Do price and volatility jump together?*

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Abstract

We consider a process X_t, which is observed on a finite time interval $[0,T]$, at discrete times $0, \Delta_n, 2\Delta_n, \cdots$. This process is an Itô semimartingale with stochastic volatility σ_t^2. Assuming that X has jumps on $[0,T]$, we derive tests to decide whether the volatility process has jumps occurring simultaneously with the jumps of X_t. There are two different families of tests, for the two possible null hypotheses (common jumps or disjoint jumps). They have a prescribed asymptotic level, as the mesh Δ_n goes to 0. We show on some simulations that these tests perform reasonably well even in the finite sample case, and we also put them in use on S&P 500 index data.

Keywords: Common jumps, tests, volatility, discrete sampling, high frequency data.

1 Introduction

Financial asset prices have two well-documented salient features: their volatility changes over time, and their trajectories can exhibit large discontinuities. Both features have nontrivial implications for risk modeling and management as the underlying asset itself is no longer sufficient to span all the available risks in it and derivatives (written on it) are typically needed. Of central importance then becomes the relationship between the price jumps and volatility. For example, if the volatility is driven by a single (Markov) diffusion process, then one can separate the management of volatility and jump risks by using first at-the-money options for the former and then out-of-the-money options for the latter. But such a simple separate management of these two risks will obviously not work if the price jumps are associated with simultaneous discontinuous changes in the level of volatility. Empirical evidence in [9] based on the behavior of close-to-maturity options written on the stock market index, suggest that this indeed might be the case. And this is exactly what we try to investigate in this paper: are price jumps accompanied by jumps in volatility?

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The link between price and volatility jumps is intrinsically associated with the observed path, and therefore we develop tests that are as much as possible independent from the underlying model. More specifically, we suppose that we have discrete observations from an arbitrary Itô semimartingale (typically the log-price) at times $i\Delta_n$ for $i = 0, 1, ..., [T/\Delta_n]$ where the time span T will stay fixed and the length of the high-frequency intervals $\Delta_n \to 0$. Under such sampling scheme, we propose tests that determine the common arrival or not of the price and volatility jumps on the discretely-observed path over $[0, T]$.

The test statistics that we construct can be intuitively described as follows. First, we identify the high-frequency price increments containing jumps as those being higher in absolute value than a truncation level which goes to zero at a certain (known) rate. Then, for the set of identified jump times we construct left and right local volatility estimators from the neighboring high-frequency price increments. Our statistics are simple sums of certain functions of the identified jumps and the associated left and right volatility estimators. Then the tests we develop are based on the different limit behavior of these statistics on the sets of common and disjoint arrival of the price and volatility jumps.

While the results in the paper are derived for general functions measuring the distance between the left and right volatility, there is one specific choice which is particularly attractive for our testing purposes and we use it in our applications. This function corresponds to the log-likelihood ratio test for deciding whether two independent samples of i.i.d. zero-mean normal variables have the same variance. The link with our analysis comes from the fact that the leading terms in the asymptotic expansions of the left and right local volatility estimators are (close to) sample averages of squared increments of a Brownian motion multiplied by the volatility level straight before and after the price jump time. The "local Gaussianity" of the high-frequency increments has been also used in [7] in a different context, i.e., for constructing various integrated measures of volatility in a continuous setting. Unlike [7], however, our analysis is for processes with jumps.

Finally, our results can be related with [6] in which we propose tests for deciding common arrival of jumps for two discretely observed processes. The major difference with that paper is that here one of the processes, namely the volatility, is not directly observed and it has to be estimated from the price increments first. This has nontrivial consequences, as it is essentially the error associated with measuring the volatility that determines the asymptotic behavior of our statistics, and it can significantly slow down their rate of converge. The intrinsic nonsymmetric nature of the price and volatility is reflected in our construction of the tests here, and this makes the statistical problem very different from the one analyzed in [6].

The paper is organized as follows. Section 2 introduces our setup and states the assumptions to be used in the rest of the paper. In Section 3 we propose statistics constructed from the high-frequency data to measure the simultaneous arrival of price and volatility jumps. In this section we also derive Central Limit Theorems associated with the statistics. Section 4 constructs our tests using the statistics of Section 3. Section 5 contains Monte Carlo evidence for the performance of the tests, while Section 6 applies our tests to real financial data. Proofs are in Section 8.

2 Setting and assumptions

We suppose throughout that our underlying process X is an Itô semimartingale on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. This means that it can be written as

$$\begin{array}{lll} X_t &=& X_0 + \int_0^t b_s ds + \int_0^t \sigma_s \, dW_s \\ && + \int_0^t \int_E (\delta(s,z) 1_{\{|\delta(t,z)|\leq 1\}})(\mu - \nu)(ds, dz) \\ && + \int_0^t \int_E (\delta(s,z) 1_{\{|\delta(t,z)|>1\}}) \mu(ds, dz). \end{array} \tag{2.1}$$

where W is a standard Brownian motion and μ is a Poisson random measure on $[0,\infty) \times E$, with $(E, \mathcal{E})$ an auxiliary measurable space, on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the predictable compensator (or intensity measure) of μ is $\nu(ds, dz) = ds \otimes \lambda(dz)$ for some given σ-finite measure λ on $(E, \mathcal{E})$. We write $c_t = (\sigma_t)^2$ for the volatility process. The processes b_t and σ_t should be progressively measurable and $\delta(\omega, t, z)$ should be a predictable function on $\Omega \times \mathbb{R}_+ \times \mathbb{E}$. We refer to [4] for all unexplained, but classical, notation.

We need some assumptions on X, and below $r \in [0, 2)$.

Assumption (H-r): *(a) The process b_t is locally bounded.*

(b) The process σ_t is càdlàg, and neither σ_t nor σ_{t-} vanish.

(c) We have $|\delta(\omega, t, x)| \leq \Gamma_t(\omega)\gamma(x)$, for a locally bounded process Γ_t and a (non-random) function $\gamma \geq 0$ satisfying $\int_E (\gamma(x)^r \wedge 1)\, \lambda(dx) < \infty$.

When $r = 2$ this is little more than X being an Itô semimartingale, except for the fact that σ_t and σ_{t-} do not vanish. When $r < 2$ it requires further that the jumps are r-summable, and the bigger r is, the weaker is the assumption. When (H-0) holds, then the jumps of X have finite activity.

Next, we make an assumption on the local behavior of σ_t. We want to accommodate two extreme cases: one is when σ_t is itself an Itô semimartingale (a quite usual assumption for stochastic volatility models), and one is when it is the sum of finitely many jumps plus a continuous process having pathwise some Hölder continuity property, such as a fractional Brownian motion. So we present an assumption which may look complicated but is satisfied by all models used so far, and implies that σ_t is càdlàg. In this assumption, v is in $(0, 1]$, and the bigger it is, the stronger is the assumption.

Assumption (K-v): *We have $\sigma_t = \Sigma(Z_t, \overline{Z}_t)$, where Σ is a C^1 function on $\mathbb{R}^2$, and Z_t and $\overline{Z}_t$ are two adapted processes with the following properties:*

(a) The process Z is an Itô semimartingale satisfying (H-2) when $v \leq 1/2$, whereas when $v > 1/2$ it satisfies (H-$1/v$) and its continuous martingale part vanishes.

(b) The process $\overline{Z}_t$ satisfies, for some locally bounded process Γ':

$$0 < s \leq 1 \;\;\Rightarrow\;\; |\overline{Z}_{t+s}(\omega) - \overline{Z}_t(\omega)| \;\leq\; \Gamma'_{t+s}(\omega)\, s^v\,. \tag{2.2}$$

3 Limit theorems for functionals of jumps and volatility

Our aim is to decide whether we have jumps of X and c occurring at the same times, and for this we make use of the following processes, where $\Delta Y_t = Y_t - Y_{t-}$ is the jump size at time t of any càdlàg process Y:

$$U(F)_t \;=\; \sum_{s\leq t} F(\Delta X_s, c_{s-}, c_s)\, 1_{\{\Delta X_s\neq 0\}}. \tag{3.1}$$

Here, F is a function on $\mathbb{R}\times\mathbb{R}_+^*\times\mathbb{R}_+^*$, where $\mathbb{R}_+^* = (0,\infty)$. The derivatives of F, when they exist, are denoted by F'_j and F''_{jk}, for $j,k=1,2,3$. The general idea will be to choose a function F which, for example, is nonnegative and $F(x,y,z)=0$ if and only if $y=z$; then $U(F)_T>0$ on the set where the two processes X and c have common jumps within the time interval $[0,T]$, and $U(F)_T=0$ elsewhere.

The process $U(F)$ is not directly observable, because we only observe $X_{i\Delta_n}$ for $i\in\mathbb{N}$. Consequently, we "approximate" it by an observable process which we presently describe. We need some notation. For any process Y we set

$$\Delta_i^n Y \;=\; Y_{i\Delta_n} - Y_{(i-1)\Delta_n}. \tag{3.2}$$

We choose two sequences $u_n>0$ and $k_n\in\mathbb{N}^*$ which serve as cutoff level and window size at stage n: we must have $u_n\to 0$ but more slowly than $\sqrt{\Delta_n}$, and $k_n\to\infty$ but more slowly than $1/\Delta_n$. To this end it is convenient to choose two exponents ϖ and ρ such that, for some constant K,

$$\frac{1}{K}\leq \frac{u_n}{\Delta_n^{\varpi}}\leq K,\quad \frac{1}{K}\leq k_n\Delta_n^{\rho}\leq K,\quad \text{with }\ 0<\varpi<\frac{1}{2},\ \ 0<\rho<1. \tag{3.3}$$

The next variables serve as "local estimators" of the volatility:

$$\widehat{c}(k_n)_i \;=\; \frac{1}{k_n\Delta_n}\sum_{j=1}^{k_n} |\Delta_{i+j}^n X|^2\, 1_{\{|\Delta_{i+j}^n X|\leq u_n\}}. \tag{3.4}$$

Note that (b) of (H-r) implies that $\Delta_i^n X\neq 0$ a.s. for all i,n, so $\widehat{c}(k_n)_i>0$ a.s. and we can set

$$U(F,k_n)_t \;=\; \sum_{i=k_n+1}^{[t/\Delta_n]-k_n} F(\Delta_i^n X, \widehat{c}(k_n)_{i-k_n-1}, \widehat{c}(k_n)_i)\, 1_{\{|\Delta_i^n X|>u_n\}}. \tag{3.5}$$

The aim of this section is to describe the asymptotic behavior of those observable processes $U(F,k_n)$.

3.1 The law of large numbers

Here we describe under which conditions on F we have $U(F,k_n)\to U(F)$. Basically, this requires that F be continuous, plus some additional conditions. However, we want to apply the result when, for example, F has the form $F(x,y,z) = 1_{\{|x|>a\}}\, g(y,z)$, where

$a > 0$, and such an F is of course not continuous: so the desired convergence does not take place, unless with probability 1 there is no jump of X with size a or $-a$. This is why we introduce the following family $\mathcal{R}$ of subsets R:

$$R \in \mathcal{R} \Leftrightarrow \begin{array}{l} \bullet\ R \text{ is open, with a finite complement} \\ \bullet\ D = \{x : \mathbb{P}\big(\exists s > 0 \text{ with } \Delta X_s = x\big) > 0\} \subset R. \end{array} \tag{3.6}$$

Theorem 3.1 *Assume (H-r) for some $r < 2$ and (K-v) and (3.3), and let F be a Borel function on $\mathbb{R} \times \mathbb{R}_+^{*2}$ which is continuous at each point of $R \times \mathbb{R}_+^{*2}$ for some $R \in \mathcal{R}$. The processes $U(F, k_n)$ converge in probability, for the Skorokhod topology, to $U(F)$, as soon as one of the following three sets of hypotheses is satisfied:*

(a) $F(x, y, z) = 0$ for $|x| \leq \varepsilon$ for some $\varepsilon > 0$;

(b) we have $r = 0$;

(c) we have $|F(x, y, z)| \leq K|x|^r(1 + y + z)$ if $|x| \leq \varepsilon$ for some $\varepsilon, K > 0$.

3.2 The central limit theorems

The above consistency result is not enough for us, and we need a central limit theorem (CLT) associated with it. Moreover, in view of the statistical applications given later, we need a joint CLT for the process $U(F, k_n)$ and for the similar process $U(F, wk_n)$ obtained by substituting k_n with wk_n for some integer $w \geq 2$.

The test function F should satisfy some smoothness conditions, in connection with the index r in (H-r), and involving another index $p \geq 1$ as well. Namely, we suppose that there exist $R \in \mathcal{R}$ and $\varepsilon \geq 0$ such that

$$\begin{array}{l} \bullet\ F \text{ is } C^1 \text{ on } R \times \mathbb{R}_+^{*2} \\ \bullet\ \frac{1}{|x|^{p-1}}\, F_1'(x, y, z) \ \text{ is locally bounded on } R \times \mathbb{R}_+^{*2} \\ \bullet\ \frac{1}{|x|^r}\, F_2'(x, y, z),\ \frac{1}{|x|^r}\, F_3'(x, y, z) \text{ are bounded on } \ [-\varepsilon, \varepsilon] \times \mathbb{R}_+^{*2} \end{array} \tag{3.7}$$

(recall that any $R \in \mathcal{R}$ contains $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$). When $\varepsilon = 0$ the last condition is empty. When $p = 1$ the second condition is empty.

We need some additional notation. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be an auxiliary space endowed with four sequences (V_p^-), (V_p^+), $(V_p'^-)$ and $(V_p'^+)$ of independent $\mathcal{N}(0, 1)$ variables. We introduce the following extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\widetilde{\Omega} \ = \ \Omega \times \Omega', \quad \widetilde{\mathcal{F}} \ = \ \mathcal{F} \otimes \mathcal{F}', \quad \widetilde{\mathbb{P}} \ = \ \mathbb{P} \otimes \mathbb{P}'.$$

Any variable or process defined on Ω or Ω' will be extended to $\widetilde{\Omega}$ in the usual way, without change of notation. We consider an arbitrary sequence $(T_p)_{p \geq 1}$ of positive stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, which exhausts the jumps of X: this means that $T_p \neq T_q$ if $T_p < \infty$ and $q \neq p$, and that for each ω the set $\{t : \Delta X_t \neq 0\}$ is contained in $\{T_p : p \geq 1\}$.

Below we assume (H-r), and F satisfies (3.7). Then the formulas

$$\left\{\begin{array}{lcl} \mathcal{U}_t &=& \sum_{p\geq 1}\Big(F_2'(\Delta X_{T_p},c_{T_p-},c_{T_p})c_{T_p-}\sqrt{2}\,V_p^- \\ && \qquad\qquad +F_3'(\Delta X_{T_p},c_{T_p-},c_{T_p})c_{T_p}\sqrt{2}\,V_p^+\Big)\,1_{\{T_p\leq t\}} \\ \mathcal{U}_t' &=& \sum_{p\geq 1}\Big(F_2'(\Delta X_{T_p},c_{T_p-},c_{T_p})c_{T_p-}\sqrt{2}\,V_p'^-) \\ && \qquad\qquad +F_3'(\Delta X_{T_p},c_{T_p-},c_{T_p})c_{T_p}\sqrt{2}\,V_p'^+)\Big)\,1_{\{T_p\leq t\}} \end{array}\right. \tag{3.8}$$

define two càdlàg adapted processes $\mathcal{U}$ and $\mathcal{U}'$ on the extended filtered space $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}})$, where $(\widetilde{\mathcal{F}}_t)$ is the smallest filtration which contains $(\mathcal{F}_t)$ and such that the variables $V_p^+,V_p^-,V_p'^+,V_p'^-$ are $\widetilde{\mathcal{F}}_{T_p}$-measurable. Moreover, conditionally on $\mathcal{F}$, these two processes are independent, with the same (conditional) laws, and are centered Gaussian martingales (hence with independent increments), and with the conditional variances

$$\begin{array}{l} \widetilde{\mathbb{E}}((\mathcal{U}_t)^2\mid\mathcal{F}) \;=\; \widetilde{\mathbb{E}}((\mathcal{U}_t')^2\mid\mathcal{F}) \;=\; B(F)_t, \quad \text{where} \\ B(F)_t \;=\; 2\sum_{s\leq t}\big(c_{s-}^2\,F_2'(\Delta X_s,c_{s-},c_s)^2+c_s^2\,F_3'(\Delta X_s,c_{s-},c_s)^2\big). \end{array} \tag{3.9}$$

Moreover, if we modify the exhausting sequence (T_p) we accordingly modify $\mathcal{U}_t$ and $\mathcal{U}_t'$, but we *do not change* their $\mathcal{F}$-conditional laws, which is the only relevant property of $(\mathcal{U},\mathcal{U}')$ for the stable convergence in law below (all these facts are proved, in a slightly different form, in [5]; we refer to [4] for the stable convergence in law).

Theorem 3.2 *Assume (H-r) for some $r<2$ and (K-v) and (3.3) with*

$$\rho \;<\; (2\varpi(2-r))\wedge\frac{2v}{1+2v}. \tag{3.10}$$

Let F satisfy (3.7) with $\varepsilon\geq 0$ when $r=0$ and $\varepsilon>0$ otherwise, and let $w\geq 2$ be an integer.

(i) If either $r=0$, or $F(x,y,z)=0$ for $|x|\leq\varepsilon$ for some $\varepsilon>0$, the two dimensional processes

$$\Big(\sqrt{k_n}\,\big(U(F,k_n)_t-U(F)_t\big),\sqrt{k_n}\,\big(U(F,wk_n)_t-U(F)_t\big)\Big). \tag{3.11}$$

converge stably in law to the process $\big(\mathcal{U},\frac{1}{w}\,(\mathcal{U}+\sqrt{w-1}\,\mathcal{U}')\big)$, in the Skorokhod sense.

(ii) Assume that $r>0$, that $F(0,y,z)=0$ and that $p>1+r/2$ in (3.7). Assume also that ρ and ϖ satisfy

$$\varpi \;<\; \frac{1}{2r},\qquad \rho \;<\; (2\varpi(p\wedge 2-r))\bigwedge\frac{2p-2-r}{r}\bigwedge\frac{2v}{1+2v} \tag{3.12}$$

(which is stronger than (3.10)). *Then for any fixed $t>0$ the variables (3.11) converge stably in law to the variables $\big(\mathcal{U}_t,\frac{1}{w}\,(\mathcal{U}_t+\sqrt{w-1}\,\mathcal{U}_t')\big)$.*

In (ii) above we do not state the "functional convergence" (stably in law), although it is probably true. For the tests we are after in the paper, we need only the finite-dimensional convergence of the above theorem.

Our second CLT is about the case when the limiting process in the first CLT vanishes. Another normalization is then needed, and also stronger smoothness assumptions on F. Namely, we assume (3.7) and

$$\begin{array}{l} \bullet\ F(x,y,z) \text{ is } C^1 \text{ in } x \text{ and } C^2 \text{ in } (y,z) \text{ on } R\times\mathbb{R}_+^{*2} \\ \bullet\ \frac{1}{|x|^r}\,F''_{ij}(x,y,z) \ \text{ for } i,j=2,3 \text{ is bounded on } \ [-\varepsilon,\varepsilon]\times\mathbb{R}_+^{*2} \end{array} \tag{3.13}$$

Of course, the limit in Theorem 3.2 may vanish under various circumstances, but for us it is enough to consider the rather simple situation where there is a Borel set $A\subset\mathbb{R}$ and some $\eta>0$ such that

$$\begin{array}{l} \bullet\ \text{either } [-\eta,\eta]\subset A \text{ or } [-\eta,\eta]\cap A=\emptyset \\ \bullet\ x\in A,\ y\in\mathbb{R}_+^* \ \Rightarrow\ F(x,y,y)=F'_2(x,y,y)=F'_3(x,y,y)=0 \\ \bullet\ x\notin A,\ y,z\in\mathbb{R}_+^* \ \Rightarrow\ F(x,y,z)=0. \end{array} \tag{3.14}$$

Then obviously $U(F)_t=\mathcal{U}_t=\mathcal{U}'_t=0$ on the set Ω_t^A on which, for all $s\leq t$, we have $\Delta\sigma_s=0$ whenever $\Delta X_s\in A\backslash\{0\}$. When $A=\mathbb{R}$ the set Ω_t^A is the set where X and σ have no common jumps on $[0,t]$.

