{aligned}
Y^4(t) \;=\; & \int_s^t \Big(4Y^3(u-)\sigma(u-)g(\mathbf{x}) + 6Y^2(u-)\sigma^2(u-)g^2(\mathbf{x}) + 4Y(u-)\sigma(u-)^3 g^3(\mathbf{x}) + \sigma^4(u-)g^4(\mathbf{x})\Big)\tilde{\mu}(du, d\mathbf{x}) \\
& + \int_s^t \sigma^4(u)du \int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 4\int_s^t Y(u)\sigma^3(u)du \int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}) \\
& + 6\int_s^t Y^2(u)\sigma^2(u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) + Y^4(s).
\end{aligned}
$$

Now I show that the first integral is of integrable variation. I demonstrate this with several inequalities. Using the integrability of $\sup_{s<u\leq t} Y^4(u)$ and $\sigma^4(s)$, combined with the Hölder's inequality we have

$$
\begin{aligned}
\mathbb{E}\Big(\int_s^t Y^3(u-)\sigma(u-)du\Big) &\leq \mathbb{E}\Big(\sup_{s<u\leq t}|Y^3(u)| \int_s^t \sigma(u-)du\Big) \\
&\leq \Big(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\Big)^{3/4} \Big(\mathbb{E}\Big(\int_s^t \sigma(u-)du\Big)^4\Big)^{1/4} \\
&\leq (t-s)^{3/4}\Big(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\Big)^{3/4} \Big(\mathbb{E}\Big(\int_s^t \sigma^4(u-)du\Big)\Big)^{1/4},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\Big(\int_s^t Y^2(u-)\sigma^2(u-)du\Big) &\leq \mathbb{E}\Big(\sup_{s<u\leq t}|Y^2(u)| \int_s^t \sigma^2(u-)du\Big) \\
&\leq \Big(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\Big)^{1/2} \Big(\mathbb{E}\Big(\int_s^t \sigma^2(u-)du\Big)^2\Big)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\Big(\int_s^t Y(u-)\sigma^3(u-)du\Big) &\leq \mathbb{E}\Big(\sup_{s<u\leq t}|Y(u)| \int_s^t \sigma^3(u-)du\Big) \\
&\leq \Big(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\Big)^{1/4} \Big(\mathbb{E}\Big(\int_s^t \sigma^3(u-)du\Big)^{4/3}\Big)^{3/4} \\
&\leq (t-s)^{1/3}\Big(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\Big)^{1/4} \Big(\mathbb{E}\Big(\int_s^t \sigma^4(u-)du\Big)\Big)^{3/4},
\end{aligned}
$$

Using these results, the fact that $\tilde{\mu}$ is a martingale random measure and the condition **S5** in the paper we have

$$
\begin{aligned}
E(r_a(0) - \mathbb{E}(r_a(0)))^4 \;=\; & aE(\sigma^4(s))\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 4a\int_0^a E(Y(s)\sigma^3(s))ds \int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}) \\
& + 6\int_0^a E(Y^2(s)\sigma^2(s))ds \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

For the first term I make use of the result in (5). For the second term I use the assumption **S5** in the paper. For the last term using again Ito's lemma we have

$$
\begin{aligned}
E(Y^2(t)\sigma^2(t)) &= E\Bigg[\bigg(\int_0^t \int_{\mathbb{R}_0^n} (2Y(s-)\sigma(s-)g(\mathbf{x}) + \sigma^2(s-)g^2(\mathbf{x}))\tilde{\mu}(ds, d\mathbf{x}) + \int_0^t \sigma^2(s)ds \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\bigg) \\
&\quad \times \bigg(\int_{-\infty}^t f(t-s)k(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}) + \int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)\Bigg].
\end{aligned}
$$

To continue further I make use of the following inequality

$$
\mathbb{E}\bigg(\int_0^t Y^2(s-)\sigma^2(s-)ds\bigg) < \infty, \tag{48}
$$

which was established above. This combined with the square-integrability of the state variable $\sigma^2(t)$ and the properties of the square-integrable martingales finally gives

$$
\begin{aligned}
E(Y^2(t)\sigma^2(t)) &= 2\int_0^t E(Y(s)\sigma(s))f(t-s)ds \int_{\mathbb{R}_0^n} g(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&\quad + E(\sigma^2(u))\int_0^t f(t-s)ds \int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&\quad + tE(\sigma^2(u))\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) \\
&\quad + \int_{-\infty}^t H^t(0,s)f(t-s)ds \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

Using again the assumption in **S5** in the paper and combining everything together I have the result for the forth central moment in (45).

