EFFICIENT ESTIMATION OF INTEGRATED VOLATILITY IN PRESENCE OF INFINITE VARIATION JUMPS

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We propose new nonparametric estimators of the integrated volatility of an Itô semimartingale observed at discrete times on a fixed time interval with mesh of the observation grid shrinking to zero. The proposed estimators achieve the optimal rate and variance of estimating integrated volatility even in the presence of infinite variation jumps when the latter are stochastic integrals with respect to locally "stable" Lévy processes, i.e., processes whose Lévy measure around zero behaves like that of a stable process. On a first step we estimate locally volatility from the empirical characteristic function of the increments of the process over blocks of shrinking length and then we sum these estimates to form initial estimators of the integrated volatility. The estimators contain bias when jumps of infinite variation are present and on a second step we estimate and remove this bias by using integrated volatility estimators formed from the empirical characteristic function of the high-frequency increments for different values of its argument. The second step debiased estimators achieve efficiency and we derive a feasible central limit theorem for them.

1. Introduction. In this paper we consider the problem of estimating the continuous part of the quadratic variation (henceforth referred to as integrated volatility) of a discretely-observed one-dimensional Itô semimartingale over a finite interval with mesh of the observation grid going to zero in the case when the observed process can contain jumps of infinite variation. Separating jumps from diffusive volatility is of central interest in finance due to the distinct role played by diffusive volatility and jumps in financial decision making, which is also reflected in the distinct risk premium demanded by investors for each of them, see e.g., [6]. Until now this problem has been well studied when jumps are of finite variation, see e.g., [2], [3], [11, 12], [8]. However, empirical results in [1] suggest that for some financial data sets jumps can be of infinite variation. This is the case we study in this paper.

In particular, we consider a one-dimensional Itô semimartingale X which is defined on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and can always be represented as

$$(1.1)\qquad \begin{aligned} X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s + \int_0^t\int_{\mathbb{R}} \delta(s,z)\,1_{\{|\delta(s,z)|\leq 1\}}\,(\underline{p} - \underline{q})(ds,dz) \\ + \int_0^t\int_{\mathbb{R}} \delta(s,z)\,1_{\{|\delta(s,z)|>1\}}\,\underline{p}(ds,dz), \end{aligned}$$

where W is a standard Brownian motion and $\underline{p}$ a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator (intensity measure) $\underline{q}(dt,dz) = dt \otimes dz$. This is the Grigelionis representation, and the specific choice of the Poisson measure $\underline{p}$ is in no way a restriction (see e.g., Theorem

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2.1.2 in [8]). Here, b and c are progressively measurable processes and δ is a predictable function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, with appropriate integrability assumptions.

The process X is observed at regularly spaced times $i\Delta_n$ for $i = 0, 1, \cdots$, within a finite time interval $[0, T]$, and without microstructure noise. Our goal is to estimate, on the basis of these observations, the so-called integrated volatility, that is

$$C_t = \int_0^t c_s\, ds, \qquad \text{where } \ c_s = \sigma_s^2, \tag{1.2}$$

for $t = T$ or more generally for all $t \in (0, T]$, with the rate $1/\sqrt{\Delta_n}$, when X contains jumps of infinite variation.

When jumps are absent, that is when $\delta \equiv 0$ (so the last two terms in (1.1) disappear), the best estimator of C_t is the *realized volatility*, or *approximate quadratic variation*:

$$\widehat{C}_t^n = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^2, \qquad \text{where } \ \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}. \tag{1.3}$$

Under very weak assumptions on b and c (namely when $\int_0^t b_s^2\, ds$ and $\int_0^t c_s^2\, ds$ are finite for all t), we have a Central Limit Theorem (CLT) with rate $\frac{1}{\sqrt{\Delta_n}}$: the processes $\frac{1}{\sqrt{\Delta_n}}(\widehat{C}_t^n - C_t)$ converge in the sense of stable convergence in law for processes, to a limit Z which is defined on an extension of the space, and which conditionally on $\mathcal{F}$ is a centered Gaussian martingale whose conditional law is characterized by its (conditional) variance

$$V_t \ := \ \mathbb{E}((Z_t)^2 \mid \mathcal{F}) \ = \ 2\int_0^t c_s^2\, ds, \tag{1.4}$$

or equivalently, we have $Z_t = \sqrt{2}\int_0^t c_s\, dW_s^{(1)}$, where $W^{(1)}$ is a Brownian motion independent of $\mathcal{F}$. Furthermore when $c_s(\omega) = c$ is a constant, or more generally when $c_t(\omega) = c(t, X_t(\omega))$ for a smooth enough function c on $\mathbb{R}_+ \times \mathbb{R}$, the estimators $\widehat{C}_t^n$ are *efficient* for any fixed time t, because in this case we have the LAN or LAMN property and V_t above is the inverse of the $\mathcal{F}$-conditional Fisher information, normalized by Δ_n. Therefore, in the general case (1.1) with $\delta \equiv 0$ we qualify the estimator $\widehat{C}_t^n$ as being *efficient*.

When jumps are present, so far there are essentially two types of results, hinging on a specification of the so-called *degree of jump activity*. To keep things simple in this introduction, and although substantial extensions can be made, we will suppose that for some $r \in [0, 2]$,

$$|\delta(\omega, t, z)|^r \wedge 1 \le J(z), \qquad \text{where } \ J \text{ is a Lebesgue-integrable function on } \mathbb{R}. \tag{1.5}$$

The smaller r above is, the stronger the assumption is, and it is (slightly) stronger than assuming $\sum_{s \le t} |\Delta X_s|^r < \infty$ for all t, where $\Delta X_s = X_s - X_{s-}$ is the size of the jump at time s. When (1.5) holds with $r = 0$ the jumps have finite activity; when (1.5) holds with $r = 1$ the jumps are (locally) summable. In the latter case we can rewrite (1.1) (up to modifying b_t) as

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s + \int_0^t \int_{\mathbb{R}} \delta(s, z)\, \underline{p}(ds, dz). \tag{1.6}$$

The supremum of all r for which (1.5) holds is the degree of jump activity, or Blumenthal-Getoor index. Then we have two cases:

1. *When* $r < 1$. In this case we have two major types of volatility estimators that enjoy a feasible CLT. The first is the *truncated realized volatility* (cf. [11, 12], [8])

$$TC(v_n)^n_t = \sum_{i=1}^{[t/\Delta_n]} (\Delta^n_i X)^2\, 1_{\{|\Delta^n_i X| \leq v_n\}}, \qquad v_n \asymp \Delta_n^{\varpi}, \tag{1.7}$$

(the last statement means that $\frac{1}{A} \leq v_n/\Delta_n^{\varpi} \leq A$ for some $A \in (1,\infty)$). $TC(v_n)^n_t$ has exactly the same limiting properties as $\widehat{C}^n$ does in the continuous case provided (1.5) holds with some $r \in [0,1)$ and $\varpi \in \big[\frac{1}{2(2-r)}, \frac{1}{2}\big)$.
The second type of jump-robust volatility estimators are the *multipower variations* (cf. [2], [3], [8]), which we do not explicitly recall here. These estimators also satisfy a CLT with rate $\frac{1}{\sqrt{\Delta_n}}$, but with a conditional variance bigger than in (1.4) (so they are rate-efficient but not variance-efficient).
2. *When* $r \geq 1$. In this case the above two types of estimators are still consistent, but when centered around C_t and appropriately scaled, they are only bounded in probability with no CLT in general and rate of convergence that is much slower than $1/\sqrt{\Delta_n}$. For example, when $r \geq 1$ the sequence $\frac{1}{\Delta_n^{\varpi(2-r)}}\,(TC(v_n)^n_t - C_t)$ is bounded in probability (when $r = 1$ the multipower variations enjoy a CLT with a bias term, see [19]).
On a more general level, we have the following general result from [9]: If we have estimators $\widehat{C}'^n_t$ such that, for some sequence $w_n \to \infty$ of numbers, the variables $w_n\,(\widehat{C}'^n_t - C_t)$ are bounded in probability in n and also when X ranges through all semimartingales of type (1.1) satisfying (1.5) with a fixed function J and also $|b_t| + c_t \leq A$ for some constant A (so w_n is a kind of "minimax" rate), we necessarily have for some constant K:

$$w_n \;\leq \begin{cases} K/\sqrt{\Delta_n} & \text{if } \; 0 \leq r \leq 1 \\ K\big(\frac{\log(1/\Delta_n)}{\Delta_n}\big)^{\frac{2-r}{2}} & \text{if } \; 1 < r < 2. \end{cases} \tag{1.8}$$

In this paper we exhibit new estimators for C_t which converge with rate $\frac{1}{\sqrt{\Delta_n}}$, and which are even variance-efficient in the sense that they satisfy the same CLT as $\widehat{C}^n_t$ does in the continuous case, when r defined in (1.5) above, i.e., the jump activity, is bigger than 1. Of course, given the result in [9], discussed in point (2) above, this is only possible under some additional assumption, namely that the "small" jumps behave like those of a stable process, or of the integral with respect to a stable-like process, with some index $\beta \in (1,2)$ (recall that in this case (1.5) holds for all $r > \beta$, but not for $r \leq \beta$). Hence here we are working in a kind of semi-parametric setting, with the (unknown) parameter β. We should point out that this "semi-parametric" setting is still quite general and covers many jump models used in empirical applications, particularly those in finance. Similar assumptions about the jumps have been also made when estimating the Blumenthal-Getoor index of jump activity in [1] and [16] among others.

The estimation method proposed in the current paper is based on estimating locally the volatility (diffusion coefficient) from the empirical characteristic function of the increments of the process over blocks of decreasing length but containing an increasing number of observations, and then summing the local volatility estimates. The separation of volatility from jumps in our method is due to the dominant role of the diffusion component of X in (the real part of) the characteristic function of the high frequency increments of the process for values of the argument that are going to infinity at the rate $1/\sqrt{\Delta_n}$, or at a slightly slower rate.

When infinite variation jumps are present, the proposed volatility estimators contain a bias which determines their rate of convergence. The bias scales differently for different values of the argument of the empirical characteristic function, used in forming our nonparametric volatility estimators, and we use this property to debias our initial volatility estimators. The debiased volatility estimators achieve the efficient rate of convergence and some of them reach the same (efficient) asymptotic variance as in (1.4).

The empirical characteristic function of high-frequency increments has been previously used in nonparametric estimation of the empirical Laplace transform of volatility in [17] as well as in [18] for estimation of the empirical Laplace transform of the stochastic scale for pure-jump semimartingales. There are two major differences between these papers and our study. First, we are interested in estimating the integrated volatility while the above cited papers consider estimation of the empirical Laplace transform of the stochastic volatility. Second, and more importantly, [17] consider jump-diffusion setting with jumps of finite variation only and [18] consider pure-jump semimartingales (i.e., processes with no diffusion). Our main contribution is rate and variance efficient estimators of integrated volatility in jump-diffusion setting with jumps of infinite variation. Finally, the empirical characteristic function in low frequency setting has been used in [13], [10] and [14] for estimating the diffusion coefficient of a Lévy process, in [7] for nonparametric estimation for a Lévy process which is a sum of a drift, a symmetric stable process and a compound Poisson process, as well as in [4] and [5] for estimation of Lévy density and jump activity in affine models.

The paper is organized as follows. In Section 2 we present the setting and state our assumptions. In Section 3 we propose our initial estimators of integrated volatility and derive a CLT for them when a bias due to the infinite variation jumps is removed from the estimators. In Section 4 we propose a way to estimate this bias and derive a feasible CLT for our debiased estimators. Section 5 contains a Monte Carlo study. Proofs are given in Section 6.

2. The setting. As mentioned before, the underlying process X is a one-dimensional Itô semimartingale on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and observed without noise at the times $i\Delta_n: \ i = 0, 1, ...$ We restrict the general form (1.1) by assuming that the jumps are a mixture of (essentially unspecified) jumps with finite variation, plus the jumps of a stochastic integral with respect to a Lévy process whose small jumps are "stable like".

We have two versions, the simplest one being as follows:

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s + \int_0^t \gamma_{s-}\,dY_s + \int_0^t \int_{\mathbb{R}} \delta(s,z)\,\underline{p}(ds,dz), \tag{2.1}$$

with Y a symmetric pure jump Lévy process with Blumenthal-Getoor index $\beta \in [0, 2)$ and the last integral being with finite variation (the precise assumptions are given below). In this version, the jumps due to Y are "symmetric" in the sense that $\int_0^t \gamma_{s-}\,dY_s$ and $-\int_0^t \gamma_{s-}\,dY_s$ have the same law, as processes. To deal with the non-symmetric case one could use a process Y which is non-symmetric. However, it is more convenient and also more general to use the following version:

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s + \int_0^t (\gamma^+_{s-}\,dY^+_s + \gamma^-_{s-}\,dY^-_s) + \int_0^t \int_{\mathbb{R}} \delta(s,z)\,\underline{p}(ds,dz), \tag{2.2}$$

with Y^+ and Y^- two independent Lévy processes with the same index β and positive jumps.

We will also require the volatility σ_t to be an Itô semimartingale, and it can thus be represented as

(2.3)
$$\sigma_t = \sigma_0 + \int_0^t b_s^\sigma\,ds + \int_0^t H_s^\sigma\,dW_s + \int_0^t H_s'^\sigma\,dW_s' + \int_0^t \int_{\mathbb{R}} \delta^\sigma(s,z)\,1_{\{|\delta^\sigma(s,z)|\le 1\}}\,(\underline{p} - \underline{q})(ds,dz) + \int_0^t \int_{\mathbb{R}} \delta^\sigma(s,z)\,1_{\{|\delta^\sigma(s,z)|>1\}}\,\underline{p}(ds,dz).$$

Most volatility models used in empirical applications satisfy (2.3), in particular models in the popular affine class.

As well known, the jumps of σ_t can, without restriction, be driven by the same Poisson measure $\underline{p}$ as X, but we need a second Brownian motion W': in the case of "pure leverage" we would have $H'^\sigma \equiv 0$ and W' is not needed; in the case of "no leverage" we rather have $H^\sigma \equiv 0$, and in the mixed case we need both W and W'.

Note that (2.1) is a special case of (2.2): indeed, if Y is a pure jump symmetric Lévy process, it can always be written as $Y = Y^+ - Y^-$ with Y^+ and Y^- being independent identically distributed and with positive jumps, so (2.2) with $\gamma^+ = \gamma$ and $\gamma^- = -\gamma$ is the same as (2.1) with γ. Therefore we only give the assumptions for (2.2). The first assumption is a structural assumption describing the driving terms $W, W', \underline{p}, Y^\pm$, the second one being a set of conditions on the coefficients implying in particular the existence of the various stochastic integrals involved above. Both assumptions involve a number r in $[0,1)$ (the same in both) and, the smaller r is, the stronger the two assumptions are.

Assumption (A): The processes W and W' are two independent Brownian motions, independent of $(\underline{p}, Y^+, Y^-)$; the measure $\underline{p}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\underline{q}(dt,dz) = dt \otimes dz$; the processes $Y^\pm$ are two independent Lévy processes with characteristics $(0,0,F^\pm)$ and positive jumps (that is, each $F^\pm$ is supported by $(0,\infty)$). Moreover, there is a number $\beta \in [1,2)$ such that the tail functions $\overline{F}^\pm(x) = F^\pm((x,\infty))$ satisfy

$$x \in (0,1] \quad \Rightarrow \quad \Big|\overline{F}^\pm(x) - \frac{1}{x^\beta}\Big| \le g(x), \tag{2.4}$$

where g is a decreasing function such that $\int_0^1 x^{r-1}\,g(x)\,dx < \infty$.

Assumption (B): We have a sequence τ_n of stopping times increasing to infinity, a sequence a_n of numbers, and a nonnegative Lebesgue-integrable function J on $\mathbb{R}$, such that the processes $b, H^\sigma, \gamma^\pm$ are càdlàg adapted, the coefficients δ, δ^σ are predictable, the processes b^σ, H'^σ are progressively measurable, and

(2.5)
$$\begin{array}{l} t < \tau_n \quad \Rightarrow \quad |\delta(t,z)|^r \wedge 1 \le a_n J(z), \quad |\delta^\sigma(t,z)|^2 \wedge 1 \le a_n J(z), \\ t < \tau_n, \quad V = b, b^\sigma, H^\sigma, H'^\sigma, \gamma^+, \gamma^- \quad \Rightarrow \quad |V_t| \le a_n, \\ V = b, H^\sigma, \gamma^+, \gamma^- \quad \Rightarrow \quad |\mathbb{E}(V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n} \mid \mathcal{F}_t)| + \mathbb{E}(|V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|^2 \mid \mathcal{F}_t) \le a_n s. \end{array}$$

Note that we do *not* require the processes $Y^\pm$ to be independent from the measure $\underline{p}$, thus allowing any kind of dependence between the jumps of X and those of σ. Intuitively, the number r in Assumptions (A) and (B) control the activity of the finite jump variation component of X as well as the degree of deviation from the stable process of $Y^\pm$ which drive the infinite jump variation component of X. Our condition in (2.4) is similar to condition

AN1 on the Lévy measure around zero in [5]. Assumptions (A) and (B) are satisfied by many parametric models for the jump component used in applications as illustrated by the following example.

Example. Suppose the jump component of X is given by a time-changed Lévy process with absolute continuous time-change, i.e., L_{T_t} where L_v is a pure-jump Lévy process with Lévy measure F satisfying (2.4) and time-change $T_t = \int_0^t a_s ds$ for a_t being strictly positive Itô semimartingale. A popular parametric example for F is that of a tempered stable process with corresponding Lévy density of the form

$$\frac{A^+ e^{-\lambda^+ x}}{|x|^{1+\beta}} 1_{\{x>0\}} + \frac{A^- e^{-\lambda^- |x|}}{|x|^{1+\beta}} 1_{\{x<0\}}, \quad A^\pm \geq 0, \quad \lambda^\pm > 0, \quad \beta \in (0,2).$$

In this case, it is not hard to show (using Theorem 2.1.2 of [8] which links integrals of random functions with respect to Poisson measure and random integer-valued measures) that Assumptions (A) and (B) (regarding the jump part of X) hold with β in Assumption (A) being the corresponding parameter in the above parametric model when $\beta \in [1,2)$ and further $r = \beta - 1 + \iota$ for $\iota > 0$ arbitrary small and $\gamma_t^+ = \left(\frac{A^+ a_t}{\beta}\right)^{1/\beta}$ and $\gamma_t^- = -\left(\frac{A^- a_t}{\beta}\right)^{1/\beta}$ (and non-zero δ in (2.2) which depends on $Y^\pm$). When $\beta \in (0,1)$ in the above parametric model, Assumptions (A) and (B) hold trivially with $\gamma_t^\pm = 0$. □

We end this section with a few comments:

1. In (2.4) there is an implicit standardization of the processes $Y^\pm$. One could replace it by

$$x \in (0,1] \quad \Rightarrow \quad \left|\overline{F}^\pm(x) - \frac{a_\pm}{x^\beta}\right| \leq g(x) \tag{2.6}$$

   for positive constants $a_\pm$. However, in this case the processes $Y'^\pm = Y^\pm / a_\pm^{1/\beta}$ satisfy (2.4) as stated, and (2.2) holds with $Y'^\pm$ and $\gamma'^\pm = a_\pm^{1/\beta}\gamma^\pm$ as well. It is more convenient in the sequel, and not a restriction, to use the standardized form (2.4).
2. By Assumption (B) and the fact that $r < 1$, the last integral in (2.2) defines a process with finite variation which is the sum of its jumps. On the other hand, $\int_0^t (\gamma_{s-}^+ \, dY_s^+ + \gamma_{s-}^- \, dY_s^-)$ has a Blumenthal-Getoor (BG) index $\beta \geq 1$ and is of infinite variation (even when $\beta = 1$, and unless γ^+ and γ^- identically vanish, of course), although still a (compensated) "pure jump" process.
3. Concerning the regularity assumptions in (B), the last part of (2.5) could be somewhat weakened (for instance we could drop it in the case of $V = H^\sigma$), but at the expense of a non trivial complication of the proofs. Since these are satisfied in virtually all models used in practice we decided to impose these assumptions here. Note also that this last part of (2.5) is satisfied as soon as the processes $b, H^\sigma, \gamma^+, \gamma^-$ are themselves Itô semimartingales with locally bounded characteristics.