When F satisfies (3.13), and with a given integer $w\geq 2$, the formulas

$$\left\{\begin{array}{l} \overline{\mathcal{U}}_t=\sum_{p\geq1}c_{T_p}^2\Big(F''_{22}(\Delta X_{T_p},c_{T_p},c_{T_p})(V_p^-)^2 \\ \qquad +2F''_{23}(\Delta X_{T_p},c_{T_p},c_{T_p})V_p^-V_p^+\Big) \\ \qquad +F''_{33}(\Delta X_{T_p},c_{T_p},c_{T_p})(V_p^+)^2\,1_{\{T_p\leq t\}} \\ \overline{\mathcal{U}}'_t=\frac{1}{w^2}\,\sum_{p\geq1}c_{T_p}^2\Big(F''_{22}(\Delta X_{T_p},c_{T_p},c_{T_p})(V_p^-+\sqrt{w-1}\,V_p'^-)^2 \\ \qquad +2F''_{23}(\Delta X_{T_p},c_{T_p},c_{T_p})(V_p^-+\sqrt{w-1}\,V_p'^-)(V_p^++\sqrt{w-1}\,V_p'^+)\Big) \\ \qquad +F''_{33}(\Delta X_{T_p},c_{T_p},c_{T_p})(V_p^++\sqrt{w-1}\,V_p'^+)^2\,1_{\{T_p\leq t\}} \end{array}\right. \tag{3.15}$$

define two càdlàg adapted processes $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$ on the extended filtered space $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq0},\widetilde{\mathbb{P}})$. Moreover, conditionally on $\mathcal{F}$, the pair $(\overline{\mathcal{U}},\overline{\mathcal{U}}')$ is a process with independent increments and finite variation on compact intervals, and with the conditional means

$$\begin{array}{rcl} \widetilde{\mathbb{E}}(\overline{\mathcal{U}}_t\mid\mathcal{F}) &=& B'(F)_t, \quad \widetilde{\mathbb{E}}(\overline{\mathcal{U}}'_t\mid\mathcal{F}) \;=\; \frac{1}{w}\,B'(F)_t, \quad \text{where} \\ B'(F)_t &=& \sum_{s\leq t}c_s^2\big(F''_{22}(\Delta X_s,c_s,c_s)+F''_{33}(\Delta X_s,c_s,c_s)\big). \end{array} \tag{3.16}$$

Here again, if we modify the exhausting sequence (T_p) we accordingly modify $\overline{\mathcal{U}}_t$ and $\overline{\mathcal{U}}'_t$, but we *do not change* their $\mathcal{F}$-conditional laws.

Theorem 3.3 *Assume (H-r) for some $r<2$ and (K-v) and (3.3) with*

$$\rho \;<\; ((2\varpi(2-r))\bigwedge\frac{2v}{1+2v}\bigwedge\frac{1}{2}. \tag{3.17}$$

Let F satisfy (3.14) for some $A\subset\mathbb{R}$, and (3.13) for $\varepsilon=0$ when $r=0$ and some $\varepsilon>0$ otherwise.

(i) If either $r=0$, or $F(x,y,z)=0$ for $|x|\le\varepsilon$ for some $\varepsilon>0$, the two dimensional variables $\big(k_nU(F,k_n)_t,k_nU(F,wk_n)_t\big)$ converge stably in law, in restriction to the set Ω_t^A, to the variable $(\overline{\mathcal{U}}_t,\overline{\mathcal{U}}_t')$.

(ii) The same holds when $r>0$, provided ρ and ϖ satisfy

$$\rho\;<\;\big(\varpi(4-r)-1\big)\bigwedge\frac{(2v)\wedge 1}{1+(2v)\wedge 1}\bigwedge\frac{1}{2}. \tag{3.18}$$

4 Construction of the tests

4.1 Preliminaries

Now we are ready to construct our tests using the limit results of the previous section. The overall interval on which the process X is observed, at times $i\Delta_n$, is $[0,T]$. In our tests the processes X and σ will not play a symmetrical role, mainly because X is observed, whereas σ is not.

Although our main concern is to test for common jumps, irrespective of their sizes, it might be useful to test also whether there are jumps of X with size in a subset A of $\mathbb{R}$, occurring at the same time as jumps of σ: for example, $A=(a,\infty)$ or $A=(-\infty,-a)$ (positive or negative jumps of X of size bigger than a only), or $A=(-\infty,-a)\cup(a,\infty)$ (jumps of X of size bigger than a).

We thus pick a subset $A\subset\mathbb{R}$ satisfying the first part of (3.14), and we are interested in the following two disjoint sets:

$$\begin{aligned}\Omega_T^{A,j} &= \{\omega:\ \exists s\in(0,T] \text{ with } \Delta X_s(\omega)\in A\backslash\{0\} \text{ and } \Delta\sigma_s(\omega)\neq 0\}\\ \Omega_T^{A,d} &= \{\omega:\ \forall s\in(0,T],\ \ \Delta X_s(\omega)\in A\backslash\{0\}\ \Rightarrow\ \Delta\sigma_s(\omega)=0,\\ &\qquad\qquad \text{and } \ \exists s\in(0,T] \text{ with } \Delta X_s(\omega)\in A\backslash\{0\}\}.\end{aligned} \tag{4.1}$$

The subscript "j" and "d" stand for "Joint" jumps and "Disjoint" jumps. One could also specify a subset A' in which the jumps of σ lie, but it requires more sophisticated CLTs than Theorems 3.1 and 3.2 and we will not consider this case here. Note that $\Omega_T^{A,d}$ is contained in the set Ω_T^A of Theorem 3.3.

Next, we recall that testing a null hypothesis "we are in a subset Ω_0" of Ω, against the alternative "we are in a subset Ω_1", with of course $\Omega_0\cap\Omega_1=\emptyset$, amounts to finding a critical (rejection) region $C_n\subset\Omega$ at stage n. The asymptotic size and asymptotic power for this sequence (C_n) of critical regions are the following numbers:

$$\begin{aligned}\alpha &= \sup\left(\limsup_n\ \mathbb{P}(C_n\mid H):\ H\in\mathcal{F},\ H\subset\Omega_0,\ \mathbb{P}(H)>0\right)\\ \beta &= \inf\left(\liminf_n\ \mathbb{P}(C_n\mid H):\ H\in\mathcal{F},\ H\subset\Omega_1,\ \mathbb{P}(H)>0\right).\end{aligned} \tag{4.2}$$

In all forthcoming tests, we fix *a priori* two sequences u_n and k_n satisfying (3.3): typically $u_n=a\Delta_n^{\varpi}$ and $k_n=[a'/\Delta_n^{\rho}]$ where $a,a'>0$ are constants. Some restrictions on ϖ and ρ will also be made, depending on the test at hand.

Finally, similar to the tests for deciding whether price and volatility jump together or not which we develop here, one can use the limit results of Section 3 to derive various other tests about the relationship between jumps in X and its volatility. Examples include: (1) testing whether all jumps in X are associated with volatility jumps, and (2) testing whether jumps in X of given sign always lead to positive (negative) volatility jumps.

4.2 Testing the null hypothesis "no common jump"

Here we take the null hypothesis to be "X and σ have no common jump" with jump size of X in A, that is $\Omega_T^{(A,d)}$, for A like in (3.14).

4.2.1 General family of tests

The idea is to use the variable $U(F)_T$ of (3.1) and its approximations $U(F,k_n)_T$ for a suitable function F, namely:

$$
\begin{array}{l}
F(x,y,z) \;=\; f(x)g(y,z), \quad \text{with} \\
\left\{\begin{array}{l} f \;\text{ is } C^1 \text{ on } \; R, \quad x\in[-\varepsilon,\varepsilon] \;\Rightarrow\; |f'(x)|\leq C|x|^{p-1} \\ x\in A\backslash\{0\} \;\Rightarrow\; f(x)>0, \quad x\notin A\backslash\{0\} \;\Rightarrow\; f(x)=0 \end{array}\right. \\
\left\{\begin{array}{l} g \;\text{ is } C^2 \text{ with bounded first and second derivatives} \\ z\neq y \;\Rightarrow\; g(y,z)>0, \quad z=y \;\Rightarrow\; g(y,z)=0 \\ g_1'(y,y)=g_2'(y,y)=0, \quad g_{11}''(y,y)+g_{22}''(y,y)>0, \end{array}\right.
\end{array}
\tag{4.3}
$$

and where $p\geq 1\vee r$. These ensure that F satisfies (3.7), (3.13) and (3.14). It also implicitly implies conditions on the set A, since $A\backslash\{0\}=f^{-1}((0,\infty))$ and f is C^1 on R, whose complement is finite.

By Theorem 3.1, we have the following convergence:

$$
U(F,k_n)_T \;\stackrel{\mathbb{P}}{\longrightarrow}\; U(F)_T \left\{\begin{array}{ll} =0 & \text{on the set } \Omega_T^{(A,d)} \\ >0 & \text{on the set } \Omega_T^{(A,j)}. \end{array}\right.
\tag{4.4}
$$

So in order to test the null hypothesis $\Omega_T^{(A,d)}$ it is natural at stage n to take a critical region of the form $C_n=\{U(F,k_n)_T>Z_n\}$ for some (possibly random) $Z_n>0$. In order to determine Z_n in such a way that the asymptotic level of the test be some α, we make use of Theorem 3.3, which says that, in restriction to the set $\Omega_T^{(A,d)}$, the variables $k_nU(F,k_n)_T$ converge stably in law to $\overline{\mathcal{U}}_T$, as defined by (3.15). Conditionally on $\mathcal{F}$, this variable is a weighted chi-square variable, with mean $B'(F)_T$ given by (3.16).

One simple, not very efficient, way to derive test with a prescribed level α makes use of Bienaymé-Tchebycheff inequality, plus the fact that by Theorem 3.1 again we can approximate the variable $B'(F)$ by $U(G,k_n)_T$, where

$$
G(x,y,z) \;=\; y^2\, f(x)\,(g_{11}''(y,z)+g_{22}''(y,z))
\tag{4.5}
$$

satisfies all the requirements of that theorem. At this point, the critical region is taken to be

$$
C_n \;=\; \Big\{U(F,k_n)_T \;>\; \frac{U(G,k_n)_T}{\alpha\, k_n}\Big\},
\tag{4.6}
$$

and the following is straightforward:

Theorem 4.1 *Assume (H-r) and (K-v), and F as in (4.3) with $p \geq r$, and choose u_n and k_n such that (3.3) and (3.18) hold. Then the critical region (4.6) has asymptotic level less than α for testing the null hypothesis $\Omega_T^{(A,d)}$, and asymptotic power 1 for the alternative $\Omega_T^{(A,j)}$.*

The actual asymptotic size of this test is usually much lower than α, because Bienaymé-Tchebycheff is a crude approximation. However we can use a Monte-Carlo simulation to better fit the size, in the spirit of [6]: we take a sequence $N_n \to \infty$, and we simulate independent $\mathcal{N}(0,1)$ variables $V_i^-(j)$ and $V_i^+(j)$ of independent $\mathcal{N}(0,1)$ variables, for $j = 1, \cdots, N_n$ and $i = 1, \cdots, [T/\Delta_n]$. Then, with the observed values of $\Delta_i^n X$, hence of the variables $\widehat{c}(k_n)_i$ as well, we set

$$\begin{aligned} \overline{\mathcal{U}}(n,j) \;=\; & \textstyle\sum_{i=k_n+1}^{[T/\Delta_n]-k_n} f(\Delta_i^n X) 1_{\{|\Delta_i^n X|>u_n\}} (\widehat{c}(k_n)_i)^2 \\ & \Big(g''_{11}(\widehat{c}(k_n)_{i-k_n-1}, \widehat{c}(k_n)_i)(V_i^-(j))^2 \\ & + g''_{22}(\widehat{c}(k_n)_{i-k_n-1}, \widehat{c}(k_n)_i)(V_i^+(j))^2 \\ & + 2g''_{12}(\widehat{c}(k_n)_{i-k_n-1}, \widehat{c}(k_n)_i) V_i^-(j) V_i^+(j) \Big). \end{aligned} \tag{4.7}$$

Next, we consider the order statistics of these simulated variables, that is $\overline{\mathcal{U}}(n)_{(1)} \geq \overline{\mathcal{U}}(n)_{(2)} \geq \cdots \geq \overline{\mathcal{U}}(n)(N_n)$ such that $\{\overline{\mathcal{U}}(n)_j : 1 \leq j \leq N_n\} = \{\overline{\mathcal{U}}(n,j) : 1 \leq j \leq N_n\}$, and we take as our critical region the following:

$$C_n \;=\; \Big\{ U(F,k_n)_T \;>\; \frac{\overline{\mathcal{U}}(n)_{([N_n \alpha])}}{k_n} \Big\}. \tag{4.8}$$

Theorem 4.2 *Assume (H-r) and (K-v), and F as in (4.3) with $p \geq r$, and choose u_n and k_n such that (3.3) and (3.18) hold. Then the critical region (4.8), constructed with any sequence N_n increasing to infinity, has asymptotic level equal to α for testing the null hypothesis $\Omega_T^{(A,d)}$, and asymptotic power 1 for the alternative $\Omega_T^{(A,j)}$.*

4.2.2 A leading example

Here we specialize A to be either $A = \mathbb{R}$ or $A = [-a,a]^c$ for some positive a, and in the first case we will need $r = 0$, that is our process X has finite activity jumps. In both cases, we end up using finite number of jumps of X (jumps of size higher than a fixed value are almost surely of finite number), therefore we consider $F(x,y,z) = f(x)g(y,z)$ with $f(x) = 1_{\{x \in A\}}$. Since for this choice $f(x)$ is discontinuous at $x = \pm a$, we need $\pm a \notin D$ (recall (3.6)) in order for (3.13) to be satisfied. Of course, D is unknown, but in the typical case when the Lévy measure of X has no atom, $D = \{0\}$ and thus any $a > 0$ works. Otherwise, we can replace $1_{\{|x|>a\}}$ by a C^1 function which is very close to this. Practically this should make no significant difference, and therefore we stick to the indicator function, with $a \notin D$. When $A = \mathbb{R}$ we set $a = 0$.

A natural choice for the function g is the following:

$$g(y,z) \;=\; 2 \log \frac{y+z}{2} - \log y - \log z. \tag{4.9}$$

This choice corresponds to the log-likelihood ratio test for testing that two independent samples of i.i.d. zero-mean normal variables have the same variance. The link with our testing comes from the fact that around a jump time the high-frequency increments of X are “approximately” i.i.d. normal.

With this choice of F, our test for common jumps becomes essentially pivotal, i.e. the limiting distribution of the test statistics depends only on the number of jumps, and is thus straightforward to implement. To see this note that in this case (3.15) writes as

$$\overline{\mathcal{U}}_T \;=\; \frac{1}{2}\sum_{p\geq 1}(V_p^+ - V_p^-)^2\, 1_{\{|\Delta X_{T_p}|>a\}}. \tag{4.10}$$

Conditionally on $\mathcal{F}$, this variable has the same law as a chi-square variable with N_T degrees of freedom, where $N_T = \sum_{p\geq 1} 1_{\{|\Delta X_{T_p}|>a\}}$. The variable N_T is not observable. However, we have

$$N_T^n \;=\; \sum_{i=1}^{[T/\Delta_n]} 1_{\{|\Delta_i^n X|>a\vee u_n\}} \;\stackrel{\mathbb{P}}{\longrightarrow}\; N_T, \tag{4.11}$$

and since these are integer-valued variables we even have $\mathbb{P}(N_T^n = N_T) \to 1$.

Therefore, denoting by $z(\alpha, n)$ the α-quantile of a chi-square variable χ_n^2 with n degrees of freedom, that is the number such that $\mathbb{P}(\chi_n^2 > z(\alpha, n)) = \alpha$, we may take the following critical region at stage n:

$$C_n \;=\; \big\{U(F, k_n)_T > \frac{z(\alpha, N_t^n)}{k_n}\big\}. \tag{4.12}$$

Theorem 4.3 *Assume (H-r) and (K-v), and F as above with either $a = 0$ if $r = 0$ or a positive and $\pm a \notin D$ if $r \geq 0$. Choose u_n and k_n such that (3.3) and (3.17) hold. Then the critical region (4.11) has asymptotic level equal to α for testing the null hypothesis $\Omega_T^{(A,d)}$, and asymptotic power 1 for the alternative $\Omega_T^{(A,j)}$.*

Note that for constructing the critical region in (4.12), we need only the critical values of a chi-square variable χ_n^2, and thus there is no need for simulation.

4.3 Testing the null hypothesis ”common jump”

Now we take the null hypothesis to be ”X and σ have common jumps” with sizes in A for X, that is $\Omega_T^{(A,j)}$, for A like in (3.14). We take an integer $w \geq 2$ and a function F satisfying (4.3), and introduce the statistics

$$S_n \;=\; \frac{U(F, wk_n)_T}{U(F, k_n)_T}. \tag{4.13}$$

If we combine Theorems 3.1 and 3.3, we first obtain

$$\begin{cases} S_n \;\stackrel{\mathbb{P}}{\longrightarrow}\; 1 & \text{on the set } \Omega_T^{(A,j)} \\ S_n \;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\; \frac{\overline{\mathcal{U}}'_T}{\overline{\mathcal{U}}_T} \;\neq\; 1 \text{ a.s.} & \text{on the set } \Omega_T^{(A,d)}, \end{cases} \tag{4.14}$$

where $\stackrel{\mathcal{L}-(s)}{\longrightarrow}$ stands for the stable convergence in law; for the second convergence we must assume that k_n satisfies (3.17), and $\overline{\mathcal{U}}'_T$ is implicitly depending on w; note that the pair $(\overline{\mathcal{U}}_T, \overline{\mathcal{U}}'_T)$ has $\mathcal{F}$-conditionally a density, implying $\overline{\mathcal{U}}'_T / \overline{\mathcal{U}}_T \neq 1$ a.s.

To determine the asymptotic level of a test based upon S_n, we make use of Theorem 3.2, which by way of the delta method shows that, in restriction to the set $\Omega_T^{(A,j)}$, the variables $\sqrt{k_n}\,(S_n - 1)$ converge stably in law to $\big(\sqrt{w-1}\,\mathcal{U}'_T - (w-1)\mathcal{U}_T\big)/wU(F)_T$. The limit is $\mathcal{F}$-conditionally centered Gaussian with variance $(w-1)B(F)_T/w(U(F)_T)^2$, recall (3.9). Hence, if

$$\begin{array}{lcl} G(x,y,z) & = & 2f(x)^2\big(y^2\,g_1'(y,z)^2 + z^2\,g_2'(y,z)^2\big) \\ V_n & = & \frac{(w-1)\,U(G,k_n)_T}{w\,k_n\,(U(F,k_n)_T)^2}, \end{array} \tag{4.15}$$

we deduce that, in restriction to the set $\Omega_T^{(A,j)}$, the variables $(S_n - 1)/\sqrt{V_n}$ converge stably in law to a standard normal variable, under (3.12) of course.

Then we may take the following critical region at stage n, where z_α denotes the symmetric α-quantile of an $\mathcal{N}(0,1)$ variable V, that is $\mathbb{P}(|V| > z_\alpha) = \alpha$.

$$C_n \;=\; \big\{|S_n - 1| > z_\alpha\,\sqrt{V_n}\big\}. \tag{4.16}$$

Theorem 4.4 *Assume (H-r) and (K-v), and F as in (4.3) with $p > 1 + r/2$. Choose u_n and k_n such that (3.3) and (3.12) hold. Then the critical region (4.16) has asymptotic level α for testing the null hypothesis $\Omega_T^{(A,j)}$.*

There is no statement about the asymptotic power for the alternative $\Omega_T^{(A,d)}$, which is any case is *not* equal to 1. Indeed, on $\Omega_T^{(A,d)}$, the variables $(S_n - 1)/\sqrt{V_n}$ converge stably in law to some limit $\mathcal{V}$ (easily constructed from $\overline{\mathcal{U}}_T$, $\overline{\mathcal{U}}'_T$, and also the variable $\overline{\mathcal{U}}_T$ associated with the function G) as soon as G satisfies the assumption of Theorem 3.3. The variable $\mathcal{V}$ is a.s. non vanishing, and the asymptotic power of our test is

$$\beta \;=\; \inf(\mathbb{P}(|\mathcal{V}| > z_\alpha \mid H) : \, H \in \mathcal{F},\, H \subset \Omega_T^{(A,d)},\, \mathbb{P}(H) > 0).$$

This quantity cannot be computed explicitly and may be close to 0, as simulations show later on.

To avoid this power problem, we can "truncate" the estimated variance V_n: Let v_n be a sequence of positive numbers (possibly random, but of course depending only on the observations at stage n), such that $v_n \to 0$ and $k_n v_n \to \infty$, and set

$$V'_n \;=\; V_n \wedge v_n.$$

Since $k_n V_n$ converges to a positive finite limit on $\Omega_T^{(A,j)}$, we have $\mathbb{P}(V_n = V'_n) \to 1$ and this truncation has no effect on the behavior of our standardized statistics under the null, and we take the following critical region:

$$C'_n \;=\; \big\{|S_n - 1| > z_\alpha\,\sqrt{V'_n}\big\}. \tag{4.17}$$

Theorem 4.5 *Assume (H-r) and (K-v), and F as in (4.3) with $p > 1 + r/2$. Choose u_n and k_n such that (3.3) and (3.17) hold. Then if $v_n \to 0$ and $k_n v_n \to \infty$, the critical region (4.17) has asymptotic level α for testing the null hypothesis $\Omega_T^{(A,j)}$, and asymptotic power 1 for the alternative $\Omega_T^{(A,d)}$.*

Remark 4.6 *Exactly as in the previous subsection, when $r = 0$ we may use the function $F(x, y, z) = g(y, z)$ given by (4.9), and $A = \mathbb{R}$. When $r > 0$ we can use $F(x, y, z) = g(y, z)\, 1_{\{|x|>a\}}$, with g as above and $a > 0$ and $A = [-a, a]^c$, provided $\pm a \notin D$. In these cases, ρ and ϖ are subject to the weaker condition (3.10) only.*

4.4 Practical aspects

The construction of the tests involves several choices to be made by the user. The first one is about the functions f and g in (4.3). A good choice seems to be $f(x) = 1_{\{|x|>a\}}$ for some $a \geq 0$ and g as given by (4.9). However this works only when (H-0) holds (a serious restriction indeed), or when $a > 0$, and in the latter case we only test for common jumps when the size of the jumps of X is bigger than a. Then the user can perform the testing for various levels of a. In addition, if jumps of certain size in X are more important, $1_{\{|x|>a\}}$ can be replaced with an appropriate weighting function for the jumps of different size. Finally, if the user wants to check cojumping including the very "small" jumps in X, then a good choice is to take $f(x) = x^2$ and $g(y, z) = h(y - z)$ where h is a C^2 function with bounded first and second derivatives, and $h(0) = h'(0) = 0$ and $h''(0) > 0$ and $h(x) > 0$ when $x \neq 0$.

