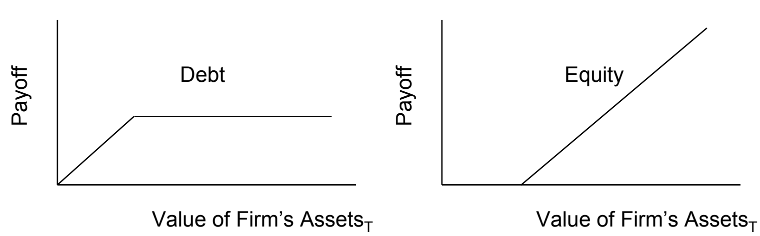
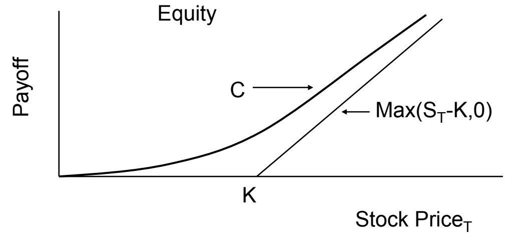
Financial Options: Pricing and Hedging

Payoff Diagrams



Valuation of distressed debt and equity-linked securities requires an understanding of financial options

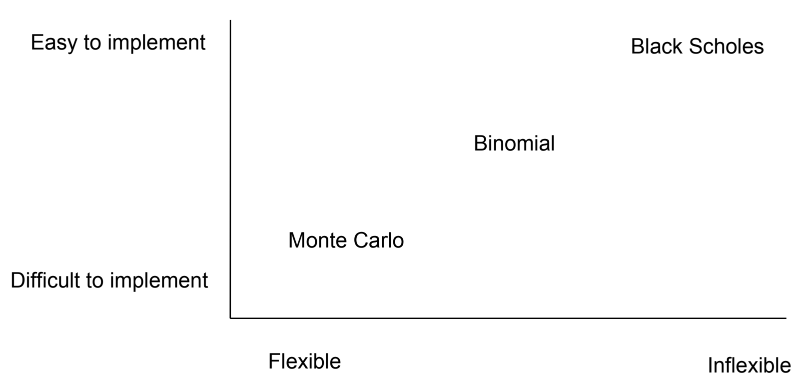
Example: Call Option



Three approaches to valuation:

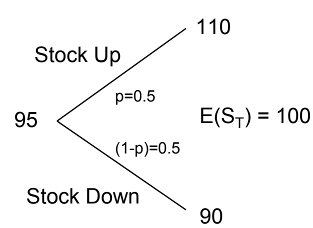
1. Binomial Model
2. Black Scholes
3. Monte-Carlo Analysis

Option Valuation Methods



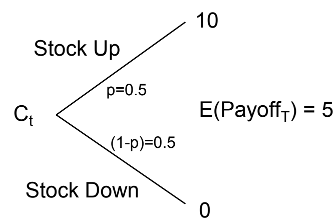
Binomial Option Pricing

Stock Price Tree



What is the value of a call option with a strike price of $100?
(Assume r = 0%)

Call Option Tree

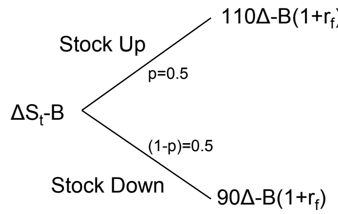


Incorrect Approach: C = (0.5)(10) + (0.5)(0) = $5.00

Why Is This Wrong?

- Simply discounting the probability weighted average of option payoffs does not properly control for risk.
- If we were operating in a “risk-neutral” world, the above approach would be fine.
  - » In a risk neutral world, the market risk premium is 0%, not 7.5%.
- But in a risk-neutral world, the stock price would be $100 (= (0.5)(110)+(0.5)(90)), not $95.

