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Financial Leverage and the Cost of Equity Capital: Teaching Note

0. Introduction

The purpose of this teaching note is to amplify on the textbook coverage of the effect of financial leverage on the cost of equity (the required return on equity from the perspective of the stockholders) --- it is assumed that the reader has read Chapters 17 & 18 of Brealey and Myers. Most texts cover the relationship between r_e, the cost of equity, and D/E, the debt to equity ratio, in the context of the MM model with no taxes, or sometimes with corporate taxes only. Then the texts go on to stress the importance of personal taxes when discussing capital structure, and discuss Miller's (1977) model showing that the MM irrelevance result (that the value of the firm is independent of the capital structure) obtains in a world with corporate and personal taxes (but no other market imperfections) as long as T*=0.

The conclusion to be drawn for financial managers is that, when formulating capital structure policy or even when estimating costs of capital for new investments, they should not ignore the personal tax implications of debt usage. But the textbooks generally stop short of providing the student with the tools to help them estimate costs of equity and debt in a corporate-and-personal tax environment.

The rest of this teaching note is organized as follows: Section 1 contains a summary of the notation used throughout the note is provided; Section 2 reviews some basic formulas assumed as background knowledge; in Section 3, the relation between debt and equity required returns in the presence of personal taxes is discussed and the capital asset pricing model is recast in a framework with personal taxes on debt and equity securities; Section 4 derives the required return on equity of a levered firm. The strength of the framework in Section 4 is that it is not dependent on any particular model of market equilibrium in securities markets, such as CAPM or APT. It derives the relation between the levered required return on equity and the unlevered required return on equity, the debt/equity ratio and the required return on debt. As a practical matter, however, if you are working within the CAPM for your cost of equity, the model derived in Section 4 can be difficult to implement. Section 5 derives the levered equity beta from the unlevered beta, the debt equity ratio and the beta of the debt. From this beta, you can calculate the levered equity required rate of return in the capital asset pricing model. Section 6 summarizes.

1. Notation

The rest of this teaching note uses the following notation:

Tax parameters

T_c	marginal corporate tax rate
τ_{pe}	marginal tax rate on equity income
τ_{pd}	marginal tax rate on debt income
T^*	effective debt tax shield per dollar of debt

Returns

r_{eU}, β_{eU}	required return and beta on equity of unlevered firm (assets)
r_{eL}, β_{eL}	required return and beta on equity of levered firm
r_d	required return on debt
r_{fd}	risk free rate on riskless debt security
r_{fe}	risk free rate on equity security with no systematic risk
$r_{d,e}$	required return on equity security if it had the same risk as a given debt security, d

2. Background information

It will be useful to begin with a few basic formulas with which the reader is assumed familiar: first, the capital asset pricing model (CAPM) security market line:

$$\boldsymbol{r_i = r_f + \beta_i[E(r_m) - r_f]} \qquad \textbf{(1)}$$

where r_f is the current risk free rate, $E(r_M)$ is the current expected rate of return on the market portfolio, β_i is the systematic risk of investment i and r_i is the required expected rate of return on invesmtent i. (1) will be modified later to incorporate the difference in personal tax rates on debt and equity instruments. Some comments are in order here: 1. r_f and $E(r_M)$ are known constant parameters in this equation, the only thing that differs between investments that causes their required rates of return to differ is beta risk. Also, while investment i is usually thought of as an equity security, it can be equity, debt, real estate, etc. We will distinguish later between the expected return on unlevered equity and levered equity; in the context of (1) we will write r_{eU} and r_{eL}, which will be a function of β_{eU} and β_{eL}, respectively. Likewise, the required rate of return on debt is r_d and is, via (1) a function of β_d.