The second choice in implementing the tests is about the sequences u_n and k_n. Here we face a natural tradeoff between efficiency and robustness. u_n and k_n should satisfy (3.10) or (3.17) when $f(x) = 1_{\{|x|>a\}}$, and (3.12) or (3.18) otherwise, depending on which test is performed. These conditions depend on the a priori unknown numbers r and v in (H-r) and (K-v). The higher the r and the lower the v are, the stricter the conditions are, and the lower the rate at which k_n can grow, i.e. the slower the rate at which $U(F, k_n)_T$ converges. Intuitively, high r makes difficult to distinguish the many small jumps from the Brownian increments, while low v means volatility is very "active" over short intervals and that makes estimation from neighboring increments "noisier".

Most stochastic volatility models imply that σ_t is an Itô semimartingale and therefore $v = \frac{1}{2}$. If in addition we assume that $r < 1$, i.e. jumps are of finite variation, then we can choose ϖ and ρ arbitrarily close to $\frac{1}{2}$, which is the optimal choice. Alternatively, if we are willing to assume only that $r \leq r_0$ for some $1 < r_0 < 2$, then we can write the conditions on ϖ and ρ with respect to r_0 and pick u_n and k_n so that they are fulfilled. One should emphasize that ϖ and ρ only give an order of magnitude, and the concrete choice of u_n and k_n when one is faced with a set of data and thus with n and Δ_n given is always a difficult question: in the Monte-Carlo study we provide some guidance on that.

The last choice to be made, for the second test, is choosing the integer w. Under the null $\Omega_T^{(A,j)}$ the normalized asymptotic $\mathcal{F}$-conditional variance of S_n takes the form $\frac{w-1}{w}\,\Phi$, where $\Phi = B(F)_Y/(U(F)_T)^2$ does not depend on w. The minimum of $\frac{w-1}{w}$ for $w \geq 2$ is achieved at $w = 2$. At the same time the effect of changing w under the alternative hypothesis is unclear and in general depends on the particular realization. For that reason we suggest to take $w = 2$ and we do so in our numerical applications without further mention. Some Monte Carlo experiments (not reported here) with $w = 4$ provide further support for this choice.

5 Monte Carlo study

In this section we check the performance of our tests on simulated data. We work with the stochastic volatility model

$$\begin{aligned} dX_t &= \sqrt{V_t^1 + V_t^2} dW_t + \alpha_0 \int_{\mathbb{R}} x\mu(dt, dx, dy), \\ dV_t^1 &= \kappa_1(\theta - V_t^1)dt + \sigma\sqrt{V_t^1} dW_t', \\ dV_t^2 &= -\kappa_2 V_t^2 dt + \alpha_1 \int_{\mathbb{R}} y\mu(dt, dx, dy) + \alpha_2 \int_{\mathbb{R}} y\mu'(dt, dy), \end{aligned} \tag{5.1}$$

where W and W' are two independent Brownian motions; the (finite activity) Poisson measures μ and μ' are independent with compensators

$$\nu(dt, dx, dy) = \frac{\lambda}{2(h-d)(u-d)} 1_{(x\in[-h;-l]\cup[l;h])} 1_{(y\in[d;u])} dtdxdy,$$

for $0 < l < h$ and $0 < d < u$ and $\nu'(dt, dy) = \frac{\lambda}{u-d} 1_{(y\in[d;u])} dtdy$. This two-factor volatility structure is found to fit high-frequency financial data very well in [8], see also references therein. The above cited study finds the continuous volatility factor to be very persistent, while the discontinuous one to be transient. This is reflected in our choice of the parameter values of κ_1 and κ_2 in the Monte Carlo settings, in an effort to make them realistically plausible for financial applications. In Table 1 we report the parameter values for all cases considered. In all of them the variance of the jumps in X is fixed and its share in the total price variation is in the range $0.2 - 0.34$, which is similar to one estimated from real financial data (see e.g. [3]). Scenarios with higher number of jumps imply that the jumps are of smaller size. The different parameter settings differ in the average number of jumps, their sizes, whether jumps are present in the volatility, and when so whether they arrive together with the jumps in X or not. The cases labeled with c and d are draws from the set $\Omega_T^{(A,d)}$, while the cases labeled with j and m are draws from the set $\Omega_T^{(A,j)}$. To ensure the latter, we discard simulations from scenarios m on which there is no common price and volatility jumps. The behavior of the tests on the discarded simulation draws is exactly as on the simulations from scenarios d.

In the simulated model we have (H-0) and (K-1/2), so we use the tests based on $f(x) = 1$ and g given by (4.9), and $A = \mathbb{R}$. Throughout, time is measured in days, and the observation length is five days, i.e. $T = 5$, which constitutes one business week. We simulate 5000 days, i.e. 1000 Monte Carlo replications. On each day we consider sampling $n = 1000$, $n = 5000$ or $n = 24000$ times, corresponding approximately to sampling every 0.5 minutes, 5 seconds or 1 second for a trading day of 6.5 hours or equivalently to sampling every 1.5 minutes, 15 seconds or 4 seconds for a trading day of 24 hours. Finally, for the

Table 1: Parameter Settings used in the Monte Carlo

Case	Parameters											
	κ_1	θ	σ	κ_2	α_0	α_1	α_2	λ	l	h	d	u
I-c	0.02	0.4	0.04	0.5	1	0	0	0.5	0.1	1.0420		
II-c	0.02	0.4	0.04	0.5	1	0	0	1.0	0.1	0.7197		
III-c	0.02	0.4	0.04	0.5	1	0	0	4.0	0.1	0.3275		
I-d	0.02	0.4	0.04	0.5	1	0	1	0.5	0.1	1.0420	0.04	0.7600
II-d	0.02	0.4	0.04	0.5	1	0	1	1.0	0.1	0.7197	0.04	0.3600
III-d	0.02	0.4	0.04	0.5	1	0	1	4.0	0.1	0.3275	0.04	0.0600
I-j	0.02	0.4	0.04	0.5	1	1	0	0.5	0.1	1.0420	0.04	0.7600
II-j	0.02	0.4	0.04	0.5	1	1	0	1.0	0.1	0.7197	0.04	0.3600
III-j	0.02	0.4	0.04	0.5	1	1	0	4.0	0.1	0.3275	0.04	0.0600
I-m	0.00	0.0	0.00	0.5	1	1	1	0.5	0.1	1.0420	0.04	0.7600
II-m	0.00	0.0	0.00	0.5	1	1	1	1.0	0.1	0.7197	0.04	0.3600
III-m	0.00	0.0	0.00	0.5	1	1	1	4.0	0.1	0.3275	0.04	0.0600

calculation of the local volatility estimators we use a window $k_n = [5 \times \Delta_n^{-0.49}]$. Our choice for the truncation parameters a and ϖ determining $u_n = a\Delta_n^{\varpi}$ is $a = 5 \times \sqrt{BV}$ and $\varpi = 0.49$ respectively, where BV denotes the bi-power variation over the day ([2, 1]). This choice of the truncation level reflects the time-variation in the volatility.

Figure 1 shows kernel density estimates of $U(F, k_n)_T / N_T^n$ and Figure 2 shows the size and power of the test for disjoint jumps. Overall the test behaves as prescribed by our asymptotic results. Not surprisingly, the size of the jumps have the strongest finite sample effect: the last row of Figure 2, corresponding to the scenarios with the smallest on average jumps, shows that for $n = 1000$ we have slight overrejection when the null is true (cases c and d) and lower power when the alternative is true (cases j and m). The size distortion disappears and the power converges to 1 as we increase the sampling frequency.

Turning to the test for common jumps, Figure 3 shows kernel density estimates of $\log(S_n)$. The statistics is centered around 0 on the samples in $\Omega_T^{(A,j)}$ (cases j and m), as predicted from our theoretical results. The distribution of $\log(S_n)$ on these samples becomes more concentrated around the true value of 0 as we increase the frequency. On the other hand, on the samples in $\Omega_T^{(A,d)}$ (cases c and d), the statistics is centered around $\log(0.5)$ and its distribution remains nearly unchanged across the different sampling frequencies (because for those samples S_n converge to a random variable and not a constant).

Figure 4 shows the size and power of the test for common jumps when we standardize $|S_n - 1|$ by V_n. The test has overall good size with the only exception being the cases with high intensity of arrival of small size jumps (last row of the figure), for which even

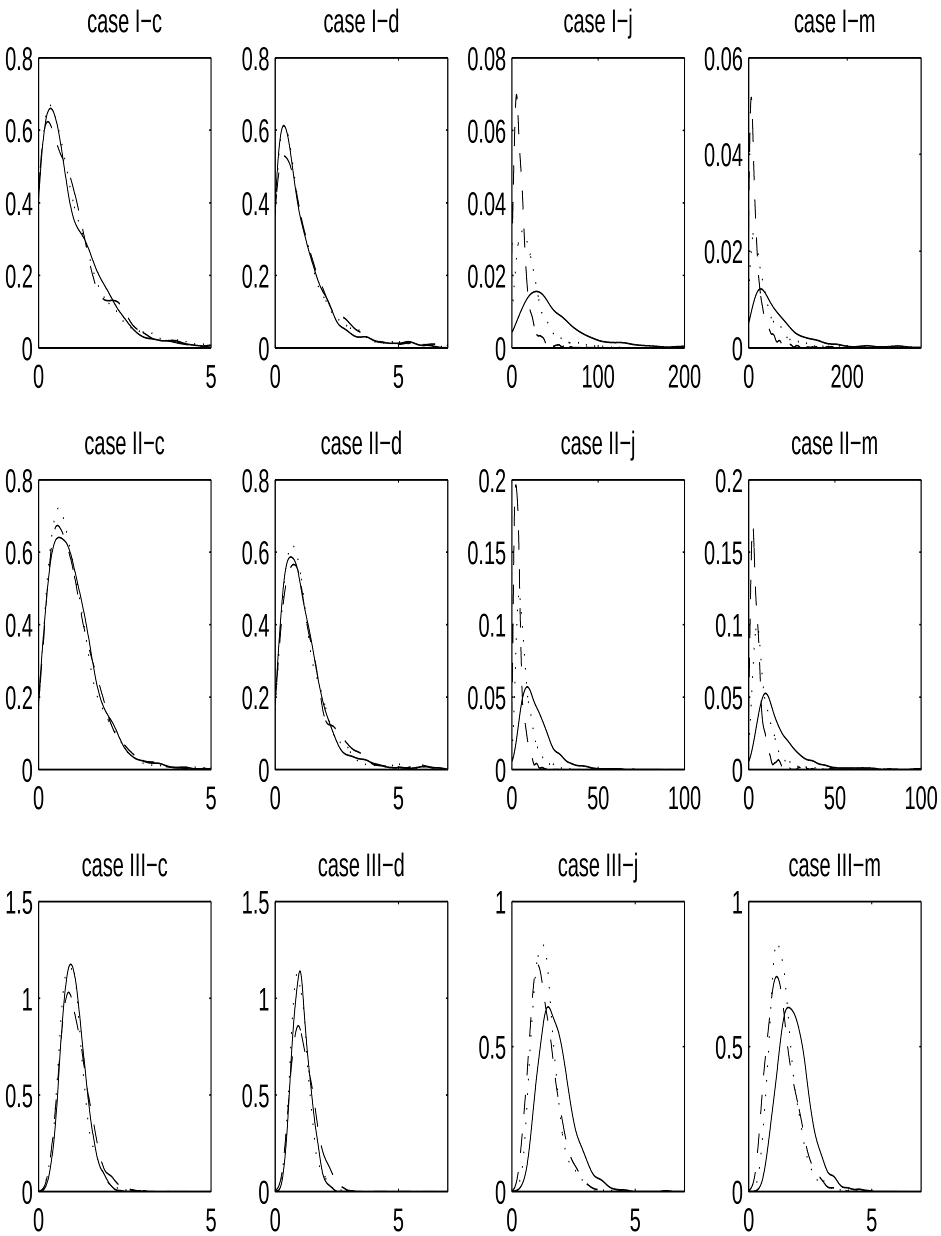

Figure 1: Kernel density estimate of $U(f, g, k_n)_T/N_t^n$ from the Monte Carlo. The dashed line corresponds to sampling frequency of $n = 1000$, the dotted line to sampling frequency of $n = 5000$ and the solid line to sampling frequency of $n = 24000$.

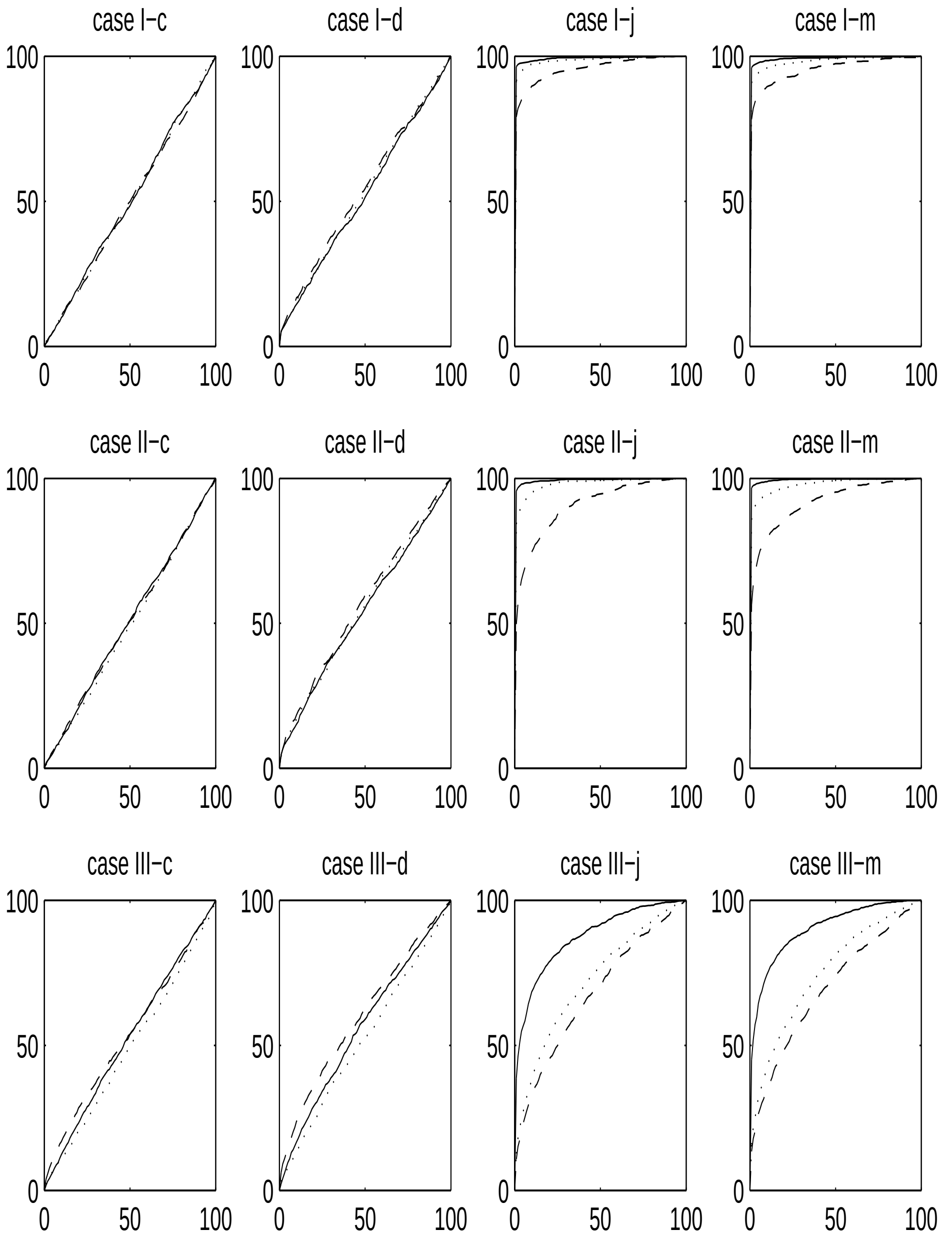

Figure 2: Size and power of the test for disjoint price and volatility jumps. The x-axis shows the nominal level of the corresponding test, while the y-axis shows the percentage of rejection in the Monte Carlo. The dashed line corresponds to sampling frequency of $n = 1000$, the dotted line to sampling frequency of $n = 5000$ and the solid line to sampling frequency of $n = 24000$.

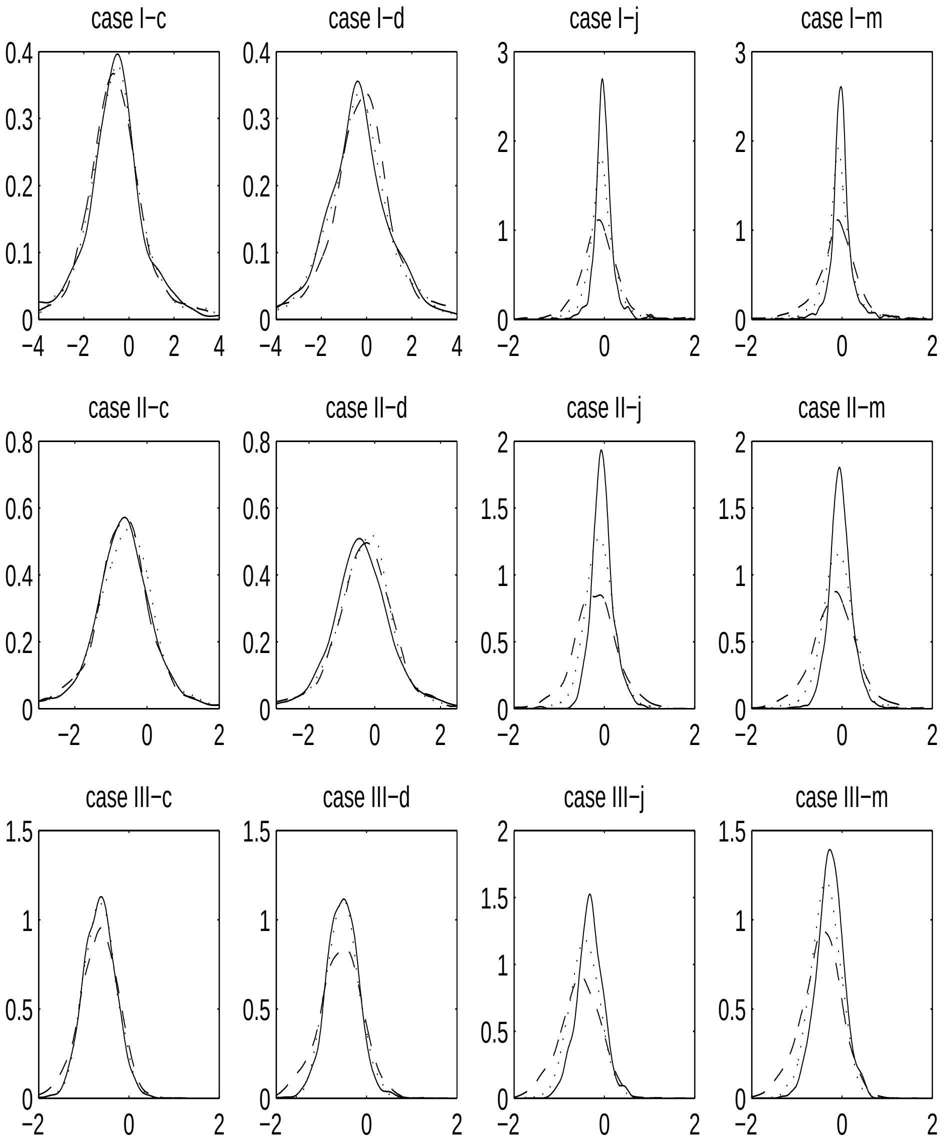

Figure 3: Kernel density estimate of $\log(S_n)$ from the Monte Carlo. The dashed line corresponds to sampling frequency of $n = 1000$, the dotted line to sampling frequency of $n = 5000$ and the solid line to sampling frequency of $n = 24000$.