3. First estimators of C_t. In this section we construct our initial estimators of C_t. These estimators are not efficient in general, but they will be used to construct efficient estimators later on.

We use the real part of the "local" (in time) empirical characteristic functions of increments, taken at point $u_n/\sqrt{\Delta_n}$ for some sequence $u_n > 0$ going to 0 slowly enough. Here, "local" means that the empirical characteristic function is constructed on windows of time length v_n or $2v_n$, where $v_n = k_n\Delta_n$ and $k_n \geq 1$ is a suitable sequence of integers going to infinity, to be specified later. We will in fact use two different versions:

(3.1)

$$\text{Symmetrized version:} \qquad L(u)^n_j = \frac{1}{k_n}\sum_{l=0}^{k_n-1}\cos(u(\Delta^n_{2jk_n+1+2l}X - \Delta^n_{2jk_n+2+2l}X)/\sqrt{\Delta_n}\,),$$

$$\text{Non-symmetrized version:} \quad L'(u)^n_j = \frac{1}{k_n}\sum_{l=0}^{k_n-1}\cos(u\Delta^n_{1+jk_n+l}X/\sqrt{\Delta_n}\,),$$

for $j \geq 1$ some integer, $u > 0$ some real and recall $\Delta^n_i X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$. $L(u)^n_j$ and $L'(u)^n_j$ are not bigger than 1, and the variables

$$\begin{aligned} \widehat{c}(u)^n_j &= -\tfrac{1}{u^2}\,\log\Big(L(u)^n_j \bigvee \tfrac{1}{\sqrt{k_n}}\Big),\\ \widehat{c}'(u)^n_j &= -\tfrac{2}{u^2}\,\log\Big(L'(u)^n_j \bigvee \tfrac{1}{\sqrt{k_n}}\Big), \end{aligned} \tag{3.2}$$

satisfy $0 \leq \widehat{c}(u)^n_j \leq \frac{\log k_n}{2u^2}$ and $0 \leq \widehat{c}'(u)^n_j \leq \frac{\log k_n}{u^2}$, and serve as local estimators of the volatility (of the average of c_t over the interval $(2jv_n, 2(j+1)v_n]$ or $(jv_n, (j+1)v_n]$, to be more precise). The associated estimators for integrated volatility are thus (recall $v_n = k_n\Delta_n$):

$$\begin{aligned} \widehat{C}(u)^n_t &= 2v_n\sum_{j=0}^{[t/2v_n]-1}\Big(\widehat{c}(u)^n_j - \tfrac{1}{u^2k_n}\,\big(\sinh(u^2\widehat{c}(u)^n_j)\big)^2\Big),\\ \widehat{C}'(u)^n_t &= v_n\sum_{j=0}^{[t/v_n]-1}\Big(\widehat{c}'(u)^n_j - \tfrac{2}{u^2k_n}\,\big(\sinh(u^2\widehat{c}'(u)^n_j/2)\big)^2\Big), \end{aligned} \tag{3.3}$$

where recall $\sinh(x) = \frac{e^x - e^{-x}}{2}$. On an intuitive level, $\widehat{C}(u)^n_t$ and $\widehat{C}'(u)^n_t$ separate volatility (of the diffusive part of X) from jumps in X by utilizing the fact that the diffusive component of X dominates the behavior of the real part of the empirical characteristic function at high-frequencies for values of the argument that are "sufficiently" away from zero. Indeed, in the simple case when $X_t = X_0 + bt + \sigma W_t + \gamma Y_t$ for Y_t a symmetric β-stable process with unit scale, we have $\log\Re(\mathbb{E}(e^{iu\Delta^n_i X/\sqrt{\Delta_n}})) = \log(\cos(ub\Delta_n^{1/2})) - \frac{u^2\sigma^2}{2} - |\gamma|^\beta u^\beta \Delta_n^{1-\beta/2}$.

The terms $\frac{1}{u^2k_n}\big(\sinh(u^2\widehat{c}(u)^n_j)\big)^2$ and $\frac{2}{u^2k_n}\big(\sinh(u^2\widehat{c}'(u)^n_j/2)\big)^2$ remove biases of higher asymptotic order in $\widehat{c}(u)^n_j$ and $\widehat{c}'(u)^n_j$, respectively, which arise due to the nonlinear transformation of $L(u)^n_j$ and $L'(u)^n_j$ in forming $\widehat{c}(u)^n_j$ and $\widehat{c}'(u)^n_j$.

We note that for any fixed n, $\lim_{u\downarrow 0}\widehat{C}'(u)^n_t = \sum_{i=1}^{[t/\Delta_n]}(\Delta^n_i X)^2$ is the realized volatility (which in presence of jumps does not estimate the integrated volatility). The robustness of our estimator $\widehat{C}'(u)^n_t$ with respect to jumps in X will result from using $u = u_n$ that is "sufficiently" far from zero, and the variance-efficiency of the corrected second-step estimators will come from the fact that $u_n \to 0$ (we make this formal in the theorems below).

For stating the asymptotic behavior of the estimators in (3.3), we need some additional notation. First, for $\beta \in (0, 2)$ we set

$$\beta > 1 \;\mapsto\; \chi'(\beta) = \int_0^\infty \frac{1-\cos y}{y^\beta}\,dy, \quad \beta > 0 \;\mapsto\; \chi(\beta) = -\beta\chi'(\beta+1) = \int_0^\infty \frac{\sin y}{y^\beta}\,dy \tag{3.4}$$

(the last integral is convergent for all $\beta > 0$, but absolutely convergent when $\beta > 1$ only). Next, with the notation $\{x\}^\beta = |x|^\beta \operatorname{sign}(x)$ for any $x \in \mathbb{R}$, we associate with the processes $\gamma^\pm$ the following (when $\chi'(\beta)$ appears below we implicitly suppose $\beta > 1$):

$$\begin{gathered} a_t = \chi(\beta)(|\gamma_t^+|^\beta + |\gamma_t^-|^\beta), \qquad a'_t = \chi'(\beta)(\{\gamma_t^+\}^\beta + \{\gamma_t^-\}^\beta), \\ A(u)_t^n = 2u^{\beta-2}\Delta_n^{1-\beta/2} \int_0^t a_s\, ds, \\ A'(u)_t^n = \tfrac{2}{u^2} \int_0^t \Big(\Delta_n^{1-\beta/2} u^\beta a_s - \log\big(\cos(\Delta_n^{1-\beta/2} u^\beta a'_s)\big)\Big)\, ds. \end{gathered} \tag{3.5}$$

Under appropriate assumptions on the sequence u_n we will see that $\widehat{C}(u_n)_T$ and $\widehat{C}'(u_n)_T$ converge to C_T, and there is an associated Central Limit Theorem with the convergence rate $1/\sqrt{\Delta_n}$. However, in the CLT there is typically a non-negligible bias due to the infinite variation jumps in X, and to account for this bias we consider the following normalized error processes

$$\begin{aligned} Z(u)_t^n &= \tfrac{1}{\sqrt{\Delta_n}} \Big(\widehat{C}(u)_t^n - C_t - A(u)_t^n\Big), \\ Z'(u)_t^n &= \tfrac{1}{\sqrt{\Delta_n}} \Big(\widehat{C}'(u)_t^n - C_t - A'(u)_t^n\Big). \end{aligned} \tag{3.6}$$

$A(u)_t^n$ and $A'(u)_t^n$ are easiest to understand in the Lévy case, i.e., when $\gamma_t^\pm$ are constants. In this case $A'(u)_1^n$ is $-\frac{2}{u^2}$ times the logarithm of the real part of the characteristic function of $\Delta_i^n L/\sqrt{\Delta_n}$, where $L = \gamma^+ L^+ + \gamma^- L^-$ and L^+ and L^- are two independent one-sided stable processes with Lévy density $\frac{\beta}{x^{\beta+1}} 1_{\{x>0\}}$, and $A(u)_1^n = A'(u)_1^n$ when $\gamma^- = -\gamma^+$. In this case of constant $\gamma_t^\pm$, taking the difference $\Delta_{i+1}^n X - \Delta_i^n X$ makes the contribution of the stochastic integrals w.r.t. $Y^\pm$ globally symmetric: the characteristic function of $\Delta_{i+1}^n L - \Delta_i^n L$ above becomes real, and this is why we put $A(u)_t^n$ instead of $A'(u)_t^n$ in the first case of (3.6). Now, $A(u)_t^n$ has a much simpler form than $A'(u)_t^n$, regarding its dependence upon u, which makes its estimation from the data, as conducted in the next section, rather easy. On the other hand, differencing increments results in a loss of information, since in the definition of $\widehat{C}(u)_t^n$ we have twice less summands than in the definition of $\widehat{C}'(u)_t^n$. (Note that the form (2.1) for X corresponds to having $\gamma^- = -\gamma^+$, hence in this case $A'(u)_t^n = A(u)_t^n$.)

In order to give a simple version of the limits below, we consider an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which supports two independent Brownian motions $W^{(1)}$ and $W^{(2)}$, independent of the σ-field $\mathcal{F}$, and on this extension we introduce the two processes

$$Z_t = \sqrt{2} \int_0^t c_s\, dW_s^{(1)}, \qquad \overline{Z}_t = \frac{1}{\sqrt{6}} \int_0^t c_s^2\, dW_s^{(2)}. \tag{3.7}$$

An equivalent characterization of the pair $(Z, \overline{Z})$ is as follows: they are defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and, conditionally on $\mathcal{F}$, they are centered continuous Gaussian martingales characterized by their (conditional) variances-covariances, as given by

$$\widetilde{\mathbb{E}}((Z_t)^2 \mid \mathcal{F}) = 2\int_0^t c_s^2\, ds, \qquad \widetilde{\mathbb{E}}(\overline{Z}_t^2 \mid \mathcal{F}) = \frac{1}{6} \int_0^t c_s^4\, ds, \qquad \widetilde{\mathbb{E}}(Z_t\, \overline{Z}_t \mid \mathcal{F}) = 0. \tag{3.8}$$

In view of the de-biasing procedure later on we need a multidimensional version of the CLT, namely the convergence for all θu_n, where θ runs through a finite subset Θ of $(0, \infty)$. We are now ready to state the main results of this section:

THEOREM 1. *Assume (A) and (B) with $r<1$, and choose k_n and u_n in such a way that*

$$k_n\sqrt{\Delta_n}\to 0,\quad k_n\Delta_n^{1/2-\varepsilon}\to\infty\ \ \forall\varepsilon>0,\quad u_n\to 0,\quad \sup_n\frac{k_n\sqrt{\Delta_n}}{u_n^4}<\infty. \tag{3.9}$$

a) We have the (functional) stable convergence in law

$$\Big(Z(u_n)^n,\Big(\frac{1}{u_n^2}(Z(\theta u_n)^n-Z(u_n)^n\Big)_{\theta\in\Theta}\Big)\ \stackrel{\mathcal{L}-s}{\Longrightarrow}\ (\sqrt{2}\,Z,(2\sqrt{2}\,(\theta^2-1)\overline{Z})_{\theta\in\Theta}). \tag{3.10}$$

b) If further $\beta>1$, we also have

$$\Big(Z'(u_n)^n,\Big(\frac{1}{u_n^2}(Z'(\theta u_n)^n-Z'(u_n)^n\Big)_{\theta\in\Theta}\Big)\ \stackrel{\mathcal{L}-s}{\Longrightarrow}\ (Z,((\theta^2-1)\overline{Z})_{\theta\in\Theta}). \tag{3.11}$$

This exhibits a kind of degeneracy. Indeed, (3.10) and (3.11) imply the following convergence ($\stackrel{\text{u.c.p.}}{\Longrightarrow}$ means convergence in probability, uniformly on each compact time interval):

$$Z(\theta u_n)^n-Z(u_n)\ \stackrel{\text{u.c.p.}}{\Longrightarrow}\ 0,\qquad Z'(\theta u_n)^n-Z'(u_n)\ \stackrel{\text{u.c.p.}}{\Longrightarrow}\ 0. \tag{3.12}$$

REMARK 2. A possible choice for k_n and u_n is $k_n\asymp 1/\sqrt{\Delta_n}\,(\log(1/\Delta_n))^x$ and $u_n\asymp 1/(\log(1/\Delta_n))^{x'}$, which satisfies (3.9) as soon as the reals x,x' are such that $0<x'\le\frac{x}{4}$.

4. Efficient Estimators of C_t. In general, the bias terms $A(u)_t^n$ or $A'(u)_t^n$ in (3.6) determine the second order behavior of the estimators $\widehat{C}(u)_t^n$ and $\widehat{C}'(u)_t^n$, thus preventing rate efficiency. In one important case, though, Theorem 1 implies that $\widehat{C}'(u)_t^n$ will be both rate and variance efficient and $\widehat{C}(u)_t^n$ will be rate efficient but with asymptotic variance somewhat larger. This is the case when the jumps in X are of finite variation, i.e., when γ^+ and γ^- are identically 0. Then (3.6) reduces to

$$Z(u)_t^n=\sqrt{\Delta_n}\,(\widehat{C}(u)_t^n-C_t),\qquad Z'(u)_t^n=\sqrt{\Delta_n}\,(\widehat{C}'(u)_t^n-C_t),$$

and Theorem 1 implies:

THEOREM 3. *Assume (A) and (B) with $\gamma^\pm\equiv 0$ and $r<1$, and choose k_n and u_n satisfying (3.9). Then the processes $Z(u_n)^n$ and $Z'(u_n)^n$ converge stably in law to $\sqrt{2}\,Z$ and Z, respecyively.*

This means, in particular, that the estimators $\widehat{C}'(u_n)_t$ are asymptotically equivalent to the truncated realized volatility $TC(v_n)_t$ of (1.7) with $v_n\asymp\Delta_n^{\varpi}$ and $\varpi\in(\frac{1}{2(2-r)}\,\frac{1}{2})$, and hence are rate and variance efficient. Thus, we provide an alternative to the truncated realized volatility which is important in applications due to the presence of tuning parameters in the construction of both jump-robust volatility estimators (ours and the truncated realized volatility).

REMARK 4. Whereas the above is a special case of Theorem 1, it is possible (although far from trivial when one allows the process σ to jump, as in (2.3)) to show that when again $\gamma^\pm\equiv 0$ and when $r=1$, and if we fix $u>0$, then the sequence $Z'(u)^n$ stably converges in law to a process $Z(u)$ which has the same description as Z above, except that the conditional variance is now

$$\widetilde{\mathbb{E}}(Z(u)_t^2\mid\mathcal{F})=8\int_0^t\Big(\frac{\sinh(u^2c_s/2)}{u^2}\Big)^2\,ds. \tag{4.1}$$

(when $u_n \to 0$ we do not know the behavior of $Z'(u_n)^n$). Hence the estimators $\widehat{C}'(u)_t^n$ are still rate efficient, but no longer variance efficient. However, the right side of (4.1) goes to $2\int_0^t c_s^2\,ds$ as $u \to 0$: so, upon choosing u small enough, one can approach variance efficiency as close as one wants to.

Note that, even without variance efficiency, the rate efficiency above plus the fact that the limit is conditionally unbiased seems to be a new result when $r = 1$. □

When the term $\int_0^t (\gamma_{s-}^+\,dY_s^+ + \gamma_{s-}^-\,dY_s^-)$ in (2.2) is present, the estimators $\widehat{C}(u_n)_t^n$ and $\widehat{C}'(u_n)_t^n$ converge to C_t at a rate arbitrarily close to $1/\Delta_n^{(2-\beta)/2}$, which up to a logarithmic term is in accordance with the minimax rate given in (1.8) (see [9]). However, this does not give us a feasible limit theorem. In this situation one can find a way of eliminating the bias term and come up with estimators with rate $1/\sqrt{\Delta_n}$ and which are even variance efficient (of course this is possible under Assumptions (A) and (B) only).

To do this, we fix the time horizon $T > 0$ and we set

$$
\begin{aligned}
\widehat{C}(u,\zeta)_T^n &= \widehat{C}(u)_T^n - \frac{(\widehat{C}(\zeta u)_T^n - \widehat{C}(u)_T^n)^2}{\widehat{C}(\zeta^2 u)_T^n - 2\widehat{C}(\zeta u)_T^n + \widehat{C}(u)_T^n}, \\
\widehat{C}'(u,\zeta)_T^n &= \widehat{C}'(u)_T^n - \frac{(\widehat{C}'(\zeta u)_T^n - \widehat{C}'(u)_T^n)^2}{\widehat{C}'(\zeta^2 u)_T^n - 2\widehat{C}'(\zeta u)_T^n + \widehat{C}'(u)_T^n}.
\end{aligned}
\tag{4.2}
$$

The new estimators above are biased-corrected analogues of $\widehat{C}(u)_T^n$ and $\widehat{C}'(u)_T^n$. Our estimation of the bias is very intuitive. It utilizes the fact that the only difference (asymptotically) in $\widehat{C}(u)_T^n$ and $\widehat{C}'(u)_T^n$ for different values of u stems from the presence of $A(u)_t^n$ and $A'(u)_t^n$. This suggests an easy way to estimate these biases from the differences of $\widehat{C}(u)_T^n$ and $\widehat{C}'(u)_T^n$ over different values of u. The next theorem derives the asymptotic behavior of $\widehat{C}(u,\zeta)_T^n$ and $\widehat{C}'(u,\zeta)_T^n$.

THEOREM 5. *Assume (A) and (B) with $r < 1$ and $C_T > 0$ a.s. Choose k_n and u_n satisfying (3.9) and any $\zeta > 1$.*

a) The variables $\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(u_n,\zeta)_T^n - C_T)$ converge stably in law to the variable $\sqrt{2}\,Z_T$, which conditionally on $\mathcal{F}$ is centered Gaussian with (conditional) variance $4\int_0^T c_s^2\,ds$.

b) Assume further that either $1 < \beta < \frac{3}{2}$, or that $\beta \geq \frac{3}{2}$ and $\gamma^+ = -\gamma^-$ identically. The variables $\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}'(u_n,\zeta)_T^n - C_T)$ converge stably in law to the variable Z_T, which conditionally on $\mathcal{F}$ is centered Gaussian with (conditional) variance $2\int_0^T c_s^2\,ds$.
In particular, this applies when (2.2) reduces to (2.1), under the only condition $1 < \beta < 2$.

