On the Analysis of Asymmetric First Price Auctions*

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Abstract

We provide a new set of tools for studying asymmetric first price auctions, connecting their equilibria to the $\rho$-concavity of the underlying type distributions, and showing how one can use surplus expressions related to symmetric auctions to bound equilibrium behavior in asymmetric auctions. We apply these tools to studying procurement auctions in which, as is common in practice, one seller is given an advantage, reflecting for example better reliability or quality. We show conditions under which for any given first price handicap auction, there is a second price auction with bonuses that outperforms it, with most of our results involving ex-post dominance.

Abstract

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1 Introduction

Asymmetric first price auctions are notoriously difficult to analyze. This paper provides new tools for understanding such auctions. We study the case of an auction with two bidders. At the heart of our results is a strong

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connection between equilibria of asymmetric first price auctions and the local \( \rho \)-concavity of the distributions over values and of each player's equilibrium interim expected surplus.\(^1\) For the symmetric case, for example, the slopes of bidding functions are one to one with the \( \rho \)-concavity of the underlying distribution of bidder types.

This connection between equilibria of first price auctions and \( \rho \)-concavity allows us to explore how increases in surplus are divided between the seller and the buyer. We give a tight answer to the question of when a better distribution over types leads to more aggressive bidding for any given type. And, we show how to use the properties of \( \rho \)-concavity to generate a new comparative static on equilibria in a symmetric setting as the number of bidders changes.

We hope these tools will have broader applicability to auction theory and mechanism design. But, for much of the paper, we focus on a specific application: we look at the choice of auction for an auctioneer who prefers to transact with one bidder over the other. An example is a procurement setting where the product of one supplier is viewed as more reliable.

Two simple mechanisms in this setting are a first price and a second price auction each with a handicap or bonus. Both types of auction are very common in practice. For example, Ariba.com, a leading internet-based provider of procurement auctions, allows its customers the option to use fixed handicaps and bonuses in combination with first and second price (open) rules. The standard request for proposal is, in the sealed bid case, equivalent to a first price handicap auction in which the handicap is set equal to the entire amount by which the auctioneer prefers his favored bidder, and similarly for the open request for proposal and the second price bonus auction. And, when quality is exogenous, auctions with quasi-linear scoring rules reduce to our setting. Examples of auctions with such scoring rules include those that the World Bank (2004a, 2004b) both uses itself and requires of its loan recipients. These have subsequently become standard for a majority of NGO's, and for a host of other international agencies, and are also widely used in corporate procurement by firms such as Boeing and by Ariba competitor Perfect Commerce.\(^2\) Because it is these simple mechanisms that are observed as a practical matter, a comparison of their merits is of substantial importance.

First price handicap auctions in this setting have an odd feature. They

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\(^1\)The local \( \rho \)-concavity of a positive function \( g \) at \( x \) is the power to which one needs to raise \( g \) to make it locally linear at \( x \).

\(^2\)We discuss examples further in the literature review, and in Section 5.
create an allocation with lots of distortion away from the symmetric case when types are unfavorable, but very little when types are favorable. This is always true at the extremes of the distribution of types, and we use the tools of $\rho$-concavity to exhibit conditions under which the distortion is in fact monotonically increasing in type. We also use these tools to show conditions under which the optimal mechanism (Myerson (1981), Riley and Samuelson (1981)) specifies a distortion that is decreasing as types become less favorable, so that the first price mechanism is getting things backward. A second price mechanism, on the other hand, creates a uniform distortion away from the symmetric cost case. Using these results, we show that for any given first price mechanism there is a second price mechanism with a well chosen bonus that outperforms it on an outcome by outcome basis.\footnote{In contrast, previous work considers the ex-ante surplus of the auctioneer.}

Designing a sensible first price handicap auction is hard, not least because simply performing the equilibrium calculations for any given handicap is a daunting task. And, given the difficulty of these calculations, the auctioneer may be skeptical that players will reliably find an equilibrium, something which may or may not work to the auctioneer’s advantage.\footnote{At least on an intuitive level, one could imagine bidders in complicated settings reasoning along the lines of the winner’s curse that the news that they have won is suggestive that they have misunderstood the situation and bid too much.} Equilibrium calculations for a second price auction, on the other hand, are trivial for both the auctioneer and the players, with potentially beneficial effects on participation.