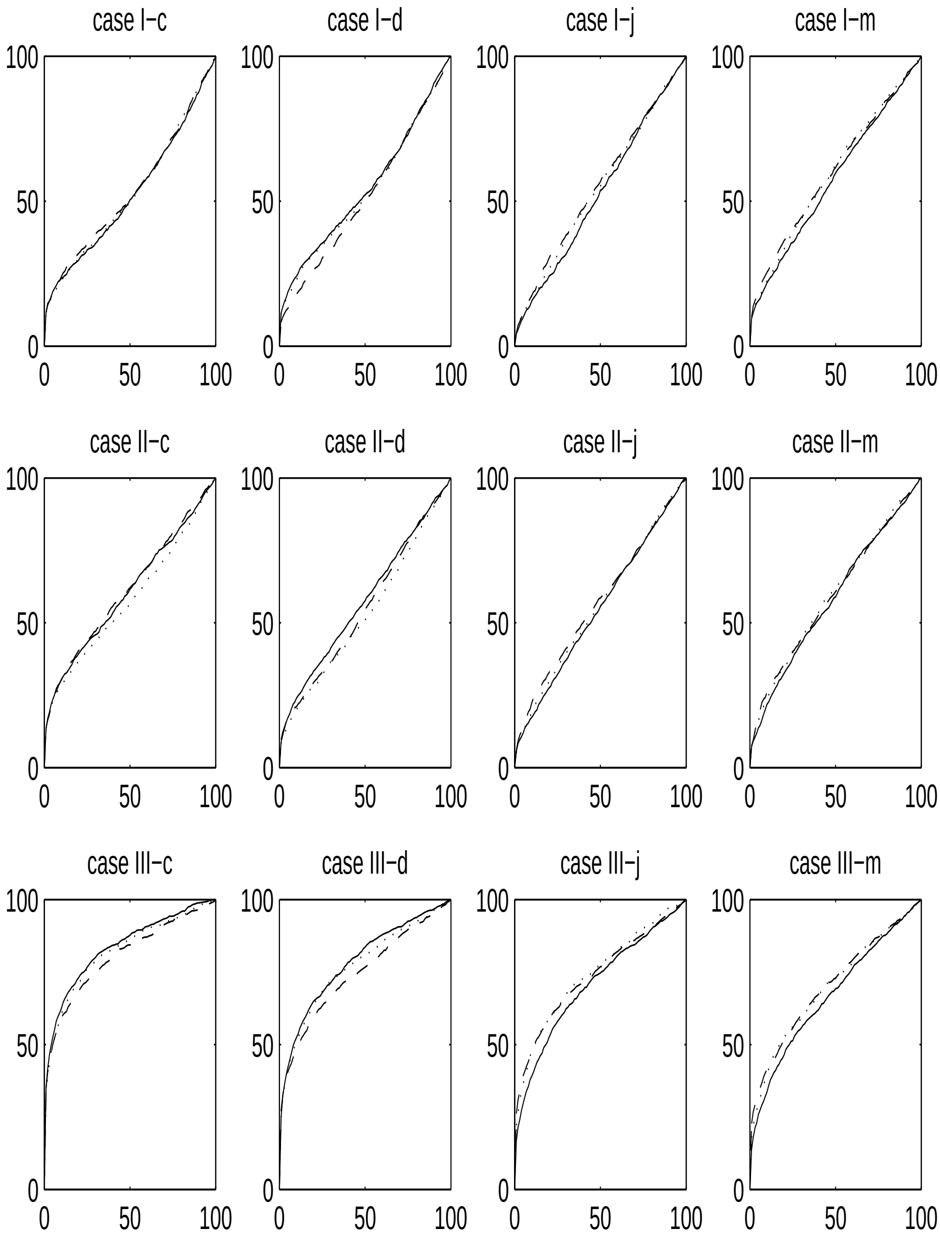

Figure 4: Size and power of the test for common price and volatility jumps with V_n used in the construction of the critical region. The x-axis shows the nominal level of the corresponding test, while the y-axis shows the percentage of rejection in the Monte Carlo. The dashed line corresponds to sampling frequency of $n = 1000$, the dotted line to sampling frequency of $n = 5000$ and the solid line to sampling frequency of $n = 24000$.

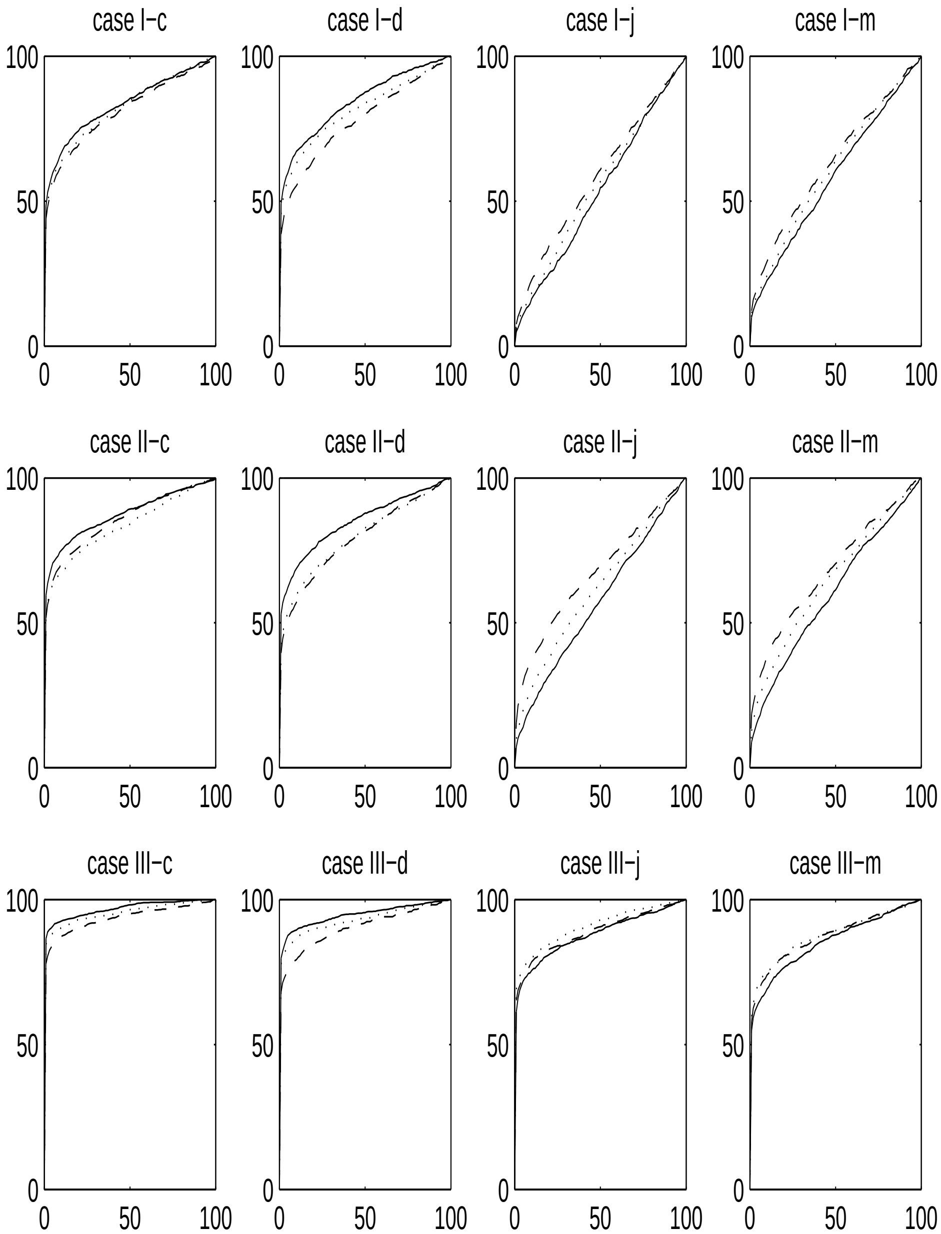

Figure 5: Size and power of the test for common price and volatility jumps with V_n' used in the construction of the critical region. The x-axis shows the nominal level of the corresponding test, while the y-axis shows the percentage of rejection in the Monte Carlo. The dashed line corresponds to sampling frequency of $n = 1000$, the dotted line to sampling frequency of $n = 5000$ and the solid line to sampling frequency of $n = 24000$.

for $n = 24000$ we have somewhat significant overrejection. On the other hand, from the first two columns of Figure 4 we can see that, when using V_n, the test has essentially no power against the considered alternatives. The lack of power is explained after Theorem 4.4.

We next performed the test with rejection region C'_n of (4.17), corresponding to the truncated variance $V'_n = V_n \wedge v_n$, and we have taken $v_n = k_n^{-0.125} \times \frac{1}{z(0.5, N_t^n)}$, where N_T^n is given by (4.11). The choice of v_n reflects the fact that on $\Omega_T^{(A,d)}$, V_n is distributed approximately as $1/\chi^2_{N_T^n}$. The results of the test with the truncated asymptotic variance are reported on Figure 5. The power against all alternatives improves in all cases, as seen from the first two columns of the figure. The cost of this is finite sample overrejection in the scenarios of frequent small jumps, i.e. the last row on Figure 5. The overrejection for cases III-j and III-m is quite big.

Overall, we conclude that the test for disjoint jumps performs well in finite samples and has relatively good power. The test for common jumps should be always performed using the truncated variance V'_n, and it can significantly overreject the null in the case of jumps of small size. Finally, as confirmed by the Monte Carlo, using coarser sampling frequencies in performing the tests leads to larger errors in estimating the left and right volatility. Therefore, our ability to distinguish small price and volatility jumps worsens in such cases. As a result, on coarser frequencies the tests will perform worse (i.e. weaker power against alternatives and possible size distortions) when jumps are small, e.g., cases III in our Monte Carlo, and there will be little effect when jumps are bigger, e.g., cases I and II considered here.

6 Empirical application

Before going to the empirical application, let us mention a crucial point. Our construction of the tests assumes that the stochastic process is observed without error, and the Monte Carlo in the previous section is conducted in this way. In financial applications at very high frequencies, e.g., seconds, the presence of microstructure noise in the prices is non-negligible. If, for example, we have an i.i.d. noise, say with a continuous bounded density ϕ, then $\frac{\Delta_n}{u_n^3}\widehat{c}(k_n)_i$ converges in probability to $\frac{2}{3}\int \phi(x)\phi(-x)dx$ for all i: so obviously our test statistics behave in a very different way than in our theorems for their limiting behavior in probability, not to speak about the CLTs. Intuitively, the microstructure noise will tend to bias downwards the estimated difference between left and right volatility, i.e., a bias in favor of no common price and volatility jumps hypothesis.

There seems to be two ways to get around the problem of microstructure noise. One is to use a coarser frequency at which the microstructure noise is considered as being negligible. Given our conclusions from the Monte Carlo, this way will inevitably sacrifice somewhat the performance of the tests when very small jumps are involved. An alternative is to develop tests which are robust against the noise, like using a pre-averaging preliminary procedure for our local volatility estimators, but this will inevitably lead to a further decrease in the rates of convergence. Furthermore such an extension of our tests, while building on the theoretical results here, asks for a significantly more involved mathematical approach which goes beyond the scope of the current paper and is thus left for future work.

In our empirical application we use one minute S&P 500 index futures data. The S&P 500 index futures contract is one of the most liquid financial instruments and thus the microstructure noise should be of little concern at the selected one minute frequency. The sample period is from January 1997 till June 2007, which has 2593 trading days. We aggregate the data into business weeks (a total of 552) and perform the tests over these periods. Our choice for F is $g(y,z)1_{\{|x|>a\}}$ with $g(y,z)$ given by (4.9) and we report results for various truncation sizes a. The choice of u_n, k_n and v_n is done exactly as in the Monte Carlo study above.

Table 2: Testing for disjoint and common price and volatility jumps for S&P 500 index data

Jump Size	# of weeks with jumps	Rejection Rate			
		Null = $\Omega_T^{(A,d)}$		Null = $\Omega_T^{(A,j)}$	
		$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 10\%$
any size	238	60.50%	64.71%	42.02%	51.26%
$> 0.2\%$	163	61.96%	65.64%	40.49%	50.31%
$> 0.3\%$	96	69.79%	70.83%	38.54%	48.96%
$> 0.4\%$	56	73.21%	73.21%	42.86%	50.00%

Note: the test for common jumps is based on C_n' in (4.17).

Table 2 reports the rejection rates of the two tests (for the conventional 5% and 10% significance levels) for various levels of the truncation size a, while Figure 6 plots the kernel density estimate of the test statistics together with rejection curves of the two tests for the case of $a = 0$. The results suggest very strongly that the jumps in the level of the S&P 500 index are accompanied by jumps in its volatility. This is further confirmed from Table 3 in which we report the percentage of weeks in which both tests suggest the observed path is in $\Omega_T^{(A,j)}$, $\Omega_T^{(A,d)}$, or disagree. Based on the results in Table 3 for the weeks in which the S&P 500 index jumps: (1) in approximately 40% of them there is strong evidence for common price and volatility jumps, (2) in around 20% of them there is evidence for disjoint jumps, and (3) for the rest of the weeks the tests are inconclusive. Given our Monte Carlo study, this last part of the sample can be explained with a lot of small jumps for which detecting common or disjoint arrival needs even higher frequencies.

7 Conclusion

In this paper we derive tests for deciding whether jumps in a stochastic process are accompanied by simultaneous jumps in its volatility using only high-frequency data of the process. Our application of the tests to S&P 500 index data indicates that most stock market jumps are associated with volatility jumps as well.

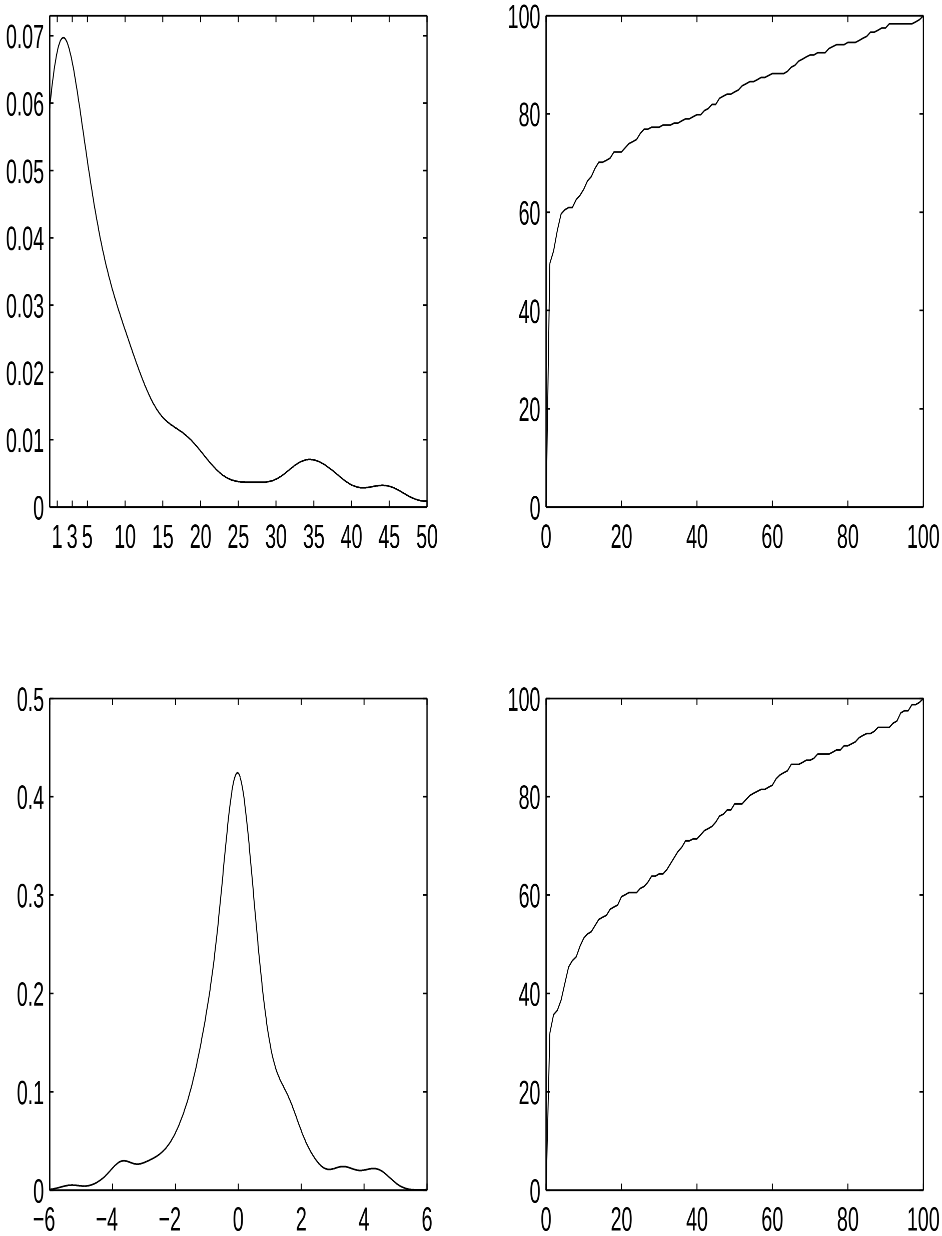

Figure 6: Test results for S&P 500 index data for truncation level $a = 0$. The top and bottom left panels show kernel density estimates of $U(f, g, k_n)_T / N_t^n$ and $\log(S_n)$ respectively. The top and bottom right panels plot empirical rejection rates against nominal size of the tests for disjoint and common jumps respectively. The latter one is based on C_n' in (4.17).

Table 3: Decision Matrix based on the two tests for S&P 500 index data

	accept $\Omega_T^{(j)}$	reject $\Omega_T^{(j)}$
accept $\Omega_T^{(d)}$	19.33%	20.17%
reject $\Omega_T^{(d)}$	38.66%	21.85%

Note: Numbers based on the two tests with 5% significance level and truncation level $a = 0$. The test for common jumps is based on C'_n in (4.17).

8 Proofs

8.1 Preliminaries

Under (H-r) and (K-v), both X and Z are Itô semimartingales, with (2.1) for X, and Z has a similar representation, in which (up to "augmenting" the Poisson measure μ) it is no restriction to assume that the Poisson measure is the same. That is, we can write

$$\begin{array}{rcl} Z_t &=& Z_0 + \int_0^t \widehat{b}_s ds + \int_0^t \widehat{\sigma}_s \, dW_s + \int_0^t \widehat{\sigma}'_s \, dW'_s \\ && + \int_0^t \int_E (\widehat{\delta}(s,z) 1_{\{|\widehat{\delta}(t,z)| \leq 1\}})(\mu - \nu)(ds, dz) \\ && + \int_0^t \int_E (\widehat{\delta}(s,z) 1_{\{|\widehat{\delta}(t,z)| > 1\}}) \mu(ds, dz) \end{array} \tag{8.1}$$

where W' is another standard Brownian motion, independent of W. Moreover we have $|\widehat{\delta}(\omega, t, z)| \leq \Gamma_t(\omega) \widehat{\gamma}(z)$, where we can always take the same process Γ_t than in (H-r) for X, as we may do for the process Γ showing in (2.2). Note also that

$$\begin{array}{lll} v \leq \frac{1}{2} & \Rightarrow & \int (\widehat{\gamma}(z)^2 \wedge 1) \, \lambda(dz) < \infty \\ v > \frac{1}{2} & \Rightarrow & \int (\widehat{\gamma}(z)^{1/v} \wedge 1) \, \lambda(dz) < \infty, \quad \widehat{\sigma} = \widehat{\sigma}' = 0. \end{array} \tag{8.2}$$

By a well known localization procedure, see for example [5], it is enough to prove all theorems of Section 3, hence also of Section 4, when in addition to the relevant assumptions (H-r) and (K-v) we have

$$|b_t| + |\sigma_t| + \frac{1}{|\sigma_t|} + |\widehat{b}_t| + |\widehat{\sigma}_t| + |\widehat{\sigma}'_t| + \Gamma_t + |X_t| + |Z_t| + |\overline{Z}_t| + \gamma(z) + \widehat{\gamma}(z) \ \leq \ C \tag{8.3}$$

for some constant C. This additional assumption will be supposed throughout. In the sequel, K is a constant which varies from line to line and may depend on C above and also on r, v, ϖ and on the function γ in (H-v), and is written K_q if it depends on an additional parameter q.

Under (8.3), we can write X as $X = X' + X''$, where

$$X''_t = \begin{cases} \int_0^t \int_E \delta(s,z)(\mu - \nu)(ds, dz) & \text{if } r > 1 \\ \int_0^t \int_E \delta(s,z) \mu(ds, dz) & \text{if } r \leq 1 \end{cases}$$

$$\begin{array}{rcl} X'_t &=& X_0 + \int_0^t b'_s ds + \int_0^t \sigma_s \, dW_s, \qquad \text{where} \\ b'_t &=& \begin{cases} b_t + \int_{\{|\delta(t,z)| > 1\}} \delta(t,z) \, \lambda(dz) & \text{if } r > 1 \\ b_t - \int_{\{|\delta(t,z)| \leq 1\}} \delta(t,z) \, \lambda(dz) & \text{if } r \leq 1. \end{cases} \end{array}$$

We also need a long series of additional notation. For each integer $m \geq 1$ we denote by $(S(m,q) : q \geq 1)$ the successive jump times of the counting (Poisson) process $\mu([0,t] \times \{z : \frac{1}{m} < \gamma(z) \leq \frac{1}{m-1}\})$. We relabel the two-parameter sequence $(S(m,q) : m,q \geq 1)$ as a single sequence $(T_p : p \geq 1)$, which clearly exhausts the jumps of X.