Some simple identities which will be used in this note include the market value balance sheet identity: V = D + E, i.e., the market value of the assets of the firm must equal the market value of the claims to the cash flows those assets generate, the value of the debt plus the value of the equity. When we discuss changes in the capital structure (may involve new issues or repurchases of debt and/or equity and net new investments) we will find it useful to think of the balance sheet in terms of "changes": letting E_o be the market value of "old" equity, E_n be the market value of "new" equity, D_o be the market value of "old" debt and D_n be the market value of "new" debt, the balance sheet constraint becomes:

$$\Delta V_L = \Delta E_o + \Delta E_n + \Delta D_o + \Delta D_n \tag{2}$$

Another important identity we will use is the cash flow identity, for which it will be useful to write a pro-forma income statement of the company:

Income Statment	Symbols
Revenues	REV
- Operating Cash Costs	-OCC
- Deprec. & oth. Non-cash chgs.	-Dep
Net operating income	NOI
- Interest expense	-Int
Profit before tax	PBT
- Taxes (@ τ_c)	- τ_c(NOI-Int)
Net Income	NI=(NOI-Int)(1-τ_c)

Assume the company's future cash flows take the form of a no-growth perpetuity and the company's capital structure contains only debt and equity. The total cash flow to shareholders equals net income *plus* depreciation and other non cash charges *less* any new investments in maintenance and growth fixed capital expenditures and working capital:

$$CFE = NI + Dep - I(growth) - I(maintenance) - \Delta WC \tag{3}$$

where CFE is cash flow to equity, Dep is depreciation and other non-cash charges, I refers to new capital investment and ΔWC is new investments in working capital.[1] If the firm is a no-growth perpetuity, then growth fixed capital expenditures and working capital increases are zero and we can make the simplifying assumption that periodic investments in maintenance (or replacement) capital expenditures maintain the capital stock of the firm --- that is, they equate with periodic depreciation. Equation (3) simplifies to CFE = NI in the no-growth case.

The cash flow identity calculates the total cash flow to both stockholders and bondholders, NI + Int in the no-growth case, and points out that it can be calculated also by adding the interest tax shield to the net income of an otherwise identical firm with no debt, i.e., NOI($1-\tau_c$) + Int τ_c:

$$NI + Int = NOI(1-\tau_c) + Int\ \tau_c \tag{4}$$

This identity is the essence of the intuition of MM with corporate taxes: total cash flows to bondholders and stockholders of a levered firm is the same as total cash flow to stockholders of an otherwise identical unlevered firm with one exception, the tax shield afforded the corporation on its interest payments.

The reader is assumed to be familiar with MM Proposition I in the presence of corporate and

[1] It should be pointed out that, in general, we would subtract out any principal repayments net of new borrowings to arrive at CFE. We are assuming the firm is in a steady state equilibrium, so the annual interest payments represent the entire annual return on the debt and the annual net income of the firm represents the anual return to the equityholders. We could not characterize the firm as a perpetuity if they were either retiring debt every year or issuing new debt every year. What we are assuming is that there are no principal repayments and no new debt issues.

personal taxes (the formula assumes a constant level of corporate debt in perpetuity):

$$\boldsymbol{V_L = V_U + T^{*}D} \qquad \textbf{(5)}$$

i.e., the value of the levered firm is equal to the value of the firm if it used no debt plus the present value of the effective tax shield offered by its debt. This latter term is calculated as T*D, where D is the market value of the debt and T* is the effective tax shield per dollar of debt, which is given by:

$$\boldsymbol{T^{*} = \left[1 - \frac{(1-\tau_c)(1-\tau_{pe})}{(1-\tau_{pd})}\right]} \qquad \textbf{(6)}$$

where τ_{pe} is the single personal tax rate on all equity income and τ_{pd} is the single personal tax rate on all debt income. When $\tau_c=\tau_{pe}=\tau_{pd}=0$, T* = 0, (5) becomes $V_L = V_U$, which is MM I with no taxes; more generally, when $\tau_{pe}=\tau_{pd}$, $T^*=\tau_c$, (5) becomes $V_L = V_U + \tau_c D$, or MM I adjusted for corporate taxes.