I finish by showing the result for the covariance of the returns.

$$
\mathrm{Cov}(r_a^2(0), r_a^2(h)) = E(r_a^2(0)r_a^2(h)) - (E(r_a^2(0)))^2
$$

Using the fact that $Y(t)$ is defined with respect to the martingale random measure $\tilde{\mu}$ we have

$$
\begin{aligned}
E(r_a^2(0)r_a^2(h)) &= E(Y^a(0)Y^a(h))^2 \\
&= E\bigg[Y^2(a)\Big(Y(h+a) - Y(h)\Big)^2\bigg] \\
&= E\bigg[Y^2(a)\Big(Y^2(h+a) - Y^2(h)\Big)\bigg].
\end{aligned}
$$

To simplify further I apply the Ito's formula and condition **S1** in the paper

$$
\begin{aligned}
Y^2(t) - Y^2(s) &= \int_s^t 2Y(u-)dY(u) + \sum_{s<u\le t} (\Delta Y(u))^2 \\
&= \int_s^t \int_{\mathbb{R}_0^n} 2Y(u-)\sigma(u-)g(\mathbf{x})\tilde{\mu}(du, d\mathbf{x}) + \int_s^t \int_{\mathbb{R}_0^n} \sigma^2(u-)g^2(\mathbf{x})\mu(du, d\mathbf{x}) \\
&= \int_s^t \int_{\mathbb{R}_0^n} \Big(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du, d\mathbf{x}) + \int_s^t \sigma^2(u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

Using this fact I can write

$$
\begin{aligned}
E(r_a^2(0)r_a^2(h)) &= E\Big[\Big(\int_0^a \int_{\mathbb{R}_0^n} \Big(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du, d\mathbf{x}) + \int_0^a \sigma^2(u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big) \\
&\quad \times\Big(\int_h^{h+a} \int_{\mathbb{R}_0^n} \Big(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du, dx) + \int_h^{h+a} \sigma^2(u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)\Big] \\
&= E\Big(\int_0^a \sigma^2(s)ds \int_h^{h+a} \sigma^2(s)ds\Big)\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&\quad + E\Big(\int_0^a \int_{\mathbb{R}_0^n} \Big(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du, d\mathbf{x}) \int_h^{h+a} \sigma^2(u)du\Big) \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

I simplify each of the two terms separately. For the first term I use the result in (1) and get

$$
\begin{aligned}
E\Big(\int_0^a \sigma^2(s)ds \int_h^{h+a} \sigma^2(s)ds\Big) &= a^2\Big(\int_0^\infty f(s)ds\Big)^2\Big(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&\quad + \int_{-\infty}^a H^a(h,u)H^a(0,u)du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

For the second term we use the representation of the integrated variance as integral against the Poisson measure μ and the square-integrability of $\sigma^2(s)$ to get

$$
\begin{aligned}
&E\Big(\int_0^a \int_{\mathbb{R}_0^n} \Big(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du, d\mathbf{x}) \int_h^{h+a} \sigma^2(u)du\Big) \\
&= E\Big(\int_0^a \int_{\mathbb{R}_0^n} \Big(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du, d\mathbf{x}) \int_{-\infty}^{h+a} \int_{\mathbb{R}_0^n} H^a(h,u)k(\mathbf{x})\mu(du, dx)\Big) \\
&= 2\int_0^a E(Y(u)\sigma(u))H^a(h,u)du \int_{\mathbb{R}_0^n} g(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + E(\sigma^2(u)) \int_0^a H^a(h,u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

Using the assumption in **S5** in the paper we get the result for the covariance of the return process in (46).