The estimator $\widehat{C}(u_n,\zeta)_T^n$ applies in all cases of Assumptions (A) and (B) and is rate efficient but not variance efficient. $\widehat{C}'(u_n,\zeta)_T^n$ is both variance and rate efficient and no prior knowledge of β is needed (except that $\beta = 1$ is excluded) whenever $\gamma^+ = -\gamma^-$ which is the case in many models. When $\gamma^+ \neq -\gamma^-$, then we can use $\widehat{C}'(u_n,\zeta)_T^n$ only when $\beta < 3/2$.

Alternatively we could iterate the de-biasing procedure and achieve rate and variance efficiency even in the asymmetric case $\gamma^+ \neq -\gamma^-$. Such an iteration also permits to replace the fourth term on the right side of (2.2) by a sum of M terms $\int_0^t (\gamma_{s-}^{m+}\,dY_s^{m+} + \gamma_{s-}^{m-}\,dY_s^{m-})$, with $Y^{m\pm}$ having Blumenthal-Getoor indices β_m with $1 \leq \beta_M < \cdots < \beta_1 < 2$, and under appropriate conditions. We leave such extensions for future work.

REMARK 6. When $\mathbb{P}(C_T > 0) < 1$ the result as stated may fail. However, a classical argument shows that it still holds *in restriction to the set* $\{C_T > 0\}$. □

5. Monte Carlo study. We test the performance of our new method of estimating integrated volatility and compare it with that of the truncated realized volatility on simulated data from the following stochastic volatility model

$$X_t = X_0+\int_0^t \sqrt{c_s}\, dW_s+\eta Y_t, \quad c_t = c_0+\int_0^t 0.03(1.0-c_s)\, ds+0.15\int_0^t \sqrt{c_s}\, dW_s', \quad \eta \geq 0, \tag{5.1}$$

where W_t and W_t' are two independent Brownian motions and Y_t is a symmetric β-stable process independent from W_t and W_t'. The volatility c_t is a square-root diffusion process, which is widely used to model stochastic volatility in financial applications. The parameters of the volatility specification are set so that the mean and persistence of volatility is similar to that in actual financial data. In particular, its mean is 1 in the stationary case. Since the key advantage of our estimation procedure is its ability to recover integrated volatility in presence of infinite variation jumps, in the Monte Carlo we experiment with values of the stability parameter of Y_t of $\beta = 1.25$, $\beta = 1.50$ and 1.75. We further vary the constant η (in the interval $[0, 2]$) which controls the relative contribution of Y_t in the total variation of X_t.

In the Monte Carlo we fix the time span to 1 day (our unit of time is a day) and we consider $1/\Delta_n = 2,400$ and $1/\Delta_n = 4,800$, which corresponds to sampling at 10 and 5 seconds, respectively, in a 6.5-hour trading day. We set $k_n = 240$ for $1/\Delta_n = 2,400$ and we increase it to $k_n = 320$ when $1/\Delta_n = 4,800$, which correspond to 10 and 15, respectively, blocks per unit of time. Experiments with more blocks per day led to very similar results.

We test in the Monte Carlo the performance of the bias-corrected estimator $\widehat{C}'(u,\zeta)^n$ defined in (4.2), whose implementation we now discuss. The choice of the tuning parameter $u_n = u$ in $\widehat{C}'(u,\zeta)^n$ plays a nontrivial role. From the asymptotic variance in (4.1), it is clear that a big or small value of u is always with respect to the level of volatility c_s. For this reason, for each time interval $[t, t+1]$ we set u for that day to $u_t^n = \frac{1}{(\log(1/\Delta_n))^{1/30}}\frac{1}{\sqrt{BV_{[t-1,t]}}}$, where

$$BV_{[t-1,t]} = \frac{\pi}{2}\sum_{i=[(t-1)/\Delta_n]+2}^{[t/\Delta_n]} |\Delta_{i-1}^n X||\Delta_i^n X|, \tag{5.2}$$

is the Bipower Variation on the unit interval $[t-1, t)$ which is a consistent estimator of $\int_{t-1}^t c_s ds$ that does not require any choice of tuning parameters. Our time-varying u_t^n is analogous to the selection of a time-varying threshold for the truncated realized volatility that is typically done (and we implement as well here). The scale factor $\frac{1}{(\log(1/\Delta_n))^{1/30}}$ is chosen so that u_t^n converges to zero very slowly as $\Delta_n \to 0$.

The bias correction term in $\widehat{C}'(u,\zeta)_T^n$ can be split into the product of two terms, as $(\widehat{C}'(\zeta u)_T^n - \widehat{C}'(u)_T^n) \times \frac{(\widehat{C}'(\zeta u)_T^n-\widehat{C}'(u)_T^n)}{\widehat{C}'(\zeta^2 u)_T^n-2\widehat{C}'(\zeta u)_T^n+\widehat{C}'(u)_T^n}$. The first term is an estimator for $A'(u)_T^n$, which is time-varying and the second is an estimator of $\frac{1}{\zeta^{\beta-2}-1}$ which depends only on the parameter β. To reduce the noise in our estimate of the bias, therefore, we use a horizon of 132 days (6 months) to estimate the second term, similar to earlier studies on estimation of the Blumenthal-Getoor index ([1] and [16]), and daily data to estimate the first term (as the limit of this term is time-varying). Also for the calculation of the second term we use a smaller value of u as this allows to capture the slope of $\widehat{C}'(u,\zeta)_T^n$ better. Overall, for a period of $T = 132$ days, our daily estimator is

$$\widehat{C}'(u_t^n,\zeta)_{[t,t+1]}^n = \widehat{C}'(u_t^n)_{[t,t+1]}^n - S_T^n\left((\widehat{C}'(\zeta u_t^n)_{[t,t+1]}^n - \widehat{C}'(u_t^n)_{[t,t+1]}^n) \wedge 0\right), \quad t = 1, ..., T-1, \tag{5.3}$$

$$S_T^n = \frac{\sum_{t=1}^T (\widehat{C}'(0.3\zeta u_t^n)_{[t,t+1]}^n - \widehat{C}'(0.3u_t^n)_{[t,t+1]}^n)}{\sum_{t=1}^T (\widehat{C}'(0.3\zeta^2 u_t^n)_{[t,t+1]}^n - 2\widehat{C}'(0.3\zeta u_t^n)_{[t,t+1]}^n + \widehat{C}'(0.3u_t^n)_{[t,t+1]}^n)} \bigwedge 0. \tag{5.4}$$

The restrictions on the sign above are finite sample restrictions with no asymptotic effect. In the calculation of the bias correction term we set $\zeta = 1.5$. Finally, if $\widehat{C}'(u_t^n, \zeta)_{[t-1,t]}^n$ is negative we repeat the calculation in (5.3) with $2u_t^n/3$ (this again has no asymptotic effect).

For the truncation realized volatility estimator $TC(v_n)^n$, which we compare below to our estimator, we set $v_n = 4\sqrt{BV_{[t-1,t]}}\Delta_n^{0.49}$, as typically done in existing work.

The results from the Monte Carlo are summarized in Figures 1 and 2. Not surprisingly, the activity of the jump component (controlled by β) and its relative share in total return variation (controlled by η) have clear impact on the ability to separate integrated variance from the jumps in X. Our volatility estimator $\widehat{C}'(u_t^n, \zeta)_{[t-1,t]}^n$ performs significantly better than the truncated variance in presence of infinite variation jumps (recall that both estimators are consistent regardless of the activity of the jumps). The superior performance of $\widehat{C}'(u_t^n, \zeta)_{[t-1,t]}^n$ is largely due to the removal of the bias in the volatility estimation that is due to the infinite variation jumps. As a result $\widehat{C}'(u_t^n, \zeta)_{[t-1,t]}^n$, unlike $TC(v_n)^n$, is essentially unbiased in all considered cases. Increasing the sampling frequency improves the performance of both estimators in all cases. We note however that the reduction of bias and MAD for $TC(v_n)^n$ for the higher jump activity case ($\beta = 1.75$) is significantly slower and this is unlike our estimator. This is consistent with the slow rate of convergence of $TC(v_n)^n$ in the case of infinite variation jumps discussed in the introduction. Overall, we conclude that our estimator provides a nontrivial improvement over existing methods for the nonparametric estimation of integrated volatility in presence of infinite variation jumps.

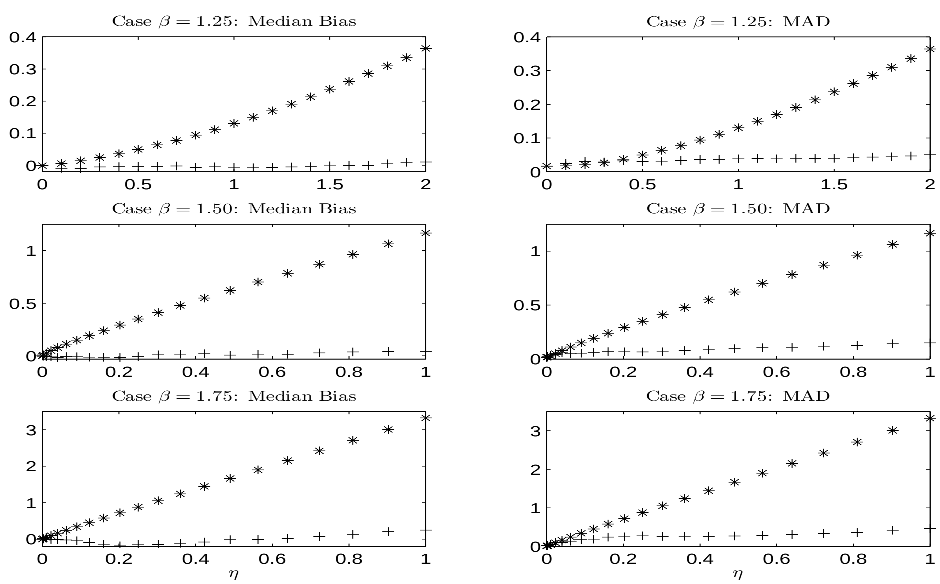

FIG 1. *Median bias and Median Absolute Deviation (MAD) around the true value,* $\int_t^{t+1} c_s ds$*, for sampling frequency* $1/\Delta_n = 2,400$. $+$ *corresponds to* $\widehat{C}'(u,\zeta)^n$ *and* $*$ *to* $TC(v_n)^n$.

6. Proofs.

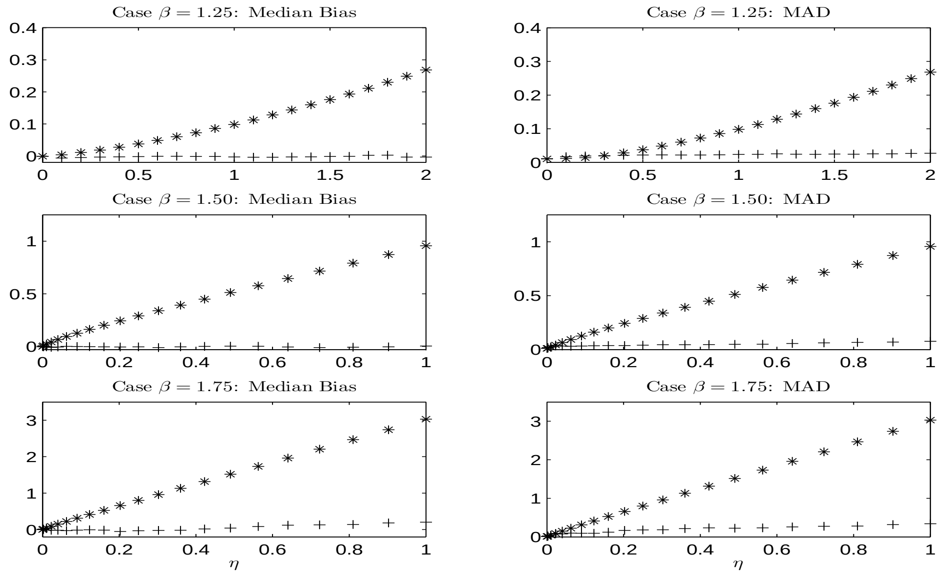

FIG 2. *Median bias and Median Absolute Deviation (MAD) around the true value,* $\int_t^{t+1} c_s ds$, *for sampling frequency* $1/\Delta_n = 4,800$. $+$ *corresponds to* $\widehat{C}'(u,\zeta)^n$ *and* $*$ *to* $TC(v_n)^n$.

6.1. *Preliminaries.* By a standard localization procedure, we may and will assume that in (B) we have $\tau_1 \equiv \infty$ and J is bounded, and also that X and σ are themselves bounded, as well as the jumps of $Y^{\pm}$. Up to modifying b^σ, we can thus rewrite (2.3) as

$$\sigma_t = \sigma_0 + \int_0^t b_s^\sigma \, ds + \int_0^t H_s^\sigma \, dW_s + \int_0^t H_s'^\sigma \, dW_s' + \int_0^t \int_{\mathbb{R}} \delta^\sigma(s,z) \, (\underline{p} - \underline{q})(ds,dz). \tag{6.1}$$

Itô's formula gives us

$$\begin{array}{l} c_t = c_0 + \int_0^t b_s^c \, ds + \int_0^t H_s^c \, dW_s + \int_0^t H_s'^c \, dW_s' + \int_0^t \int_{\mathbb{R}} \delta^c(s,z) \, (\underline{p} - \underline{q})(ds,dz) \\ \text{where } \left\{ \begin{array}{l} b_t^c = 2\sigma_t b_t^\sigma + (H_t^\sigma)^2 + (H_t'^\sigma)^2 + \int_{\mathbb{R}} \delta^\sigma(t,z)^2 \, dz \\ H_t^c = 2\sigma_t \, H_t^\sigma, \quad H_t'^c = 2\sigma_t \, H_t'^\sigma, \quad \delta^c(t,z) = 2\sigma_{t-}\delta^\sigma(t,z) + \delta^c(t,z)^2 \end{array} \right. \end{array} \tag{6.2}$$

and we can thus strengthen and complement (2.5) as follows:

$$\begin{array}{l} |\delta(t,z)|^r \leq J(z), \qquad |\delta^\sigma(t,z)|^2 \leq J(z), \qquad |\delta^c(t,z)|^2 \leq J(z) \\ |X_t| + |\sigma_t| + c_t + |b_t| + |b_t^\sigma| + |H_t^\sigma| + |H_t'^\sigma| + |b_t^c| + |H_t^c| + |H_t'^c| + |\gamma_t^\pm| + |\Delta Y_t^\pm| \leq K \\ V = X, c, \sigma, b, \gamma^+, \gamma^-, H^\sigma, H^c \quad \Rightarrow \quad |\mathbb{E}(V_{t+s} - V_t \mid \mathcal{F}_t)| + \mathbb{E}(|V_{t+s} - V_t|^2 \mid \mathcal{F}_t) \leq Ks. \end{array} \tag{6.3}$$

Here K is a constant, and below K and ϕ_n will denote a constant and a sequence of (non random) numbers going to 0 as $n \to \infty$, all these changing from line to line. They may depend on the characteristics of X and on the powers for which the forthcoming estimates are stated. Moreover in the theorem to be proven the arguments u in $\widehat{C}(u)_t^n$ or $\widehat{C}'(u)_t^n$ are $u = \theta u_n \to 0$, where θ varies in a *fixed* set $\Theta \subset (0,\infty)$: hence in the sequel we *implicitly assume* $u \in (0,1]$.

Upon replacing $g(x)$ by $g(1)+1$ when $x > 1$ we get (2.4) for all $x \in (0,\infty)$. We loose the fact that g is decreasing, but it is still decreasing on $(0,1]$, hence $x^{r-1}g(x)$ as well because

$r \leq 1$, and the property $\int_0^1 x^{r-1} g(x)\,dx < \infty$ implies $x^r g(x) \to 0$ as $x \to 0$. Summarizing and recalling $\beta \geq 1$, we have

$$x > 0 \;\Rightarrow\; |\overline{F}^{\pm}(x) - \frac{1}{x^\beta}| \leq g(x), \quad \overline{F}^{\pm}(x) \leq \frac{K}{x^\beta}, \quad \text{and} \quad \lim_{x \to 0} x^r g(x) = 0. \tag{6.4}$$

Below we unify the proofs of the claims (a) and (b). This is at the expense of somewhat cumbersome notation, but it saves a lot of space because the proofs are totally similar. To this end we introduce a number κ which takes the value 1 if we deal with the non-symmetrized version and the value 2 when we consider the symmetrized version. We set

$$\begin{array}{c} L(1,u)_j^n = L'(u)_j^n, \qquad L(2,u)_j^n = L(u)_j^n, \qquad \widehat{c}(1,u)_j^n = \widehat{c}'(u)_j^n, \qquad \widehat{c}(2,u)_j^n = \widehat{c}(u)_j^n \\ \widehat{C}(\kappa,u)_t^n = \kappa v_n \sum\limits_{j=0}^{[t/\kappa v_n]-1} \big(\widehat{c}(\kappa,u)_j^n - \frac{2}{\kappa u^2 k_n}\,(\sinh(\kappa u^2 \widehat{c}(\kappa,u)_j^n/2))^2\big) \end{array} \tag{6.5}$$

(so $\widehat{C}(1,u)_t^n = \widehat{C}'(u)_t^n$ and $\widehat{C}(2,u)_t^n = \widehat{C}(u)_t^n$), and also (recall that when $\kappa = 1$ we suppose $\beta > 1$, so the quantities below are well defined)

$$\begin{array}{c} A(1,u)_t^n = A'(u)_t^n, \qquad\qquad A(2,u)_t^n = A(u)_t^n \\ Z(\kappa,u)_t^n = \frac{1}{\sqrt{\Delta_n}}\Big(\widehat{C}(\kappa,u)_t^n - C_t - A(\kappa,u)_t^n\Big). \end{array} \tag{6.6}$$

Next, recalling the notation (3.5), we set

$$\begin{array}{c} U(\kappa,u)_t = e^{-\kappa u^2 c_t/2}, \qquad \overline{U}(\kappa,u)_t^n = e^{-\kappa \Delta_n^{1-\beta/2} u^\beta\, a_t} \\ \widehat{U}(1,u)_t^n = \cos(\Delta_n^{1-\beta/2} u^\beta a'_t), \qquad \widehat{U}(2,u)_t^n = 1 \\ \mathcal{U}(\kappa,u)_t^n = U(\kappa,u)_t\, \overline{U}(\kappa,u)_t^n\, \widehat{U}(\kappa,u)_t^n). \end{array} \tag{6.7}$$