Correct Approach: Replicating Portfolio

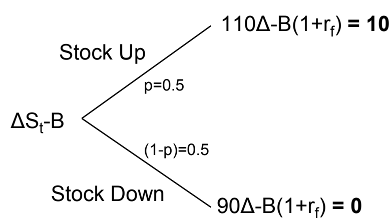


Replicating Portfolio:

1. Buy Δ shares of stock
2. Borrow $B

Cost of Portfolio = $\Delta S_t - B$

Correct Approach: Replicating Portfolio



Two Equations:

1. 10Δ-B(1+rf) = 10
2. 90Δ-B(1+rf) = 0

Solving yields Δ=0.5 and B=45

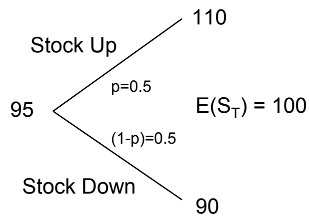
No Arbitrage: Law of One Price

- The replicating portfolio has payoffs that are identical to payoffs from the call option in every state of the world (e.g. stock price up versus stock price down).
- Therefore, to prevent arbitrage, the price of the replicating portfolio must equal the price of the call option.
- $C = \Delta S_t - B = (0.5)(95) - 45 = \$2.50$ (not \$5.00)

Alternative Approach: Risk Neutral Probabilities

- Notice that the Replicating Portfolio approach did not require knowledge of investors' risk aversion
- Therefore, the Replicating Portfolio approach will work for all levels of risk aversion
- In particular, it will work when investors are risk-neutral (e.g. market risk premium = 0%, not 7.5%)
- Risk-neutral probabilities are those that would have to exist to support observed prices in a risk-neutral world

Stock Price Tree

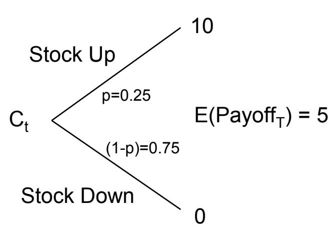


The risk-neutral probability, p, satisfies the following equation:

$$95 = [(p)(110)+(1-p)(90)]e^{-rt}$$

$$p = 0.25$$

Call Option Tree



$$C = [(0.25)(10) + (0.75)(0)]e^{-rt} = \$2.50$$

(recall r=0% by assumption)

Multi-period Binomial Trees

- Desired features of tree
  - » We would like it to be recombining
    - – The stock price if it first goes up and then goes down should be the same as the stock price if it first goes down and then goes up
  - » We would like the resulting distribution of stock returns to be similar to observed distributions (e.g. approximately normal)
- Define the following (μ=expected stock return, F=standard deviation of stock return,Δt=size of time step,*=dividend yield)

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{e^{(r-\delta)} - d}{u - d}$$

10-Period Example

- Value a call option with the following parameters:
  - » Time until maturity = 1 year
  - » Standard deviation of stock returns = 30%
  - » Strike price = $100
  - » Stock price = $95
  - » Dividend yield = 1%

Stock Price Tree

Current Stock Price	$95.00		Sigma*Sqrt(T)	0.0949
Dividend Yield	1.00%		Probability Up	0.4974
Volatility	30.00%		Probability Dow n	0.5026
Risk-free rate	5.00%		Up	1.0995
Time Step (years)	0.1		Dow n	0.9095
Strike Price	100			
Call Option Value	10.90		Delta	0.54

p=[exp((r-divyld)dt)-d]/[u-d]

=(1-p)

exp(sigma*sqrt(dt))

exp(-sigma*sqrt(dt))

	2/7/2001	3/15/2001	4/21/2001	5/27/2001	7/3/2001	8/8/2001	9/14/2001	10/20/2001	11/26/2001	1/1/2002	2/7/2002
	0	1	2	3	4	5	6	7	8	9	10
0	95.00	104.45	114.85	126.28	138.84	152.66	167.85	184.56	202.92	223.12	245.32
1		86.40	95.00	104.45	114.85	126.28	138.84	152.66	167.85	184.56	202.92
2			78.58	86.40	95.00	104.45	114.85	126.28	138.84	152.66	167.85
3				71.47	78.58	86.40	95.00	104.45	114.85	126.28	138.84
4					65.00	71.47	78.58	86.40	95.00	104.45	114.85
5						59.12	65.00	71.47	78.58	86.40	95.00
6							53.77	59.12	65.00	71.47	78.58
7								48.90	53.77	59.12	65.00
8									44.48	48.90	53.77
9										40.45	44.48
10											36.79