We think of this paper as contributing to the Wilson agenda of finding and justifying simple mechanisms. An implication of our results is that in many settings, the auctioneer can without cost ignore first price mechanisms in favor of searching for a good second price mechanism. In related work (Mares and Swinkels 2010a), we show how the tools of $\rho$-concavity make this search practical, allowing the auctioneer to translate a surprisingly small amount of information about the underlying type distributions into a recommended handicap that performs well even compared to the optimal mechanism. While our model surely does not capture all procurement settings, it seems a fair approximation to many in which first price mechanisms are used. There is serious reason for the users of these mechanisms to reconsider their choice.

Our conditions are most easily satisfied when the only source of asymmetry across bidders is via the handicap rule, or equivalently, if one player’s distribution over types is a constant shift of the other’s, a case we will argue is quite common in practice. But, we also make considerable headway when
the underlying distributions over types differ across bidders.

A good entry point into the general literature on asymmetric first price auctions is Maskin and Riley (2000a, 2000b). Our results on the relative merits of first and second price auctions are related too, and at first glance contradict, Maskin and Riley (2000a) and Kirkegaard (2010b) both of whom argue the superiority of first price auctions in a class of settings that intersects our significantly. We discuss the connection in detail in Section 6.4.

Lebrun (1996, 1998, 1999) has done extensive work characterizing equilibria in asymmetric first price auctions. The equilibrium in these auctions is unique under log-concave distributions (Maskin and Riley (2003), Lebrun (2006)). A key result on existence of monotone strategies in first price auctions is provided by Reny and Zamir (2004).5

Myerson (1981) and Riley and Samnuelson (1981) begin a long discussion of implementation with asymmetric cost distributions. McAfee and McMillan (1988) argue that one doesn’t want to always buy from the low-cost bidder. In Che (1993), suppliers have different costs and can provide goods of different qualities. Suppliers submit a bid \((p,q)\) which is evaluated via a quasi-linear scoring rule \(S(p,q)\). The optimal scoring rule distorts quality downward, and can be implemented by either the first or second score rules.6,7 Manelli and Vincent (1995) also study procurement settings. In their model, low costs are correlated with low quality. Our setting differs from these in that we assume that the buyer has a fixed and known relative valuation for purchasing from the two sellers.

Investigating the effects of affirmative action programs in procurement, Corns and Schotter (1999) conducted experiments which show that small percentage bonuses improve the price performance of auctions in asymmetric environments.8 Shachat and Swarthout (2003) provide further experimental evidence that the optimal English auction with a bonus outperforms a standard sealed bid request for proposal, in a setting with uniformly distributed

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5 For a further summary of the literature on asymmetric first price auctions, see Krishna (2002) and the references therein.

6 Naegelen (2002) extends Che’s result by allowing for an exogenous preference for one bidder. Branco (1997) adds common value aspects and correlation to costs. Asker and Cantillion (2006) expand the results to multi-dimensional quality and discuss the connection to scoring rules.

7 In Gauza and Pechlivanos (2000), the buyer, who chooses the design of the object to be procured, chooses a design which favors one firm, but then use the mechanism to “recapture” that advantage. If the buyer must use a symmetric mechanism, he chooses a design that increases homogeneity, and thus competition.

8 Flambard and Perrigne (2006) provide empirical evidence from Canadian snow removal procurement auctions supporting this conclusion.
costs. Cabral and Greenstein (1990) discuss the empirical implications of favored bidding in federal procurement, while Wolfstetter and Lengweiler (2006) analyze favoritism and corruption, in the context of handicap auctions. Marion (2006, 2007) empirically estimates the price effect of favoring disadvantaged bidders through proportional bonuses and points out that bid preferences can have significant negative participation effects on non-favored bidders.