□

4 Moments of Power Variation Statistics for the JDSV Models with CARMA(2,1) kernel

Theorem 3 *For the jump-diffusion JDSV model with CARMA(2,1) kernel assume that the conditions* **H1**-**H4** *in the paper are satisfied and set* $\alpha = 0$*. Also let*

$$b_0 \geq -\max\{\rho_1, \rho_2\} > 0.$$

Then we have

$$\mathbb{E}(RV_\delta(t)) = \frac{b_0}{\rho_1\rho_2}\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) + \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{49}$$

$$\mathbb{E}(FV_\delta(t)) = \int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 6M\frac{\delta^2 b_0}{\rho_1\rho_2}\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) +$$
$$3M\bigg(\frac{1-e^{\rho_1\delta}}{\rho_1^3}\frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2} + \frac{1-e^{\rho_2\delta}}{\rho_2^3}\frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2} + \frac{\delta b_0^2}{\rho_1^2\rho_2^2}\bigg)\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + 3M\frac{\delta^2 b_0^2}{\rho_1^2\rho_2^2}\bigg(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2 +$$
$$6M\bigg(\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1\delta}-1}{\rho_1^2} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2\delta}-1}{\rho_2^2} + \frac{\delta b_0}{\rho_1\rho_2}\bigg)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + 3M\delta^2\bigg(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\bigg)^2, \tag{50}$$

$$Var(RV_\delta(t)) =$$
$$2\bigg(\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1\delta M}-Me^{\rho_1\delta}+M-1}{\rho_1^2} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2\delta M}-Me^{\rho_2\delta}+M-1}{\rho_2^2}\bigg)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x})$$
$$+\bigg(\frac{b_0^2-\rho_1^2}{\rho_1^3(\rho_1^2-\rho_2^2)}(e^{\rho_1\delta M}-Me^{\rho_1\delta}+M-1) + \frac{b_0^2-\rho_2^2}{\rho_2^3(\rho_2^2-\rho_1^2)}(e^{\rho_2\delta M}-Me^{\rho_2\delta}+M-1)\bigg)\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x})$$
$$+ M\mathbb{E}(r_\delta^4(0)) - M\bigg(\mathbb{E}(r_\delta^2(0))\bigg)^2. \tag{51}$$

For $i = 1, 2...$ we have

$$\mathbb{E}(r_1(t)RV_\delta(t+i))$$
$$= \bigg(\frac{b_0+\rho_1}{(\rho_1-\rho_2)\rho_1^2}e^{\rho_1(i-\delta M)}(e^{\rho_1\delta M}-1)^2 + \frac{b_0+\rho_2}{(\rho_2-\rho_1)\rho_2^2}e^{\rho_2(i-\delta M)}(e^{\rho_2\delta M}-1)^2\bigg)\int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}), \tag{52}$$

$$Cov(RV_\delta(t), RV_\delta(t-i)) = C_1 e^{\rho_1(i-\delta M)} + C_2 e^{\rho_2(i-\delta M)}, \tag{53}$$

where

$$C_1 = \frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{(e^{\rho_1\delta M}-1)^2}{\rho_1^2}\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + \frac{(b_0^2-\rho_1^2)(e^{\rho_1\delta M}-1)^2}{2\rho_1^3(\rho_1^2-\rho_2^2)}\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) \tag{54}$$

and

$$C_2 = \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{(e^{\rho_2\delta M}-1)^2}{\rho_2^2}\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + \frac{(b_0^2-\rho_2^2)(e^{\rho_2\delta M}-1)^2}{2\rho_2^3(\rho_2^2-\rho_1^2)}\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}). \tag{55}$$

Proof: First I compute the following expressions associated with the CARMA(2,1) kernel

$$\int_0^\infty f(s)ds = \frac{b_0}{\rho_1\rho_2},$$

$$\int_0^a H^a(0,u)du = \int_0^a\int_0^s f(s-u)duds = \frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1 a}-1}{\rho_1^2} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2 a}-1}{\rho_2^2} + \frac{ab_0}{\rho_1\rho_2},$$

$$\int_{-\infty}^a (H^a(0,u))^2du = \frac{1-e^{\rho_1 a}}{\rho_1^3}\frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2} + \frac{1-e^{\rho_2 a}}{\rho_2^3}\frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2} + \frac{ab_0^2}{\rho_1^2\rho_2^2},$$

$$\int_{-\infty}^{a} H^a(h,u)H^a(0,u)du = \frac{(1-e^{\rho_1 a})^2(b_0^2-\rho_1^2)}{2\rho_1^3(\rho_1^2-\rho_2^2)}e^{\rho_1(h-a)} + \frac{(1-e^{\rho_2 a})^2(b_0^2-\rho_2^2)}{2\rho_2^3(\rho_2^2-\rho_1^2)}e^{\rho_2(h-a)},$$