Call Price Tree

	2/7/2001	3/15/2001	4/21/2001	5/27/2001	7/3/2001	8/8/2001	9/14/2001	10/20/2001	11/26/2001	1/1/2002	2/7/2002
	0	1	2	3	4	5	6	7	8	9	10
0	10.90	15.84	22.48	31.08	41.77	54.52	69.16	85.49	103.51	123.39	145.32
1		6.12	9.43	14.20	20.80	29.56	40.58	53.69	68.51	84.87	102.92
2			2.90	4.81	7.80	12.34	18.95	28.01	39.56	53.01	67.85
3				1.05	1.89	3.38	5.93	10.17	16.86	26.65	38.84
4					0.22	0.44	0.89	1.80	3.64	7.35	14.85
5						0.00	0.00	0.00	0.00	0.00	0.00
6							0.00	0.00	0.00	0.00	0.00
7								0.00	0.00	0.00	0.00
8									0.00	0.00	0.00
9										0.00	0.00
10											0.00

=[(p)(Call Up)+(1-p)(Call Down)]*exp(-rT)

Max(ST-K,0)

Black Scholes Option Value

$$C = S_t e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_t / K) + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

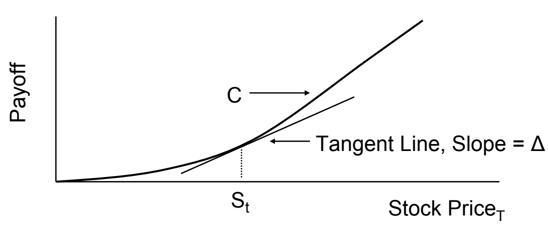
- Variables
  - » S_t: Current stock price
  - » *: Dividend yield (annual $ dividend/current stock price)
  - » T: Time until expiration
  - » N(): Cumulative normal function
  - » K: Exercise price
  - » R: Risk-free rate
  - » F: Volatility (standard deviation of stock returns)
- Assumptions
  - » Frictionless financial markets
  - » Assets are perfectly divisible
  - » No restrictions on short sales
  - » Borrow and lend at the same interest rate
  - » Interest rate is constant
  - » Stock prices follow a continuous path -- i.e. there are no jumps
  - » Stock's expected return and the variance are constant over the life of the option
  - » The continuously compounded return over any period is normally distributed:

Example

- 10-Period Binomial
  - » Call Price = $10.90
  - » Delta = 0.54
- 200-Period Binomial Model
  - » Call Price = $10.76
  - » Delta = 0.54
- Black-Scholes Model
  - » Call Price = $10.75
  - » Delta = 0.54

Delta Hedging

Delta Hedging: Convex Payoff

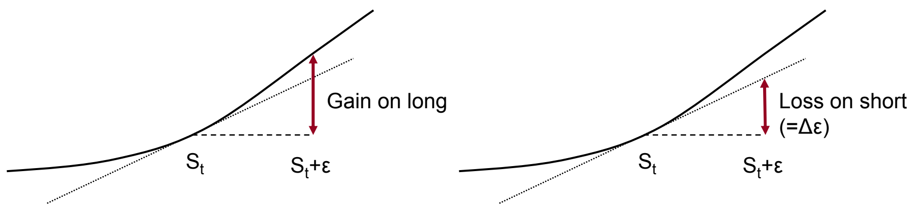


Portolio:

1. Long 1Call Option
2. Short Δ shares of stock:

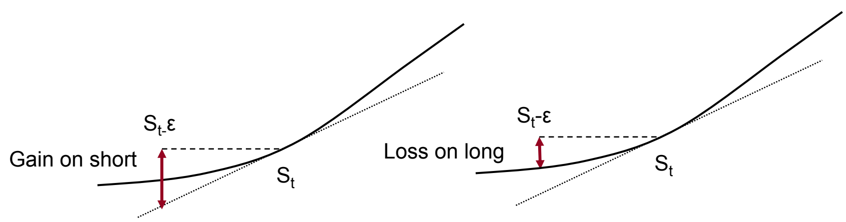
$$\text{Call}\,\Delta = \frac{\partial C}{\partial S_t} = e^{-\delta T} N(d_1)$$