3. Personal Taxes and the Relation between Debt and Equity Returns

Traditional models of capital market equilibrium such as the capital asset pricing model assume there are no personal taxes on investments and, implicitly, that if there were, all investments would be taxed at the same marginal rate. Capital structure models with corporate and personal taxes are based on the observation that debt securities and equity securities have different tax characteristics: most debt income comes in the form of cash interest payments, which are immediately taxable to a taxable holder of the debt security. On the other hand, equity income may be in the form of taxable cash dividends (to a taxable individual stockholder), realized capital gains which are taxed at the same rate as dividends (under the 1986 Tax Reform Act, but at a lower rate under pre-1987 tax code and under the new tax code) and, importantly, unrealized capital gains which are not taxed until the security is sold (and may never be taxed if the security is passed to the stockholder's estate at death).

There are also important tax options available in equities which are not as common with debt securities. In particular, while realized capital gains are taxable, realized capital losses are deductible from income; if a stockholder holds a portfolio of securities, he or she will usually have securities to sell to recognize losses to reduce the tax burden of selling "winners." While the wash sale rule precludes the stockholder from taking the loss for tax purposes if the same securities are purchased within six months of the sale, other similar securities (similar risk, size, etc.) can be purchased instead. This strategy allows for dynamic tax reduction opportunities with equities which is difficult to accomplish with debt instruments, because the bulk of the income from holding debt securities is in the form of interest payments.

The net effect is that the marginal tax rate of the marginal holder of common stocks, τ_{pe}, is believed to be somewhat smaller than the marginal tax rate of the marginal holder of corporate debt, τ_{pd}. T* is difficult to measure empirically, but estimates between 20% and 28% under the current tax code seem reasonable.

How does this affect the relative rates of return required by holders of debt securities and equity securities? Basically, we assume that all equity income is taxed at τ_{pe} and that all debt income is taxed at τ_{pd}. Consider the risk free rate typically used in practical applications of the CAPM is the rate on a debt security such as a government bill or bond, the after-personal tax rate of return on which is $(1-\tau_{pd})r_{fd}$, where r_{fd} indicates that the risk free instrument is a debt instrument. We can imagine a risk-free equity security (with no systematic risk); its return after personal taxes would be

$(1-\tau_{pe})r_{fe}$, where r_{fe} indicates that this security is an equity security. In equilibrium, the after-personal tax rates of return on these two securities should be the same:

$$\begin{gathered}(1-\tau_{pe})r_{fe} = (1-\tau_{pd})r_{fd} \\ \textit{or, equivalently,} \\ r_{fe} = r_{fd}\frac{(1-\tau_{pd})}{(1-\tau_{pe})}\end{gathered} \tag{7}$$

Similarly, if two securities (one debt and one equity) had the same systematic risks, they should earn the same after-personal tax rate of return. Let r_d be the pre-personal tax required rate of return on a risky debt security and let $r_{d,e}$ be the pre-personal tax required rate of return on an equity security with the same systematic risk as the debt security in question. $r_{d,e}$, by the same reasoning applied above, would be given by

$$r_{d,e} = \left[\frac{(1-\tau_{pd})}{(1-\tau_{pe})}\right] r_d \tag{8}$$

In the context of the capital asset pricing model, to estimate the required return on an equity security, we would use r_{fe} in place of the traditional debt-oriented risk free rate:

$$r_e = r_{fe}+\beta_e[E(r_M)-r_{fe}] \tag{9}$$

4. Personal taxes and the cost of equity capital

The following assumptions will be maintained throughout the derivation in this section:

1. the future cash flows of the company are in the form of a no-growth perpetuity;
2. the debt of the company is not necessarily riskless, it may have some systematic risk;

3. the company issues only debt and equity;
4. and, other than taxes, there are no market imperfections.