Since $0 \leq c_t \leq K$ and $0 \leq a_t \leq K$ and $|a'_t| \leq K$, and assuming n large enough to have $\Delta_n^{1-\beta} u^\beta |a'_t| \leq \frac{1}{2}$ for all t and $u \in (0,1]$, we see that, for some $\chi \in (0,1)$,

$$\chi \leq U(\kappa,u)_t \leq 1, \quad \chi \leq \overline{U}(\kappa,u)_t^n \leq 1, \quad \chi \leq \widehat{U}(\kappa,u)_t^n \leq 1, \quad \chi \leq \mathcal{U}(\kappa,u)_t^n \leq 1. \tag{6.8}$$

Moreover, Itô's formula yields

$$\begin{array}{l} U(\kappa,u)_t = U(\kappa,u)_0 + \int_0^t b_s^{U(\kappa,u)}\,ds + \int_0^t H_s^{U(\kappa,u)}\,dW_s + \int_0^t H_s'^{U(\kappa,u)}\,dW'_s \\ \qquad\qquad + \int_0^t \int_{\mathbb{R}} \delta^{U(\kappa,u)}(s,z)\,(\underline{p} - \underline{q}\,)(ds,dz) \\ \text{where} \left\{ \begin{array}{l} b_t^{U(\kappa,u)} = -\frac{\kappa u^2}{2}\, U(\kappa,u)_t\, b_t^c + \frac{\kappa^2 u^4}{8}\, U(\kappa,u)_t\big((H_t^c)^2 + (H_t'^c)^2\big) \\ \qquad\qquad + U(\kappa,u)_t \int_{\mathbb{R}} \big(e^{-\kappa u^2 \delta^c(t,z)/2} - 1 + \frac{\kappa u^2}{2}\,\delta^c(t,z)\big) dz \\ H_t^{U(\kappa,u)} = -\frac{\kappa u^2}{2}\, U(\kappa,u)_t\, H_t^c, \qquad H_t'^{U(\kappa,u)} = -\frac{\kappa u^2}{2}\, U(\kappa,u)_t\, H_t'^c \\ \delta^{U(\kappa,u)}(t,z) = U(\kappa,u)_{t-}\,\big(e^{-\kappa u^2 \delta^c(t,z)/2} - 1\big). \end{array} \right. \end{array} \tag{6.9}$$

Therefore we have for all $u \in (0,1]$:

$$|b_t^{U(\kappa,u)}| + |H_t^{U(\kappa,u)}| + |H_t'^{U(\kappa,u)}| \leq K u^2, \qquad |\delta^{U(\kappa,u)}(t,z)|^2 \leq K u^2 J(z). \tag{6.10}$$

Since $\big||x|^\beta - |y|^\beta - \beta\,\{y\}^{\beta-1}(x-y)\big| \leq K|x-y|^\beta$ for $x,y \in \mathbb{R}$ when $1 \leq \beta < 2$, and a similar estimate for $\{x\}^\beta - \{y\}^\beta$, and since $|\gamma_t^{\pm}| \leq K$, the last part of (6.3) implies for $s \in [0,1]$ and $q \geq 2$:

$$\begin{array}{l} |\mathbb{E}(a_{t+s} - a_t \mid \mathcal{F}_t)| + |\mathbb{E}(a'_{t+s} - a'_t \mid \mathcal{F}_t)| \leq K s^{\beta/2} \\ \mathbb{E}(|a_{t+s} - a_t|^q + |a'_{t+s} - a'_t|^q \mid \mathcal{F}_t) \leq K s^{1 \wedge (q\beta/2)}. \end{array} \tag{6.11}$$

Using $|e^x - e^y - e^y(x-y)| \leq (x-y)^2$ for $x, y \leq 0$ and a similar estimate for the cosine function when $\kappa = 1$, we deduce for all $u > 0$ and $q \geq 2$:

(6.12)
$$\begin{array}{l}
|\mathbb{E}(U(\kappa,u)_{t+s} - U(\kappa,u)_t \mid \mathcal{F}_t)| \leq K u^2 s, \qquad \mathbb{E}(|U(\kappa,u)_{t+s} - U(\kappa,u)_t|^q \mid \mathcal{F}_t) \leq K u^{2q} s \\
|\mathbb{E}(\overline{U}(\kappa,u)^n_{t+s} - \overline{U}(\kappa,u)^n_t \mid \mathcal{F}_t)| + |\mathbb{E}(\widehat{U}(\kappa,u)^n_{t+s} - \widehat{U}(\kappa,u)^n_t \mid \mathcal{F}_t)| \leq K \Delta_n^{1-\beta/2} u^\beta s^{\beta/2} \\
\mathbb{E}(|\overline{U}(\kappa,u)^n_{t+s} - \overline{U}(\kappa,u)^n_t|^q + |\widehat{U}(\kappa,u)^n_{t+s} - \widehat{U}(\kappa,u)^n_t|^q \mid \mathcal{F}_t) \leq K \Delta_n^{q(1-\beta/2)} u^{q\beta} s.
\end{array}$$

In turn, since $xy - zw = (x-z)(y-w) + z(y-w) + w(x-z)$, this yields

(6.13)
$$\begin{array}{l}
|\mathbb{E}(\mathcal{U}(\kappa,u)^n_{t+s} - \mathcal{U}(\kappa,u)^n_t \mid \mathcal{F}_t)| \leq K\big(u^2 s + \Delta_n^{1-\beta/2} u^\beta s^{\beta/2}\big) \\
\mathbb{E}(|\mathcal{U}(\kappa,u)^n_{t+s} - \mathcal{U}(\kappa,u)^n_t|^q \mid \mathcal{F}_t) \leq K\, s\, (u^{2q} + \Delta_n^{q(1-\beta/2)} u^{q\beta}) \\
\mathbb{E}(|\mathcal{U}(\kappa,u)^n_{t+s} - \mathcal{U}(\kappa,u)^n_t - (U(\kappa,u)_{t+s} - U(\kappa,u)_t)\overline{U}(\kappa,u)^n_t\, \widehat{U}(\kappa,u)^n_t|^q \mid \mathcal{F}_t) \leq K\, s\, \Delta_n^{q(1-\beta/2)} u^{q\beta}.
\end{array}$$

We end this preliminary subsection with another set of notation, with again $\kappa = 1, 2$.

$$\begin{array}{c}
\rho'(1)^n_i = \frac{1}{\sqrt{\Delta_n}}\, \sigma_{(i-1)\Delta_n} \Delta^n_i W, \qquad \rho'(2)^n_i = \frac{1}{\sqrt{\Delta_n}}\, \sigma_{(i-1)\Delta_n} (\Delta^n_i W - \Delta^n_{i+1} W) \\
\rho''(1)^n_i = \frac{1}{\sqrt{\Delta_n}}\, (\gamma^+_{(i-1)\Delta_n} \Delta^n_i Y^+ + \gamma^-_{(i-1)\Delta_n} \Delta^n_i Y^-) \\
\rho''(2)^n_i = \frac{1}{\sqrt{\Delta_n}}\, (\gamma^+_{(i-1)\Delta_n} (\Delta^n_i Y^+ - \Delta^n_{i+1} Y^+) + \gamma^-_{(i-1)\Delta_n} (\Delta^n_i Y^- - \Delta^n_{i+1} Y^-)) \\
\rho(\kappa)^n_i = \rho'(\kappa)^n_i + \rho''(\kappa)^n_i, \qquad \overline{\rho}(1)^n_i = \frac{1}{\sqrt{\Delta_n}}\, \Delta^n_i X, \qquad \overline{\rho}(2)^n_i = \frac{1}{\sqrt{\Delta_n}}\, (\Delta^n_i X - \Delta^n_{i+1} X).
\end{array}$$

(6.14)
$$\begin{array}{l}
\xi(\kappa,u)^{w,n}_j = \left\{ \begin{array}{ll}
\frac{1}{k_n} \sum_{l=0}^{k_n-1} \big(\cos(u\rho(\kappa)^n_{1+\kappa j k_n + \kappa l}) - \mathcal{U}(\kappa,u)^n_{\kappa(jk_n+l)\Delta_n} \big) & \text{if } w = 1 \\
\frac{1}{k_n} \sum_{l=0}^{k_n-1} \big(\cos(u\overline{\rho}(\kappa)^n_{1+\kappa j k_n + \kappa l}) - \cos(u\rho(\kappa)^n_{1+\kappa j k_n + \kappa l}) \big) & \text{if } w = 2 \\
\frac{1}{k_n} \sum_{l=0}^{k_n-1} (\mathcal{U}(\kappa,u)^n_{\kappa(jk_n+l)\Delta_n} - \mathcal{U}(\kappa,u)^n_{\kappa j v_n}) & \text{if } w = 3.
\end{array} \right. \\
\xi(\kappa,u)^n_j = \frac{1}{\mathcal{U}(\kappa,u)^n_{\kappa j v_n}} \sum_{w=1}^3 \xi(\kappa,u)^{w,n}_j \\
\Omega(\kappa,u)_{n,t} = \big\{ \sup_{j=0,\cdots,[t/\kappa v_n]-1} |\xi(\kappa,u)^n_j| \leq \frac{1}{2} \big\}.
\end{array}$$

Note that, by virtue of (6.8),

$$|\xi(\kappa,u)^{w,n}_j| \leq K, \qquad |\xi(\kappa,u)^n_j| \leq K. \tag{6.15}$$

Finally, let us mention that below we assume (3.9). This implies the following properties, which will be used many times below for various values of the reals w_j below:

$$\frac{k_n^{w_1}\, v_n^{w_2}\, \Delta_n^{w_3}}{u_n^{w_4}} \to 0 \qquad \text{if} \quad \left\{ \begin{array}{l} \text{either} \quad w_3 > \frac{w_1 - w_2}{2} \\ \text{or} \quad w_4 < 4(w_1 + w_2), \quad w_3 \geq \frac{w_1 - w_2}{2}. \end{array} \right. \tag{6.16}$$

6.2. *The scheme of the proof.* We have the sequence u_n and $\beta \in [1,2)$ with further $\beta > 1$ when we deal with (b) of Theorem 1, hence when $\kappa = 1$. Below, θ always belongs to a finite set $\Theta \subset (0,\infty)$ which, without loss of generality, contains 1. We set

$$\begin{array}{c}
\Omega(\kappa)_{n,t} = \cap_{\theta \in \Theta}\, \Omega(\kappa, \theta u_n)_{n,t}, \qquad a^n_t = \Delta_n^{1-\beta/2}\, a_t, \qquad a'^n_t = \left\{ \begin{array}{ll} \Delta_n^{1-\beta/2}\, a'_t & \text{if } \beta > 1 \\ 0 & \text{if } \beta = 1 \end{array} \right. \\
f_{\kappa,u}(x) = \big(\sinh(\kappa u^2 x/2)\big)^2, \quad h_{1,u}(x,x') = \frac{2}{u^2}\big(u^\beta x - \log\big(\cos(u^\beta x')\big)\big), \quad h_{2,u}(x,x') = 2u^{\beta-2} x.
\end{array}$$

Because c_t, a_t, a'_t are bounded, we have the estimates (with f' and f'' the first two derivatives of f):

$$\begin{gathered} f_{\kappa,u}(c_t) + |f'_{\kappa,u}(c_t)| + |f''_{\kappa,u}(c_t)| \leq Ku^4 \\ |u^2 x| \leq K \quad \Rightarrow \quad |u^2 f'_{\kappa,u}(x)| + |f''_{\kappa,u}(x)| \leq Ku^4 \\ \Delta_n \leq Ku^2 \quad \Rightarrow \quad |h_{\kappa,u}(a^n_t, a'^n_t)| \leq Ku^{\beta-2}\Delta_n^{1-\beta/2} \leq K. \end{gathered} \tag{6.17}$$

and also

$$-\frac{2}{\kappa u^2}\, \log \mathcal{U}(\kappa,u)^n_t = c_t + h_{\kappa,u}(a^n_t, a'^n_t), \qquad A(\kappa,u)^n_t = \int_0^t h_{\kappa,u}(a^n_s, a'^n_s)\, ds. \tag{6.18}$$

1) The key step of the proof is as follows. By construction we have $L(\kappa,u)^n_j = \mathcal{U}(\kappa,u)^n_{jv_n}(1+\xi(\kappa,u)^n_j)$. Moreover, we have $\mathcal{U}(\kappa,\theta u_n)^n_t \geq \chi > 0$ by (6.8) and there is a non random integer n_0 such that $k_n \geq 4/\chi^2$ for $n \geq n_0$, implying $L(\kappa,\theta u_n)^n_j \geq 1/\sqrt{k_n}$ for all $j \leq [t/v_n]-1$ such that $1+\xi(\kappa,\theta u_n)^n_j \geq \frac{1}{2}$. Hence we deduce from (6.18) that

$$\begin{aligned} &n \geq n_0,\ \omega \in \Omega(\kappa)_{n,t} \quad \Rightarrow \\ &\qquad \widehat{c}(\kappa,\theta u_n)^n_j = c_{\kappa j v_n} + h_{\kappa,\theta u_n}(a^n_{\kappa j v_n}, a'^n_{\kappa j v_n}) - \tfrac{2}{\kappa(\theta u_n)^2}\, \log(1+\xi(\kappa,\theta u_n)^n_j). \end{aligned} \tag{6.19}$$

Another key point is as such: on the set $\Omega_{n,t}$ and again for $n \geq n_0$, we can expand $\log(1+x)$ around 0 and $f_{\kappa,u}$ around $c_{\kappa j\Delta_n}$ to obtain

$$\begin{aligned} &\big|\widehat{c}(\kappa,\theta u_n)^n_j - c_{\kappa j v_n} - h_{\kappa,\theta u_n}(a^n_{\kappa j v_n}, a'^n_{\kappa j v_n}) + \tfrac{2}{\kappa(\theta u_n)^2}\, \xi(\kappa,\theta u_n)^n_j - \tfrac{1}{\kappa(\theta u_n)^2}\, |\xi(\kappa,\theta u_n)^n_j|^2\big| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \leq \tfrac{K}{u_n^2}\, |\xi(\kappa,\theta u_n)^n_j|^3 \\ &\big|f_{\kappa,\theta u_n}(\widehat{c}(\kappa,\theta u_n)^n_j) - f_{\theta u_n}(c_{\kappa j v_n}) + \tfrac{2}{\kappa(\theta u_n)^2}\, f'_{\kappa,\theta u_n}(c_{\kappa j v_n})\xi(\kappa,\theta u_n)^n_j\big| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \leq K(u_n^\beta \Delta_n^{1-\beta/2} + |\xi(\kappa,\theta u_n)^n_j|^2), \end{aligned}$$

where for the last estimate we have used (6.17) and the fact that $|\widehat{c}(\kappa,\theta u_n)^n_j| \leq K/u_n^2$ (by the first estimate, plus again (6.17) and (6.15)), hence $|u_n^2 f'_{\kappa,\theta u_n}(x)| + |f''_{\kappa,\theta u_n}(x)| \leq Ku_n^4$ for all x between $\widehat{c}(\kappa,\theta u_n)^n_j$ and $c_{\kappa j v_n}$. In turn, this and (6.16) yield on the set $\Omega(\kappa)_{n,t}$ and for $n \geq n_0$ again:

$$\begin{aligned} &\Big|\Big(\widehat{c}(\kappa,\theta u_n)^n_j - c_{\kappa j v_n} - h_{\kappa,\theta u_n}(a^n_{\kappa j v_n}, a'^n_{\kappa j v_n}) - \tfrac{2}{\kappa k_n\,(\theta u_n)^2}\, f_{\kappa,\theta u_n}(\widehat{c}(\kappa,\theta u_n)^n_j)\Big) \\ &\quad -\tfrac{2}{\kappa(\theta u_n)^2}\big(\tfrac{2}{\kappa k_n(\theta u_n)^2}\, f'_{\kappa,\theta u_n}(c_{\kappa j v_n}) - 1\big)\xi(\kappa,\theta u_n)^n_j \\ &\quad -\tfrac{1}{\kappa(\theta u_n)^2}\,\big(|\xi(\kappa,\theta u_n)^n_j|^2 - \tfrac{2}{k_n}\, f_{\kappa,\theta u_n}(c_{\kappa j v_n})\big)\Big| \\ &\qquad\qquad \leq K\Big(\tfrac{|\xi(\kappa,\theta u_n)^n_j|^2}{k_n u_n^2} + \tfrac{|\xi(\kappa,\theta u_n)^n_j|^3}{u_n^2} + \tfrac{\Delta_n^{1-\beta/2}}{k_n u_n^{2-\beta}}\Big). \end{aligned} \tag{6.20}$$

2) Recalling (6.6) and (6.18), we can write

$$Z(\kappa,\theta u_n)^n = V^{\kappa,n,\theta} + V'^{\kappa,n,\theta} + V''^{\kappa,n,\theta}, \qquad \text{where} \tag{6.21}$$

$$\begin{aligned} V^{\kappa,n,\theta}_t &= -\tfrac{1}{\sqrt{\Delta_n}}\, \textstyle\int_{\kappa v_n([t/\kappa v_n]-1)}^t \big(c_s + h_{\kappa,\theta u_n}(a^n_s, a'^n_s)\big)\, ds \\ V'^{\kappa,n,\theta}_t &= -\sum_{j=0}^{[t/\kappa v_n]-1} \tfrac{1}{\sqrt{\Delta_n}}\, \textstyle\int_{\kappa j v_n}^{\kappa(j+1)v_n} \big(c_s - c_{\kappa j v_n} + \big(h_{\kappa,\theta u_n}(a^n_s, a'^n_s) - h_{\kappa,\theta u_n}(a^n_{\kappa j v_n}, a'^n_{\kappa, j v_n})\big)\big)\, ds \\ V''^{\kappa,n,\theta}_t &= \sum_{j=0}^{[t/\kappa v_n]-1} \tfrac{\kappa v_n}{\sqrt{\Delta_n}}\, \Big(\widehat{c}(\kappa,\theta u_n)^n_j - c_{\kappa j v_n} h_{\kappa,\theta u_n}(a^n_{\kappa j v_n}, a'^n_{\kappa j v_n}) - \tfrac{2}{\kappa k_n(\theta u_n)^2}\, f_{\kappa,\theta u_n}(\widehat{c}(\kappa,\theta u_n)^n_j)\Big). \end{aligned}$$

Let us also introduce the following processes:

$$\begin{aligned}
\overline{V}_t^{\kappa,n,\theta} &= \sum_{j=0}^{[t/\kappa v_n]-1} \frac{2v_n}{(\theta u_n)^2\sqrt{\Delta_n}} \Big(\frac{2}{\kappa k_n(\theta u_n)^2}\, f'_{\kappa,\theta u_n}(c_{\kappa j v_n}) - 1\Big)\xi(\kappa,\theta u_n)_j^n \\
\overline{V}_t'^{\kappa,n,\theta} &= \sum_{j=0}^{[t/\kappa v_n]-1} \frac{v_n}{(\theta u_n)^2\sqrt{\Delta_n}} \big(|\xi(\kappa,\theta u_n)_j^n|^2 - \tfrac{2}{k_n}\, f_{\kappa,\theta u_n}(c_{\kappa j v_n})\big) \\
R_t^{\kappa,n,\theta} &= \sum_{j=0}^{[t/\kappa v_n]-1} \frac{v_n}{\sqrt{\Delta_n}} \Big(\frac{|\xi(\kappa,\theta u_n)_j^n|^2}{k_n u_n^2} + \frac{|\xi(\kappa,\theta u_n)_j^n|^3}{u_n^2} + \frac{\Delta_n^{1-\beta/2}}{k_n u_n^{2-\beta}}\Big).
\end{aligned}$$

By virtue of (6.20) we then obtain

$$\big|\, Z(\kappa,\theta u_n)_s^{n,l} - V_s^{\kappa,n,\theta} - V_s'^{\kappa,n,\theta} - \overline{V}_s^{\kappa,n,\theta} - \overline{V}_s'^{\kappa,n,\theta}\big| \leq K R_t^{\kappa,n,\theta} \qquad \text{on } \Omega(\kappa)_{n,t}, \ \text{ for all } s\leq t.$$

Therefore, Theorem 1 follows from the next four lemmas, where Z and $\overline{Z}$ are as in (3.7):

LEMMA 7. *We have* $\mathbb{P}((\Omega(\kappa)_{n,t})^c) \to 0$.