Delta Hedging Profits: Stock Price Increase



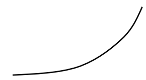
Gain on Long > Loss on Short

Delta Hedging Profits: Stock Price Decrease



Gain on Short > Loss on Long

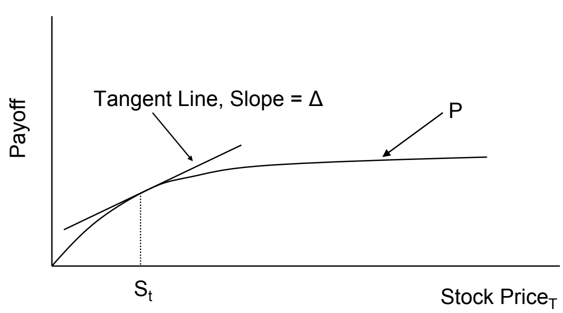
Convexity



- Positive profits are made when:
  - » Stock price increases
  - » Stock price decreases
- Since convexity is good, you must pay for it
  - » Option premium
  - » Time decay
- Convex payoff schedules have positive "gamma"

$$\text{Call}\,\Gamma = \frac{\partial^2 C}{\partial S_t^2} = \frac{\partial \Delta}{\partial S_t} = \frac{e^{-\delta T} N'(d_1)}{S_t \sigma \sqrt{T}}$$

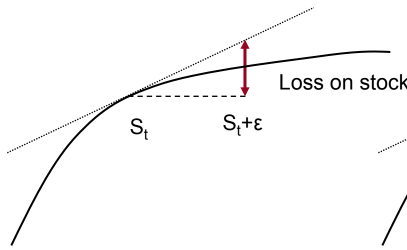
Delta Hedging: Concave Payoff

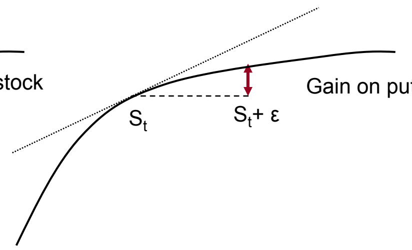


Portfolio:

1. Short one put option
2. Short Δ shares of stock

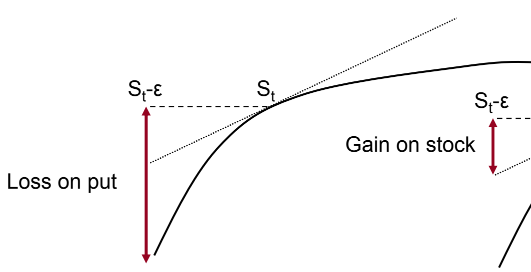
Delta Hedging Profits: Stock Price Increase

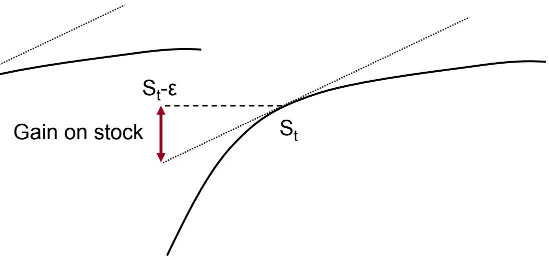




Gain on put < Loss on stock

Delta Hedging Profits: Stock Price Decrease





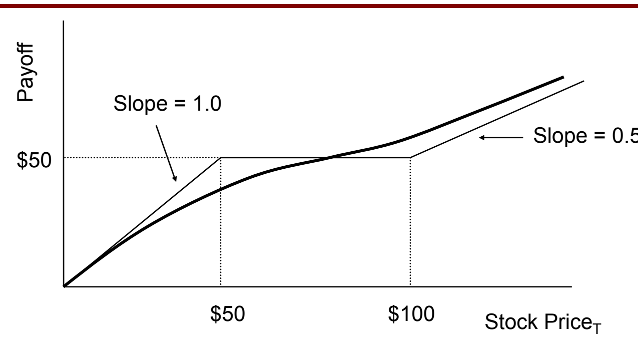
Loss on put > Gain on stock

Concavity

- Negative profits are made when:
  - » Stock price increases
  - » Stock price decreases
- Unattractive features of concavity are compensated by
  - » Option premium (you receive the premium)
  - » Time decay
- Concave payoff schedules have negative “gamma”

$$\text{Call}\,\Gamma = \frac{\partial^2 C}{\partial S_t^2} = \frac{\partial \Delta}{\partial S_t} = \frac{e^{-\delta T} N'(d_1)}{S_t \sigma \sqrt{T}}$$