What we are interested in is the return that is required by stockholders on their investment and expressing that required rate of return as a function of the leverage of the company. If we think of a dollar of equity being invested into the company's operations, then the dollar could come from retained earnings (old equity) or from a new issue, and equals $\Delta E_o + \Delta E_n$. In the perpetuity case, shareholders receive a marginal return of ΔNI on this investment. Therefore the quantity we wish to study is $\Delta NI/(\Delta E_o+\Delta E_n)$.

The total incremental investment may have been larger (or smaller) than this amount, to the extent to which more debt was issued (or was retired); that is, the investment amount, ΔI, has to equal the net investment raised via debt and equity.

Let us begin with the cash flow identity (4). First we need to restate this to include both personal and corporate taxes. The first term on the left-hand side of (4) is the return to shareholders, which is taxed at τ_{pe}; the second term on the left-hand side of (4) is the return to bondholders, which is taxed at τ_{pd}. We can derive the personal-tax version of the right hand side of (4) as follows:

$$NI(1-\tau_{pe}) + Int(1-\tau_{pd}) = (Rev - OCC - Dep - Int)(1-\tau_c)(1-\tau_{pe}) + Int(1-\tau_{pd})$$

Separating the terms in Int, and recognizing that Rev - OCC - Dep = NOI, we have (4a)[2]

$$NI(1-\tau_{pe}) + Int(1-\tau_{pd}) = NOI(1-\tau_c)(1-\tau_{pe}) + Int[(1-\tau_{pd}) - (1-\tau_c)(1-\tau_{pe})] \qquad (4a)$$

Subtracting the last term on the right-hand side from both sides of (4a) and isolating the NOI term:

$$NOI(1-\tau_c)(1-\tau_{pe}) = NI(1-\tau_{pe}) + Int(1-\tau_{pd}) - Int[(1-\tau_{pd}) - (1-\tau_c)(1-\tau_{pe})]$$

$$= NI(1-\tau_{pe}) + Int(1-\tau_c)(1-\tau_{pe})$$

[2]Keep in mind that NOI and Int (and, hence, NI) are each random variables (i.e., the returns to the assets and the returns to debt and equity holders are all, to some extent uncertain --- risky).

Expressed in terms of *changes*, we have

$$\Delta NOI(1-\tau_c)(1-\tau_{pe}) \;=\; \Delta NI(1-\tau_{pe}) \;+\; \Delta Int(1-\tau_c)(1-\tau_{pe}) \tag{10}$$

The intuition behind (10) is that the after-personal tax returns to stockholders of the levered firm (first term on the right hand side of (10)) equal the after-personal tax returns to stockholders of an otherwise identical all-equity firm (the left hand side of (10)) minus the after-personal tax value of the additional interest paid net of tax shields *to the stockholders of the levered firm*.

Next we use (5) for the value of the levered firm, reexpressed in terms of changes: $\Delta V_L = \Delta V_U + T^*\Delta D$. The value of the unlevered firm in the perpetuity case with personal and corporate taxes would be:

$$V_U \;=\; \frac{E(NOI)(1-\tau_c)(1-\tau_{pe})}{r_{eU}(1-\tau_{pe})} \tag{11}$$

The logic of (11) is that, for the unlevered firm, net income equals net operating income after corporate tax, which is all taxed at the personal equity rate. A perpetuity of this amount would be discounted at the required rate of return on an all equity security, r_{eU}, but because it is a stream of after-personal tax cash flows, it must be discounted at the *after-personal tax* required rate of return on equity of the unlevered firm, $r_{eU}(1-\tau_{pe})$. Next, the market value of the debt in equation (5) is the present value (at the *after-personal tax* required rate of return on debt) of the perpetuity of *after-personal tax* interest payments:

$$T^*D \;=\; T^*\frac{Int(1-\tau_{pd})}{r_d(1-\tau_{pd})} \;=\; \frac{T^*Int}{r_d} \tag{12}$$