$$\int_{0}^{a} H^a(h,u)du = \frac{b_0+\rho_1}{\rho_1-\rho_2}\left(\frac{1-e^{\rho_1 a}}{\rho_1}\right)^2 e^{\rho_1(h-a)} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\left(\frac{1-e^{\rho_2 a}}{\rho_2}\right)^2 e^{\rho_2(h-a)},$$

$$\int_{-\infty}^{a} H^a(h,u)du = \frac{b_0+\rho_1}{\rho_2-\rho_1}\frac{e^{\rho_1 a}-1}{\rho_1^2}e^{\rho_1(h-a)} + \frac{b_0+\rho_2}{\rho_1-\rho_2}\frac{e^{\rho_2 a}-1}{\rho_2^2}e^{\rho_2(h-a)}.$$

The expressions for the mean of RV and FV follow from

$$\mathbb{E}(RV_\delta(t)) = M\mathbb{E}(r_\delta^2(s)) \qquad\qquad \mathbb{E}(FV_\delta(t)) = M\mathbb{E}(r_\delta^4(s)),$$

and the expressions for the variance and the forth central moment of the return process given in equations (9)-(11) respectively.

For the covariance and the variance of the RV I first compute the covariance between the squared demeaned high-frequency returns using the quantities for the CARMA(2,1) kernel calculated above and the result (12) in Theorem 1

$$\begin{aligned}
&\mathrm{Cov}(r_a^2(0), r_a^2(h)) = \\
&\left(\frac{b_0+\rho_1}{\rho_1-\rho_2}\left(\frac{1-e^{\rho_1 a}}{\rho_1}\right)^2 e^{\rho_1(h-a)} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\left(\frac{1-e^{\rho_2 a}}{\rho_2}\right)^2 e^{\rho_2(h-a)}\right)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&+\left(\frac{(1-e^{\rho_1 a})^2(b_0^2-\rho_1^2)}{2\rho_1^3(\rho_1^2-\rho_2^2)}e^{\rho_1(h-a)} + \frac{(1-e^{\rho_2 a})^2(b_0^2-\rho_2^2)}{2\rho_2^3(\rho_2^2-\rho_1^2)}e^{\rho_2(h-a)}\right)\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}$$

Then I use the following expression for the variance and covariance of RV

$$\mathrm{Cov}(RV_\delta(t), RV_\delta(t-i)) = \sum_{k=-(M-1)}^{M-1}(M-|k|)\mathrm{Cov}(r_\delta^2(0), r_\delta^2(i-k\delta)), \qquad \text{for } i=0,1,.... \tag{56}$$

To simplify further the expressions for the covariance between RV and past intraday returns, the covariance and variance of the RV I make use of the following expressions for arbitrary ρ

$$\sum_{k=-(M-1)}^{M-1}(M-|k|)e^{\rho(i-\delta k)} = M\sum_{k=-(M-1)}^{M-1}e^{\rho(i-\delta k)} - \left(\sum_{k=1}^{M-1}ke^{\rho(i-\delta k)} + \sum_{k=1}^{M-1}ke^{\rho(i+\delta k)}\right),$$

and

$$\sum_{k=1}^{M-1}ke^{\rho\delta k} = \frac{(M-1)e^{(M+1)\rho\delta} - Me^{M\rho\delta} + e^{\rho\delta}}{(e^{\rho\delta}-1)^2}. \tag{57}$$

Therefore

$$\sum_{k=-(M-1)}^{M-1}(M-|k|)e^{\rho(i-\delta k)} = e^{\rho\delta}\frac{(e^{\rho\delta M}-1)^2}{(e^{\rho\delta}-1)^2}e^{\rho(i-\delta M)}. \tag{58}$$

□

Theorem 4 *For the pure-jump JDSV model with CARMA(2,1) kernel, assume that the conditions* **S1**-**S4** *in the paper are satisfied. Also let*

$$b_0 \geq -\max\{\rho_1, \rho_2\} > 0.$$

Then if $\alpha = 0$ we have

$$\mathbb{E}(RV_\delta(t)) = \frac{b_0}{\rho_1\rho_2}\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)G(dx), \tag{59}$$