Portfolios of Options

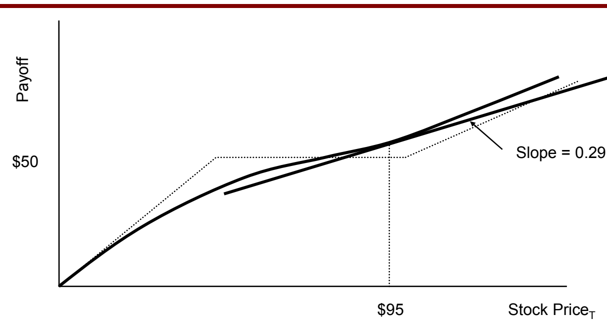


Replicating Portfolio:
1. Long one share of stock
2. Short one call option with strike of $50
3. Long 0.5 call options with strike of $100

Example

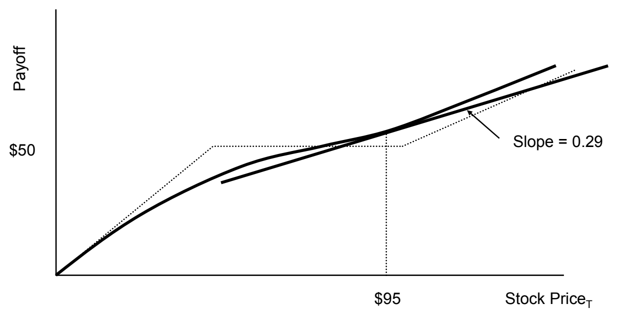
- Assumptions
  - » S_t = $95 (current stock price)
  - » * = 1.0% (dividend yield)
  - » T = 1.0 years (time until expiration)
  - » r = 5.0% (risk-free rate)
  - » F = 30% (standard deviation of stock returns)

Portfolios of Options



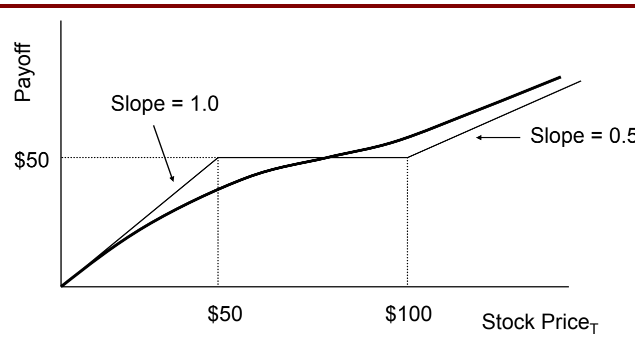
	Quantity	Price	Delta	Gamma	Quantity X Price	Quantity X Delta	Quantity X Gamma
Stock		95.00	1.00	-			
Short Call, X=$50		(46.57)	(0.98)	(0.0007)			
Long Call, X=$100		10.75	0.54	0.0138			
Total							

Portfolios of Options



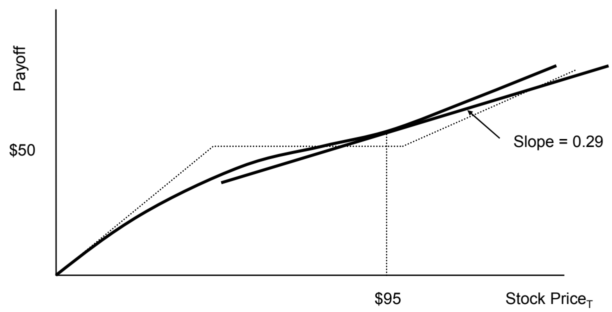
	Quantity	Price	Delta	Gamma	Quantity X Price	Quantity X Delta	Quantity X Gamma
Stock	1.00	95.00	1.00	-	95.00	1.00	-
Short Call, X=$50	1.00	(46.57)	(0.98)	(0.0007)	(46.57)	(0.98)	(0.0007)
Long Call, X=$100	0.50	10.75	0.54	0.0138	5.38	0.27	0.0069
Less PV of Dividends	1.00	(0.91)	-	-	(0.91)	-	-
Total					52.89	0.29	0.0061

Alternative Replicating Portfolio



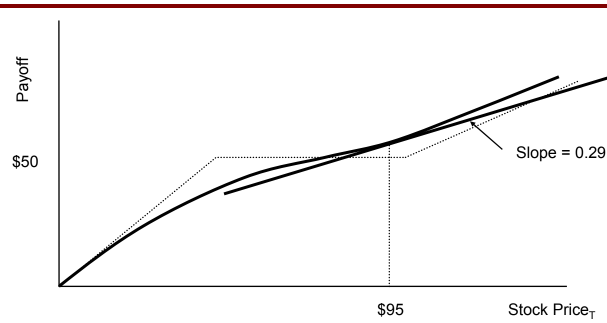
Alternative Replicating Portfolio:
1.
2.
3.