Plugging in from (11) and (12) into (5) and expressing all terms as changes:

$$\Delta V_L = \frac{\Delta E(NOI)(1-\tau_c)(1-\tau_{pe})}{r_{eU}(1-\tau_{pe})} + \frac{T^* \Delta Int}{r_d} \quad (13)$$

Substituting the right hand side of (10) in for $\Delta E(NOI)(1-\tau_c)(1-\tau_{pe})$ in (13):

$$\Delta V_L = \frac{\Delta NI(1-\tau_{pe}) + \Delta Int(1-\tau_c)(1-\tau_{pe})}{r_{eU}(1-\tau_{pe})} + \frac{T^* \Delta Int}{r_d} \quad (14)$$

We also know from (2) that $\Delta V_L = \Delta E_o + \Delta E_n + \Delta D_o + \Delta D_n$. If we assume that none of the old debt is repurchased and assume that the risk of the old debt is unchanged, then $\Delta D_o = 0$ and we can simply refer to ΔD_n as ΔD, which is the additional new debt. Then we have

$$\Delta V_L = (\Delta E_o + \Delta E_n) + \Delta D \quad (15)$$

Equating (14) and (15) yields:

$$(\Delta E_o + \Delta E_n) + \Delta D = \frac{\Delta NI(1-\tau_{pe}) + \Delta Int(1-\tau_c)(1-\tau_{pe})}{r_{eU}(1-\tau_{pe})} + \frac{T^* \Delta Int}{r_d} \quad (16)$$

Cancelling the $(1-\tau_{pe})$ terms in the first term on the right-hand side of (16), noting that $\Delta Int = r_d \Delta D$, multiplying the second fraction on the right-hand side of (16) by r_{eU}/r_{eU}, then combining terms on the right-hand side of (16), we get:

$$(\Delta E_o + \Delta E_n) + \Delta D = \frac{\Delta NI + r_d \Delta D(1-\tau_c) + r_{eU} T^* \Delta D}{r_{eU}} \quad (17)$$

Subtracting ΔD from each side of (17) and combining it in the fraction on the right hand side of (17) yields:

$$\Delta E_o + \Delta E_n = \frac{\Delta NI + r_d \Delta D(1-\tau_c) + r_{eU} T^* \Delta D - r_{eU} \Delta D}{r_{eU}} \tag{18}$$

Multiply across by r_{eU}:

$$r_{eU}(\Delta E_o + \Delta E_n) = \Delta NI + r_d \Delta D(1-\tau_c) + r_{eU} T^* \Delta D - r_{eU} \Delta D \tag{19}$$

Isolating ΔNI on the left hand side yields:

$$\Delta NI = r_{eU}(\Delta E_o + \Delta E_n) + r_{eU} \Delta D - r_{eU} T^* \Delta D - r_d \Delta D(1-\tau_c) \tag{20}$$

Dividing all terms in (20) by $\Delta E_o + \Delta E_n$ and consolidating terms in $r_{eu}\Delta D$, we have

$$\frac{\Delta NI}{\Delta E_o + \Delta E_n} = r_{eU} + r_{eU}(1-T^*)\left(\frac{\Delta D}{\Delta E_o + \Delta E_n}\right) - r_d(1-\tau_c)\left(\frac{\Delta D}{\Delta E_o + \Delta E_n}\right) \tag{21}$$

Notice that the left-hand side of (21) is our definition of r_{EL}. Assuming the company follows a target capital structure policy, then in the long run $\Delta D/(\Delta E_o + \Delta E_n)$ will equal the target debt to equity ratio, D/E. Substituting these into (20) gives us the relation between the cost of equity for the levered company, the cost of equity for the unlevered company, the cost of debt of the levered company and the debt-to-equity ratio of the company.