$$\begin{aligned}
E(FV_\delta(t)) \;=\; & \left(\frac{b_0^2-\rho_1^2}{2\rho_1(\rho_1^2-\rho_2^2)}+\frac{b_0^2-\rho_2^2}{2\rho_2(\rho_2^2-\rho_1^2)}\right)\int_{R_0^n} k^2(x)G(dx)\int_{R_0^n} g^4(x)G(dx)\\
&+\frac{b_0^2}{\rho_1^2\rho_2^2}\left(\int_{R_0^n} k(x)G(dx)\right)^2\int_{R_0^n} g^4(x)G(dx)\\
&+6M\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{(e^{\delta\rho_1}-1-\delta\rho_1)}{\rho_1^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+6M\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{(e^{\delta\rho_2}-1-\delta\rho_2)}{\rho_2^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+3\frac{\delta b_0^2}{\rho_1^2\rho_2^2}\left(\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\right)^2\\
&+6M\frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2}\frac{\delta\rho_1+1-e^{\delta\rho_1}}{2\rho_1^3}\int_{R_0^n} k^2(x)G(dx)\left(\int_{R_0^n} g^2(x)G(dx)\right)^2\\
&+6M\frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2}\frac{\delta\rho_2+1-e^{\delta\rho_2}}{2\rho_2^3}\int_{R_0^n} k^2(x)G(dx)\left(\int_{R_0^n} g^2(x)G(dx)\right)^2,
\end{aligned} \tag{60}$$

$$\begin{aligned}
Var(RV_\delta(t)) =& 2\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1\delta M}-Me^{\rho_1\delta}+M-1}{\rho_1^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+2\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2\delta M}-Me^{\rho_2\delta}+M-1}{\rho_2^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+\frac{b_0^2-\rho_1^2}{\rho_1^3(\rho_1^2-\rho_2^2)}(e^{\rho_1\delta M}-Me^{\rho_1\delta}+M-1)\int_{R_0^n} k^2(x)G(dx)\left(\int_{R_0^n} g^2(x)G(dx)\right)^2\\
&+\frac{b_0^2-\rho_2^2}{\rho_2^3(\rho_2^2-\rho_1^2)}(e^{\rho_2\delta M}-Me^{\rho_2\delta}+M-1)\int_{R_0^n} k^2(x)G(dx)\left(\int_{R_0^n} g^2(x)G(dx)\right)^2\\
&+ME(r_\delta^4(0)-M\left(E(r_\delta^2(0))\right)^2,
\end{aligned} \tag{61}$$

and for $i = 1, 2...$

$$Cov(RV_\delta(t), RV_\delta(t-i)) = C_1e^{\rho_1(i-\delta M)} + C_2e^{\rho_2(i-\delta M)}, \tag{62}$$

where

$$
\begin{aligned}
C1 =& \frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{(e^{\rho_1\delta M}-1)^2}{\rho_1^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+\frac{(b_0^2-\rho_1^2)(e^{\rho_1\delta M}-1)^2}{2\rho_1^3(\rho_1^2-\rho_2^2)}\int_{R_0^n} k^2(x)G(dx)\Big(\int_{R_0^n} g^2(x)G(dx)\Big)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
C2 =& \frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{(e^{\rho_2\delta M}-1)^2}{\rho_2^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+\frac{(b_0^2-\rho_2^2)(e^{\rho_2\delta M}-1)^2}{2\rho_2^3(\rho_2^2-\rho_1^2)}\int_{R_0^n} k^2(x)G(dx)\Big(\int_{R_0^n} g^2(x)G(dx)\Big)^2.
\end{aligned}
$$

Proof: The theorem is a direct consequence of the result in Theorem 2, the definitions of realized variance and forth power variation, the results used in the proof of Theorem 3, combined with the following additional properties associated with the CARMA(2,1) kernel.

$$
\begin{aligned}
cov(r_a^2(0), r_a^2(h)) \quad =& \quad \Bigg[\frac{(1-e^{\rho_1 a})^2(b_0^2-\rho_1^2)}{2\rho_1^3(\rho_1^2-\rho_2^2)}e^{\rho_1(h-a)}+\frac{(1-e^{\rho_2 a})^2(b_0^2-\rho_2^2)}{2\rho_2^3(\rho_2^2-\rho_1^2)}e^{\rho_2(h-a)}\Bigg]\\
&\times\int_{R_0^n} k^2(x)G(dx)\Big(\int_{R_0^n} g^2(x)G(dx)\Big)^2\\
&+\Bigg[\Big(\frac{b_0+\rho_1}{\rho_1-\rho_2}\Big)\Big(\frac{1-e^{\rho_1 a}}{\rho_1}\Big)^2 e^{\rho_1(h-a)}+\Big(\frac{b_0+\rho_2}{\rho_2-\rho_1}\Big)\Big(\frac{1-e^{\rho_2 a}}{\rho_2}\Big)^2 e^{\rho_2(h-a)}\Bigg]\\
&\times\frac{b_0}{\rho_1\rho_2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx),
\end{aligned} \tag{63}
$$