LEMMA 8. *We have* $\frac{1}{u_n^2}\, V^{\kappa,n,\theta} \stackrel{u.c.p.}{\Longrightarrow} 0$ *and* $\frac{1}{u_n^2}\, V'^{\kappa,n,\theta} \stackrel{u.c.p.}{\Longrightarrow} 0$.

LEMMA 9. *We have* $\frac{1}{u_n^2}\, R_t^{\kappa,n,\theta} \stackrel{u.c.p.}{\Longrightarrow} 0$ *and* $\frac{1}{u_n^2}\, \overline{V}'^{\kappa,n,\theta} \stackrel{u.c.p.}{\Longrightarrow} 0$.

LEMMA 10. *The processes* $\big(\overline{V}^{\kappa,n,1}, (\frac{1}{u_n^2}\,(\overline{V}^{\kappa,n,\theta} - \overline{V}^{\kappa,n,1}))_{\theta\in\Theta}\big)$ *converge stably in law to the limit* $\big(\kappa^{1/2}Z, (\kappa^{3/2}(\theta^2-1)\overline{Z})_{\theta\in\Theta}\big)$, *provided* $\beta>1$ *when* $\kappa=1$.

6.3. *Proofs of Lemmas 7–10.* We begin with Lemma 8, which is simple to prove:

PROOF OF LEMMA 8. By the boundedness of c_t and the property $\Delta_n \leq K u_n^2$, we deduce from (6.17) that $|V_t^{\kappa,n,\theta}| \leq K \frac{v_n}{\sqrt{\Delta_n}}$, which is $\mathrm{o}(u_n^2)$ by (3.9), hence the first claim. Next, we have $h_{2,\theta u_n}(a_s^n, a_s'^n) - h_{2,\theta u_n}(a_w, a'_w) = \frac{2\Delta_n^{1-\beta/2}}{(\theta u_n)^{2-\beta}}\,(a_s - a_w)$ and also, as soon as $(\theta u_n)^\beta |a_w'^n| \leq \frac{1}{2}$ (hence for all n large enough),

$$\begin{aligned}
\Big| & h_{1,\theta u_n}(a_s^n, a_s'^n) - h_{1,\theta u_n}(a_w, a'_w) \\
& -\tfrac{2\Delta_n^{1-\beta/2}}{(\theta u_n)^{2-\beta}}\Big(a_s - a_w - (a'_s - a'_w)\tan(\Delta_n^{1-\beta/2}(\theta_l u_n)^\beta a'_w)\Big)\Big| \leq K\Delta_n^{2-\beta}u_n^{2\beta-2}\,|a'_s - a'_w|^2.
\end{aligned}$$

Hence (6.3) for $V=c$ and (6.11) imply that the jth summand ζ_j^n in the definition of $V_t'^{\kappa,n,\theta}$ satisfies in all cases

$$\begin{aligned}
|\mathbb{E}(\zeta_j^n \mid \mathcal{F}_{\kappa j v_n})| &\leq \tfrac{K}{\sqrt{\Delta_n}}\Big(v_n^2 + v_n^{1+\beta/2}\Delta_n^{1-\beta/2}u_n^{\beta-2}\Big) = \mathrm{o}(v_n\, u_n^2) \\
\mathbb{E}((\zeta_j^n)^2 \mid \mathcal{F}_{\kappa j v_n}) &\leq \tfrac{K v_n^3}{\Delta_n}\Big(1 + \Delta_n^{2-\beta}u_n^{2\beta-4}\Big) = \mathrm{o}(v_n u_n^4),
\end{aligned}$$

where the last two estimates follow from (6.16). Then a classical argument yields the second claim. □

The other lemmas need quite many preliminary results. Below, to ease notation we simply write u_n instead of θu_n.

LEMMA 11. *Recalling (6.14), we have for all $q \geq 2$:*

$$\begin{array}{l} \big|\mathbb{E}(\cos(u_n\overline{\rho}(\kappa)_i^n) - \cos(u_n\rho(\kappa)_i^n) \mid \mathcal{F}_{(i-1)\Delta_n})\big| \leq u_n^4\sqrt{\Delta_n}\,\phi_n \\ \mathbb{E}(|\cos(u_n\overline{\rho}(\kappa)_i^n) - \cos(u_n\rho(\kappa)_i^n)|^q \mid \mathcal{F}_{(i-1)\Delta_n}) \leq u_n^4\sqrt{\Delta_n}\,\phi_n. \end{array} \tag{6.22}$$

PROOF. 1) We begin the proof with the case $\kappa = 1$. Letting $\overline{X}_t = \int_0^t \int_{\mathbb{R}} \delta(s,z)\,\underline{p}(ds,dz)$, we have $u_n\overline{\rho}(1)_i^n = \sum_{k=1}^4 \theta(k)_i^n$, where

$$\begin{array}{l} \theta(1)_i^n = u_n\rho(1)_i^n, \qquad \theta(2)_i^n = \frac{u_n}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})\,dW_s + u_n\sqrt{\Delta_n}\,b_{(i-1)\Delta_n} \\ \theta(3)_i^n = \frac{u_n}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} (b_s - b_{(i-1)\Delta_n})\,ds \\ \theta(4)_i^n = \frac{u_n}{\sqrt{\Delta_n}} \Big(\Delta_i^n\overline{X} + \int_{(i-1)\Delta_n}^{i\Delta_n} (\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+)\,dY_s^+ + \int_{(i-1)\Delta_n}^{i\Delta_n} (\gamma_s^- - \gamma_{(i-1)\Delta_n}^-)\,dY_s^-\Big). \end{array}$$

We also write $\overline{\theta}(k)_i^n = \sum_{m=1}^k \theta(m)_i^n$, so

$$\cos(u_n\rho(1)_i^n) = \cos(\overline{\theta}(1)_i^n), \qquad \cos(u_n\overline{\rho}(1)_i^n) = \cos(\overline{\theta}(4)_i^n). \tag{6.23}$$

2) In this step we prove the following estimates, for any $w \geq 2$ and $\varepsilon > 0$:

$$\begin{array}{l} \mathbb{E}(|\theta(k)_i^n|^w \mid \mathcal{F}_{(i-1)\Delta_n}) \leq \begin{cases} Ku_n^w\Delta_n^{1-w/2} & \text{if } k=1 \\ Ku_n^w\Delta_n & \text{if } k=2 \\ Ku_n^w\Delta_n^{1+w/2} & \text{if } k=3, \end{cases} \\ \mathbb{E}(|\theta(4)_i^n| \wedge 1 \mid \mathcal{F}_{(i-1)\Delta_n}) \leq u_n^4\sqrt{\Delta_n}\,\phi_n \\ \mathbb{E}((|u_n\rho''(\kappa)_i^n| \wedge 1)^2 \mid \mathcal{F}_{(i-1)\Delta_n}) \leq K\Delta_n^{1-\varepsilon-\beta/2}. \end{array} \tag{6.24}$$

We classically have $\mathbb{E}(|\Delta_i^n W|^w \mid \mathcal{F}_{(i-1)\Delta_n}) \leq K\Delta_n^{w/2}$, whereas $\mathbb{E}(|\Delta_i^n Y^{\pm}|^w \mid \mathcal{F}_{(i-1)\Delta_n}) \leq K\Delta_n$ by Lemma 2.1.5 of [8] (because $Y^{\pm}$ has bounded jumps), yielding Case $k=1$. Cases $k=2,3$ follow from (6.3).

For Case $k=4$ it is enough to prove the result for each of the three summands in the definition of $\theta(4)_i^n$. For the first summand $\Delta_i^n\overline{X}$, we observe that $|u_n\Delta_i^n\overline{X}/\sqrt{\Delta_n}| \wedge 1 \leq K(|\Delta_i^n\overline{X}/\sqrt{\Delta_n}| \wedge 1)$. Then we apply Corollary 2.1.9-(c) of [8] with $q = \frac{1}{2}$ and $s = \Delta_n$ and r as in (A) and (B) and $p = 1$, to obtain

$$\mathbb{E}\Big(\frac{|u_n\Delta_i^n\overline{X}|}{\sqrt{\Delta_n}} \bigwedge 1 \mid \mathcal{F}_{(i-1)\Delta_n}\Big) \leq \Delta_n^{1-r/2}\phi_n. \tag{6.25}$$

The other two summands are treated analogously, and we consider only one of them, say $\alpha_i^n = \int_{(i-1)\Delta_n}^{i\Delta_n} (\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+)\,dY_s^+$. We observe that the jump measure of Y^+, say $\underline{p}'$, is Poisson with compensator $\underline{q}'(dt,dz) = dt \otimes F^+(dz)$, and $\alpha_i^n = \Delta_i^n(\delta * (\underline{p}' - \underline{q}'))$ if we take $\delta(t,z) = (\gamma_{t-}^+ - \gamma_{(i-1)\Delta_n}^+)z\,1_{\{t>(i-1)\Delta_n\}}$. The notation (2.1.35) of [8] for $\delta * (\underline{p}' - \underline{q}')$ becomes

$$\begin{array}{l} \widehat{\delta}(p,a)_{(i-1)\Delta_n,i\Delta_n} = \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} |\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+|^p\,ds \int_0^{a/|\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+|} z^p\,F^+(dz) \\ \widehat{\delta}'(p)_{(i-1)\Delta_n,i\Delta_n} = \widehat{\delta}(p,1)_{(i-1)\Delta_n,j\Delta_n} + \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} |\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+|\,ds \int_{1/|\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+|}^{\infty} z\,F^+(dz) \\ \widehat{\delta}''(p)_{(i-1)\Delta_n,i\Delta_n} = \widehat{\delta}(p,1)_{(i-1)\Delta_n,j\Delta_n} + \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \overline{F}^+(1/|\gamma_s^+ - \gamma_{(i-1)\Delta_n}^+|)\,ds, \end{array}$$

and we observe that, since γ^+ is bounded and F^+ is supported by $[0,A]$ for some finite A, necessarily $\widehat{\delta}'(p)_{(i-1)\Delta_n,i\Delta_n} \leq K\widehat{\delta}''(p)_{(i-1)\Delta_n,i\Delta_n}$. (6.4) yields $\int_0^x z^p\,F^+(dz) \leq Kx^{p-\beta}$ when $p>\beta$, hence

$$\begin{array}{l} \widehat{\delta}(p,a)_{(i-1)\Delta_n,i\Delta_n} \leq Ka^{p-\beta}\frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}|\gamma^+_s-\gamma^+_{(i-1)\Delta_n}|^\beta\,ds \\ \widehat{\delta}'(p)_{(i-1)\Delta_n,i\Delta_n} \leq K\widehat{\delta}''(p)_{(i-1)\Delta_n,i\Delta_n} \leq \frac{K}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}|\gamma^+_s-\gamma^+_{(i-1)\Delta_n}|^\beta\,ds. \end{array} \tag{6.26}$$

Since $1\leq\beta<2$ we then use Lemma 2.1.6 of [8] with $q=\frac{1}{2}$ and $r=p\in(\beta,2]$ and $s=\Delta_n$. Since $\mathbb{E}(|\gamma^+_{(i-1)\Delta_n+s}-\gamma^+_{(i-1)\Delta_n}|^\beta\mid\mathcal{F}_{(i-1)\Delta_n})\leq Ks^{\beta/2}$ by (6.3), we obtain for $p>\beta$:

$$\mathbb{E}\Big(\Big(\frac{|u_n\Delta_i^n(\delta*(\underline{p}'-\underline{q}'))|}{\sqrt{\Delta_n}}\bigwedge 1\Big)^p\mid\mathcal{F}_{(i-1)\Delta_n}\Big)\leq K\Delta_n^{1-(p-\beta)/4}. \tag{6.27}$$

We then apply Hölder's inequality to get

$$\mathbb{E}\Big(\frac{|u_n\alpha_i^n|}{\sqrt{\Delta_n}}\bigwedge 1\mid\mathcal{F}_{(i-1)\Delta_n}\Big)\leq K\Delta_n^{1/p-1/4+\beta/4p}.$$

Under (6.16), both this and (6.25) are smaller than $u_n^4\sqrt{\Delta_n}\,\phi_n$, upon choosing p close enough to β above. Hence (6.24) holds for $k=4$.

Finally the last estimate in (6.24) is obtained exactly as above, upon taking $\gamma^+_{(i-1)\Delta_n}$ instead of $\gamma^+_{t-}-\gamma^+_{(i-1)\Delta_n}$, so the bounds in (6.26) become $Ka^{p-\beta}$ and K, and the one in (6.27) is $K\Delta_n^{1-(p+\beta)/4}$. We then apply the latter with p close enough to β, and the result follows.

3) Since $|\cos(x+y)-\cos(x)|\leq 1\wedge|y|\wedge(|xy|+y^2)$ and $|\cos(x+y)-\cos(x)-y\sin(x)|\leq Ky^2$, we deduce from (6.24) and Cauchy-Schwarz inequality that

$$\begin{array}{l} w\geq 1 \quad\Rightarrow\quad \mathbb{E}(|\cos(\overline{\theta}(4)_i^n)-\cos(\overline{\theta}(3)_i^n)|^w\mid\mathcal{F}_{(i-1)\Delta_n})\leq u_n^4\sqrt{\Delta_n}\,\phi_n \\ \mathbb{E}(|\cos(\overline{\theta}(3)_i^n)-\cos(\overline{\theta}(2)_i^n)|^2\mid\mathcal{F}_{(i-1)\Delta_n})+\mathbb{E}(|\cos(\overline{\theta}(2)_i^n)-\cos(\theta(1)_i^n)|^2\mid\mathcal{F}_{(i-1)\Delta_n})\leq Ku_n^2\Delta_n \\ \mathbb{E}(|\cos(\overline{\theta}(3)_i^n)-\cos(\overline{\theta}(2)_i^n)|\mid\mathcal{F}_{(i-1)\Delta_n})\leq Ku_n^2\Delta_n \\ \mathbb{E}(|\cos(\overline{\theta}(2)_i^n)-\cos(\theta(1)_i^n)-\theta(2)_i^n\,\sin(\theta(1)_i^n)|\mid\mathcal{F}_{(i-1)\Delta_n})\leq Ku_n^2\Delta_n. \end{array}$$

This with $w=2$ and (6.23) and $\sqrt{\Delta_n}=\mathrm{o}(u_n^2)$ yield the second estimate (6.22) for $q=2$, hence for all $q\geq 2$ because $|\cos x|\leq 1$, and also (with $w=1$ above) that, for the first estimate, it only remains to prove that

$$|\mathbb{E}(\theta(2)_i^n\,\sin(\theta(1)_i^n)\mid\mathcal{F}_{(i-1)\Delta_n})|\leq u_n^4\sqrt{\Delta_n}\,\phi_n.$$

Now, we have $|\sin(\theta(1)_i^n)-\sin(u_n\rho'(1)_i^n)|\leq K(|u_n\rho''(1)_i^n|\wedge 1)$ and thus

$$\mathbb{E}(|\,\theta(2)_i^n(\sin(\theta(1)_i^n)-\sin\,(u_n\rho'(1)_i^n)|\mid\mathcal{F}_{(i-1)\Delta_n})\leq Ku_n\Delta_n^{1-\varepsilon/2-\beta/4}=u_n^4\sqrt{\Delta_n}\,\phi_n$$

by Cauchy-Schwarz inequality and (6.24), and where the last equality comes from (6.16), upon choosing $\varepsilon<\frac{2-\beta}{4}$. Hence, it remains to prove that

$$|\,\mathbb{E}(\theta(2)_i^n\,\sin\,(u_n\rho'(1)_i^n)\mid\mathcal{F}_{(i-1)\Delta_n})|\leq u_n^4\sqrt{\Delta_n}\,\phi_n.$$

4) Recalling (6.2), we set

$$V_t=\int_0^t H_s'^\sigma\,dW_s'+\int_0^t\int_E\delta^\sigma(s,z)\,(\underline{p}-\underline{q})(ds,dz).$$

We have the decomposition $\theta(2)^n_i = -\sum_{j=1}^5 \mu(j)^n_i$, where

$$\begin{aligned}
\mu(1)^n_i &= u_n\sqrt{\Delta_n}\, b_{(i-1)\Delta_n} \\
\mu(2)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\int_{(i-1)\Delta_n}^{s} b^\sigma_t\, dt\Big)\, dW_s \\
\mu(3)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}}\, H^\sigma_{(i-1)\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} (W_s - W_{(i-1)\Delta_n})\, dW_s \\
\mu(4)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\int_{(i-1)\Delta_n}^{s} (H^\sigma_t - H^\sigma_{(i-1)\Delta_n})\, dW_t\Big)\, dW_s \\
\mu(5)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} (V_s - V_{(i-1)\Delta_n})\, dW_s,
\end{aligned}$$

and it thus suffices to prove that, for $j = 1, 2, 3, 4, 5$:

$$\big|\,\mathbb{E}(\mu(j)^n_i\, \sin\big(u_n\rho'(1)^n_i\big) \mid \mathcal{F}_{i-1\Delta_n})\big| \leq K u_n^4 \sqrt{\Delta_n}\, \phi_n. \tag{6.28}$$

First, $\mathbb{E}\big(\mu(j)^n_i\, \sin\big(u_n\rho'(1)^n_i\big) \mid \mathcal{F}_{(i-1)\Delta_n}\big) = 0$ for $j = 1, 3$ follows from the fact that in these cases the variable whose conditional expectation is taken is a function of $(\omega, (W_{(i-1)\Delta_n+t} - W_{(i-1)\Delta_n})_{t\geq 0})$ which is $\mathcal{F}_{(i-1)\Delta_n}$-measurable in ω and odd in the second argument. Second, we have $\mathbb{E}((\mu(j)^n_i)^2 \mid \mathcal{F}_{(i-1)\Delta_n}) \leq K u_n^2 \Delta_n^2$ for $j = 2, 4$ (use (6.3)), implying (6.28) for $j = 2, 4$ by Cauchy-Schwarz inequality and (6.16).