$$r_{eL} = r_{eU} + [r_{eU}(1-T^*) - r_d(1-\tau_c)]\frac{D}{E} \tag{22}$$

We are not assuming the debt is risk free, so r_d will not, in general, equal r_{fd}. But, we can use equation (8) to express r_d as a funtion of tax rates and the pre-personal tax required return on an equity security with the same systematic risk as the above debt security:

$$\boldsymbol{r_d} = \left[\frac{(1-\tau_{pe})}{(1-\tau_{pd})}\right] \boldsymbol{r_{d,e}} \tag{23}$$

Substituting for r_d from (23) into (22) and recognizing from (6) that $(1-\tau_{pe})(1-\tau_c)/(1-\tau_{pd}) = 1-T^*$, we derive our relevering formula in the presence of corporate and personal taxes:

$$\boldsymbol{r_{eL}} = \boldsymbol{r_{eU}} + [(\boldsymbol{r_{eU}} - \boldsymbol{r_{d,e}})(1-\boldsymbol{T}^*)]\frac{\boldsymbol{D}}{\boldsymbol{E}} \tag{24}$$

The levered cost of equity is an increasing function of the debt/equity ratio of the firm, given its underlying business risk (which determines r_{eU}). Given riskless debt, the slope of the curve is constant, i.e., the relation is linear and depends on the assumed tax regime. If there are no personal taxes, $T^* = \tau_c$ and the slope is $(1-\tau_c)(r_{eU}-r_{fe})$; if there are no taxes at all, then we have MM propostion II: $r_{eL} = r_{eU} + [r_{eU} - r_{fe}](D/E)$. With risky debt, as the debt/equity ratio increases, $r_{d,e}$ will increase, reducing the slope of the relevering formula.

5. CAPM method of estimating the levered cost of equity

The strength of the method derived in Section 4 is that it is not dependent on the particular model of market equilibrium employed (although if the estimate of r_{eu} is based on a particular model, then so is your relevered cost of equity). The downside of the method derived in Section 4 is that it explicitly makes use of $r_{d,e}$ and/or r_{fe}, which, while theoretically sound, are rarely used in practice in CAPM or APT formulas.

The method discussed below will assume that the CAPM is the underlying model of capital market equilibrium and will derive the relation between unlevered and relevered *betas* in the presence of personal and corporate taxes and in the general case of risky debt. We will again assume a perpetuity no-growth model of the firm.

In an after-personal tax CAPM framework, the only thing that matters to investors is after-personal tax returns[3]. As such we must characterize what we mean by the after-personal tax realized rate of return on unlevered equity, levered equity and debt. Also, investors only care about the covariance between after-personal tax rates of return on securities and the after-personal tax rate of return on the market.

The unlevered firm has no debt, no interest and its annual after-personal tax unlevered equity cash flow is simply NOI$(1-\tau_c)(1-\tau_{pe})$. (Remember, NOI is a random variable.) This cash flow represents the realized return on the equity because it is a perpetuity model. Assuming the equity value of unlevered firm at the beginning of the period was E_U (which equals V_U because the firm is all-equity), the realized rate of return after personal taxes on unlevered equity is given by $R_{eu}(1-\tau_{pe})$:[4]

$$\boldsymbol{R_{eU}(1-\tau_{pe}) = \frac{NOI(1-\tau_c)(1-\tau_{pe})}{E_U}} \qquad \textbf{(25)}$$

Let $R_m(1-\tau_{pe})$ be the realized after-personal tax rate of return on the market portfolio, then the unlevered beta of the firm is defined as the covariance between after tax R_{eU} and R_m, i.e.,

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[3]As the adage goes, it's not what you make, but what you keep.