$$
\int_0^\infty f^2(s)ds = \frac{b_0^2-\rho_1^2}{2\rho_1(\rho_1^2-\rho_2^2)}+\frac{b_0^2-\rho_2^2}{2\rho_2(\rho_2^2-\rho_1^2)},
$$

$$
\int_{-\infty}^u H^u(0,s)f(u-s)ds = \frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2}\frac{1-e^{\rho_1 u}}{2\rho_1^2}+\frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2}\frac{1-e^{\rho_2 u}}{2\rho_2^2}.
$$

□

5 Moments of the Affine Jump-Diffusion Model

Theorem 5 *In the affine jump-diffusion model assume that $W(t)$, $B_1(t)$ and $B_2(t)$ are independent Brownian motions. Further assume that the parameters controlling the two variance factors satisfy*

$$
\kappa_i > 0, \qquad \theta_i > 0 \qquad \text{and} \qquad \sigma_i^2 \le 2\kappa_i\theta_i, \qquad \text{for } i = 1,2.
$$

Then for $\alpha = 0$

$$\mathbb{E}(RV_\delta(t)) = (\theta_1 + \theta_2) + \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{64}$$

$$\begin{aligned} Cov(RV_\delta(t), RV_\delta(t-i)) &= e^{-\kappa_1(i-\delta M)}\left(\frac{1-e^{-\kappa_1\delta M}}{\kappa_1}\right)^2\frac{\sigma_1^2\theta_1}{2\kappa_1} \\ &\quad + e^{-\kappa_2(i-\delta M)}\left(\frac{1-e^{-\kappa_2\delta M}}{\kappa_2}\right)^2\frac{\sigma_2^2\theta_2}{2\kappa_2}, \end{aligned} \tag{65}$$

$$\begin{aligned} Var(RV_\delta(t)) &= \frac{e^{-\kappa_1\delta M} - Me^{-\kappa_1\delta} + M - 1}{\kappa_1^2}\frac{\sigma_1^2\theta_1}{\kappa_1} + \frac{e^{-\kappa_2\delta M} - Me^{-\kappa_2\delta} + M - 1}{\kappa_2^2}\frac{\sigma_2^2\theta_2}{\kappa_2} \\ &\quad + M\mathbb{E}(r_\delta^4(0)) - M\left(\mathbb{E}(r_\delta^2(0))\right)^2, \end{aligned} \tag{66}$$

$$\begin{aligned} \mathbb{E}(FV_\delta(t)) &= 3M\delta^2(\theta_1+\theta_2)^2 + 6M\delta^2(\theta_1+\theta_2)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\ &\quad + M\delta\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3M\delta^2\left(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\right)^2 \\ &\quad + 3M\frac{\sigma_1^2\theta_1}{\kappa_1^3}(\delta\kappa_1 - 1 + e^{-\kappa_1\delta}) + 3M\frac{\sigma_2^2\theta_2}{\kappa_2^3}(\delta\kappa_2 - 1 + e^{-\kappa_2\delta}). \end{aligned} \tag{67}$$

Proof: By an application of Ito's theorem for i=1,2 we have

$$de^{\kappa_i t}V_i(t) = e^{\kappa_i t}\kappa_i\theta_i dt + \sigma_i e^{\kappa_i t}\sqrt{V_i(t)}dB_i(t),$$

therefore for $t \geq s$

$$\mathbb{E}(V_i(t)|\mathcal{F}_s) = e^{-\kappa_i(t-s)}V_i(s) + \theta_i(1 - e^{-\kappa_i(t-s)}). \tag{68}$$