When $m \geq 1$ we denote by $\mathcal{T}_m$ the set of all p's such that $T_p = S(m',q)$ for some $q \geq 1$ and $m' \in \{1, \cdots, m\}$. We set $I(n,i) = ((i-1)\Delta_n, i\Delta_n]$ and

$$
\begin{aligned}
i(n,p) &= \text{the unique integer such that } \ T_p \in I(n, i(n,p)) \\
J(n,m) &= \{i(n,p) : p \in \mathcal{T}_m\}, \qquad J'(n,m) \;=\; \mathbb{N}^* \backslash J(n,m) \\
\Omega_{n,t,m} &= \cap_{p \neq q,\, p,q \in \mathcal{T}_m} \{T_p > t, \text{ or } T_p > 3k_n\Delta_n \text{ and } |T_p - T_q| > 6k_n\Delta_n.\}
\end{aligned}
$$

We have

$$
\lim_{n \to \infty} \mathbb{P}(\Omega_{n,t,m}) \;=\; 1. \tag{8.4}
$$

When $m \in \mathbb{N}$ we also set

$$
\left.
\begin{aligned}
&A_m \;=\; \{z : \gamma(z) \leq 1/m\}, \qquad \gamma_m \;=\; \textstyle\int_{A_m} \gamma(z)^r \, \lambda(dz) \\
&b'(m)_t \;=\; \begin{cases} b'_t - \int_{(A_m)^c} \delta(t,z)\, \lambda(dz) & \text{if } \ r > 1 \\ b'_t & \text{if } \ r \leq 1 \end{cases} \\
&X'(m)_t = X_0 + \textstyle\int_0^t b'(m)_s \, ds + \int_0^t \sigma_s \, dW_s \\
&Y(m)_t = \textstyle\int_0^t \int_{(A_m)^c} \delta(s,z) \mu(ds,dz) \\
&X''(m)_t \;=\; \begin{cases} \int_0^t \int_{A_m} \delta(s,z)\, (\mu - \nu)(ds,dz) & \text{if } \ r > 1 \\ \int_0^t \int_{A_m} \delta(s,z)\, \mu(ds,dz) & \text{if } \ r \leq 1 \end{cases} \\
&\overline{Y}(m) \;=\; X'(m) + X''(m) \;=\; X - Y(m).
\end{aligned}
\right\} \tag{8.5}
$$

Note that $A_0 = E$, $b'(0) = b'$, $Y(0) = 0$, $X'(0) = X'$ and $X''(0) = X''$. When $r \leq 1$, we can also define those quantities when $m = \infty$, in which case $A_\infty = \{z : \gamma(z) = 0\}$, $b'(\infty) = b'$, $Y(\infty) = X''$, $X'(\infty) = X'$ and $X''(\infty) = 0$.

Next, similar to (3.4), we put

$$
\eta(k_n)_i \;=\; \frac{1}{k_n\Delta_n} \sum_{j=1}^{k_n} |\Delta^n_{i+j} W|^2. \tag{8.6}
$$

This notation, as well as (3.4), is extended for convenience to the case where $i \leq 0$, with the convention that $\Delta^n_i Y = 0$ when $i \leq 0$ for any process Y. Finally, we set

$$
\begin{aligned}
&\widehat{c}(k_n, p-) = \widehat{c}(k_n)_{i(n,p)-k_n-1}, && \widehat{c}(k_n, p+) = \widehat{c}(k_n)_{i(n,p)} \\
&\eta(k_n, p-) \;= \eta(k_n)_{i(n,p)-k_n-1}, && \eta(k_n, p+) = \eta(k_n)_{i(n,p)} \\
&\kappa(k_n, p-) = \sqrt{k_n}\, \big(\widehat{c}(k_n, p-) - c_{T_p-}\big), && \kappa(k_n, p+) = \sqrt{k_n}\, \big(\widehat{c}(k_n, p+) - c_{T_p}\big) \\
&\kappa'(k_n, p-) = \sqrt{k_n}\, \big(\eta(k_n, p-) - 1\big), && \kappa'(k_n, p+) = \sqrt{k_n}\, \big(\eta(k_n, p+) - 1\big).
\end{aligned}
$$

8.2 Estimates

We proceed here to recalling or proving a number of useful estimates. As said before, we always assume (H-r) and (K-v) and (8.3). Mostly, these estimates are conditional with respect to a possibly larger filtration than $(\mathcal{F}_t)$. So we fix $m \in \mathbb{N}$, and denote by $\mu^{(m)}$ and $\mu'^{(m)}$ the restrictions of the measure μ to the sets $\mathbb{R}_+ \times A_m$ and $\mathbb{R}_+ \times (A_m)^c$ respectively. These are two independent Poisson measures, independent of W and W' as well. We denote by $\mathcal{G}_m$ the σ-field generated by the measure $\mu'^{(m)}$, and by $(\mathcal{F}_t^{(m)})$ the smallest filtration containing $(\mathcal{F}_t)$ and such that $\mathcal{F}_0^{(m)}$ contains $\mathcal{G}_m$.

We set $D_m = \{(\omega, s) : \mu'^{(m)}(\omega, \{s\} \times E) = 1\}$, which is also the union of the graphs of the stopping times T_p for $p \in \mathcal{T}_m$. Then we define the process

$$Z(m)_t \;=\; Z_t - \sum_{s \le t} \Delta Z_s \, 1_{D_m}(s).$$

Due to the independence of W, W', $\mu^{(m)}$ and $\mu'^{(m)}$, the processes W and W' and the measure $\mu^{(m)}$ are still Wiener processes and a Poisson random measure, relative to the filtration $(\mathcal{F}_t^{(m)})$. Hence $X'(m)$ and $X''(m)$ are Itô semimartingales, with the same form as in (8.5) (we can replace μ and ν by $\mu^{(m)}$ and its deterministic compensator, because of the presence of 1_{A_m}), and relative to the filtration $(\mathcal{F}_t^{(m)})$. In the same way $Z(m)$ is still of the form (8.1), driven by W, W' and $\mu^{(m)}$ (instead of μ), relatively to $(\mathcal{F}_t^{(m)})$ (and up to replacing $\widehat{b}_t$ by $\widehat{b}(m)_t = \widehat{b}_t - \int_{(A_m)^c} \widehat{\delta}(t,z) \, 1_{\{|\widehat{\delta}(t,z)| \le 1\}} \, \lambda(dz)$, which is still bounded).

1 - Estimates on σ. The latter property, together with (8.3) and classical estimates and the fact that $\widehat{\sigma}_t = \widehat{\sigma}'_t = 0$ identically when $v > 1/2$ imply that for any $p \ge 1$:

$$\mathbb{E}(\sup_{s \le t} |Z(m)_{R+s} - Z(m)_R|^p \mid \mathcal{F}_R^{(m)}) \le \begin{cases} K_p \, t^{(p/2) \wedge 1} & \text{if } v \le 1/2 \\ K_p \, t^{(pv) \wedge 1} & \text{if } v > 1/2 \end{cases} \tag{8.7}$$

for any finite $(\mathcal{F}_t^{(m)})$-stopping time R. Since Z and $\overline{Z}$ stay in a compact set, we have

$$|\sigma_{t+s} - \sigma_t| \;\le\; K(|Z_{t+s} - Z_t| + |\overline{Z}_{t+s} - \overline{Z}_t|).$$

Moreover, $Z_t - Z_s = Z(m)_t - Z(m)_s$ if $s < t$ and $(s,t] \cap D_m = \emptyset$. If R is a finite $(\mathcal{F}_t^{(m)})$-stopping, the set $\{(R, R+t] \cap D_m = \emptyset\}$ belongs to $\mathcal{F}_0^{(m)}$, so (2.2) and (8.7) yield

$$\mathbb{E}(\sup_{s \le t} |\sigma_{R+s} - \sigma_R|^p \mid \mathcal{F}_R^{(m)}) \le K t^{(pv) \wedge 1} \;\text{ on }\; \big\{(R, R+t] \cap D_m = \emptyset\big\}. \tag{8.8}$$

2 - Estimates on X. The following classical estimates use (8.3) and $|b'(m)_t| \le K m^{(r-1)^+}$. Below, $q > 0$ and $p \ge r$ and i is an integer, possibly random but $\mathcal{F}_0^{(m)}$-measurable, and we have

$$\begin{array}{l}
\mathbb{E}\big(|\Delta_i^n W|^q \mid \mathcal{F}^{(m)}_{(i-1)\Delta_n}\big) \;\leq\; K_q \Delta_n^{q/2} \\
\mathbb{E}\big(|\Delta_i^n X'(m)|^q \mid \mathcal{F}^{(m)}_{(i-1)\Delta_n}\big) \leq K_q \Delta_n^{q/2}(1+\Delta_n^{q/2}\, m^{q(r-1)^+}) \\
\mathbb{E}\big(|\Delta_i^n X''(m)|^p \mid \mathcal{F}^{(m)}_{(i-1)\Delta_n}\big) \leq \left\{ \begin{array}{ll} \frac{K_p\Delta_n\gamma_m}{m^{p-r}}\,\big(1+(\Delta_n m^r)^{(p-1)^+}\big) & \text{if } r\leq 1 \\ \frac{K_p\Delta_n\gamma_m}{m^{p-r}}\,\big(1+(\Delta_n m^r)^{(p-2)^+/2}\big) & \text{if } r>1 \end{array}\right. \\
\mathbb{E}\big(\big|\Delta_i^n X'(m) - \sigma_{(i-1)\Delta_n}\Delta_i^n W\big|^q \,|\, \mathcal{F}^{(m)}_{(i-1)\Delta_n}\big) \leq K\big(\Delta_n^{q/2+(qv)\wedge 1} + \Delta_n^q\, m^{q(r-1)^+}\big) \\
\qquad\qquad\qquad\qquad \text{on the set } \{I(n,i)\cap D_m=\emptyset\}.
\end{array} \tag{8.9}$$

Next, we also have for $p \geq r$:

$$\mathbb{E}\big(|\Delta_i^n X''(m)|^p \wedge u_n^p \mid \mathcal{F}^{(m)}_{(i-1)\Delta_n}\big) \leq K u_n^{p-r}\Delta_n\gamma_m. \tag{8.10}$$

These estimates hold when $m=0$ as well (in which case $\mathcal{F}^{(0)}_t = \mathcal{F}_t$ and i is not random, and $Y(0)=0$). In particular, in this case we deduce

$$\mathbb{E}\big(|\Delta_i^n X|^2 \mid \mathcal{F}_{(i-1)\Delta_n}\big) \;\leq\; K\Delta_n. \tag{8.11}$$

Next, with any measurable subset A of E we consider the increasing process $G(A)_t = \int_0^t \int_A \gamma(z)\mu(ds,dz)$. This process is infinite for all $t>0$ if $\int_A \gamma(z)\lambda(dz)=\infty$, and otherwise is a Lévy process, and known estimates on Lévy processes yield for all $q>0$:

$$\mathbb{E}((G(A)_t)^q) \;\leq\; K_q\Big(t\int_A \gamma(z)^q\lambda(dz) + \Big(t\int_A \gamma(z)\lambda(dz)\Big)^{q\vee 1}\Big). \tag{8.12}$$

(Since γ is bounded, when $q\leq 1$ the right side above is smaller than $K_q t\int_A \gamma(z)^q\lambda(dz)$.) Since $|\Delta_i^n Y(m)| \leq \Delta_i^n G(A_m^c)$, we deduce (for $i\geq 1$ not random):

$$q\geq r \;\;\Rightarrow\;\; \mathbb{E}\big(|\Delta_i^n Y(m)_i^n|^q \mid \mathcal{F}_{(i-1)\Delta_n}\big) \;\leq\; K_q\big(\Delta_n + (\Delta_n m^{(r-1)^+})^{q\vee 1}\big). \tag{8.13}$$

3 - Estimates on $\widehat{c}(k_n)_i$. Below, $i\geq 1$ is a non-random integer. First (8.11) yields

$$\mathbb{E}\big(\widehat{c}(k_n)_i \mid \mathcal{F}_{i\Delta_n}\big) \;\leq\; K. \tag{8.14}$$

We need also estimates on the difference $\widehat{c}(k_n)_i - c_t$ for suitable times t. If S is a $\mathcal{F}_0^{(m)}$-measurable positive finite time and $i\geq 1$ an $\mathcal{F}_0^{(m)}$-measurable random integer, the sets

$$\begin{array}{l}
\Omega(m,n,S,i)_+ = \big\{(i-1)\Delta_n < S \leq i\Delta_n,\; (S, S+(k_n+1)\Delta_n]\cap D_m=\emptyset\big\} \\
\Omega(m,n,S,i)_- = \big\{(i-1)\Delta_n < S \leq i\Delta_n,\; (S-(k_n+2)\Delta_n, S)\cap D_m=\emptyset\big\}
\end{array}$$

are $\mathcal{F}_0^{(m)}$-measurable, and we have:

Lemma 8.1 *Assume (H-r) and (K-v) and (8.3). Let $q=1$ or $q=2$, and assume (3.3) with also*

$$\begin{array}{lll}
q=1 & \Rightarrow & \rho < \frac{2v}{1+2v} \bigwedge (2\varpi(2-r)) \\
q=2 & \Rightarrow & \rho < \frac{(2v)\wedge 1}{1+(2v)\wedge 1} \bigwedge \big(\varpi(4-r)-1\big).
\end{array} \tag{8.15}$$

Then there is a sequence $\alpha_n(q) \to 0$ such that, for $m \geq 0$ and any $\mathcal{F}_0^{(m)}$-measurable variables S and i as above, we have:

$$\begin{array}{l} \mathbb{E}\big(|\widehat{c}(k_n)_i - c_S\,\eta(k_n)_i|^q \mid \mathcal{F}_S^{(m)}\big) \;\leq\; \frac{K_m\,\alpha_n(q)}{k_n^{q/2}} \quad \text{on } \Omega(m,n,S,i)_+ \\ \mathbb{E}\big(|\widehat{c}(k_n)_{i-k_n-1} - c_{S-}\,\eta(k_n)_{i-k_n-1}|^q \mid \mathcal{F}_{(i-k_n-1)\Delta_n}^{(m)}\big) \\ \qquad\qquad\qquad\qquad \leq\; \frac{K_m\,\alpha_n(q)}{k_n^{q/2}} \quad \text{on } \Omega(m,n,S,i)_- \end{array} \tag{8.16}$$

and also

$$\begin{array}{lll} \mathbb{E}\big(|\widehat{c}(k_n)_i - c_S|^q \mid \mathcal{F}_S^{(m)}\big) \;\leq\; \frac{K_m}{k_n^{q/2}} & \text{on} & \Omega(m,n,S,i)_+ \\ \mathbb{E}\big(|\widehat{c}(k_n)_{i-k_n-1} - c_{S-}|^q \mid \mathcal{F}_{(i-k_n-1)\Delta_n}^{(m)}\big) \leq \frac{K_m}{k_n^{q/2}} & \text{on} & \Omega(m,n,S,i)_- \end{array} \tag{8.17}$$

Moreover, as soon as $r < 2$, and under (3.3) only, we have

$$\begin{array}{ll} \widehat{c}(k_n)_i \stackrel{\mathbb{P}}{\longrightarrow} c_S \ \text{ on} & \Omega(m,n,S,i)_+ \\ \widehat{c}(k_n)_{i-k_n-1} \stackrel{\mathbb{P}}{\longrightarrow} c_{S-} \ \text{ on} & \Omega(m,n,S,i)_-. \end{array} \tag{8.18}$$

Proof. We will prove for example the second claims of (8.16), (8.13) and (8.18) (the first ones are slightly easier). On the set $\Omega(m,n,S,i)_-$ the variable $\widehat{c}(k_n)_{i-k_n-1}$ is equal to the variable $\widehat{c}'(k_n)_{i-k_n-1}$ associated in the same way with the process $\overline{Y}(m)$.

The following estimate, for all $x,y,z \in \mathbb{R}$, $u > 0$, $w > 0$, is straightforward:

$$\begin{array}{l} \Big||x+y+z|^2 1_{\{|x+y+z| \leq u\}} - x^2\Big|^q \\ \leq\; K_q\Big((y \wedge u)^{2q} + z^{2q} + |x|^q(|y| \wedge u)^q + |x|^q|z|^q + \frac{|x|^{(2+w)q}}{u^{wq}}\Big). \end{array}$$

This will be applied with $x = \sigma_{(j-1)\Delta_n}\Delta_j^n W$ and $y = \Delta_j^n X''(m)$ and $z = \Delta_j^n X'(m) - \sigma_{(j-1)\Delta_n}\Delta_j^n W$ (so $\Delta_j^n \overline{Y}(m) = x+y+z$), and $u = u_n$ and w such that $w(1-2\varpi) \geq 2$, and when $j = i-k_n-1, i-k_n, \cdots, i-1$: using Hölder's inequality, we deduce from (8.9) and (8.10) and the boundedness of σ_t, and after some calculation, that in this case

$$\begin{array}{l} \mathbb{E}\Big(\big|(\Delta_j^n \overline{Y}(m))^2\, 1_{\{|\Delta_j^n \overline{Y}(m)| \leq u_n\}} - c_{(j-1)\Delta_n}(\Delta_j^n W)^2\big|^q \mid \mathcal{F}_{(i-k_n-1)\Delta_n}^{(m)}\Big) \\ \qquad\qquad\qquad\qquad \leq\; K_{m,\theta}\big(\Delta_n^{1+(2q-r)\varpi} + \Delta_n^{q+(qv)\wedge\theta}\big) \end{array}$$

for any $\theta \in (0,1)$, on the set $\Omega(m,n,S,i)_-$, because $I(n,j) \cap D_m = \emptyset$.

Next, we write $|c_{(j-1)\Delta_n} - c_{S-}| \leq |c_{(j-1)\Delta_n} - c_{j\Delta_n}| + |c_{j\Delta_n} - c_{S-}|$, and we apply (8.8) and (8.9) and either Hölder's inequality plus the boundedness of σ_t, or successive conditioning, to get for j and θ as above:

$$\begin{array}{l} \mathbb{E}\big(|c_{(j-1)\Delta_n} - c_{S-}|^q\,(\Delta_j^n W)^{2q} \mid \mathcal{F}_{(i-k_n-1)\Delta_n}^{(m)}\big) \\ \leq\; K_\theta\big(\Delta_n^q(k_n\Delta_n)^{(qv)\wedge 1} + \Delta_n^{q+(qv)\wedge\theta}\big). \end{array}$$

These estimates, together with the definition of $\widehat{c}'(k_n)_{i-k_n-1}$ and $\eta(k_n)_{i-k_n-1}$, yield

$$\mathbb{E}\big(|\widehat{c}'(k_n)_{i-k_n-1} - c_{S-}\,\eta(k_n)_{i-k_n-1}|^q \mid \mathcal{F}_{(i-k_n-1)\Delta_n}^{(m)}\big) \;\leq\; K_{m,\theta}\,a(q)_n$$

on the set $\Omega(m,n,S,i)_-$, where $a(q)_n = \Delta_n^{1+(2q-r)\varpi-q} + \Delta_n^{((qv)\wedge 1)(1-\rho)} + \Delta_n^{(qv)\wedge\theta}$. Then (3.3) and a proper choice of θ show that $a(q)_n k_n^{q/2} \to 0$ for $q=1,2$, under (8.12), and $a(1)_n \to 0$ as soon as $r<2$. This in particular gives the second part of (8.16).

Finally (8.17) and (8.18) follow from the above, from the boundedness of the process c_t, and from the following property: if R is any $(\mathcal{F}_t^{(m)})$-stopping time and $i\Delta_n \geq R$, then

$$\mathbb{E}\big((\eta(k_n)_i-1)^2 \mid \mathcal{F}_R^{(m)}\big) \;=\; 2/k_n.$$

This readily follows from the fact that $\eta(k_n)_i$ is independent of $\mathcal{F}_R^{(m)}$ and given by (8.6).□

8.3 The stable convergence of $\widehat{c}(k_n)_i$

From now on, the integer $w\geq 2$ is fixed. The aim of this subsection is to prove the following stable convergence:

Proposition 8.2 *As soon as (H-r), (K-v), (8.3) and (8.15) for $q=1$ hold, the sequence of variables*

$$\big(\big(\kappa(k_n,p-),\kappa(k_n,p+),\kappa(wk_n,p-),\kappa(wk_n,p+)\big)_{p\geq 1} \tag{8.19}$$

converges stably in law as $n\to\infty$ (for the product topology on $\mathbb{R}^{\mathbb{N}^*}$) to

$$\begin{aligned}\Big(c_{T_p-}\sqrt{2}\,V_p^-, c_{T_p}\sqrt{2}\,V_p^+, c_{T_p-}\sqrt{\tfrac{2}{w}}(V_p^- + \sqrt{w-1}\,V_p'^-),\\ c_{T_p}\sqrt{\tfrac{2}{w}}(V_p^+ + \sqrt{w-1}\,V_p'^+)\Big)_{p\geq 1},\end{aligned} \tag{8.20}$$

where the variables $V_p^-, V_p^+, V_p'^-, V_p'^+$ are defined on an extension of the original space $(\Omega,\mathcal{F},\mathbb{P})$ and are all independent and $\mathcal{N}(0,1)$-distributed, and independent of $\mathcal{F}$.