[4]The common notation for returns is to refer to pre-tax returns as R and after-tax returns as $R(1-\tau_p)$. I maintain that convention.

$$\beta_{eU} = COV(R_{eU}(1-\tau_{pe}), R_m(1-\tau_{pe})). \qquad (25a)$$

Plugging in from (25) into (25a) yields (25b)

$$\boldsymbol{\beta_{eU} = COV\left(\frac{NOI(1-\tau_c)(1-\tau_{pe})}{E_U}\ ,\ R_m(1-\tau_{pe})\right)} \qquad (25b)$$

Similarly, the after-personal tax realized return on the levered equity and the covariance of the after-personal tax return on levered equity with the market is given in (26):

$$\boldsymbol{R_{eL}(1-\tau_{pe}) = \frac{NI(1-\tau_{pe})}{E_L}}$$

$$\boldsymbol{\beta_{eL} = COV\left(\frac{NI(1-\tau_{pe})}{E_L}\ ,\ R_m(1-\tau_{pe})\right)} \qquad \textbf{(26)}$$

The after-personal tax return on the debt of the levered firm and the beta of the after-personal tax return on the debt with the market is given in (27):

$$\boldsymbol{R_d(1-\tau_{pd}) = \frac{Int}{D_L}(1-\tau_{pd})}$$

$$\boldsymbol{\beta_d = COV\left(\frac{Int}{D_L}(1-\tau_{pd}),\ R_m(1-\tau_{pe})\right)} \qquad \textbf{(27)}$$

(Remember, that in (26) and (27), NI and Int are each random variables.)

Now we will use the definition of β_{eL} in (26) to derive the relation between β_{eL} and β_{eU}, the D/E ratio and β_d. Expanding the NI term in (26) by applying (10) to it, we get the following:

$$\beta_{eL} = COV\left(\frac{NOI(1-\tau_c)(1-\tau_{pe}) - Int(1-\tau_c)(1-\tau_{pe})}{E_L},\ R_m(1-\tau_{pe})\right)$$

Remembering that NOI and Int are each random variables and using the fact that the covariance operator is linear, we can break the right-hand side into two covariances:

$$\beta_{eL} = COV\left(\frac{NOI(1-\tau_c)(1-\tau_{pe})}{E_L},\ R_m(1-\tau_{pe})\right) - COV\left(\frac{Int\ (1-\tau_c)(1-\tau_{pe})}{E_L},\ R_m(1-\tau_{pe})\right)$$

Next, multiply the NOI term in the first covariance term by V_U/V_U and multiply the Int term in the second covariance term by $(1-\tau_{pd})D_L/(1-\tau_{pd})D_L$.

$$\beta_{eL} = COV\left(\frac{(V_U)NOI(1-\tau_c)(1-\tau_{pe})}{(V_U)(E_L)},\ R_m(1-\tau_{pe})\right) - COV\left(\frac{Int\ (1-\tau_c)(1-\tau_{pe})(1-\tau_{pd})D_L}{(D_L)(1-\tau_{pd})(E_L)},\ R_m(1-\tau_{pe})\right)$$

Next, we want to turn the first covariance term into the unlevered beta multiplied by some constants and the second covariance term into the debt beta multiplied by some constants. Because V_U and E_L are constants (they are today's values, not random variables in the covariance) and the linearity of the covariance operator we can extract a V_U/E_L term from the first covariance term. Similarly, we can extract a $[(1-\tau_c)(1-\tau_{pe})/(1-\tau_{pd})]D_L/E_L$ term from the second covariance term. We thus rewrite the above equation as:

$$\beta_{eL} = \left(\frac{V_U}{E_L}\right) COV\left(\frac{NOI(1-\tau_c)(1-\tau_{pe})}{V_U},\ R_m(1-\tau_{pe})\right) - \left[\frac{(1-\tau_c)(1-\tau_{pe})}{(1-\tau_{pd})}\right]\left(\frac{D_L}{E_L}\right) COV\left(\frac{Int\ (1-\tau_{pd})}{D_L},\ R_m(1-\tau_{pe})\right)$$