Section 2 presents the model and basic equilibrium characterization. Section 3 discusses \( \rho \)-concavity and its relation to first price auctions. Section 4 presents a key technical result about the slope of the allocation function. Section 5 looks at the geometry of allocations when type densities are symmetric (or differ by a shift) and increasing. Section 6 uses these results to look at the choice between first and second price mechanisms. Sections 7 and 8 extend the analysis to more general cost distributions. Section 9 concludes. Some illuminating proofs are kept in main text. The balance are in the appendix.

2 Model and Basic Characterization

In line with our leading application, we consider a setting in which a buyer faces sellers 1 and 2 with costs \( c_1 \) and \( c_2 \). The results carry over exactly to a standard auction setting with one seller and two buyers. Costs are independent, from cumulatives \( F_1 \) and \( F_2 \), where \( F_i \) has density \( f_i \) which is log-concave and twice continuously differentiable. We assume costs have finite support and normalize the support of \( F_2 \) without loss of generality to \([0, 1]\). The support of \( F_1 \) is \([c_1, c_1]\).\(^9\) The reverse cumulative is \( \bar{F}_i = 1 - F_i \). We assume that \( \frac{f_1}{F_1} \) is strictly increasing. This is very mild.

The buyer’s utility from purchasing from \( i \) is \( v_i - p \), where \( p \) is the transaction price. Let \( \Delta = v_1 - v_2 \) be the amount by which 1 is preferred to 2. We assume the buyer knows \( \Delta \) at the time he commits to the rules of the auction.\(^{10}\) Without loss of generality, we assume \( \Delta \geq 0 \).

In a First Price Handicap Auction with handicap \( A \) (FPHA\(_A\)) bidders 1 and 2 draw costs independently from \( F_1 \) and \( F_2 \), and submit bids \( b_1, b_2 \).

\(^9\)Given the normalization of the support of \( F_2 \), it can be that \( c_1 < 0 \). This leads to no problems: shifting both distributions and the buyer’s value by a constant does not change any of our results.

\(^{10}\)Rezende (2004) asks the rather interesting question of whether the buyer should want to know \( \Delta \). In a setting where the buyer cannot commit to a mechanism, the answer can easily be no.
Bidder 1 wins if and only if $b_1 < b_2 + A$.\textsuperscript{11} The winner receives their bid. Player $i$ is restricted to bid at most $c_i$, so that bidders cannot receive a payment larger than their highest possible cost.\textsuperscript{12} We consider Bayesian equilibria in pure, continuous, and strictly increasing strategies, and in which $b_i \geq c_i$. Primitives for this can be found in Reny and Zamir (2004) and Jackson and Swinkels (2005).

For the purposes of interpreting our results, consider two related settings. In a First Price Bonus Auction with bonus $A$, 1 and 2 draw costs independently from $F_i$ and submit bids $b_1, b_2$. Bidder 1 wins if and only if $b_1 < b_2$. If 2 wins, he receives $b_2$. If 1 wins, he receives $b_1 + A$. Player 1 is restricted to bid at most $c_1 - A$. In a First Price Auction with Cost Shift $A$, 1 and 2 draw costs independently and submit bids $b_1, b_2$. Bidder 1 draws his cost from $F_1$, Bidder 2 draws his cost from $F_2$. Bidder 1 wins if and only if $b_1 < b_2$. The winner receives their bid, and player 1 is again restricted to bid at most $c_1 - A$.

Then, an allocation is an equilibrium in one of these auctions if and only if its obvious translations are equilibrium allocations in the others.\textsuperscript{13} So, in studying auctions with handicaps, we are also studying auctions with bonuses and auctions in which type distributions differ in that one is shifted from the other by a constant.