For analyzing the case $j = 5$ we use the representation theorem for martingales of the Brownian filtration. This implies that the variable $\sin(u_n\rho'(1)^n_i)$, whose $\mathcal{F}_{(i-1)\Delta_n}$-conditional expectation vanishes, has the form $\int_{i\Delta_n}^{(i+1)\Delta_n} L^n_s\, dW_s$ for some process L^n, adapted to the filtration $(\mathcal{F}^W_t)_{t\geq 0}$ generated by the process W, hence

$$\mathbb{E}\big(\mu(5)^n_i\, \sin(u_n\rho'(1)^n_i) \mid \mathcal{F}_{(i-1)\Delta_n}\big) = \frac{u_n}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}((V_s - V_{(i-1)\Delta_n})\, L^n_s \mid \mathcal{F}_{(i-1)\Delta_n})\, ds.$$

Since further the martingale V is orthogonal to W, and by using once more the representation theorem (so $L^n_s = \mathbb{E}(L^n_s \mid \mathcal{F}^n_{i-1}) + \int_{(i-1)\Delta_n}^{s} L'^n_t\, dW_t$ for $s \geq i\Delta_n$), we deduce $\mathbb{E}((V_s - V_{(i-1)\Delta_n})\, L^n_s \mid \mathcal{F}_{(i-1)\Delta_n}) = 0$, hence $\mathbb{E}\big(\mu(5)^n_i\, \sin(u_n\rho'^n_i) \mid \mathcal{F}_{(i-1)\Delta_n}\big) = 0$ and (6.28) holds for $j = 5$. This ends the proof for the case $\kappa = 1$.

5) When $\kappa = 2$, we do as above, with a few changes: First $u_n\overline{\rho}(2)^n_i = \sum_{k=1}^4 \theta(k)^n_i$, where

$$\begin{aligned}
\theta(1)^n_i &= u_n\rho(2)^n_i \\
\theta(2)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \Big(\int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})\, dW_s - \int_{i\Delta_n}^{(i+1)\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})\, dW_s\Big) \\
\theta(3)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \Big(\int_{(i-1)\Delta_n}^{i\Delta_n} (b_s - b_{s+\Delta_n})\, ds - \int_{i\Delta_n}^{(i+1)\Delta_n} (b_s - b_{s+\Delta_n})\, ds\Big) \\
\theta(4)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \Big(\Delta^n_i \overline{X} - \Delta^n_{i+1} \overline{X} + \int_{(i-1)\Delta_n}^{i\Delta_n} (\gamma^+_s - \gamma^+_{(i-1)\Delta_n})\, dY^+_s - \int_{i\Delta_n}^{(i+1)\Delta_n} (\gamma^+_s - \gamma^+_{(i-1)\Delta_n})\, dY^+_s \\
&\qquad + \int_{(i-1)\Delta_n}^{i\Delta_n} (\gamma^-_s - \gamma^-_{(i-1)\Delta_n})\, dY^-_s - \int_{i\Delta_n}^{(i+1)\Delta_n} (\gamma^-_s - \gamma^-_{(i-1)\Delta_n})\, dY^-_s\Big).
\end{aligned}$$

The estimates (6.24) remain trivially valid, as well as Step 3. In Step 4 we use the decomposition $\theta(2)^n_i = -\sum_{j=2}^5 \mu(j)^n_i$, where

$$\begin{aligned}
\mu(2)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \Big(\int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\int_{(i-1)\Delta_n}^{s} b^\sigma_t\, dt\Big)\, dW_s - \int_{i\Delta_n}^{(i+1)\Delta_n} \Big(\int_{(i-1)\Delta_n}^{s} b^\sigma_t\, dt\Big)\, dW_s\Big) \\
\mu(3)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}}\, H^\sigma_{(i-1)\Delta_n} \Big(\int_{(i-1)\Delta_n}^{i\Delta_n} (W_s - W_{(i-1)\Delta_n})\, dW_s - \int_{i\Delta_n}^{(i+1)\Delta_n} (W_s - W_{(i-1)\Delta_n})\, dW_s\Big) \\
\mu(4)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \Big(\int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\int_{(i-1)\Delta_n}^{s} (H^\sigma_t - H^\sigma_{(i-1)\Delta_n})\, dW_t\Big)\, dW_s \\
&\qquad - \int_{(i\Delta_n}^{(i+1)\Delta_n} \Big(\int_{(i-1)\Delta_n}^{s} (H^\sigma_t - H^\sigma_{(i-1)\Delta_n})\, dW_t\Big)\, dW_s\Big) \\
\mu(5)^n_i &= \tfrac{u_n}{\sqrt{\Delta_n}} \Big(\int_{(i-1)\Delta_n}^{i\Delta_n} (V_s - V_{(i-1)\Delta_n})\, dW_s - \int_{i\Delta_n}^{(i+1)\Delta_n} (V_s - V_{(i-1)\Delta_n})\, dW_s\Big)
\end{aligned}$$

(the term $\mu(1)^n_i$ no longer shows up). The rest of proof carries over without modification. $\square$

LEMMA 12. *We have for all $q \geq 2$, and if $u'_n \asymp u_n$:*

$$\big|\,\mathbb{E}(\cos(u_n\rho(\kappa)^n_i) \mid \mathcal{F}_{(i-1)\Delta_n}) - \mathcal{U}(\kappa,u_n)^n_{(i-1)\Delta_n}\big| \leq \phi_n\, u_n^4\, \sqrt{\Delta_n}. \tag{6.29}$$

$$\begin{aligned}\Big|\,&\mathbb{E}(\cos(u_n\rho(\kappa)^n_i)\,\cos(u'_n\rho(\kappa)^n_i) \mid \mathcal{F}_{(i-1)\Delta_n}))\\ &\qquad -\tfrac{1}{2}\left(\mathcal{U}(\kappa,u_n+u'_n)^n_{(i-1)\Delta_n} + \mathcal{U}(\kappa,|u_n-u'_n|)^n_{(i-1)\Delta_n}\right)\Big| \leq \phi_n u_n^4\sqrt{\Delta_n}\end{aligned} \tag{6.30}$$

$$\mathbb{E}(|\cos(u_n\rho(\kappa)^n_i) - \mathcal{U}(\kappa,u_n)^n_{(i-1)\Delta_n}|^q \mid \mathcal{F}_{(i-1)\Delta_n})) \leq K u_n^4. \tag{6.31}$$

PROOF. 1) The variables $\Delta^n_i W/\sqrt{\Delta_n}$, $\Delta^n_i Y^+/\sqrt{\Delta_n}$ and $\Delta^n_i Y^-/\sqrt{\Delta_n}$ are independent one from another and from $\mathcal{F}_{(i-1)\Delta_n}$, with characteristic functions $\exp(-u^2/2)$ and $\exp(-G^{\pm}_n(u) - iH^{\pm}_n(u))$, where

$$G^{\pm}_n(y) = \Delta_n \int_0^1 \Big(1-\cos\frac{xy}{\sqrt{\Delta_n}}\Big)\,F^{\pm}(dx), \qquad H^{\pm}_n(y) = \Delta_n \int_0^1 \Big(\frac{xy}{\sqrt{\Delta_n}} - \sin\frac{xy}{\sqrt{\Delta_n}}\Big)\,F^{\pm}(dx).$$

Analogously, the characteristic functions of $(\Delta^n_i W - \Delta^n_{i+1}W)/\sqrt{\Delta_n}$ and $(\Delta^n_i Y^{\pm} - \Delta^n_{i+1}Y^{\pm})/\sqrt{\Delta_n}$ are $\exp(-u^2)$ and $\exp(-2G^{\pm}_n(u))$. Therefore by the definition of $\rho(\kappa)^n_i$, and since $\sigma_{(i-1)\Delta_n}$ and $\gamma^{\pm}_{(i-1)\Delta_n}$ are $\mathcal{F}_{(i-1)\Delta_n}$-measurable, we have

$$\begin{aligned}\mathbb{E}(\cos(u_n\rho(1)^n_i) \mid \mathcal{F}_{(i-1)\Delta_n})) &= U(1,u_n)_{(i-1)\Delta_n}\, e^{-G^+_n(u_n\gamma^+_{(i-1)\Delta_n}) - G^-_n(u_n\gamma^-_{(i-1)\Delta_n})}\\ &\qquad\times\cos\left(H^+_n(u_n\gamma^+_{(i-1)\Delta_n}) + H^-_n(u_n\gamma^-_{(i-1)\Delta_n})\right)\\ \mathbb{E}(\cos(u_n\rho(2)^n_i) \mid \mathcal{F}_{(i-1)\Delta_n})) &= U(2,u_n)_{(i-1)\Delta_n}\, e^{-2G^+_n(u_n\gamma^+_{(i-1)\Delta_n}) - 2G^-_n(u_n\gamma^-_{(i-1)\Delta_n})}.\end{aligned} \tag{6.32}$$

2) In this step we analyze the behavior of $G^{\pm}_n(y)$ when $y \in (0,A]$ for some $A > 0$. Let $\zeta_n = \Delta_n^{\eta}$ for some $\eta \in (0,\frac{1}{2})$, to be chosen later, so that $\zeta_n \to 0$ and $\zeta'_n = \zeta_n y/\sqrt{\Delta_n} \to \infty$. Using (6.4), we first see that

$$0 \leq \int_{\zeta_n}^1 \Big(1-\cos\frac{xy}{\sqrt{\Delta_n}}\Big)\,F^{\pm}(dx) \leq 2\overline{F}^{\pm}(\zeta_n) \leq \frac{K}{\zeta_n^{\beta}}.$$

Next, Fubini's theorem and a change of variable yield

$$\int_0^{\zeta_n} \Big(1-\cos\frac{xy}{\sqrt{\Delta_n}}\Big)\,F^{\pm}(dx) = \int_0^{\zeta'_n} \overline{F}^{\pm}\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)\,\sin(z)\,dz - \int_0^{\zeta'_n} \overline{F}^{\pm}(\zeta_n)\,\sin(z)\,dz,$$

and the absolute value of the last term above is again smaller than K/ζ_n^{β} because $\left|\int_0^x \sin z\,dz\right| \leq 2$ for all x. To evaluate the first term we use (6.4) again to get

$$\Big|\int_0^{\zeta'_n} \overline{F}^{\pm}\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)\,\sin(z)\,dz - \frac{y^{\beta}}{\Delta_n^{\beta/2}}\,\chi(\beta)\Big| \leq \Big|\frac{y^{\beta}}{\Delta_n^{\beta/2}}\int_{\zeta'_n}^{\infty}\frac{\sin z}{z^{\beta}}\,dz\Big| + \int_0^{\zeta'_n} g\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)\,dz.$$

We have $\int_x^\infty \frac{\sin z}{z^\beta}\,dz = \frac{\cos x}{x^\beta} - \beta\int_x^\infty \frac{\cos z}{z^{1+\beta}}\,dz$ by integration by parts, yielding $\big|\int_x^\infty \frac{\sin z}{z^\beta}\,dz\big| \leq 2/x^\beta$. We also have

$$\int_0^{\zeta'_n} g\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)\,dz = \frac{y}{\sqrt{\Delta_n}}\int_0^{\zeta_n} g(z)\,dz \leq \frac{y}{\sqrt{\Delta_n}}\,\zeta_n^{1-r}\int_0^{\zeta_n}\frac{g(z)}{z^{1-r}}\,dz = \frac{y}{\sqrt{\Delta_n}}\,\zeta_n^{1-r}\phi_n$$

because $\zeta_n\to 0$. Putting all these together yields

$$\big|\,G_n^\pm(y) - \Delta_n^{1-\beta/2}y^\beta\chi(\beta)\big| \leq \frac{K\Delta_n}{\zeta_n^\beta} + \sqrt{\Delta_n}\,y\,\zeta_n^{1-r}\,\phi_n \leq K\Delta_n^{1-\eta\beta} + y\Delta_n^{1/2+\eta(1-r)}\phi_n$$

for all $y>0$, and also (trivially) when $y=0$. Now, we take $\eta = \frac{1}{2(1-r+\beta)}$ and use (6.16) to deduce

$$\big|G_n^+(u_n\gamma^+_{(i-1)\Delta_n}) + G_n^-(u_n\gamma^-_{(i-1)\Delta_n}) - \Delta_n^{1-\beta/2}u_n^\beta\,a_{(i-1)\Delta_n}\big| \leq u_n^4\sqrt{\Delta_n}\,\phi_n.$$

Using once more $|e^x-e^y|\leq|x-y|\wedge 1$ if $x,y\leq 0$, and recalling the definition of $\overline{U}(u)^n_t$, we deduce

$$\big|\,e^{-\kappa(G_n^+(u_n\gamma^+_{i-1)\Delta_n})+G_n^-(u_n\gamma^-_{i-1)\Delta_n}))} - \overline{U}(\kappa,u_n)^n_{(i-1)\Delta_n}\big| \leq u_n^4\sqrt{\Delta_n}\,\phi_n. \tag{6.33}$$

3) Next we analyze $H_n^\pm(y)$: this is for the case when $\kappa=1$, hence $\beta>1$. The following estimates are easy consequences of (6.4):

$$0<z\leq 1 \quad\Rightarrow\quad \int_0^z x^3\,F^\pm(dx)\leq Kz^{3-\beta},\qquad \int_z^1 x\,F^\pm(dx)\leq Kz^{1-\beta}.$$

With ζ_n and ζ'_n as in the previous step, and we have

$$0\leq\int_{\zeta_n}^1\Big(\frac{xy}{\sqrt{\Delta_n}}-\sin\frac{xy}{\sqrt{\Delta_n}}\Big)F^\pm(dx)\leq\frac{y}{\sqrt{\Delta_n}}\,\overline{F}^\pm(\zeta_n)\leq\frac{Ky}{\sqrt{\Delta_n}\,\zeta_n^\beta},$$

$$\int_0^{\zeta_n}\Big(\frac{xy}{\sqrt{\Delta_n}}-\sin\frac{xy}{\sqrt{\Delta_n}}\Big)F^\pm(dx) = \int_0^{\zeta'_n}\overline{F}^\pm\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)(1-\cos z)\,dz - \int_0^{\zeta'_n}\overline{F}^\pm(\zeta_n)\,(1-\cos z)\,dz,$$

and the absolute value of the last term above is smaller than $Ky\zeta'_n/\zeta_n^\beta$. We also have

$$\Big|\int_0^{\zeta'_n}\overline{F}^\pm\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)(1-\cos z)\,dz - \frac{y^\beta}{\Delta_n^{\beta/2}}\,\chi'(\beta)\Big| \leq \frac{y^\beta}{\Delta_n^{\beta/2}}\int_{\zeta'_n}^\infty\frac{1-\cos z}{z^\beta}\,dz + \int_0^{\zeta'_n} g\Big(\frac{z\sqrt{\Delta_n}}{y}\Big)\,dz.$$

As seen before, the last term above is less than $\frac{y}{\sqrt{\Delta_n}}\,\zeta_n^{1-r}\phi_n$, whereas $\int_x^\infty\frac{1-\cos z}{z^\beta}\,dz\leq K/x^{\beta-1}$. Putting all these together, plus $\zeta'_n = y\zeta_n/\sqrt{\Delta_n}$, yields for $y>0$:

$$\big|H_n^\pm(y) - \Delta_n^{1-\beta/2}\,y^\beta\chi'(\beta)\big| \leq \frac{Ky\sqrt{\Delta_n}}{\zeta_n^\beta}\leq Ky\Delta_n^{1/2-\beta\eta}.$$

The same holds with $-|y|^\beta$ and $|y|$ instead of y^β and y when $y<0$, and it trivially holds for $y=0$. Since $|\cos x-\cos y|\leq 2|x-y|(|x-y|+|y|)$ for all x,y, we obtain

$$\Big|\cos\big(H_n^+(u_n\gamma^+_{i-1)\Delta_n})+H_n^-(u_n\gamma^-_{i-1)\Delta_n})\big)-\widehat{U}(1,u_n)^n_{(i-1)\Delta_n}\Big| \leq K\big(u_n^2\Delta_n^{1-2\beta\eta}+u_n^{1+\beta}\Delta_n^{3/2-\beta/2-\beta\eta}\big).$$

In view of (6.16), and upon choosing $\eta > 0$ small enough, we deduce that

$$\Big|\cos\big(H_n^+(u_n\gamma^+_{i-1)\Delta_n}) + H_n^-(u_n\gamma^-_{i-1)\Delta_n})\big) - \widehat{U}(1,u_n)^n_{i-1)\Delta_n}\Big| \le u_n^4\sqrt{\Delta_n}\,\phi_n. \tag{6.34}$$

4) At this stage, (6.29) is an easy consequence of (6.7), (6.32), (6.33) and (6.34). Since

$$\cos(u_n\rho(\kappa)^n_i)\,\cos(u'_n\rho(\kappa)^n_i) = \frac{1}{2}\,\big(\cos((u_n+u'_n)\rho(\kappa)^n_i) + \cos(|u_n-u'_n|\rho(\kappa)^n_i)\big),$$

(6.30) follows from (6.29).

Finally, since $|\cos x| \le 1$ and $|\mathcal{U}(\kappa,u)^n_t| \le 1$, it is enough to prove (6.31) for $q=2$. Since $(\cos x)^2 = \frac{1}{2}\,(1+\cos(2x))$, an application of (6.29) and (6.30) shows that the left side of (6.31) is, up to a remainder term of size smaller than $\phi_n u_n^4\sqrt{\Delta_n}$, equal to

$$\frac{1}{2}\,\big(\mathcal{U}(\kappa,2u_n)^n_{(i-1)\Delta_n} - 2(\mathcal{U}(\kappa,u_n)^n_{(i-1)\Delta_n})^2 + 1\big).$$

An expansion near 0 of the function $u\mapsto \mathcal{U}(\kappa,u)^n$ in (6.7) yields that the above is smaller than $K(u_n^4 + \Delta_n^{1-\beta/2}u_n^\beta)$, which in turn is smaller than Ku_n^4 by (6.16). This yields (6.31). $\square$

Below, we use the simplifying notation

$$V(\kappa,u,u')^n_t = \mathcal{U}(\kappa,u+u')^n_t + \mathcal{U}(\kappa,|u-u'|)^n_t - 2\mathcal{U}(\kappa,u)^n_t\,\mathcal{U}(\kappa,u')^n_t. \tag{6.35}$$

LEMMA 13. *For all $q\ge 2$, and if $u'_n \asymp u_n$, we have*

$$\begin{aligned}
&|\mathbb{E}(\xi(\kappa,u_n)^{1,n}_j \mid \mathcal{F}_{\kappa j v_n})| \le u_n^4\sqrt{\Delta_n}\,\phi_n\\
&|\mathbb{E}(\xi(\kappa,u_n)^{1,n}_j\,\xi(\kappa,u'_n)^{1,n}_j \mid \mathcal{F}_{\kappa j v_n}) - \tfrac{1}{2k_n}\,V(\kappa,u_n,u'_n)^n_{\kappa j v_n}| \le \phi_n u_n^4\,\sqrt{\Delta_n}\\
&\mathbb{E}(|\xi(\kappa,u_n)^{1,n}_j|^q \mid \mathcal{F}_{\kappa j v_n}) \le Ku_n^4/k_n^{q/2}
\end{aligned} \tag{6.36}$$

$$\begin{aligned}
&|\mathbb{E}(\xi(\kappa,u_n)^{2,n}_j \mid \mathcal{F}_{\kappa j v_n})| \le u_n^4\sqrt{\Delta_n}\,\phi_n\\
&\mathbb{E}(|\xi(\kappa,u_n)^{2,n}_j|^q \mid \mathcal{F}_{\kappa j v_n}) \le u_n^4\sqrt{\Delta_n}\,\phi_n\,/k_n^{q/2}
\end{aligned} \tag{6.37}$$

$$\begin{aligned}
&|\mathbb{E}(\xi(\kappa,u_n)^{3,n}_j \mid \mathcal{F}_{\kappa j v_n})| \le u_n^4\sqrt{\Delta_n}\,\phi_n\\
&\mathbb{E}(|\xi(\kappa,u_n)^{3,n}_j|^q \mid \mathcal{F}_{\kappa j v_n}) \le Kv_n(u_n^{2q} + \Delta_n^{q(1-\beta/2)}u_n^{q\beta}).
\end{aligned} \tag{6.38}$$

PROOF. In the proof, and for simplicity, we denote by $\zeta(l,w)_n$ the lth summand in the definition of $\xi(\kappa,u_n)^{w,n}_j$, for $w=1,2,3$.