Proof. *Step 1)* It is enough to prove the convergence of any finite sub-family of indices p. In other words, instead of considering the infinite sequence indexed by $p\geq 1$ in (8.19) and (8.20), we can fix an arbitrarily large integer P and consider the families indexed by $p\in\{1,\cdots,P\}$. All p smaller than P are in some $\mathcal{T}_m$, and we consider the set Ω_n on which for any $p\leq P$ and any $q\in\mathcal{T}$ we have $T_p > 3k_n\Delta_n$ and $|T_p-T_q| > 6k_n$. Obviously, $\mathbb{P}(\Omega_n)\to 1$ as $n\to\infty$.

Now we will apply Lemma 8.1 with $S=T_p$ for $p\leq P$, and $i=i(n,p)$: then S and i are $\mathcal{F}_0^{(m)}$-measurable, and the set Ω_n is included into both $\Omega(m,n,T_p,i(n,p))_+$ and $\Omega(m,n,T_p,i(n,p))_-$. Since $\mathbb{P}(\Omega_n)\to 1$, we deduce from this lemma that

$$\begin{aligned}\sqrt{k_n}\,\big(\widehat{c}(k_n,p-) - c_{T_p-}\eta(k_n,p-)\big) &\stackrel{\mathbb{P}}{\longrightarrow} 0\\ \sqrt{k_n}\,\big(\widehat{c}(k_n,p+) - c_{T_p}\eta(k_n,p+)\big) &\stackrel{\mathbb{P}}{\longrightarrow} 0\end{aligned} \tag{8.21}$$

Step 2) Now we set

$$\begin{aligned}\chi^n &= \big(\kappa'(k_n,p-),\kappa'(k_n,p+),\kappa'(wk_n,p-),\kappa'(wk_n,p+)\big)_{1\leq p\leq P}\\ \chi &= \Big(\sqrt{2}\,V_p^-, \sqrt{2}\,V_p^+, \sqrt{\tfrac{2}{w}}(V_p^- + \sqrt{w-1}\,V_p'^-),\\ &\qquad \sqrt{\tfrac{2}{w}}(V_p^+ + \sqrt{w-1}\,V_p'^+)\Big)_{1\leq p\leq P}.\end{aligned} \tag{8.22}$$

By (8.21), we are left to prove that the variables χ^n stably converge in law to χ. Taking into account that χ is independent of $\mathcal{F}$, this amounts to proving

$$\mathbb{E}(U\ f(\chi^n)) \ \to\ \mathbb{E}(U)\,\mathbb{E}(f(\chi)), \tag{8.23}$$

where U is any bounded $\mathcal{F}$-measurable variable and f is continuous bounded.

In fact, if $(\mathcal{G}_t^{(m)})$ denotes the smallest filtration to which W is adapted and such that $\mathcal{G}_m \subset \mathcal{G}_0^{(m)}$, each χ^n is $\mathcal{G}_\infty^{(m)}$-measurable. So, up to substituting U with $E(U \mid \mathcal{G}_\infty)$ above, it is clearly enough to prove (8.23) when U is $\mathcal{G}_\infty^{(m)}$-measurable.

Step 3) We introduce some further notation: first the set $F_n = \cup_{1\leq p\leq P}((T_p-(wk_n+1)\Delta_n)^+, T_p+(wk_n+1)\Delta_n]$, which is a random $\mathcal{G}_0^{(m)}$-measurable set, and second the processes

$$\overline{W}_t^n \ =\ \int_0^t 1_{F_n}(s)\,dW_s, \qquad \overline{W}'^n \ =\ W-\overline{W}^n$$

(those are well defined because W is a $(\mathcal{G}_t^{(m)})$-Brownian motion). The σ-fields $\mathcal{H}_n$ generated by $\mathcal{G}_0^{(m)}$ and all variables $\overline{W}_t'^n$ increase with n, and $\bigvee_n \mathcal{H}_n = \mathcal{G}_\infty^{(m)}$. Therefore it is enough to prove (8.23) when U is $\mathcal{H}_q$-measurable for some q: to see this, let U be $\mathcal{G}_\infty^{(m)}$-measurable; set $U_q=\mathbb{E}(U\mid\mathcal{H}_q)$; if (8.23) holds for each U_q, it also holds for U because $U_q\to U$ in $\mathbb{L}^1(\mathbb{P})$.

The set Ω_n of Step 1 is $\mathcal{G}_m$-measurable, hence $\mathcal{H}_q$-measurable for all q. Since $\mathbb{P}(\Omega_n)\to 1$ it is enough to prove that for any bounded $\mathcal{H}_q$-measurable variable U,

$$\mathbb{E}\big(U\,1_{\Omega_n}\,f(\chi^n)\big) \ \to\ \mathbb{E}(U)\,\mathbb{E}(f(\chi)), \tag{8.24}$$

Step 4) We introduce a double sequence $(N(p,i): p,i\geq 1)$ of i.i.d. $\mathcal{N}(0,1)$ variables on some auxiliary probability space. Then, define the variables

$$\begin{aligned}
\zeta(k_n,p-) &= \tfrac{1}{\sqrt{k_n}}\textstyle\sum_{i=1}^{k_n}(N(p,i)^2-1)\\
\zeta(k_n,p+) &= \tfrac{1}{\sqrt{k_n}}\textstyle\sum_{i=wk_n+1}^{(w+1)k_n}(N(p,i)^2-1)\\
\zeta(wk_n,p-) &= \tfrac{1}{\sqrt{wk_n}}\textstyle\sum_{i=1}^{wk_n}(N(p,i)^2-1)\\
\zeta(wk_n,p+) &= \tfrac{1}{\sqrt{wk_n}}\textstyle\sum_{i=wk_n+1}^{2wk_n}(N(p,i)^2-1).
\end{aligned}$$

Observe that in restriction to the set Ω_n the variable χ^n involves increments of W which are different for different values of p, and are increments of the process $\overline{W}^n$ above, which is independent of $\overline{W}'^n$. Therefore if q is fixed, for any $n\geq q$ and in restriction to the $\mathcal{H}_q$-measurable set Ω_n, the $\mathcal{H}_q$-conditional distribution of the variable χ^n of (8.22) is exactly the law of

$$\zeta_n \ =\ \big(\zeta(k_n,p-),\zeta(k_n,p+),\zeta(wk_n,p-),\zeta(wk_n,p+)\big)_{1\leq p\leq P}.$$

This means that the left side of (8.24) for $n\geq q$ is equal to $\mathbb{E}(U1_{\Omega_n})\,\mathbb{E}(f(\zeta_n))$.

At this stage, we see that (8.24) amounts to proving that ζ_n converges in law to the variable χ given in the second half of (8.22). This is an obvious consequence of the $4P$-dimensional ordinary central limit theorem. □

8.4 Proof of Theorem 3.1

1) As said before we assume (H-r) and (K-v) and (8.3). If $m \geq 1$ and $J(n,m,t) = J'(n,m) \cap \{k_n+1, k_n+2, \cdots, [t/\Delta_n]-k_n\}$ and $\mathcal{T}_m(n,t) = \{p \in \mathcal{T}_m : T_p \leq \Delta_n[t/\Delta_n]\}$, we have

$$\begin{array}{l} t \leq T \quad \Rightarrow \quad U(F,k_n)_t \; = \; \widetilde{U}^n(m)_t + \overline{U}^n(m)_t \quad \text{on the set } \Omega_{n,T,m}, \text{ where} \\ \widetilde{U}^n(m)_t = \sum_{p\in\mathcal{T}_m(n,t)} F(\Delta^n_{i(n,p)}X, \widehat{c}(k_n,p-), \widehat{c}(k_n,p+)) 1_{\{\Delta^n_{i(n,p)}X|>u_n\}} \\ \overline{U}^n(m)_t = \sum_{i\in J(n,m,t)} F(\Delta^n_i\overline{Y}(m), \widehat{c}(k_n)_{i-k_n-1}, \widehat{c}(k_n)_i)\, 1_{\{|\Delta^n_i\overline{Y}(m)|>u_n\}}. \end{array} \tag{8.25}$$

The sum defining $\widetilde{U}^n(m)_t$ has a bounded number of summands, as n varies. We also have for $p \in \mathcal{T}_m$:

$$\begin{array}{l} \Delta^n_{i(n,p)}X \to \Delta X_{T_p}, \quad \mathbb{P}(|\Delta^n_{i(n,p)}\overline{Y}(m)| > u_n/2) \to 0 \\ \Delta^n_{i(n,p)}X = \Delta X_{T_p} + \Delta^n_{i(n,p)}\overline{Y}(m) \;\; \text{on} \;\; \Omega_{n,t,m} \end{array} \tag{8.26}$$

(use (8.9) and $\sqrt{\Delta_n}/u_n \to 0$ for the second property). We have $\mathbb{P}(\Delta X_{T_p} \in R) = 1$ by (3.6) and F is continuous on $R \times \mathbb{R}^{*2}_+$. Since $\widehat{c}(k_n,p-) \stackrel{\mathbb{P}}{\longrightarrow} c_{T_p-}$ and $\widehat{c}(k_n,p+) \stackrel{\mathbb{P}}{\longrightarrow} c_{T_p}$ on $\Omega_{n,t,m} \cap \{T_p \leq t\}$ (use (8.18) with $S = T_p$), the pth summand in $\widetilde{U}^n(m)_t$ converges to $F(\Delta X_{T_p}, c_{T_p-}, c_{T_p}) 1_{\{\Delta X_{T_p}\neq 0\}}\, 1_{\{T_p\leq t\}}$ in probability. Therefore we have the following convergence in probability, for the Skorokhod topology:

$$\widetilde{U}^n(m)_t \stackrel{\mathbb{P}}{\longrightarrow} \widetilde{U}(m)_t \; = \; \sum_{p\in\mathcal{T}_m} F(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\, 1_{\{T_p\leq t\}}. \tag{8.27}$$

2) Next, we show the result in case (a). Pick $m > 2/\varepsilon$. Since $|\Delta\overline{Y}(m)_s| \leq 1/m$, for any $t > 0$ we have $|\Delta^n_i\overline{Y}(m)| \leq 2/m$ for all $i \leq [t/\Delta_n]$, on a set Ω^n_t whose probability goes to 1. On Ω^n_t we have $\overline{U}^n(m)_s = 0$ for all $s \leq t$, because of the property of F, which also implies $\widetilde{U}(m) = U(F)$ identically. Then the result readily follows from (8.27).

3) Next, we show the result in case (b). The notation (8.5) is also valid for $m = \infty$, and (8.27) holds for $m = \infty$ (the right side is a finite sum) and $\widetilde{U}(\infty) = U(F)$. Since $\overline{Y}(\infty) = X'(\infty)$, it follows from the second part of (8.9) (which also holds with $m = \infty$ when $r = 0$) that $\mathbb{P}(\Delta^n_i\overline{Y}(\infty)| > u_n) \leq K_q\Delta_n^{q/2}u_n^{-q}$, which is smaller than $K\Delta_n^2$ if $q = \frac{4}{1-2\varpi}$. So Borel-Cantelli Lemma yields that, for each t, we have $|\Delta^n_i\overline{Y}(\infty)| \leq u_n$ for all $i \leq [t/\Delta_n]$, hence $\overline{U}^n(\infty)_s = 0$ for $s \leq t$, when n is large enough. We then conclude as above.

4) It remains to consider the case (c). First, $|F(\Delta X_s, c_{s-}, c_s)| \leq K|\Delta X_s|^r$ as soon as $|\Delta X_s| \leq \varepsilon$ (recall that c_s is bounded). Since $\sum_{s\leq t} |\Delta X_s|^r < \infty$ a.s. for all t, whereas $|\Delta X_s| \leq 1/m$ when s differs from all T_p for $p \in \mathcal{T}_m$, we deduce from the dominated convergence theorem that $\widetilde{U}(m) \to U(F)$ a.s., locally uniformly in time as $m \to \infty$. Therefore by (8.27) it remains to prove that for all $t > 0$,

$$\eta > 0 \quad \Rightarrow \quad \lim_{m\to\infty} \; \limsup_{n\to\infty} \; \mathbb{P}\Big(\sup_{s\leq t} |\overline{U}^n(m)_s| > \eta\Big) \; = \; 0. \tag{8.28}$$

On the one hand, as in the previous step we deduce from (8.9) and from $|\Delta X''(m)_s| \leq 1/m$ that, if $m > 4/\varepsilon$, we have $|\Delta^n_i X'(m)| \leq u_n/2$ and $|\Delta^n_i X''(m)| \leq \varepsilon/2$ for all $i \leq [t/\Delta_n]$, when n is large enough. On the other hand, our assumption on F yields that if $|x|\le u_n/2$ and $|x'|\le\varepsilon/2$, then $|F(x+x',y,z)|1_{\{|x+x'|>u_n\}}\le K|x'|^r(1+y+z)$ as soon as $u_n\le\varepsilon$. Hence for any given t, and outside a set $\Omega'_{n,t,m}$ satisfying $\mathbb{P}(\Omega'_{n,t,m})\to 1$ as $n\to\infty$, we have $|\overline{U}^n(m)_s|\le H^n(m)_t$ for all $s\le t$, where

$$H^n(m)_t \;=\; K\sum_{i=k_n+1}^{[t/\Delta_n]-k_n}|\Delta_i^n X''(m))|^r\,\big(1+\widehat{c}(k_n)_{i-k_n-1}+\widehat{c}(k_n)_i)\big).$$

Therefore we are left to show that for all t,

$$\lim_{m\to\infty}\;\sup_n\;\mathbb{E}\big(H^n(m)_t\big)\;=\;0. \tag{8.29}$$

The estimates (8.13) and (8.14) and successive conditioning yield that

$$\mathbb{E}\big(|\Delta_i^n X''(m))|^r\,\big(1+\widehat{c}(k_n)_{i-k_n-1}+\widehat{c}(k_n)_i)\big)\;\le\;K\,\Delta_n\,\gamma_m.$$

Since $\gamma_m\to 0$ as $m\to\infty$, we deduce (8.29) and Theorem 3.1 is proved.

8.5 Proof of Theorem 3.2

We need many steps, and as before we assume (H-r) and (K-v), and also (8.3).

Step 1) We use the notation (8.25) of the previous proof when we deal with k_n and write instead $\widetilde{U}'^n(m)$ and $\overline{U}'^n(m)$ when we deal with wk_n. We also use $\widetilde{U}(m)_t$, as defined in (8.27), and

$$\widehat{U}^n(m)_t \;=\; \overline{U}^n(m)_t-\sum_{p\ge1,\,p\notin\mathcal{T}_m}F(\Delta X_{T_p},c_{T_p-},c_{T_p})1_{\{T_p\le t\}}$$

and $\widehat{U}'(m)$ is the same with $\overline{U}'^n(m)$ instead of $\overline{U}^n(m)$. We have

$$\widetilde{U}^n(m)_t-\widetilde{U}(m)_t\;=\;\sum_{p\in\mathcal{T}_m}\zeta_p^n,\qquad \widetilde{U}'^n(m)_t-\widetilde{U}(m)_t\;=\;\sum_{p\in\mathcal{T}_m}\zeta_p'^n,$$

where

$$\begin{aligned}\zeta_p^n &= F(\Delta_{i(n,p)}^n X,\widehat{c}(k_n,p-),\widehat{c}(k_n,p+))\,1_{\{\Delta_{i(n,p)}^n X|>u_n\}}\,1_{\{T_p\le\Delta_n[t/\Delta_n]\}}\\ &\qquad -F(\Delta X_{T_p},c_{T_p-},c_{T_p})1_{\{\Delta X_{T_p}\ne0\}}\,1_{\{T_p\le t\}}\\ \zeta_p'^n &= F(\Delta_{i(n,p)}^n X,\widehat{c}(wk_n,p-),\widehat{c}(wk_n,p+))\,1_{\{\Delta_{i(n,p)}^n X|>u_n\}}\,1_{\{T_p\le\Delta_n[t/\Delta_n]\}}\\ &\qquad -F(\Delta X_{T_p},c_{T_p-},c_{T_p})1_{\{\Delta X_{T_p}\ne0\}}\,1_{\{T_p\le t\}}.\end{aligned}$$

We also set

$$\begin{aligned}\overline{\zeta}_p^n &= \Big(F_2'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(k_n,p-)\\ &\qquad +F_3'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(k_n,p+)\Big)1_{\{\Delta X_{T_p}\ne0\}}\\ \overline{\zeta}_p'^n &= \Big(F_2'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(wk_n,p-)\\ &\qquad +F_3'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(wk_n,p+)\Big)1_{\{\Delta X_{T_p}\ne0\}}.\end{aligned} \tag{8.30}$$

Step 2) In this step we prove that

$$\Big(\sqrt{k_n}\,\big(\widetilde{U}^n(m)-\widetilde{U}(m)\big),\sqrt{k_n}\,\big(\widetilde{U}'^n(m)-\widetilde{U}(m)\big)\Big) \stackrel{\mathcal{L}-s}{\Longrightarrow} \big(\mathcal{U}(m),\tfrac{1}{w}\,(\mathcal{U}(m)+\sqrt{w-1}\,\mathcal{U}'(m))\big), \tag{8.31}$$

(stable functional convergence in law), where $\mathcal{U}(m)$ and $\mathcal{U}'(m)$ are as described in (3.8), except that the sum is taken over the $p\in\mathcal{T}_m$ only. By Proposition 8.2, we have

$$\sum_{p\in\mathcal{T}_m}\big(\overline{\zeta}_p^n,\overline{\zeta}_p'^n\big)\,1_{\{T_p\le t\}} \stackrel{\mathcal{L}-s}{\Longrightarrow} \big(\mathcal{U}(m)_t,\frac{1}{\sqrt{w}}\,(\mathcal{U}(m)_t+\sqrt{w-1}\,\mathcal{U}'(m)_t)\big),$$

whereas note the normalization in $\kappa(wk_n,p\pm)$ is by $\sqrt{wk_n}$. Hence proving (8.31) amounts to show that for each $p\in\mathcal{T}_m$ we have

$$\sqrt{k_n}\,\zeta_p^n-\overline{\zeta}_p^n \stackrel{\mathbb{P}}{\longrightarrow} 0,\qquad \sqrt{wk_n}\,\zeta_p'^n-\overline{\zeta}_p'^n \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{8.32}$$

in restriction to each set $\{T_p\le t\}$. We will prove for example the first property. We have $\mathbb{P}(\Delta_n[t/\Delta_n]<T_p\le t)\to 0$ and (8.4) and (8.26), implying that the set $\{|\Delta_{i(n,p)}^nX|>u_n\}$ converges in probability to the set $\{\Delta X_{T_p}\neq 0\}$. Therefore it is enough to show that

$$\begin{aligned}&\sqrt{k_n}\,\Big(F(\Delta_{i(n,p)}^nX,\widehat{c}(k_n,p-),\widehat{c}(k_n,p+))-F(\Delta X_{T_p},c_{T_p-},c_{T_p})\Big)\\ &-\Big(F_2'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(k_n,p-)+F_3'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(k_n,p+)\Big) \stackrel{\mathbb{P}}{\longrightarrow} 0.\end{aligned}$$

The sequences $\kappa(k_n,p\pm)^n$ are bounded in probability and $\Delta X_{T_p}\in R$ a.s., so (3.7) and Taylor's formula yield

$$\begin{aligned}&\sqrt{k_n}\,\big(F(\Delta X_{T_p},\widehat{c}(k_n,p-),\widehat{c}(k_n,p+))-F(\Delta X_{T_p},c_{T_p-},c_{T_p})\big)\\ &-F_2'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(k_n,p-)-F_3'(\Delta X_{T_p},c_{T_p-},c_{T_p})\kappa(k_n,p+) \stackrel{\mathbb{P}}{\longrightarrow} 0.\end{aligned}$$

So in fact it is enough to prove that

$$\begin{aligned}\sqrt{k_n}\,\Big(&F(\Delta_{i(n,p)}^nX,\widehat{c}(k_n,p-),\widehat{c}(k_n,p+))\\ &-F(\Delta X_{T_p},\widehat{c}(k_n,p-),\widehat{c}(k_n,p+))\Big) \stackrel{\mathbb{P}}{\longrightarrow} 0.\end{aligned} \tag{8.33}$$

Since $\Delta X_{T_p}\in R$ a.s. and the two sequences $\widehat{c}(k_n,p-)$ and $\widehat{c}(k_n,p+)$ are tight in $(0,\infty)$, the first part of (3.7) yields that (8.33) will hold if $\sqrt{k_n}\,|\Delta_{i(n,p)}^nX-\Delta X_{T_p}| \stackrel{\mathbb{P}}{\longrightarrow} 0$. Therefore (8.33) follows from the facts that $k_n\Delta_n\to 0$ and that the sequence $\frac{1}{\sqrt{\Delta_n}}\,|\Delta_{i(n,p)}^nX-\Delta X_{T_p}|$ is bounded in probability, the latter coming for example from Lemma 8.5 of [5]. This ends the proof of (8.33), hence of (8.31).