Another application of Ito's theorem gives for $i = 1, 2$

$$de^{2\kappa_i t}V_i^2(t) = e^{2\kappa_i t}(2\kappa_i\theta_i + \sigma_i^2)V_i(t)dt + 2\sigma_i e^{2\kappa_i t}V_i(t)\sqrt{V_i(t)}dB_i(t),$$

and therefore for $t \geq s$

$$\mathbb{E}(V_i^2(t)|\mathcal{F}_s) = e^{-2\kappa_i(t-s)}V_i^2(s) + (2\kappa_i\theta_i + \sigma_i^2)\int_s^t e^{2\kappa_i(u-t)}\mathbb{E}(V_i(u)|\mathcal{F}_s)du,$$

and from here

$$\begin{aligned} \mathbb{E}(V_i^2(t)|\mathcal{F}_s) &= e^{-2\kappa_i(t-s)}V_i^2(s) + \left(\frac{2\kappa_i\theta_i + \sigma_i^2}{\kappa_i}\right)(e^{-\kappa_i(t-s)} - e^{-2\kappa_i(t-s)})V_i(t) \\ &\quad + \left(\frac{(2\kappa_i\theta_i + \sigma_i^2)\theta_i}{2\kappa_i}\right)(1 - e^{-\kappa_i(t-s)})^2. \end{aligned} \tag{69}$$

The first two unconditional moments of $V_i(t)$, provided it is covariance stationary process (which is the case under the conditions on the parameters in the theorem) are

$$\mathbb{E}(V_i(t)) = \theta_i \qquad \text{and} \qquad \mathbb{E}(V_i^2(t)) = \frac{(2\kappa_i\theta_i + \sigma_i^2)\theta_i}{2\kappa_i}.$$

For further use we also derive the following

$$\begin{aligned}\mathbb{E}(V_i(u)V_i(s)) &= \mathbb{E}(V_i(u)\mathbb{E}(V_i(s)|\mathcal{F}_u)) \\ &= e^{-\kappa_i(s-u)}\frac{\sigma_i^2\theta_i}{2\kappa_i} + \theta_i^2, \qquad \text{for } s \geq u,\ i = 1, 2.\end{aligned} \tag{70}$$

The mean of the squared returns over a period with length a is

$$\begin{aligned}\mathbb{E}(r_a^2(t)) &= \mathbb{E}\Big(\int_t^{t+a} V_i(s)ds\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\Big)^2 \\ &= a(\theta_1 + \theta_2) + a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).\end{aligned}$$

From here and the definition of realized variance, follows the expression in (64).

For the forth power of the return over period with length a, we have

$$\begin{aligned}\mathbb{E}(r_a^4(t)) &= \mathbb{E}\Big(\int_t^{t+a}\sqrt{V(t)}dW(t)\Big)^4 + 6\mathbb{E}\Big(\int_t^{t+a}\sqrt{V(t)}dW(t)\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\Big)^2 \\ &\quad + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\Big)^4 \\ &= 3\mathbb{E}\Big(\Big(\int_t^{t+a} V(s)ds\Big)^2\Big) + 6\mathbb{E}\Big(\int_t^{t+a} V(s)ds\Big)\mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\Big)^2 \\ &\quad + a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2 \\ &= 3\mathbb{E}\Big(\Big(\int_t^{t+a} V(s)ds\Big)^2\Big) + 6a^2(\theta_1 + \theta_2)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\ &\quad + a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2,\end{aligned} \tag{71}$$

where I made use of the fact that conditional on the volatility the continuous component of the return process is Gaussian.

To continue further I derive the mean of the squared integrated variance. I use the independence and stationarity of the two volatility factors as well as the result in (70) to get

$$\begin{aligned}\mathbb{E}\Big(\Big(\int_t^{t+a} V(s)ds\Big)^2\Big) &= \int_0^a\int_0^a \mathbb{E}(V(s)V(u)))dsdu \\ &= 2\int_0^a\int_u^a \mathbb{E}(V_1(s)V_1(u)))dsdu + 2\int_0^a\int_u^a \mathbb{E}(V_2(s)V_2(u)))dsdu + 2a^2\theta_1\theta_2 \\ &= a^2(\theta_1 + \theta_2)^2 + \frac{\sigma_1^2\theta_1}{\kappa_1^3}(e^{-\kappa_1 a} + a\kappa_1 - 1) + \frac{\sigma_2^2\theta_2}{\kappa_2^3}(e^{-\kappa_2 a} + a\kappa_2 - 1).\end{aligned}$$

From here and the definition of the forth variation its mean is easily deduced.