Portfolios of Options



	Quantity	Price	Delta	Gamma	Quantity X Price	Quantity X Delta	Quantity X Gamma
		47.62	-	0.0000			
		(0.08)	0.01	(0.00)			
		10.75	0.54	0.01			
Total							

Portfolios of Options



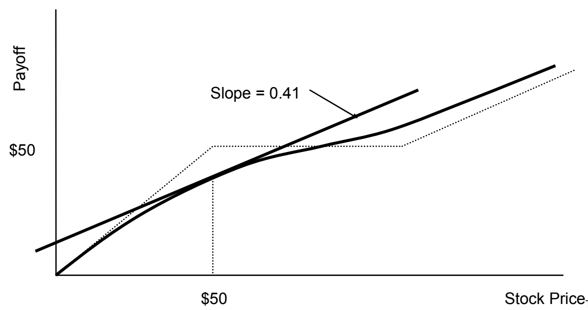
	Quantity	Price	Delta	Gamma	Quantity X Price	Quantity X Delta	Quantity X Gamma
Long risk-free bond	1.00	47.62	-	0.0000	47.62	-	-
Short Put, X=$50	1.00	(0.08)	0.01	(0.00)	(0.08)	0.01	(0.0007)
Long Call, X=$100	0.50	10.75	0.54	0.01	5.38	0.27	0.0069
Total					52.92	0.28	0.0061

Gamma Hedging

Example

- Assumptions
  - » S_t = \$50 (current stock price)
  - » * = 1.0% (dividend yield)
  - » T = 1.0 years (time until expiration)
  - » r = 5.0% (risk-free rate)
  - » F = 30% (standard deviation of stock returns)

Portfolios of Options: Negative Gamma



	Quantity	Price	Delta	Gamma	Quantity X Price	Quantity X Delta	Quantity X Gamma
Stock	1.00	50.00	1.00	0.0000	50.00	1.00	0.0000
Short Call, X=$50	1.00	(6.81)	(0.61)	(0.0253)	(6.81)	(0.61)	(0.0253)
Long Call, X=$100	0.50	0.11	0.02	0.0034	0.05	0.01	0.0017
Less PV of Dividends	1.00	(0.92)	-	0.0000	(0.92)	-	0.0000
Total					42.33	0.41	(0.0236)

Hedging Negative Gamma

- To eliminate the negative gamma in a portfolio, add a security with positive gamma
- In the above example, we can buy X put options with a strike price of $50 (long put has positive gamma)
  - » Set X such that the positive gamma from the puts exactly offsets the negative portfolio gamma

$$X\Gamma_{Put} = 0.0236$$
$$\Gamma_{Put} = .0253(\text{from Black} - \text{Scholes})$$
$$\Rightarrow X = 0.93 \text{ Puts}$$

  - » Now adjust short to remain delta neutral

$$\Delta_{Put} = -0.38(\text{from Black} - \text{Scholes})$$
$$.93\Delta_{Put} = -0.35$$
$$\Rightarrow \Delta_{Portfolio} + .93\Delta_{Put} = 0.41 - 0.35 = 0.06$$

Hedging Gamma-Summary

- Delta and Gamma Neutral Portfolio consists of
  - » Long Original Portfolio
  - » Long 0.93 Put Options With a Strike = $50 and T-t = 1 year
  - » Short 0.05 Shares of Stock

	Delta	Gamma
Long Portfolio	0.41	-0.0236
Long .93 Puts	-0.36	0.0236
Short .05 Shares	-0.05	0.0000
Total	0.00	0.0000

Summary

- Big Picture
  - » The payoff from most securities and transactions can be viewed as a portfolio of options
  - » Value the security or transaction can be accomplished by summing the values of the individual options
- Mechanics
  - » Simple Black-Scholes often provides a very good estimate of value and Greeks
  - » Binomial models are easy to build and can easily handle most bells and whistles
  - » Path dependency is trickier…Monte Carlo methods are better suited for path-dependent situations