Step 3) Here we prove (i). Suppose first that $F(x,y,z)=0$ for $|x|\le\varepsilon$ for some $\varepsilon>0$, and take $m>2/\varepsilon$. As in the previous theorem we then have $U(F)=\widetilde{U}(m)$ and $\mathcal{U}=\mathcal{U}(m)$ and $\mathcal{U}'=\mathcal{U}'(m)$, whereas $U(F,k_n)_s=\widetilde{U}(m)_s$ for all $s\le t$ on a set Ω_t^n having $\mathbb{P}(\Omega_t^n)\to 1$. The result follows from (8.31).

Next we assume $r=0$. Again as in the previous proof, we argue with $m=\infty$: we have $U(F)=\widetilde{U}(\infty)$ and $\mathcal{U}=\mathcal{U}(\infty)$ and $\mathcal{U}'=\mathcal{U}'(\infty)$, whereas $U(F,k_n)_s=\widetilde{U}(\infty)_s$ for all $s\leq t$ on a set $\Omega_t'^n$ having $\mathbb{P}(\Omega_t'^n)\to 1$. Then the result follows as before.

Step 4) Now we assume $r>0$. By (3.9) and the boundedness of c_t, we have

$$\widetilde{\mathbb{E}}\big(|\mathcal{U}_t-\mathcal{U}(m)_t|^2\mid\mathcal{F}\big)\ \leq\ K\sum_{s\leq t}|\Delta X_s|^{2r}\,1_{\{|\Delta X_s|\leq 1/m\}}$$

as soon as $m\geq 1/\varepsilon$. This goes to 0 a.s. as $m\to\infty$, because of (H-r), and it follows that $\mathcal{U}(m)\ \stackrel{\text{u.c.p.}}{\longrightarrow}\mathcal{U}$ (convergence in probability, locally uniformly in time). In the same way, we have $\mathcal{U}'(m)\ \stackrel{\text{u.c.p.}}{\longrightarrow}\mathcal{U}'$. Therefore, it remains to prove that for all $t,\eta>0$,

$$\lim_{m\to\infty}\ \limsup_{n\to\infty}\ \mathbb{P}\big(\sqrt{k_n}\,|\widehat{U}^n(m)_t|>\eta\big)\ =\ 0, \tag{8.34}$$

and the same for $\widehat{U}'^n(m)$. We will prove (8.34) only. Observe that, with the simplifying notation $c_i^n=c_{i\Delta_n}$ and $c_i'^n=c_{(i-1)\Delta_n}$, we have $\widehat{U}^n(m)=\sum_{j=1}^2\overline{V}(m,j)^n+\sum_{j=1}^3 V(m,j)^n$, where $V(m,j)_t^n=\sum_{i\in J(n,m,t)}\zeta(m,j)_i^n$ and, with $J'(n,m,t)=\{i:1\leq i\leq[t/\Delta_n]\}\cap J(n,m,t)^c$,

$$\begin{array}{rcl}
\overline{V}(m,1)_t^n &=& -\sum_{i\in J'(n,m,t)}\sum_{s\in I(n,i)}F(\Delta\overline{Y}(m)_s,c_{s-},c_s)\\
\overline{V}(m,2)_t^n &=& -\sum_{0<s\leq k_n\Delta_n}F(\Delta X_s,c_{s-},c_s)\\
&& \quad -\sum_{[t/\Delta_n]-k_n)\Delta_n<s\leq t}F(\Delta X_s,c_{s-},c_s)\\
\zeta(m,1)_i^n &=& \big(F(\Delta_i^n\overline{Y}(m),\widehat{c}(k_n)_{i-k_n-1},\widehat{c}(k_n)_i)\\
&& \quad -F(\Delta_i^n\overline{Y}(m),c_i'^n,c_i^n)\big)1_{\{\Delta_i^n\overline{Y}(m)|>u_n\}}\\
\zeta(m,2)_i^n &=& F(\Delta_i^n\overline{Y}(m),c_i'^n,c_i^n)\,1_{\{\Delta_i^n\overline{Y}(m)|>u_n\}}\\
&& \quad -\sum_{s\in I(n,i)}F(\Delta\overline{Y}(m)_s,c_i'^n,c_i^n)\\
\zeta(m,3)_i^n &=& \sum_{s\in I(n,i)}\big(F(\Delta\overline{Y}(m)_s,c_i'^n,c_i^n)-F(\Delta\overline{Y}(m)_s,c_{s-},c_s)\big).
\end{array}$$

In view of (8.4) we are thus left to prove the existence of sets $\Omega(n,m,t,j)$ and $\overline{\Omega}(n,m,t,j)$ satisfying for all $m\geq 2/\varepsilon$

$$\lim_{n\to\infty}\ \mathbb{P}(\Omega(n,m,t,j))\ =\ 1,\qquad \lim_{n\to\infty}\ \mathbb{P}(\overline{\Omega}(n,m,t,j))\ =\ 1, \tag{8.35}$$

such that, for $j=1,2$ and $j=1,2,3$ respectively,

$$\lim_{m\to\infty}\ \limsup_{n\to\infty}\ \sqrt{k_n}\ \mathbb{E}\big(1_{\overline{\Omega}(n,m,t,j)}|\overline{V}(m,j)_t^n|\big)\ =\ 0 \tag{8.36}$$

$$\lim_{m\to\infty}\ \limsup_{n\to\infty}\ \sqrt{k_n}\ \mathbb{E}\Big(1_{\Omega(n,m,t,j)}\sum_{i=1}^{[t/\Delta_n]}|\zeta(m,j)_i^n|\Big)\ =\ 0. \tag{8.37}$$

Step 5) In this step we prove (8.36). In view of the second part of (3.7) and of $F(0,y,z)=0$ and (8.3) we have when $m>1/\varepsilon$:

$$\sum_{s\in I(n,i)}|F(\Delta\overline{Y}(m)_s,c_{s-},c_s)|\ \leq\ a(n,i)\ =\ K\sum_{s\in I(n,i)}|\Delta\overline{Y}(m)_s|^p.$$

Moreover, we have the following estimate, for all i possibly random but $\mathcal{F}_0^{(m)}$-measurable:

$$\mathbb{E}(a(n,i)\mid \mathcal{F}_0^{(m)}) \;\leq\; K\Delta_n\int_{A_m}\gamma(z)^p\,\lambda(dz)\;\leq\; K\Delta_n\gamma_m. \tag{8.38}$$

Since $k_n\Delta_n\to 0$ the set $\overline{\Omega}(n,m,t,2)=\{D_m\cap[0,k_n\Delta_n]=\emptyset, D_m\cap[t-(k_n+1)\Delta_n,t]=\emptyset\}$ satisfies (8.35), and on this set we have $|\overline{V}(m,2)_t^n|\leq\sum_{i=1}^{k_n}a(n,i)+\sum_{i=[t/\Delta_n]-k_n}^{[t/\Delta_n]+1}a(n,i)$. Then (8.36) for $j=2$ readily follows from (8.38) and the property $k_n^{3/2}\Delta_n\leq K$, see (3.12).

Now we consider the case $j=1$. We have $|\overline{V}(m,1)_t^n|\leq\sum_{i\in J'(n,m,t)}^{k_n}a(n,i)$. The successive integers in $J'(n,m,t)$ are $\mathcal{F}_0^{(m)}$-measurable, and the number of them is a Poisson variable independent of the $a(n,i)$'s and with some parameter $\alpha(m,t)$ (exploding with m). Then $\mathbb{E}(|\overline{V}(m,1)_t^n|)\leq K\alpha(m,t)\Delta_n$, and (8.36) for $j=1$ holds with $\overline{\Omega}(n,m,t,1)=\Omega$.

Step 6) In this step we prove (8.37) for $j=1$. The sets

$$\Omega(n,m,t,1) \;=\; \cap_{i\leq[t/\Delta_n]}\{|\Delta_i^n\overline{Y}(m)|\leq 2/m,\;|\Delta_i^nX'(m)|\leq u_n/2\} \tag{8.39}$$

satisfy the first part of (8.35) because $|\Delta\overline{Y}(m)_s|\leq 1/m$ and $\mathbb{P}(|\Delta_i^nX'(m)|>u_n/2)\leq K_m\Delta_n^2$ (use (8.9) for this). When $m\geq 2/\varepsilon$, (3.7) yields that $|\zeta(m,1)_i^n|\leq\zeta(m,4)_i^n$ on the set $\Omega(n,m,t,1)$ and for all $i\leq[t/\Delta_n]$, where

$$\zeta(m,4)_i^n \;=\; K|\Delta_i^nX''(m)|^r\big(|\widehat{c}(k_n)_{i-k_n-1}-c_i'^n|+|\widehat{c}(k_n)_i)-c_i^n|\big).$$

Then it remains to prove that (8.37) holds for $j=4$ and $\Omega(n,m,t,4)=\Omega$.

Apply (8.17) with $m=0$ and $S=(i-1)\Delta_n$ or $S=i\Delta_n$ (so $\Omega(0,n,S,i)_\pm=\Omega$) to get

$$\mathbb{E}(|\widehat{c}(k_n)_{i-k_n-1}-c_i'^n|)\;\leq\;\frac{K}{\sqrt{k_n}},\qquad \mathbb{E}(|\widehat{c}(k_n)_i-c_i^n|\mid\mathcal{F}_{i\Delta_n})\;\leq\;\frac{K}{\sqrt{k_n}}. \tag{8.40}$$

Moreover (8.9) gives $\mathbb{E}(|\Delta_i^nX''(m)|^r\mid\mathcal{F}_{(i-1)\Delta_n})\leq K\Delta_n\gamma_m$. Then by successive conditioning we obtain $\mathbb{E}(\zeta(m,4)_i^n)\leq K\Delta_n\gamma_m/\sqrt{k_n}$. Since $\gamma_m\to 0$ as $m\to\infty$ we deduce (8.37).

Step 7) Now we prove (8.37) for $j=3$, with $\Omega(n,m,t,3)=\Omega$. We suppose that $m\geq 1/\varepsilon$, so $|\Delta\overline{Y}(m)_s|\leq\varepsilon$ and (3.7) yields that $|\zeta(m,3)_i^n|\leq K(\zeta(m,5)_i^n+\zeta(m,6)_i^n)$, where

$$\begin{aligned}\zeta(m,5)_i^n &= \textstyle\sum_{s\in I(n,i)}|\Delta\overline{Y}(m)_s|^r\,|c_{s-}-c_i'^n|\\ \zeta(m,6)_i^n &= \textstyle\sum_{s\in I(n,i)}|\Delta\overline{Y}(m)_s|^r\,|c_i^n-c_s|.\end{aligned}$$

So it is enough to prove (8.37) for $j=5,6$. The case $j=5$ is simple: the process $|c_{s-}-c_i'^n|1_{s>(i-1)\Delta_n}$ is predictable, hence

$$\begin{aligned}\mathbb{E}(\zeta(m,5)_i^n) &= \mathbb{E}\Big(\int_{I(n,i)}\int_{A_m}|c_{s-}-c_i'^n|\,|\delta(s,z)|^r\,\mu(ds,dz)\Big)\\ &= \mathbb{E}\Big(\int_{I(n,i)}ds\int_{A_m}|c_{s-}-c_i'^n|\,|\delta(s,z)|^r\,\lambda(dz)\Big)\\ &\leq \gamma_m\int_{I(n,i)}\mathbb{E}(|c_{s-}-c_i'^n|)ds\;\leq\;K\Delta_n^{1+v}\gamma_m,\end{aligned}$$

where the last inequality comes from (8.8) with $m = 0$ and $R = (i-1)\Delta_n$. Then (8.37) for $j = 5$ follows because $\Delta_n^v \sqrt{k_n} \to 0$ by (3.12).

For $j = 6$ we use again (8.8) with $m = 0$ and $R = T_p$ below to get

$$\begin{aligned}\mathbb{E}(\zeta(m,6)_i^n) &= \sum_{p\geq 1} \mathbb{E}\big(|\Delta\overline{Y}(m)_{T_p}|^r\, |c_i^n - c_{T_p}|\, 1_{I(n,i)}(T_p)\big) \\ &\leq K\Delta_n^v \sum_{p\geq 1} \mathbb{E}\big(|\Delta\overline{Y}(m)_{T_p}|^r\, 1_{I(n,i)}(T_p)\big) \\ &\leq K\Delta_n^v \mathbb{E}\Big(\sum_{s\in I(n,i)} |\Delta\overline{Y}(m)_s|^r\Big) \;\leq\; K\Delta_n^{1+v}\gamma_m,\end{aligned}$$

and we conclude as above.

Step 8) Now we start proving (8.37) for $j = 2$. Set

$$\begin{aligned}\zeta(m,7)_i^n &= F(\Delta_i^n\overline{Y}(m), c_i'^n, c_i^n)\, 1_{\{\Delta_i^n\overline{Y}(m)|>u_n\}} \\ &\quad - \textstyle\sum_{s\in I(n,i)} F(\Delta\overline{Y}(m)_s, c_i'^n, c_i^n) 1_{\{\Delta\overline{Y}(m)_s|>u_n\}}.\end{aligned}$$

If $m \geq 1/\varepsilon$, we deduce from (3.7) and the boundedness of c_t that

$$|\zeta(m,2)_i^n - \zeta(m,7)_i^n| \;\leq\; K \sum_{s\in I(n,i)} |\Delta\overline{Y}(m)_s|^p\, 1_{\{|\Delta\overline{Y}(m)_s|\leq u_n\}}.$$

Therefore

$$\mathbb{E}(|\zeta(m,2)_i^n - \zeta(m,7)_i^n|) \;\leq\; K\Delta_n \int_{\{z:\gamma(z)\leq u_n\}} \gamma(z)^p \lambda(dz) \;\leq\; K\Delta_n^{1+\varpi(p-r)}\gamma_m.$$

Taking (3.12) into consideration, we deduce that

$$\lim_{n\to\infty} \sqrt{k_n}\; \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} |\zeta(m,2)_i^n - \zeta(m,7)_i^n|\Big) \;=\; 0,$$

and thus we are left to prove (8.37) for $j = 7$.

Step 9) In this auxiliary step we fix $m > 2/\varepsilon$, and also some $l \in (1, 1/2r\varpi)$ (this is possible by (3.12)). We write $q_n = [(u_n)^{-l}]$ and we suppose that n is big enough for having $1/q_n < u_n < 1/m$. We complement the notation (8.5) with

$$\begin{aligned}&A_n' \;=\; A_m \cap (A_{q_n})^c, \qquad Y_t^n = \textstyle\int_0^t \int_{A_n'} \delta(s,z)\mu(ds,dz) \\ &b_t^n \;=\; \begin{cases} -\int_{A_n'} \delta(t,z)\lambda(dz) & \text{if } r > 1 \\ 0 & \text{if } r \leq 1 \end{cases} \qquad B_t^n = \textstyle\int_0^t b_s^n\, ds \\ &\overline{Y}^n = \overline{Y}(m) - Y^n \;=\; X'(q_n) + X''(q_n) + B^n \\ &N_t^n = \mu([0,t]\times A_n'), \quad H(n,i) = \big\{|\Delta_i^n\overline{Y}^n| \leq \tfrac{u_n}{2}\big\} \cap \big\{\Delta_i^n N^n \leq 1\big\}.\end{aligned} \tag{8.41}$$

First, N^n is a Poisson process with parameter $\lambda(A_n') \leq K\gamma_m q_n^r$, hence

$$\mathbb{P}(\Delta_i^n N^n \geq 2 \mid \mathcal{F}_{(i-1)\Delta_n}^{(m)}) \;\leq\; K\Delta_n^{2-2rl\varpi}\gamma_m. \tag{8.42}$$

Second, upon observing that $\Delta_n q_n^r \leq K$ (because $rl\varpi \leq 1$) and $|b_t^n| \leq q_n^{r-1}\gamma_m$ when $r>1$ and $b_t^n = 0$ if $r \leq 1$, that

$$\iota \geq r \quad \Rightarrow \quad \mathbb{E}(|\Delta_i^n \overline{Y}^n|^\iota \mid \mathcal{F}^{(q_n)}_{(i-1)\Delta_n}) \;\leq\; K_\iota\big(\Delta_n^{\iota/2} + \Delta_n^{1+l\varpi(\iota-r)}\gamma_m\big). \tag{8.43}$$

This applied with $\iota = \frac{4}{1-2\varpi} \vee \frac{1+lr\varpi}{\varpi(l-1)}$ and Markov's inequality yield

$$\mathbb{P}(|\Delta_i^n \overline{Y}^n| > u_n/2) \;\leq\; K\Delta_n^2. \tag{8.44}$$

Next, on the set $H(n,i)$, we have $|\Delta_i^n \overline{Y}^n| \leq u_n/2$ and $|\Delta_i^n Y^n| \leq 1/m$, and also $|\Delta \overline{Y}(m)_s| \leq u_n$ for all $s \in I(n,i)$, except when $\Delta_i^n N^n = 1$ for a single value of s for which $\Delta \overline{Y}(m)_s = \Delta_i^n Y^n$ (whose absolute value may be smaller or greater than u_n). In other words, on $H(n,i)$ we have

$$\begin{aligned}\zeta(m,7)_i^n \;&=\; \Big(F(\Delta_i^n Y^n + \Delta_i^n \overline{Y}^n, c_i'^n, c_i^n)\, 1_{\{|\Delta_i^n Y^n + \Delta_i^n \overline{Y}^n| > u_n\}} \\ &\quad -F(\Delta_i^n Y^n, c_i'^n, c_i^n)\, 1_{\{|\Delta_i^n Y^n| > u_n\}}\Big)\, 1_{\{|\Delta_i^n Y^n| \leq 1/m,\, |\Delta_i^n \overline{Y}^n| \leq u_n/2\}}.\end{aligned}$$

The following estimate, when $u \in (0,1/m)$ and $y,z \in (0,M]$ for some M (this will be the bound of the process c_t) and $x,x' \in \mathbb{R}$ with $|x| \leq 1/m$ and $|x'| \leq u/2$, is easy to prove, upon using (3.7):

$$\big|F(x+x',y,z)1_{\{|x+x'|>u\}} - F(x,y,z)1_{\{|x|>u\}}\big| \leq K\big(|x|^{p-1}\,|x'| + (|x| \wedge u)^p\big).$$

Therefore, on the set $H(n,i)$ again we have

$$|\zeta(m,7)_i^n| \;\leq\; K\big(|\Delta_i^n Y^n|^{p-1}|\Delta_i^n \overline{Y}^n| + (|\Delta_i^n Y^n| \wedge u_n)^p\big). \tag{8.45}$$

The process Y^n satisfies the same estimate than $X''(m)$ in (8.10), hence since $p \geq r$:

$$\mathbb{E}\big((|\Delta_i^n Y^n| \wedge u_n)^p \mid \mathcal{F}^{(m)}_{(i-1)\Delta_n}\big) \;\leq\; K\Delta_n u_n^{p-r}\gamma_m \;\leq\; K\Delta_n^{1+(p-r)\varpi}\gamma_m. \tag{8.46}$$

On the other hand, we can apply (8.43) with $\iota = 2$ and Cauchy-Schwarz inequality to obtain $\mathbb{E}\big(|\Delta_i^n \overline{Y}^n| \mid \mathcal{F}^{(q_n)}_{(i-1)\Delta_n}\big) \leq K\sqrt{\Delta_n}$. We also have $|\Delta_i^n Y^n| \leq \Delta_i^n G(A_n')$ (see before (8.12) for this notation), and $\Delta_i^n G(A_n')$ is $\mathcal{F}_0^{(q_n)}$-measurable. Therefore, in view of (8.12) applied with the power $(p-1) \vee r$ and Hölder's inequality, and upon applying $(r \vee 1)(1-(r-1)^+ l\varpi) \geq 1$, and with the notation $q = 1 \wedge \frac{p-1}{r}$, we see that

$$\begin{aligned}\mathbb{E}\big(|\Delta_i^n Y^n|^{p-1}|\Delta_i^n \overline{Y}^n|\big) \;&=\; \mathbb{E}\Big(|\Delta_i^n Y^n|^{p-1}\,\mathbb{E}\big(|\Delta_i^n \overline{Y}^n| \mid \mathcal{F}^{(q_n)}_{(i-1)\Delta_n}\big)\Big) \\ &\leq\; K\sqrt{\Delta_n}\,\mathbb{E}\big(|\Delta_i^n Y^n|^{p-1}\big) \;\leq\; K\Delta_n^{1/2+q}\,\gamma_m^q.\end{aligned}$$

Hence by (8.45) and (8.46), we deduce

$$\mathbb{E}\big(|\zeta(m,7)_i^n|\, 1_{H(n,i)}\big) \;\leq\; K\gamma_m^q\,\big(\Delta_n^{1+(p-r)\varpi} + \Delta_n^{1/2+q}\big). \tag{8.47}$$

Step 10) Now we are ready to prove the result for $j = 7$. We take $\Omega(n,m,t,7) = \cap_{1\leq i\leq [t/\Delta_n]} H(n,i)$, which by (8.42) and (8.44) satisfies

$$\mathbb{P}(\Omega(n,m,t,7)^c) \;\leq\; \sum_{i=1}^{[t/\Delta_n]} \mathbb{P}(H(n,i)^c) \;\leq\; Kt\Delta_n^{1-2rl\varpi},$$

hence (8.35) because $2rl\varpi < 1$. Finally,

$$\mathbb{E}\big(1_{\Omega(n,m,t,7)} \sum_{i=1}^{[t/\Delta_n]} |\zeta(m,7)_i^n|\big) \leq \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}(|\zeta(m,7)_i^n|\, 1_{H(n,i)}),$$

so (8.47) shows that (8.35) holds, provided the sequences $\Delta_n^{(p-r)\varpi}\sqrt{k_n}$ and $\Delta_n^{q-1/2}\sqrt{k_n}$ are bounded. These amount to having $2(p-r) \geq \rho$ and $2q-1 \geq \rho$, which are implied by (3.12).