Given $FPHA_A$, define the allocation function $\phi$ by

$$\beta_1 (\phi(c_2)) = \beta_2 (c_2) + A.$$ 

If $c_1 > \phi(c_2)$, then 2 wins in equilibrium, while if $c_1 < \phi(c_2)$, then 1 wins.\textsuperscript{14} Define $\psi$ as the inverse of $\phi$.\textsuperscript{15}

We now turn to a more detailed examination of the equilibrium bid and allocation functions. We will assume in what follows that $\bar{c}_1 - A \leq 1$, so that bidder 2, for the range of costs $[\bar{c}_1 - A, 1]$ has no chance of winning. The case $\bar{c}_1 - A \geq 1$ is similar with some obvious changes. For an arbitrary

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\textsuperscript{11}Ties are zero probability in equilibrium and the tie breaking rule is inessential (see Jackson and Swinkels (2005)).
\textsuperscript{12}Depending on the relation between $\bar{c}_1 - A$ and 1, one of these restrictions will be irrelevant.
\textsuperscript{13}See Mares and Swinkels (2008) for further detail.
\textsuperscript{14}Since the equilibrium is strictly increasing, $\phi$ is well-defined and increasing.
\textsuperscript{15}Here and in a number of analogous situations that follow, it can be the case that for some $c_2$, $\beta_1 (c_1) < \beta_2 (c_2) + A$ for all $c_1$. In this event, define $\phi_{FP} (c_2) = \bar{c}_1$. Similarly, (although this will not in fact occur in this instance) if it were the case that for some $c_2$, $\beta_1 (c_1) > \beta_2 (c_2) + A$ for all $c_1$, one would define $\phi_{FP} (c_2) = \underline{c}_1$. 

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positive valued function \( g \), let

\[ W_g(c) = \frac{gg''}{(g')^2}(c). \]

**Theorem 1** Assume that \( \bar{c}_1 - A < 1 \). Let \( \beta_1, \beta_2 \) be an equilibrium of \( FPHA_A \). Then \( \phi(0) = c_1 \), and \( \phi(\bar{c}_1 - A) = \bar{c}_1 \). Surpluses with cost \( c \) are

\[ S_1(c) = \int_c^{\bar{c}_1} \Phi_2(\psi(s))ds \text{ and } S_2(c) = \int_c^{\bar{c}_1 - A} \Phi_1(\phi(s))ds. \]

(1)

If \( f_1 \) and \( f_2 \) are \( C^k \), then \( \beta_1, \beta_2, \phi \) are \( C^{k+1}[0,1-A] \). On their domains

\[ \beta'_1(c) = W_{S_1}(c) > 0, \quad \beta'_2(c) = W_{S_2}(c) > 0, \]

(2)

and

\[ \phi'(c) = \frac{\beta'_2(c)}{\beta'_1(\phi(c))} = \frac{S_1(\phi(c))}{S_2(c)} \frac{f_2(c)}{f_1'(\phi(c))} \frac{f_2(c)}{f_1(\phi(c))} > 0. \]

(3)

Unlike in a symmetric auction, \( S_1 \) and \( S_2 \) (and hence \( \beta_1 \) and \( \beta_2 \)) are not functions of primitives, but also depend on the entire behavior of \( \phi \) and \( \psi \) after \( c \). It is this that makes the analysis of asymmetric first price auctions difficult.

### 3 First Price Auctions and Local Concavity

Given Theorem 1, a better understanding of either the bid functions or the allocation of any given \( FPHA_A \) comes down to getting a better handle on \( W_{S_1} \) and \( W_{S_2} \). It is here that the tools of \( \rho \)-concavity first come into their own.

For an arbitrary positive valued function \( g \) with support wlog \([0,1]\) the **local \( \rho \)-concavity of \( g \) at \( c \)** is defined as

\[ \rho_g(c) \equiv 1 - W_g(c). \]

To see the motivation for the terminology, note that it is straightforward that \( \frac{g'}{T} \) is concave at \( c \) if and only if \( t \leq \rho_g(c) \).

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\( ^{16} \)For us, the object of primary interest will be \( W_g \). To draw connections to the mathematical literature, it is helpful to work in many places with \( \rho_g \).