Upon expanding the product $\xi(\kappa,u_n)^{1,n}_j\xi(\kappa,u'_n)^{1,n}_j$, (6.29) and (6.30) and successive conditioning yield

$$\Big|\mathbb{E}(\xi(\kappa,u_n)^{1,n}_j\xi(\kappa,u'_n)^{1,n}_j \mid \mathcal{F}_{\kappa j v_n}) - \frac{1}{2k_n^2}\sum_{l=0}^{k_n-1}\mathbb{E}(V(\kappa,u_n,u'_n)^n_{\kappa(jk_n+l)\Delta_n} \mid \mathcal{F}_{\kappa j v_n})\Big| \le \phi_n\,u_n^4\,\sqrt{\Delta_n}.$$

The first part of (6.16) and (6.13) also yield for $l\le k_n$:

$$|\mathbb{E}(V(\kappa,u_n,u'_n)^n_{\kappa(jk_n+l)\Delta_n} - V(\kappa,u_n,u'_n)^n_{\kappa j v_n} \mid \mathcal{F}_{\kappa j v_n})| \le K(u_n^2 v_n + u_n^\beta\Delta_n^{1-\beta/2}v_n^{\beta/2}) \le \phi_n u_n^4\sqrt{\Delta_n},$$

the last estimate coming from (6.16). We deduce the second part of (6.36). Next, (6.29) and (6.31) yields $|\mathbb{E}(\zeta(l,1)_n \mid \mathcal{F}^n_{jk_n+l})| \leq \phi_n u_n^4 \sqrt{\Delta_n}$ and $\mathbb{E}(|\zeta(l,1)_n|^q \mid \mathcal{F}^n_{jk_n+l}) \leq K u_n^4$, so we have the first part of (6.36), and also the last part by the Burkholder-Gundy and Hölder inequalities.

(6.37) is a simple consequence of (6.22), plus Burkholder-Gundy inequality again. Finally (6.13) yields

$$\begin{aligned} &\big|\mathbb{E}(\zeta(l,3)_n \mid \mathcal{F}_{\kappa j v_n})\big| \leq K\big(u_n^2 v_n + \Delta_n^{1-\beta/2} u_n^\beta v_n^{\beta/2}\big) \\ &\mathbb{E}(|\zeta(l,3)_n|^q \mid \mathcal{F}_{\kappa j v_n}) \leq K v_n\big(u_n^{2q} + \Delta_n^{q(1-\beta/2)} u_n^{q\beta}\big). \end{aligned}$$

Then (6.16) yields (6.38). □

LEMMA 14. *For all $q \geq 2$, and if $u'_n \asymp u_n$, we have*

$$\begin{aligned} &|\mathbb{E}(\xi(\kappa,u_n)^n_j \mid \mathcal{F}_{\kappa j v_n})| \leq u_n^4 \sqrt{\Delta_n}\, \phi_n \\ &|\mathbb{E}(\xi(\kappa,u_n)^n_j\, \xi(\kappa,u'_n)^n_j \mid \mathcal{F}_{\kappa j v_n}) - \tfrac{1}{2k_n} \tfrac{V(\kappa,u_n,u'_n)^n_{\kappa j v_n}}{\mathcal{U}(\kappa,u_n)^n_{\kappa j v_n}\, \mathcal{U}(\kappa,u'_n)^n_{\kappa j v_n}}| \leq u_n^4 \sqrt{\Delta_n}\, \phi_n \\ &\mathbb{E}(|\xi(\kappa,u_n)^n_j|^q \mid \mathcal{F}_{\kappa j v_n}) \leq K\big(\tfrac{u_n^4}{k_n^{q/2}} + u_n^{2q} v_n + \Delta_n^{q(1-\beta/2)} u_n^{q\beta} v_n\big). \end{aligned} \tag{6.39}$$

PROOF. In view of (6.8) and of the previous lemma, the first and last parts of (6.39) are obvious. For the second part, by virtue of the second estimate in (6.36), it is enough to prove that

$$|\mathbb{E}(\xi(\kappa,u_n)^{z,n}_j\, \xi(\kappa,u'_n)^{w,n}_j \mid \mathcal{F}_{\kappa j v_n})| \leq u_n^4 \sqrt{\Delta_n}\, \phi_n.$$

for all $z,w = 1,2,3$ but $z = w = 1$. This property follows from Cauchy-Schwarz inequality and all estimates in the previous lemma with $q = 2$, except when $z = 1$ and $w = 3$ or $z = 3$ and $w = 1$.

We will examine the case $z = 1$ and $w = 3$, the other one being analogous. We have

$$\xi(\kappa,u'_n)^{3,n}_j = \frac{1}{k_n} \sum_{l=0}^{k_n-2} (k_n - l - 1)\,\big(\mathcal{U}(\kappa,u'_n)^n_{\kappa(jk_n+1+l)\Delta_n} - \mathcal{U}(\kappa,u'_n)^n_{\kappa(jk_n+l)\Delta_n}\big),$$

yielding

$$\begin{gathered} \xi(\kappa,u_n)^{1,n}_j \xi(\kappa,u'_n)^{3,n}_j = \tfrac{1}{k_n^2} \textstyle\sum_{l=0}^{k_n-1} \sum_{l'=0}^{k_n-2} (k_n - l' - 1)\alpha^n_{l,l'}, \quad \text{where} \\ \alpha^n_{l,l'} = \big(\cos(u_n \rho(\kappa)^n_{1+\kappa j k_n + \kappa l}) - \mathcal{U}(\kappa,u_n)^n_{(\kappa(jk_n+l')\Delta_n)}\big)\big(\mathcal{U}(\kappa,u'_n)^n_{\kappa(jk_n+1+l')\Delta_n} - \mathcal{U}(\kappa,u'_n)^n_{\kappa(jk_n+l')\Delta_n}\big) \end{gathered}$$

and it is thus enough to prove that $a^n_{l,l'} = \mathbb{E}(\alpha^n_{l,l'} \mid \mathcal{F}_{\kappa j v_n})$ satisfies

$$|a^n_{l,l'}| \leq \begin{cases} u_n^4 \frac{\sqrt{\Delta_n}}{k_n}\, \phi_n & \text{if } l \neq l' \\ u_n^4 \sqrt{\Delta_n}\, \phi_n & \text{if } l = l'. \end{cases} \tag{6.40}$$

If $l < l'$, and since $|\mathcal{U}(\kappa,u_n)^n_t| \leq 1$, (6.13) with $s = \Delta_n$ and the first part of (6.16) give us

$$|\mathbb{E}(\alpha^n_{l,l'} \mid \mathcal{F}_{\kappa(jk_n+l')\Delta_n})| \leq K \Delta_n u_n^\beta |\cos(u_n \rho(\kappa)^n_{1+jk_n+l}) - \mathcal{U}(\kappa,u_n)^n_{\kappa(jk_n+l)\Delta_n}|.$$

Then (6.31) and the Cauchy-Schwarz inequality yield $|a^n_{l,l'}| \leq K \Delta_n u_n^{2+\beta}$, so (6.16) again implies (6.40). If $l > l'$ (6.29) yields

$$|\mathbb{E}(\alpha^n_{l,l'} \mid \mathcal{F}_{\kappa(jk_n+l)\Delta_n})| \leq \phi_n u_n^4 \sqrt{\Delta_n}\, |\mathcal{U}(\kappa,u'_n)^n_{\kappa(jk_n+1+l')\Delta_n} - \mathcal{U}(\kappa,u'_n)^n_{\kappa(jk_n+l')\Delta_n}|,$$

and (6.13) with $s=\kappa\Delta_n$ and Cauchy-Schwarz inequality yield $|a^n_{l,l'}|\leq u_n^4\Delta_n\phi_n$, hence (6.40).

For $l=l'$, upon using (6.8) and the last part of (6.13), plus (6.31) and the Cauchy-Schwarz inequality and (6.16), we see that it is enough to prove (6.40) with $\alpha^n_{l,l}$ replaced by

$$\begin{array}{l}\alpha'^n_{l,l}=\big(\cos(u_n\rho(\kappa)^n_{1+\kappa jk_n+\kappa l})-\mathcal{U}(\kappa,u_n)^n_{(\kappa(jk_n+l)\Delta_n}\big)\,\zeta^n_{\kappa(jk_n+l)}\\ \qquad\text{where}\qquad \zeta^n_i=U(\kappa,u'_n)^n_{(i+\kappa)\Delta_n}-U(\kappa,u'_n)^n_{i\Delta_n}.\end{array}$$

The same type of argument, now based on the first part of (6.12) and (6.24), plus the property $|\cos(u_n\rho(\kappa)^n_i)-\cos(u_n\rho'(\kappa)^n_i)|\leq|u_n\rho''(\kappa)^n_i|\wedge 1$, shows that we can even replace $\alpha'^n_{l,l}$ by

$$\alpha''^n_{l,l}=\psi^n_{\kappa(jk_n+l)}\zeta^n_{\kappa(jk_n+l)},\qquad\text{where}\qquad \psi^n_i=\cos(u_n\rho'(\kappa)^n_{1+i})-\mathcal{U}(\kappa,u_n)^n_{i\Delta_n}.$$

Observe that $\zeta^n_i=\sum_{w=1}^4\beta(w)^n_i$, where

$$\begin{array}{l}\beta(1)^n_i=\int_{i\Delta_n}^{(i+\kappa)\Delta_n}b_s^{U(\kappa,u'_n)}\,ds,\qquad \beta(2)^n_i=H_{i\Delta_n}^{U((\kappa,u'_n)}(W_{(i+\kappa)\Delta_n}-W_{i\Delta_n})\\ \beta(3)^n_i=\int_{i\Delta_n}^{(i+\kappa)\Delta_n}(H_s^{U(\kappa,u'_n)}-H_{i\Delta_n}^{U(\kappa,u'_n)})\,dW_s\\ \beta(4)^n_i=\int_{i\Delta_n}^{(i+\kappa)\Delta_n}H_s'^{U(\kappa,u'_n)}\,dW'_s+\int_{i\Delta_n}^{(i+\kappa)\Delta_n}\int_{\mathbb{R}}\delta^{U(\kappa,u'_n)}(s,z)\,(\underline{p}-\underline{q}\,)(ds,dz).\end{array}$$

By (6.10) we have $|\beta(1)^n_i|\leq K\Delta_n u_n'^2$. Combining (6.3), (6.9) and (6.12), we easily check that $\mathbb{E}(|\beta(3)^n_i|\mid\mathcal{F}_{(i-1)\Delta_n})\leq Ku_n'^2\Delta_n$, hence

$$w=1,3,\quad 0\leq l\leq k_n-1\ \Rightarrow\ \mathbb{E}(|\psi^n_i\,\beta(w)^n_i|\mid\mathcal{F}^n_i)\leq Ku_n^4\sqrt{\Delta_n}\,\phi_n.$$

A parity argument (as in Step 4 of the proof of Lemma 11) shows that $\mathbb{E}(\psi^n_i\beta(2)^n_i|\mid\mathcal{F}_{(i-1)\Delta_n})=0$ for all i. Finally, with $\mathcal{G}^W=\sigma(W_s:s\geq 0)$, the independence between W and $(W',\underline{p})$ implies that $\mathbb{E}(\beta(4)^n_i\mid\mathcal{F}_{(i-1)\Delta_n}\vee\mathcal{G}^W)=0$, whereas ψ^n_i is $\mathcal{F}_{(i-1)\Delta_n}\vee\mathcal{G}^W$-measurable, hence $\mathbb{E}(\psi^n_i\beta(4)^n_i\mid\mathcal{F}_{(i-1)\Delta_n})=0$. All these partial results give us the needed estimate for $|\mathbb{E}(\alpha^n_{l,l}\mid\mathcal{F}_{\kappa(jk_n+l)\Delta_n})|$, and the proof is complete. □

LEMMA 15. *For any square-integrable martingale M and any random variables ζ^n_j such that $|\zeta^n_j|\leq K$ and each ζ^n_j is $\mathcal{F}_{\kappa jv_n}$-measurable, and for all $t>0$, we have*

$$\frac{v_n}{u_n^4\sqrt{\Delta_n}}\sum_{j=0}^{[t/\kappa v_n]-1}\mathbb{E}\big((M_{\kappa(j+1)v_n}-M_{\kappa jv_n})\,\zeta^n_j\,\xi(\kappa,u_n)^n_j\mid\mathcal{F}_{\kappa jv_n}\big)\ \xrightarrow{\mathbb{P}}\ 0.\tag{6.41}$$

PROOF. It suffices to prove the result if we replace $\xi(\kappa,u_n)^n_j$ above by $\xi(\kappa,u_n)^{w,n}_j$, for $w=1,2,3$, and in this case we denote by $R^{w,n}_t$ the normalized sum in (6.41).

When $w=2,3$ we use the following argument: the properties of ζ^n_j and Cauchy-Schwarz inequality yield

$$\mathbb{E}(|R^{w,n}_t|)\leq\frac{Kv_n}{u_n^4\sqrt{\Delta_n}}\Big(\sum_{j=0}^{[t/\kappa v_n]-1}\mathbb{E}(|\xi(\kappa,u_n)^{w,n}_j|^2)\sum_{j=0}^{[t/\kappa v_n]}\mathbb{E}((M_{\kappa(j+1)v_n}-M_{\kappa jv_n})^2)\Big)^{1/2},\tag{6.42}$$

and the last sum is equal to $\mathbb{E}((M_{\kappa v_n([t/\kappa v_n]-1)}-M_0)^2)$, which is bounded. Then it is enough to show that $\frac{v_n^2}{u_n^8\Delta_n}\sum_{j=0}^{[t/\kappa v_n]-1}\mathbb{E}(|\xi(\kappa,u_n)^{w,n}_j|^2)\to 0$, which follows from Lemma 13 and (6.16).

When $w=1$, we write $\xi(\kappa,u_n)^{1,n}_j=\psi'^n_j+\psi''^n_j$, where $\psi'^n_j=\frac{1}{k_n}\sum_{l=0}^{k_n-1}\eta^n_{1+\kappa(jk_n+l)}$ and $\eta^n_i=\cos(u_n\rho(\kappa)^n_i)-\mathbb{E}(\cos(u_n\rho(\kappa)^n_i)\mid\mathcal{F}_{(i-1)\Delta_n})$, and we are left to prove that (6.41) holds with $\xi(\kappa,u_n)^n_j$ replaced by ψ'^n_j and by ψ''^n_j. In both cases we denote by $R'^{1,n}_t$ and $R''^{1,n}_t$ the corresponding normalized sums. For proving $R''^{1,n}_t\xrightarrow{\mathbb{P}}0$ we proceed as above, that is, we have (6.42) with ψ''^n_j instead of $\xi(\kappa,u_n)^{w,n}_j$, whereas $|\psi''^n_j|\le\phi_nu_n^4\sqrt{\Delta_n}$ by (6.29), hence the result holds.

For $R'^{1,n}_t$, we observe that, by successive conditioning,

$$\begin{gathered}R'^{1,n}_t=\frac{v_n}{u_n^4k_n\sqrt{\Delta_n}}\sum_{j=0}^{[t/\kappa v_n]-1}\zeta^n_j\sum_{l=0}^{k_n-1}\mathbb{E}(\overline{\eta}^n_{1+\kappa(k_n+l)}\mid\mathcal{F}_{\kappa jv_n})\\ \text{where}\quad \overline{\eta}^n_i=M^n_i\,\cos(u_n\rho(\kappa)^n_i),\qquad M^n_i=M_{(i-1+\kappa)\Delta_n}-M_{(i-1)\Delta_n}.\end{gathered}$$

As above, $\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}((M^n_i)^2)\le K$ and $|\cos(u_n\rho(\kappa)^n_i)-\cos(u_n\rho'(\kappa)^n_i)|\le K(|u_n\rho''(\kappa)^n_i|\wedge1)$. Hence if

$$\overline{R}'^{1,n}_t=\frac{v_n}{u_n^4k_n\sqrt{\Delta_n}}\sum_{j=0}^{[t/\kappa v_n]-1}\zeta^n_j\sum_{l=0}^{k_n-1}\mathbb{E}(\overline{\eta}'^n_{1+\kappa(k_n+l)}\mid\mathcal{F}_{\kappa jv_n}),\quad\text{where}\;\;\overline{\eta}'^n_i=M^n_i\,\cos(u_n\rho'(\kappa)^n_i),$$

by (6.24) and Cauchy-Schwarz inequality and $|\zeta^n_j|\le K$, we have for all $\varepsilon>0$ arbitrarily small:

$$\mathbb{E}(|R'^{1,n}_t-\overline{R}'^{1,n}_t|)\le\frac{K\sqrt{t}\,v_n\Delta_n^{-\varepsilon/2-\beta/4}}{u_n^4k_n\sqrt{\Delta_n}}\Big(\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}((M^n_i)^2)\Big)^{1/2}\le\frac{K\sqrt{t}\,\Delta_n^{1/2-\varepsilon/2-\beta/4}}{u_n^4}\;\to\;0$$

(use (6.16) again). Now, by classical arguments, it suffices to show that $\overline{R}'^{1,n}_t\xrightarrow{\mathbb{P}}0$ when M is orthogonal to W, or is equal to W itself. In the second case, we clearly have $\mathbb{E}(\overline{\eta}'^n_i\mid\mathcal{F}^n_{i-1})=0$. In the first case, we have the same by an application of Itô's formula. So $\overline{R}'^{1,n}_t=0$ in all cases, and the proof is complete. □

At this stage, we can prove Lemmas 7, 9 and 10.