8.6 Proof of Theorem 3.3

Step 1) We assume (H-r) and (K-v) and (8.3). Recalling (2.1) and (8.2), we set $\overline{\delta}(t,z) = \delta(t,z)\, 1_{\{\delta(t,z)\notin A\}\cup\{\widehat{\delta}(t,z)=0\}}$, and define $\overline{X}$ by (2.1) with $\overline{\delta}$ instead of δ. This process satisfies (H-r) as well, and coincides with X on the interval $[0,t]$, in restriction to the set Ω_t^A. Hence the variables $U(F,k_n)_t$ and $U(F)_t$ and $\overline{\mathcal{U}}_t$ and $\overline{\mathcal{U}}'_t$ are the same on Ω_t^A, whether computed using X or $\overline{X}$. So it is enough to prove the result for the process $\overline{X}$. Or, in other words, we can assume throughout that

$$\Delta X_s \in A\backslash\{0\} \quad\Rightarrow\quad \Delta\sigma_s = 0 \quad \text{identically.} \tag{8.48}$$

We use the same arguments as in the previous proof, and the same notation, except that the variable $\overline{\zeta}_p^n$ of (8.30) should be replaced by

$$\begin{aligned}\overline{\zeta}_p^n = \tfrac{1}{2}\Big(& F''_{22}(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\kappa(k_n,p-)^2 + F''_{33}(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\kappa(k_n,p+)^2 \\ &+2F''_{23}(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\kappa(k_n,p+)\kappa(k_n,p-)\Big)\, 1_{\{\Delta X_{T_p}\neq 0\}}\end{aligned}$$

and the same for $\overline{\zeta}_p^{\prime n}$ with wk_n instead of k_n.

Step 2) In this step we prove that

$$\big(k_n\widetilde{U}^n(m), k_n\widetilde{U}'^n(m)\big) \stackrel{\mathcal{L}-s}{\Longrightarrow} \big(\overline{\mathcal{U}}(m), \overline{\mathcal{U}}'(m)\big), \tag{8.49}$$

where $\overline{\mathcal{U}}(m)$ and $\overline{\mathcal{U}}'(m)$ are as in (3.15), except that the sum is taken over the $p \in \mathcal{T}_m$ only. By Proposition 8.2, we have

$$\Big(\sum_{p\in\mathcal{T}_m} \overline{\zeta}_p^n\, 1_{\{T_p\leq t\}}, \sum_{p\in\mathcal{T}_m} \overline{\zeta}_p^{\prime n}\, 1_{\{T_p\leq t\}}\Big) \stackrel{\mathcal{L}-s}{\Longrightarrow} \big(\overline{\mathcal{U}}(m)_t, w\overline{\mathcal{U}}'(m)_t\big),$$

so proving (8.31) amounts to show that for each $p \in \mathcal{T}_m$ and on each set $\{T_p \leq t\}$ we have

$$k_n\,\zeta_p^n - \overline{\zeta}_p^n \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad wk_n\,\zeta_p^{\prime n} - \overline{\zeta}_p^{\prime n} \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{8.50}$$

We prove only the first property, which (like in Theorem 3.2; note that here $F(\Delta X_{T_p}, c_{T_p-}, c_{T_p}) = 0$ by (3.14) and (8.48)) amounts to showing convergence of

$$k_n F(\Delta^n_{i(n,p)} X, \widehat{c}(k_n,p-), \widehat{c}(k_n,p+)) - \tfrac{1}{2}\Big(F''_{22}(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\kappa(k_n,p-)^2 + 2F''_{23}(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\kappa(k_n,p+)\kappa(k_n,p-) + F''_{33}(\Delta X_{T_p}, c_{T_p-}, c_{T_p})\kappa(k_n,p+)^2\Big)$$

to 0 in probability. Upon using again (3.14) and (8.48), we deduce from Taylor's formula and the tightness of the sequences $\kappa(k_n,p\pm)$ that, on the set $\{\Delta X_{T_p} \in R\}$ which has probability 1, the variables

$$k_n F(\Delta X_{T_p}, \widehat{c}(k_n,p-), \widehat{c}(k_n,p+)) - \tfrac{1}{2}\Big(F''_{22}(\Delta X_{T_p}, c_{T_p}, c_{T_p})\kappa(k_n,p-)^2 + 2F''_{23}(\Delta X_{T_p}, c_{T_p}, c_{T_p})\kappa(k_n,p-)\kappa(k_n,p+) + F''_{33}(\Delta X_{T_p}, c_{T_p}, c_{T_p})\kappa(k_n,p+)^2\Big)$$

go to 0 in probability. Hence the first part of (8.50) will follow if we show

$$k_n\Big(F(\Delta^n_{i(n,p)} X, \widehat{c}(k_n,p-), \widehat{c}(k_n,p+)) - F(\Delta X_{T_p}, \widehat{c}(k_n,p-), \widehat{c}(k_n,p+))\Big) \stackrel{\mathbb{P}}{\longrightarrow} 0.$$

This is proved exactly as (8.33), except that here we use the property $k_n\sqrt{\Delta_n} \to 0$.

Step 3) The proof of (i) follows from (8.49) in exactly the same way as in Step 3 of the proof of Theorem 3.2.

Step 4) Now we start proving (ii), so $r > 0$. We can suppose that A contains a neighborhood of 0, otherwise we are in the second situation of case (i). Hence we may take $\varepsilon > 0$ in (3.13) such that also $[-\varepsilon, \varepsilon] \subset A$. Similar to (3.16), and by the boundedness of c_t and (3.13), we have if $m \geq 1/\varepsilon$:

$$\widetilde{\mathbb{E}}\big(|\overline{\mathcal{U}}_t - \overline{\mathcal{U}}(m)_t| \mid \mathcal{F}\big) \;\leq\; K \sum_{s\leq t} |\Delta X_s|^r \, 1_{\{|\Delta X_s| \leq 1/m\}}$$

This goes to 0 a.s. as $m \to \infty$, because of (H-r), so $\overline{\mathcal{U}}(m) \stackrel{\text{u.c.p.}}{\longrightarrow} \overline{\mathcal{U}}$, and also $\overline{\mathcal{U}}'(m) \stackrel{\text{u.c.p.}}{\longrightarrow} \overline{\mathcal{U}}'$. Then it remains to prove that for all $t, \eta > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(k_n\, |\widehat{U}^n(m)_t| > \eta\big) \;=\; 0. \tag{8.51}$$

and the same for $\widehat{U}'^n(m)$. We will prove (8.51) only.

Because of our assumptions we have here $\widehat{U}^n(m) = \overline{U}^n(m)$. Then, in view of the definition (8.25), and since the sets $\Omega(n,m,t,1)$ of (8.39) satisfy (8.35), it is enough to prove that

$$\lim_{m\to\infty} \limsup_{n\to\infty} k_n\, \mathbb{E}\Big(1_{\Omega(n,m,t,1)} \sum_{i=1}^{[t/\Delta_n]} |\zeta(m,1)^n_i|\Big) \;=\; 0, \tag{8.52}$$

where

$$\zeta(m,1)^n_i = F(\Delta^n_i \overline{Y}(m), \widehat{c}(k_n)_{i-k_n-1}, \widehat{c}(k_n)_i) 1_{\{|\Delta^n_i \overline{Y}(m)| > u_n\}}.$$

On $\Omega(n,m,t,1)$, when $m > 2/\varepsilon$, for all $i \leq [t/\Delta_n]$ we have $|\Delta^n_i \overline{Y}(m)| \leq \varepsilon$ and also $|\Delta^n_i \overline{Y}(m)| \leq 2|\Delta^n_i X''(m)|$ when further $|\Delta^n_i \overline{Y}(m)| > u_n$. Then, using (3.13) and a Taylor expansion around $(\Delta_i^n \overline{Y}(m), c_{i\Delta_n}, c_{i\Delta_n})$ and since $F(x,y,y) = F_2'(x,y,y) = F_3'(x,y,y) = 0$ for all x, y, we see that

$$|\zeta(m,1)_i^n| \leq K\big(\zeta(m,2)_i^n + \zeta(m,3)_i^n\big) \quad \text{on } \Omega(n,m,t,1) \text{ and for } i \leq [t/\Delta_n],$$

where

$$\begin{aligned} \zeta(m,2)_i^n &= |\Delta_i^n X''(m)|^r \big(|\widehat{c}(k_n)_{i-k_n-1} - c_{(i-1)\Delta_n}|^2 + |\widehat{c}(k_n)_i - c_{i\Delta_n}|^2\big) \\ \zeta(m,3)_i^n &= |\Delta_i^n X''(m)|^r \, |\Delta_i^n c|^2. \end{aligned}$$

Hence we are left to prove that, for $j = 2, 3$, we have

$$\lim_{m\to\infty} \ \limsup_n \ k_n \,\mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} \zeta(m,j)_i^n\Big) \ = \ 0 \tag{8.53}$$

Step 5) On the one hand, successive conditioning, plus the third estimate in (8.9) with $p = r$, plus (8.17) with $m = 0$ and $q = 2$, yield $\mathbb{E}(\zeta(m,2)_i^n) \leq K\Delta_n \gamma_m / k_n$. Then (8.53) for $j = 2$ follows. For $j = 3$ we will prove the stronger statement, for m large enough:

$$\lim_n \ k_n \,\mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} \zeta(m,j)_i^n\Big) \ = \ 0. \tag{8.54}$$

Therefore, we fix $m \geq 2/\varepsilon$ below.

First, suppose that $r \leq 1$. Then $X''(m)_t = \sum_{s\leq t} \Delta X''(m)_s$, and since $|x + x'|^r \leq |x|^r + |x'|^r$ and $c_s = c_{s-}$ when $\Delta X(m)_s'' \neq 0$ (recall $m \geq 2/\varepsilon$ and (8.48)), we have $\zeta(m,3)_i^n \leq \zeta(m,4)_i^n$, where

$$\zeta(m,4)_i^n = \sum_{s\in I(n,i)} |\Delta \overline{Y}(m)_s|^r \, |c_{s-} - c_{(i-1)\Delta_n}|^2 + \sum_{s\in I(n,i)} |\Delta \overline{Y}(m)_s|^r \, |c_{i\Delta_n} - c_s|^2.$$

Then exactly as in Step 7 of Theorem 3.2, and upon using (8.8) with $p = 2$ instead of $p = 1$, we obtain $\mathbb{E}(\zeta(m,3)_i^n) \leq K\Delta_n^{1+(2v)\wedge 1}$. Then (8.54) holds for $j = 4$, hence for $j = 3$, by (3.18).

It remains to consider the case $r > 1$. We take $l = 1/r\varpi$, and we use the notation $q_n = [(u_n)^{-l}]$ and (8.41), which we complement as follows:

$$Z(5)^n \ = \ B^n, \quad Z(6)^n \ = \ X''(q_n), \quad Z(7)^n \ = \ Y^n,$$

so $X''(m) = \sum_{j=5}^7 Z(j)^n$, and we associate the variables

$$\zeta(m,j)_i^n \ = \ |\Delta_i^n Z(j)|^r \, |\Delta_i^n c|^2.$$

It is thus enough to prove (8.54) when $j = 5, 6, 7$. First, we have $|\Delta_i^n Z(5)^n| \leq K\Delta_n^{1-l\varpi(r-1)} \gamma_m$ and thus by (8.8) we get $\mathbb{E}(\zeta(m,5)_i^n) \leq K\Delta_n^{r-(r-1)rl\varpi+(2v)\wedge 1}$, which equals $K\Delta_n^{1+(2v)\wedge 1}$, and (8.54) for $j = 5$ holds, by (3.18). Next, (8.9) applied with q_n instead of m implies that for any $p \geq 2$ we have $\mathbb{E}(|\Delta_i^n Z(6)|^p) \leq K_p \Delta_n^{p/r}$ (use again $rl\varpi = 1$). Then by (8.8) and Hölder's inequality we see that $\mathbb{E}(\zeta(m,6)_i^n) \leq K_\theta \Delta_n^{1+(2v)\wedge\theta}$ for any $\theta \in (1/2,1)$. Then again, upon taking θ close to 1, we have (8.54) for $j=6$.

Finally, we set $Y(n,i)_t = \sum_{(i-1)\Delta_n < s \leq t} |\Delta Y_s^n|$ for $t \in I(n,i)$. Observe that

$$\begin{aligned} |\Delta_i^n Z(7)^n|^r &\leq Y(n,i)_{\Delta_n}^r = \sum_{s\in I(n,i)} \big((Y(n,i)_{s-} + |\Delta Y_s^n|)^r - Y(n,i)_{s-}^r\big) \\ &\leq K \sum_{s\in I(n,i)} \big(|\Delta Y_s^n|^r + Y(n,i)_{s-}^{r-1} |\Delta Y_s^n|\big). \end{aligned}$$

Since $|\Delta Y^n| \leq |\Delta \overline{Y}(m)|$, it follows that $\zeta(m,7)_i^n \leq K(\zeta(m,4)_i^n + \zeta(m,8)_i^n + \zeta(m,9)_i^n)$, where $\zeta(m,4)_i^n$ is as in the case $r \leq 1$ and

$$\begin{aligned} \zeta(m,8)_i^n &= \textstyle\sum_{s\in I(n,i)} Y(n,i)_{s-}^{r-1}\, |\Delta Y_s^n|\, |c_{s-} - c_{(i-1)\Delta_n}|^2 \\ \zeta(m,9)_i^n &= \textstyle\sum_{s\in I(n,i)} Y(n,i)_{s-}^{r-1}\, |\Delta Y_s^n|\, |c_{i\Delta_n} - c_s|^2. \end{aligned}$$

We have seen that (8.54) is satisfied for $j=4$ (this is irrespective of the value of r). For proving it for $j=8$ and $j=9$ we use the same argument as in Step 7 of Theorem 3.2 again, thus getting:

$$\begin{aligned} \mathbb{E}(\zeta(m,8)_i^n) &\leq \gamma_m \textstyle\int_{I(n,i)} \mathbb{E}\big(Y(n,i)_{s-}^{r-1}\, |c_{s-} - c_{(i-1)\Delta_n}|^2\big)\, ds \\ \mathbb{E}(\zeta(m,9)_i^n) &\leq K\Delta_n^{(2v)\wedge 1}\, \mathbb{E}\Big(\textstyle\sum_{s\in I(n,i)} Y(n,i)_{s-}^{r-1}\, |\Delta Y_s^n|\Big) \\ &\leq K\Delta_n^{(2v)\wedge 1}\, \mathbb{E}\Big(\sup_{s\leq i\Delta_n} (Y(n,i)_s)^r\Big). \end{aligned}$$

Note that $Y(n,i)$ has the same structure as $X''(q_n)$ does in case $r \leq 1$, so although $r > 1$ here we have, as in the first part of the third estimate in (8.9),

$$p \geq r \quad \Rightarrow \quad \mathbb{E}\Big(\sup_{s\leq i\Delta_n} (Y(n,i)_s)^p\Big) \leq K_p\big(\Delta_n^{1+(p-r)l\varpi} + \Delta_n^{p+r(p-1)l\varpi}\big) \leq K_p \Delta_n^{p/r}.$$

Applying (8.8) and Hölder's inequality yields $\mathbb{E}(\zeta(m,8)_i^n) \leq K_\theta \Delta_n^{r+(2v)\wedge\theta}$ for any $\theta \in (1/2,1)$, whereas obviously $\mathbb{E}(\zeta(m,9)_i^n) \leq K_\theta \Delta_n^{1+2v}$. Then (8.54) holds for $j=8$ and $j=9$.

8.7 Proof of the results on the tests

Proof of Theorem 4.1. Theorems 3.1 and 3.3 yield that, in restriction to $\Omega_T^{(A,d)}$, the variables $k_n U(F,k_n)_T / U(G,k_n)_T$ converge stably to a positive variable $\mathcal{V}$ which, conditionally on $\mathcal{F}$, has mean 1. Hence if $H \subset \Omega_T^{(A,d)}$ and with C_n given by (4.6), we have $\limsup_n \mathbb{P}(C_n \cap H) \leq \widetilde{\mathbb{P}}(H \cap \{\mathcal{V} \geq 1/\alpha\})$, which is smaller than $\alpha\mathbb{P}(H)$ because $\widetilde{\mathbb{E}}(\mathcal{V} \mid \mathcal{F}) = 1$, and the result for the asymptotic level follows. Since $k_n U(F,k_n)_T / U(G,k_n)_T \stackrel{\mathbb{P}}{\longrightarrow} \infty$ on the set $\Omega_T^{(A,j)}$ by Theorem 3.1, the asymptotic power is clearly 1. □

Proof of Theorem 4.2. We will be very sketchy here. By localization we may assume (8.3).

First, we can suppose that the simulated variables $V_i^{\pm}(j)$ are defined on our auxiliary space $(\Omega', \mathcal{F}', \mathbb{P}')$, so that the $\overline{\mathcal{U}}(n,j)$'s are defined on the extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. Then we can reproduce the proof of Theorem 4.4 of [6] to obtain that, if $Z_n \stackrel{\mathbb{P}}{\longrightarrow} Z$ are $\mathcal{F}$-measurable variables, we have

$$\widetilde{\mathbb{P}}(\overline{\mathcal{U}}(n,1) > Z_n \mid \mathcal{F}) \stackrel{\mathbb{P}}{\longrightarrow} \widetilde{\mathbb{P}}(\overline{\mathcal{U}}_T > Z \mid \mathcal{F}). \tag{8.55}$$

The only slightly different point is that we need here $\mathbb{E}((\widehat{c}(k_n)_i)^2 \mid \mathcal{F}_{(i-1)\Delta_n}) \leq K$. This does not follow from (8.14), but it does from (8.17) applied with $q = 2$, because by hypothesis (8.15) holds.

Then, using (8.55) and that $k_n U(F,k_n)_T \stackrel{\mathcal{L}-(s)}{\longrightarrow} \overline{\mathcal{U}}_T$ on the set $\Omega_T^{(A,d)}$, we can reproduce the proof of Theorem 5.1, Part (c), of [6], and we obtain the claim about the asymptotic level. In the course of this proof it is also shown that $\mathcal{F}$-conditionally the variables $\overline{\mathcal{U}}_{([N_n\alpha])}$ converge in law to the unique variable $Z(\alpha)$ such that $\widetilde{\mathbb{P}}(\overline{\mathcal{U}}_T > Z(\alpha) \mid \mathcal{F}) = \alpha$, from which $\overline{\mathcal{U}}_{([N_n\alpha])} \stackrel{\mathbb{P}}{\longrightarrow} Z(\alpha)$ follows.

Finally $k_n U(F,k_n)_T \stackrel{\mathbb{P}}{\longrightarrow} \infty$ on $\Omega_T^{(A,j)}$. This and $\overline{\mathcal{U}}_{([N_n\alpha])} \stackrel{\mathbb{P}}{\longrightarrow} Z(\alpha)$, yields that $\widetilde{\mathbb{P}}(C_n \cap \Omega_T^{(A,j)}) \to \mathbb{P}(\Omega_T^{(A,j)}$. Hence the asymptotic power equals 1. □

Proof of Theorem 4.3. The proof is the same as for Theorem 4.1, with the following changes: we now have $\mathbb{P}(C_n \cap H) \to \alpha\mathbb{P}(H)$ because $k_n U(F,k_n)_T$ converges stably in law on $\Omega_T^{(A,d)}$ to a chi-square variable with N_T degrees of freedom, independent of $\mathcal{F}$, and $N_T^n = N_T$ for n large enough. This gives that the asymptotic level is α, and for the asymptotic power we use the fact that $k_n U(F,k_n)_T \stackrel{\mathbb{P}}{\longrightarrow} \infty$ and $N_T < \infty$ on the set $\Omega_T^{(A,j)}$. □

Proof of Theorem 4.4. The result readily follows from the stable convergence in law of $(S_n - 1)/\sqrt{V_n}$ to a standard normal. □

Proof of Theorem 4.5. Since $V_n' = V_n$ for all n large enough, on the set $\Omega_T^{(A,j)}$, only the claim about the power needs a proof. Now, $V_n' \to 0$, and we have the second part of (4.14) on $\Omega_T^{(A,d)}$: that the asymptotic power equals 1 is now obvious. □

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