\( ^{17} \)Standard concavity is thus equivalent to local \( \rho \)-concavity everywhere at least 1, while log-concavity is equivalent to local \( \rho \)-concavity everywhere at least 0.
Let $G(c) = \int_c^1 g(s) \, ds$, and let

$$\bar{\rho}_g(c) = \max_{s \in [c,1]} \rho_g(s) \quad \text{and} \quad \underline{\rho}_g(c) = \min_{s \in [c,1]} \rho_g(s).$$

Then, we have the following extension of Prekopa (1971,1973) and Borell (1975).

**Theorem 2** Let $\rho_g > -1$. Let $g$ be decreasing on some $[\bar{c},1]$ and $g(1) = 0$. Then,

$$\frac{\bar{\rho}_g(c)}{1 + \bar{\rho}_g(c)} \leq \rho_\bar{G}(c) \leq \frac{\underline{\rho}_g(c)}{1 + \underline{\rho}_g(c)} \quad (4)$$

for all $c \in [\bar{c},1]$.

Anderson and Renault (2003) obtain a very similar result in studying Cournot and other oligopoly settings.\textsuperscript{19} The fact that they find $\rho$-concavity (in their case of a demand function) at the heart of their results strengthens our belief in the importance of this approach to auction theory, since along the lines of Bulow and Roberts (1989), there is a strong relationship between a demand curve in a monopoly problem and a type distribution in an mechanism setting.

A first easy application of Theorem 2 lets us look at how cost changes are shared between the seller and the buyer in a symmetric setting.

**Proposition 1** Consider a symmetric standard FPA with $F_1 = F_2 = F$, and $A = 0$. If $f$ is increasing, then $\beta'(c) \leq \frac{1}{2}$ for all $c$. If $f$ is decreasing, then $\beta'(c) \geq \frac{1}{2}$ for all $c$.

To see this, note that for the symmetric case, $S_1(c) = S_2(c) = \int_c^1 \bar{F}(s) \, ds$, and so $\beta'(c) = W_{\bar{F}}(c) = 1 - \rho_{\bar{F}}(c)$. If $f$ is increasing, then $\bar{F}$ is concave and so $\rho_{\bar{F}} \geq 1$, and so by Theorem 2,

$$W_{\bar{F}} = 1 - \rho_{\bar{F}} \leq \frac{1}{2}. \quad (5)$$

\textsuperscript{18}We will assume that any $g$ we deal with is sufficiently well behaved that $W_g(1) \equiv \lim_{x \to 1} W_g(x)$ is well defined in the extended real line. We will also assume that when $g(1) = 0$, $W_g(1)$ is finite. This is a very mild condition: see Lemma 9 in the Appendix for a discussion.

\textsuperscript{19}Our result differs from Anderson and Renault only in that we can restrict ourselves to the maximum and minimum $\rho$-concavity to one side of the point in question. Weyl and Fabinger (2009) also use a similar result in studying an oligopoly setting. They point out that an expression of the form $W$ shows up pretty early in economic analysis: Cournot uses it in his 1838 study of the oligopoly problem. For other applications of $\rho$-concavity in economic theory see Caplin and Nalebuff (1991a, 1991b).
Similarly, if \( f \) is decreasing, then \( \rho_F \leq 1 \), from which \( \rho_{fF} \leq \frac{1}{2} \), and so \( \beta' (c) \geq \frac{1}{2} \).

Consider two symmetric first price auctions with cost distributions \( F \) and \( G \), and with equilibrium bid functions \( \beta_F (c) \) and \( \beta_G (c) \). Under what conditions can one say that bidding is more or less aggressive with \( G \) than with \( F \)? A partial answer follows.

**Proposition 2** Let \( F \) with support \([a, 1] \), \( a \geq 0 \), and \( G \) with support \([0, 1] \) be related by \( G (c) = F (\gamma (c)) \), where \( \gamma : [0, 1] \rightarrow [a, 1] \) satisfies \( \gamma (c) \geq c \).