The first covariance term, recognizing that $V_U=E_U$ for unlevered firm, is the unlevered equity beta in (25b); the second covariance term is the debt beta in (27). The bracketed term on the right hand side is, by (6), equal to (1-T*). Substituting $V_U = (E_L + D_L - T^*D_L)$ from (5) and (2) in the first term on the right hand side, we get:[5]

$$\beta_{eL} = \left(\frac{E_L}{E_L} + \frac{D_L}{E_L} (1-T^*) \right) \beta_{eU} - (1-T^*)\left(\frac{D_L}{E_L}\right) \beta_d$$

Collapsing terms multiplied by $(1-T^*)(D_L/E_L)$ yields:

$$\beta_{eL} = \beta_{eU} + (1-T^*) \left(\frac{D_L}{E_L}\right) (\beta_{eU} - \beta_d) \qquad \textbf{(28)}$$

Notice the similarity between (28) and (24). If required returns on debt and equity securities are generated by the CAPM, these equations are perfectly consistent with one another. Either method would be fine.

What if, however, you believe that the CAPM is a reasonable model of required returns on equity, but its usefulness in estimating required returns on debt securities is less accurate: perhaps debt securities being less liquid than equities results in a premium in their expected returns unrelated to their beta risk. We may use (28) to unlever and relever equity betas and use the traditional capital asset pricing model to estimate required return on equity for our application, then estimate r_d via a method we think is more applicable than CAPM. This method is discussed in the Cost of Capital

[5] Equation (5) is, again, assuming perpetuity debt. This is consistent with our perpetuity assumptions made throughout this section.

teaching note.

6. Conclusions

The purpose of this teaching note is to provide some of the technical derivations behind using formulas such as (24) or (28) for unlevering/relevering costs of equity or, in the CAPM context, equity betas. We apply these formulas in the cost of capital teaching note and valuation teaching note.[6]

This series of teaching notes underscores some important interaction between areas of theoretical/empirical research in corporate finance and the application of corporate finance in practice, especially in the valuation arena. First, academic work to date, both theoretical and empirical, suggests that the tax impact on financial structure decisions must take an integrated view of taxes: that is, it must consider both the corporate issuers taxes and the taxes (both implicit and explicit) paid by its capital providers.[7] Whether we are using a WACC-oriented valuation approach or an APV-oriented valuation approach and irrespective of what equilibrium model is being used to estimate costs of equity, the personal-and-corporate tax integration enters into our analysis via unlevering/relevering formulas used to calculate the cost of equity.

And yet, I'm not aware of any practitioner framework that explicitly makes use of the personal-and-corporate tax framework, especially in cost of capital estimation. Even most textbooks

[6] Respectively, Thompson, *Cost of Capital Notes: Teaching Note*, Fall, 1995, and Thompson, *Teaching Note: Valuation Using the Adjusted Present Value (APV) Method vs. Adjusted Discount Rate (ADR) Method (Theory)*, Spring, 1994.

[7] This is a central theme of Sholes and Wolfson's **Taxes and Business Strategy: A Planning Approach**, 1992.

in corporate finance abandon the integrated tax framework once they start a discussion of adjusted present value vs. adjusted discount rate approaches to valuation.

Second, unlevering/relevering formulas used in practice[8] almost always ignore the fact that the debt of their companies is not riskless. We allow for risky debt in our derivation of unlevering/relevering formulas (24) and (28). In the Cost of Capital note, we discuss further the implications of omitting the consideration of debt betas on both WACC method and APV method valuations.

In our other teaching notes, we make use of the formulas derived herein for the following applications: 1. to estimate unlevered betas of a projects/divisions/etc. derived from peer group companies' estimated levered betas. These unlevered betas can be used directly in an APV valuation of the project/division/etc; 2. to estimate the levered cost of equity for a project/division/etc. to be used in a WACC valuation for the project/division/etc.

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[8]See, for example, texts by Damodaran such as **Damodaran on Valuation**, 1994, or **Investment Valuation**, 1996, the Salomon Bros. article *The Financial Executive's Guide to the Cost of Capital.* Earlier versions of these teaching notes made this faux pas as well.

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