Proof of Lemma 7. Using (6.39) with $q>2$ and $u_n\to0$ and (6.16) and Markov inequality yields

$$\mathbb{P}((\Omega(\kappa,\theta u_n)_{n,t})^c)\le\sum_{j=0}^{[t/\kappa v_n]-1}\mathbb{P}\big(|\xi(\kappa,\theta u_n)^n_j|>\frac12\big)\le2^{-q}\sum_{j=0}^{[t/\kappa v_n]}\mathbb{E}(|\xi(\kappa,\theta u_n)^n_j|^q)\le Kt\phi_n,$$

hence the claim because $\Omega(\kappa)_{n,t}$ is a finite union of sets $\Omega(\kappa,\theta u_n)_{n,t}$. □

Proof of Lemma 9. The claim $\frac{1}{u_n^2}R^{\kappa,n,\theta}_t\xrightarrow{\mathbb{P}}0$ readily follows from (6.16) and from the last part of (6.39) with $q=2$ and $q=3$.

For the second claim, we set

$$\zeta^n_j=\frac{v_n}{u_n^4\sqrt{\Delta_n}}\big((\xi(\kappa,\theta u_n)^n_j)^2-\frac{2}{k_n}\,f_{\kappa,\theta u_n}(c_{\kappa jv_n})\big),\qquad\zeta'^n_j=\mathbb{E}(\zeta^n_j\mid\mathcal{F}_{\kappa jv_n}).$$

By a standard martingale argument, and since ζ_j^n is $\mathcal{F}_{\kappa(j+1)v_n}$-measurable, it is enough to show that

$$\sum_{j=0}^{[t/\kappa v_n]-1} \zeta_j'^n \stackrel{\text{u.c.p.}}{\Longrightarrow} 0, \qquad \sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}(|\zeta_j^n|^2) \stackrel{\mathbb{P}}{\longrightarrow} 0. \tag{6.43}$$

Recall that $f_{\kappa,\theta u}(x) \leq K u^4$ when $|x| \leq K$, hence (6.39) yields

$$\mathbb{E}(|\zeta_j^n|^2) \leq \frac{K v_n^2}{\Delta_n}\Big(\frac{1}{k_n^2 u_n^4} + v_n + v_n \Delta_n^{4-2\beta} u_n^{4\beta-8}\Big).$$

The right side above is easily seen to be $o(v_n)$ by (6.16), hence the second part of (6.43). For the first part we use (6.39) again and also (6.35) and $\mathcal{U}(\kappa,0)_t^n = 1$ to observe that it suffices to prove that

$$\frac{v_n}{u_n^4 k_n \sqrt{\Delta_n}} \sum_{i=0}^{[t/\kappa v_n]-1} \Big(\frac{\mathcal{U}(\kappa,2\theta u_n)_{\kappa j v_n}^n + 1 - 2(\mathcal{U}(\kappa,\theta u_n)_{\kappa j v_n}^n)^2}{2(\mathcal{U}(\kappa,\theta u_n)_{\kappa j v_n}^n)^2} - 2 f_{\kappa,\theta u_n}(c_{\kappa j v_n})\Big) \stackrel{\text{u.c.p.}}{\Longrightarrow} 0. \tag{6.44}$$

Now we recall that $|\mathcal{U}(\kappa,\theta u_n)_t^n - U(\kappa,\theta u_n)_t| \leq K u_n^\beta \Delta_n^{1-\beta/2}$ and $1/\mathcal{U}(\kappa,\theta u_n)_t^n \leq K$: since we have $\Delta_n^{1-\beta/2} u_n^{\beta-4}/k_n\sqrt{\Delta_n} \to 0$ by (6.16), we can thus substitute $\mathcal{U}(\kappa,\theta u_n)^n$ in (6.44) with $U(\kappa,\theta u_n)$. But in this case, and by definition of $f_{\kappa,u}$, each summand is identically 0, hence (6.44) is proved. □

Proof of Lemma 10. Set $\Theta' = \Theta\backslash\{1\}$ and

$$\alpha_j^{\kappa,n,\theta} = \frac{2v_n}{(\theta u_n)^2\sqrt{\Delta_n}}\Big(\frac{2}{\kappa k_n (\theta u_n)^2}\, f'_{\kappa,\theta u_n}(c_{\kappa j v_n}) - 1\Big)$$

and

$$\widehat{Y}_t^{\kappa,n,\theta} = \sum_{j=0}^{[t/\kappa v_n]-1} \zeta_j^{\kappa,n,\theta}, \qquad \zeta_j^{\kappa,n,\theta} = \begin{cases} \alpha_j^{\kappa,n,1}\xi(\kappa,u_n)_j^n & \text{if } \theta = 1 \\ \frac{1}{u_n^2}\,(\alpha_j^{\kappa,n,\theta}\xi(\kappa,\theta u_n)_j^n - \alpha_j^{\kappa,n,1}\xi(\kappa,u_n)_j^n) & \text{if } \theta \in \Theta' \end{cases}$$

The claim of the lemma is then equivalent to saying that $(\widehat{Y}^{\kappa,n,\theta})_{\theta\in\Theta}$ converges stably in law to $\big(\kappa^{1/2}Z, (\kappa^{1/3}(\theta^2-1)\overline{Z})_{\theta\in\Theta'}\big)$.

We observe that the variable $\zeta_j^{\kappa,n,\theta}$ is $\mathcal{F}_{\kappa(j+1)v_n}$-measurable, whereas (6.39) and (6.16) and Lemma 15 imply, for all $t > 0$ and all square-integrable martingale M:

$$\sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}(\zeta_j^{\kappa,n,\theta} \mid \mathcal{F}_{\kappa j v_n}) \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}((\zeta_j^{\kappa,n,\theta})^4 \mid \mathcal{F}_{j v_n}) \stackrel{\mathbb{P}}{\longrightarrow} 0$$
$$\sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}(\zeta_j^{\kappa,n,\theta}(M_{\kappa(j+1)v_n} - M_{\kappa j v_n}) \mid \mathcal{F}_{\kappa j v_n}) \stackrel{\mathbb{P}}{\longrightarrow} 0.$$

Hence Theorem 2.2.15 of [8] shows that it remains to prove the following convergences:

(6.45)
$$\sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}(\zeta_j^{\kappa,n,\theta}\,\zeta_j^{\kappa,n,\theta'} \mid \mathcal{F}_{\kappa j v_n}) \stackrel{\mathbb{P}}{\longrightarrow} \Gamma_t^{\kappa,\theta,\theta'} = \begin{cases} 2\kappa \int_0^t c_s^2\,ds & \text{if } \theta = \theta' = 1 \\ 0 & \text{if } \theta = 1 \neq \theta' \\ \frac{\kappa^3}{6}\,(\theta^2-1)(\theta'^2-1)\int_0^t c_s^4\,ds & \text{if } \theta,\theta' \in \Theta'. \end{cases}$$

Recalling $|f'_{\kappa,\theta u_n}(c_t)| \leq K u_n^4$, it is enough to show that $\Gamma_t^{\kappa,n,\theta,\theta'} \xrightarrow{\mathbb{P}} \Gamma_t^{\kappa,\theta,\theta'}$, where

$$\Gamma_t^{\kappa,n,\theta,\theta'} = \begin{cases} \frac{4v_n^2}{u_n^4\Delta_n} \sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}((\xi(\kappa,u_n)_j)^2 \mid \mathcal{F}_{\kappa j v_n}) & \text{if } \theta=\theta'=1 \\ \frac{4v_n^2}{u_n^6\Delta_n} \sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}\Big(\xi(\kappa,u_n)_j^n\big(\frac{\xi(\kappa,\theta' u_n)_j^n}{\theta'^2} - \xi(\kappa,u_n)_j^n\big) \mid \mathcal{F}_{\kappa j v_n}\Big) & \text{if } \theta=1\neq\theta' \\ \frac{4v_n^2}{u_n^8\Delta_n} \sum_{j=0}^{[t/\kappa v_n]-1} \mathbb{E}\Big(\big(\frac{\xi(\kappa,\theta u_n)_j^n}{\theta^2} - \xi(\kappa,u_n)_j^n\big) & \\ \qquad\qquad \times\big(\frac{\xi(\kappa,\theta' u_n)_j^n}{\theta'^2} - \xi(\kappa,u_n)_j^n\big) \mid \mathcal{F}_{\kappa j v_n}\Big) & \text{if } \theta,\theta'\in\Theta'. \end{cases}$$

We then apply (6.39) again, plus $v_n = k_n\Delta_n$ and the fact that $v_n/u_n^4\sqrt{\Delta_n} \to 0$ by (3.9), and conclude that it is enough to show $\Gamma_t'^{\kappa,n,\theta,\theta'} \xrightarrow{\mathbb{P}} \Gamma_t^{\kappa,\theta,\theta'}$, where

$$\Gamma_t'^{\kappa,n,\theta,\theta'} = \begin{cases} \frac{2v_n}{u_n^4} \sum_{j=0}^{[t/\kappa v_n]-1} \frac{V(\kappa,u_n,u_n)_{\kappa j v_n}^n}{(\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n)^2} & \text{if } \theta=\theta'=1 \\ \frac{2v_n}{u_n^6} \sum_{j=0}^{[t/\kappa v_n]-1} \Big(\frac{V(\kappa,u_n,\theta' u_n)_{\kappa j v_n}^n}{\theta'^2\,\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n\,\mathcal{U}(\kappa,\theta' u_n)_{\kappa j v_n}^n} - \frac{V(\kappa,u_n,u_n)_{\kappa j v_n}^n}{(\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n)^2}\Big) & \text{if } \theta=1\neq\theta' \\ \frac{2v_n}{u_n^8} \sum_{j=0}^{[t/\kappa v_n]-1} \Big(\frac{V(\kappa,\theta u_n,\theta' u_n)_{\kappa j v_n}^n}{\theta^2\theta'^2\,\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n\,\mathcal{U}(\kappa,\theta' u_n)_{\kappa j v_n}^n} + \frac{V(\kappa,u_n,u_n)_{\kappa j v_n}^n}{(\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n)^2} & \\ \quad - \frac{V(\kappa,u_n,\theta u_n)_{\kappa j v_n}^n}{\theta^2\,\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n\,\mathcal{U}(\kappa,\theta' u_n)_{\kappa j v_n}^n} - \frac{V(\kappa,u_n,\theta' u_n)_{\kappa j v_n}^n}{\theta'^2\,\mathcal{U}(\kappa,u_n)_{\kappa j v_n}^n\,\mathcal{U}(\kappa,\theta' u_n)_{\kappa j v_n}^n}\Big) & \text{if } \theta,\theta'\in\Theta'. \end{cases}$$

If we denote $\Gamma_t''^{\kappa,n,\theta,\theta'}$ the same as above, with $\mathcal{U}(\kappa,u)^n$ and $V(\kappa,u,u')^n$ substituted with $U(\kappa,u)$ and $U(\kappa,u+u')+U(\kappa,|u-u'|)-2U(\kappa,u)U(\kappa,u')$, and upon using (6.8) and $|\mathcal{U}(\kappa,u)_t^n - U(\kappa,u)_t| \leq K u^\beta \Delta_n^{1-\beta/2}$ and the same argument as in the proof of the previous lemma, we see that it remains to prove $\Gamma_t''^{\kappa,n,\theta,\theta'} \xrightarrow{\mathbb{P}} \Gamma_t^{\kappa,\theta,\theta'}$. The form (6.7) of $U(\kappa,u)$ allows us to check that indeed

$$\Gamma_t''^{\kappa,n,\theta,\theta'} = \begin{cases} \frac{8v_n}{u_n^4} \sum_{j=0}^{[t/\kappa v_n]-1} f_{\kappa,u_n}(c_{\kappa j v_n}) & \text{if } \theta=\theta'=1 \\ \frac{8v_n}{u_n^6} \sum_{j=0}^{[t/\kappa v_n]-1} \Big(\frac{f_{\kappa,u_n\sqrt{\theta'}}(c_{\kappa j v_n})}{\theta'^2} - f_{\kappa,u_n}(c_{\kappa j v_n})\Big) & \text{if } \theta=1\neq\theta' \\ \frac{8v_n}{u_n^8} \sum_{j=0}^{[t/\kappa v_n]-1} \Big(\frac{f_{\kappa,u_n\sqrt{\theta\theta'}}(c_{\kappa j v_n})}{\theta^2\theta'^2} + f_{\kappa,u_n}(c_{\kappa j v_n}) & \\ \qquad - \frac{f_{\kappa,u_n\sqrt{\theta}}(c_{\kappa j v_n})}{\theta^2} - \frac{f_{\kappa,u_n\sqrt{\theta'}}(c_{\kappa j v_n})}{\theta'^2}\Big) & \text{if } \theta,\theta'\in\Theta'. \end{cases}$$

Observing that $|f_{\kappa,y}(x) - \frac{\kappa^2}{4}\,y^4x^2 - \frac{\kappa^4}{48}\,y^8x^4| \leq Ky^{12}$ for all x,y within an arbitrary compact set, we readily obtain $\Gamma_t''^{\kappa,n,\theta,\theta'} \xrightarrow{\mathbb{P}} \Gamma_t^{\kappa,\theta,\theta'}$ from a Riemann sum approximation and $u_n \to 0$. This completes the proof. $\square$

6.4. *Proof of Theorem 5.* At this stage, Theorem 5 is the only result left to be proven. In view of (3.5) and by expanding $x \mapsto \log(\cos x)$ near 0 and using the boundedness of the process a'_t, we get the following bound, uniform in $u \in (0,1]$:

$$|A'(u)_t^n - A(u)_t^n| \;\leq\; Ktu^{2\beta-2}\,\Delta_n^{2-\beta},$$

implying

$$\frac{1}{u_n^2\sqrt{\Delta_n}}\,(A'(\theta u_n)^n - A(\theta u_n)^n) \overset{\text{u.c.p.}}{\Longrightarrow} 0$$

if $\beta < \frac{3}{2}$ because of (6.16). Recall also that $A'(u)^n = A(u)^n$ when $\gamma^+ + \gamma^- = 0$ identically. Henceforth, if we put

$$\widetilde{Z}(\kappa,u)^n_t = \frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(\kappa,u)^n_t - C_t - A(u)^n_t),$$

we have the following consequence of Theorem 1: Under the assumptions of this theorem, then

$$\Big(\widetilde{Z}(\kappa,u_n)^n, (\frac{1}{u_n^2}\,(\widetilde{Z}(\kappa,\theta u_n) - \widetilde{Z}(\kappa,u_n)^n))_{\theta\in\Theta}\Big) \overset{\mathcal{L}-s}{\Longrightarrow} (\kappa^{1/2}Z, (\kappa^{3/2}(\theta^2-1)\overline{Z})_{\theta\in\Theta}) \tag{6.46}$$

for $\kappa = 2$, and also for $\kappa = 1$ when either $1 < \beta < \frac{3}{2}$ or $\beta \geq \frac{3}{2}$ and $\gamma^+ + \gamma^- = 0$ identically, that is, under the conditions of Theorem 5.

We choose a number $\zeta > 1$, and observe that $\widehat{C}(u,\zeta)^n_T = \widehat{C}(2,u,\zeta)^n_T$ and $\widehat{C}'(u,\zeta)^n_T = \widehat{C}(1,u,\zeta)^n_T$, where

$$\widehat{C}(\kappa,u,\zeta)^n_T = \widehat{C}(\kappa,u)^n_T - \frac{(\widehat{C}(\kappa,\zeta u)^n_T - \widehat{C}(\kappa,u)^n_T)^2}{\widehat{C}(\kappa,\zeta^2 u)^n_T - 2\widehat{C}(\kappa,\zeta u)^n_T + \widehat{C}(\kappa,u)^n_T}.$$

By the definition of $A(u)^n$ we have $A(\zeta u)^n_t = \zeta^{\beta-2}A(u)^n_t$. Hence, with $\eta = \zeta^{\beta-2} - 1$, we get

$$\begin{aligned}&\widehat{C}(\kappa,u_n,\zeta)^n_T = C_T + A(u_n)^n_T + \sqrt{\Delta_n}\,\widetilde{Z}(\kappa,u_n)^n_T - \frac{\big(\eta A(u_n)^n_T + u_n^2\sqrt{\Delta_n}\,\Phi_n\big)^2}{\eta^2 A(u_n)^n_T + u_n^2\sqrt{\Delta_n}\,\Phi'_n}, \quad \text{where}\\ &\Phi_n = \tfrac{1}{u_n^2}\,(\widetilde{Z}(\kappa,\zeta u)^n_T - \widetilde{Z}(\kappa,u)^n_T), \quad \Phi'_n = \tfrac{1}{u_n^2}\,(\widetilde{Z}(\kappa,\zeta^2 u)^n_T - 2\widetilde{Z}(\kappa,\zeta u)^n_T + \widetilde{Z}(\kappa,u)^n_T).\end{aligned} \tag{6.47}$$

Now, (6.46) applied with $\Theta = \{1,\zeta,\zeta^2\}$ yields

$$(\widetilde{Z}(\kappa,u_n)^n_T, \Phi_n, \Phi'_n) \overset{\mathcal{L}-s}{\longrightarrow} (\kappa^{1/2}Z_T, \kappa^{3/2}(\zeta^2-1)\overline{Z}_T, \kappa^{3/2}(\zeta^2-1)^2\overline{Z}_T). \tag{6.48}$$

Recall also that $A(u)^n_t = u^{\beta-2}\Delta_n^{1-\beta/2}A_t$, where $A_t = 2\int_0^t a_s\,ds$. We then single out two cases:

First, on the set $\{A_T = 0\}$, we have

$$\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(\kappa,u_n,\zeta)^n_T - C_T) = \widetilde{Z}(\kappa,u_n)^n_T + u_n^2\,\frac{\Phi_n^2}{\Phi'_n}$$

and (6.48) shows that the ratio Φ_n^2/Φ'_n converges in law to $\kappa^{3/2}\overline{Z}_T$ ($\mathcal{F}$-conditionally Gaussian with positive variance, hence non-vanishing almost surely). Since $u_n \to 0$, another application of (6.48) readily yields that, in restriction to the set $\{A_T = 0\}$, the variables $\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(\kappa,u_n,\zeta)^n_T - C_T)$ converge stably in law to $\kappa^{1/2}Z_T$.

Second, we look at what happens on the set $\{A_T > 0\}$, on which we have by a simple calculation:

$$\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(\kappa,u_n,\zeta)^n_T - C_T) = \widetilde{Z}(\kappa,u_n)^n_T + \frac{u_n^2(\Phi'_n - 2\eta\Phi_n)A_T + u_n^{6-\beta}\Delta_n^{(\beta-1)/2}\Phi_n^2}{\eta^2 A_T + u_n^{4-\beta}\Delta_n^{(\beta-1)/2}\Phi'_n}.$$

Then (6.48) again yields that, in restriction to the set $\{A_T > 0\}$, the variables $\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(\kappa,u_n,\zeta)^n_T - C_T)$ converge stably in law to $\kappa^{1/2}Z_T$.

So indeed $\frac{1}{\sqrt{\Delta_n}}\,(\widehat{C}(\kappa,u_n,\zeta)^n_T - C_T) \overset{\mathcal{L}-s}{\longrightarrow} \kappa^{1/2}Z_T$ on Ω, which ends the proof of Theorem 5.

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