Next I derive the covariance of the squared returns over period with length a, which is in turn used for deriving the covariance and the variance of the realized variance. For $h = a, 2a, ...$

$$Cov(r_a^2(t), r_a^2(t+h)) = \mathbb{E}(r_a^2(t) r_a^2(t+h)) - (\mathbb{E}(r_a^2(t)))^2.$$

$$\begin{aligned}
\mathbb{E}(r_a^2(t) r_a^2(t+h)) &= \mathbb{E}\Big[\Big(\int_t^{t+a} \sqrt{V(s)} dW(s) + \int_t^{t+a} \int_{\mathbb{R}_0^n} g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x})\Big)^2 \\
&\quad \times \Big(\int_{t+h}^{t+h+a} \sqrt{V(s)} dW(s) + \int_{t+h}^{t+h+a} \int_{\mathbb{R}_0^n} g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x})\Big)^2\Big] \\
&= \mathbb{E}\Big(\int_t^{t+a} V(s) ds \int_{t+h}^{t+h+a} V(s) ds\Big) + 2a^2(\theta_1 + \theta_2) \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}) \\
&\quad + a^2 \Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x})\Big)^2,
\end{aligned} \tag{72}$$

where I made use of the independence of the continuous and the jump component of the returns, as well as the time-homogeneity of the Poisson random measure μ.

Therefore

$$\begin{aligned}
Cov(r_a^2(t), r_a^2(t+h)) &= \mathbb{E}\Big(\int_t^{t+a} V(s) ds \int_{t+h}^{t+h+a} V(s) ds\Big) - (\theta_1 + \theta_2)^2 \\
&= \mathbb{E}\Big(\int_t^{t+a} V_1(s) ds \int_{t+h}^{t+h+a} V_1(s) ds\Big) - a^2 \theta_1^2 \\
&\quad + \mathbb{E}\Big(\int_t^{t+a} V_2(s) ds \int_{t+h}^{t+h+a} V_2(s) ds\Big) - a^2 \theta_2^2.
\end{aligned} \tag{73}$$

Thus we need the covariance of the integrated variance. By the law of the iterated expectation we have

$$\mathbb{E}\Big(\int_t^{t+a} V_i(s) ds \int_{t+h}^{t+h+a} V_i(s) ds\Big) = \mathbb{E}\Big(\int_t^{t+a} V_i(s) ds \mathbb{E}\Big(\int_{t+h}^{t+h+a} V_i(s) ds \Big| \mathcal{F}_{t+a}\Big)\Big),$$

and using (68)

$$\mathbb{E}\Big(\int_{t+h}^{t+h+a} V_i(s) ds \Big| \mathcal{F}_{t+a}\Big) = \frac{V_i(t+a)}{\kappa_i}(e^{-\kappa_i(h-a)} - e^{-\kappa_i h}) + a\theta_i - \theta_i \frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}.$$

Therefore

$$\begin{aligned}
\mathbb{E}\Big(\int_t^{t+a} V_i(s) ds \int_{t+h}^{t+h+a} V_i(s) ds\Big) &= \mathbb{E}\Big(\int_t^{t+a} V_i(s) V_i(t+a) ds\Big)\Big(\frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}\Big) \\
&\quad + a\theta_i \Big(a\theta_i - \theta_i \frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}\Big).
\end{aligned} \tag{74}$$

Using the result in (70) we have

$$\mathbb{E}\left(\int_t^{t+a} V_i(s)ds \int_{t+h}^{t+h+a} V_i(s)ds\right) = \left(\frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}\right)\left(\frac{1-e^{-\kappa_i a}}{\kappa_i}\right)\frac{\sigma_i^2\theta_i}{2\kappa_i} + a^2\theta_i^2.$$

From here follows

$$Cov(r_a^2(t), r_a^2(t+h)) = e^{-\kappa_1(h-a)}\left(\frac{1-e^{-\kappa_1 a}}{\kappa_1}\right)^2\frac{\sigma_1^2\theta_1}{2\kappa_1} + e^{-\kappa_2(h-a)}\left(\frac{1-e^{-\kappa_2 a}}{\kappa_2}\right)^2\frac{\sigma_2^2\theta_2}{2\kappa_2}.$$

Then using the expression for the variance and the covariance in equation (56), as well as (57)-(58) we deduce the expressions in the theorem for the variance and the covariance of the realized variance.

□