1. If \( W_{fF} \) is decreasing\(^{20} \) and \( \gamma \) is convex then for all \( c \in [a, 1] \),
   \[
   \beta'_G (c) \leq \beta'_F (c) \quad \text{and} \quad \beta_G (c) \geq \beta_F (c).
   \]

2. If \( W_{fF} \) is increasing and \( \gamma \) is concave then for all \( c \in [a, 1] \),
   \[
   \beta'_G (c) \geq \beta'_F (c) \quad \text{and} \quad \beta_G (c) \leq \beta_F (c).
   \]

One way of thinking about \( G \) and \( F \) is that one first draws a cost according to \( G \), and then, to arrive at \( F \), suffers a cost penalty given by \( \gamma (c) - c \). Concavity (convexity) of \( \gamma \) says that incremental cost increases in the original draw have lower (higher) incremental impact on final costs when costs are higher than when they are lower.\(^{21} \)

Under (1), while costs are stochastically lower under \( G \), bids for any given cost are higher, and less of any given cost saving for a bidder shows up in a more aggressive bid.\(^{22} \) Under (2), the buyer benefits both from the stochastically lower costs implied by \( G \) and from more aggressive bidding.

The proof follows from (2) combined with the following observation about \( \rho \)-concavity, and the observation that \( \beta_G (1) = \beta_F (1) = 1 \).

**Lemma 1** If \( W_{fF} \) is increasing (decreasing), \( \gamma \) is concave (convex), and \( \gamma (c) \geq c \) for all \( c \in [a, 1] \), then \( W_{fF \gamma} (c) \geq (\leq) W_{fF} (c) \) for all \( c \in [a, 1] \).

As a final quick application, let us depart momentarily from the two bidder setting.

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\(^{20}\)We will use and discuss monotonicity conditions on \( W_{fF} \) later in Section 8.

\(^{21}\)When \( \gamma \) is concave, \( G \) dominates \( F \) in the convex transform order, and conversely when \( \gamma \) is convex. See Shaked and Shanthikumar (1994). See Ganuza and Penalva (2009) for an auction application.

\(^{22}\)The buyer is better off under \( G \) than under \( F \), with the effect through better costs dominating the effect through less aggressive bidding.
Proposition 3 Fix $F$, and let $\beta_n$ be the symmetric first price equilibrium bid function with $n$ bidders. Then $\beta_n'(c)$ increases in $n$ and $\beta_n(c)$ decreases in $n$ for all $c$.

That $\beta_n(c)$ is decreasing in $n$ is well known, as $\beta_n(c)$ is the expectation of the lowest cost from $n-1$ bidders. But, as Proposition 2 illustrates, how incremental cost improvements are shared is less obvious. The proof relies on first observing that as for the two bidder case, the symmetric equilibrium bid with $n$ bidders is characterized by $\beta_n'(c) = 1 - \rho F^{n-1}(c)$. The result is then implied by the following property of $\rho$-concavity, and by the observation that $\beta_n(1) = 1$ for all $n$.

**Lemma 2** Consider $g$ decreasing and log-concave and $\alpha > 1$. Then $\rho_f g^\alpha(c) \leq \rho_f g(c)$.

In particular, to compare $\beta_{n+1}'(c)$ and $\beta_n'(c)$, set $\alpha = \frac{n}{n-1}$ and $g(c) = F^{n-1}(c)$.

4 A Key Stepping Stone

We now dive a little deeper into the geometry of $\phi$. We will establish a connection between the curvature of $\phi$ at specific points and the underlying $\rho$-concavity properties of $F_1$ and $F_2$. In subsequent sections, we show how to apply this result.

**Theorem 3** Let $r$ be an interior global minimum of $\phi'$. Assume that $\phi'(r) \leq 1$, that $\liminf_{c \to \infty} \phi'(c) > 1$, and that $\frac{f_1'}{F_1} (\phi(r)) - \frac{f_2'}{F_2} (r) \geq 0$. Let

$$H(r) = \left( \frac{1}{W_f F_1 (\phi(r))} - 2 \right) \left( \frac{f_1'}{F_1} (\phi(r)) - \frac{f_2'}{F_2} (r) \right) + \frac{f_2'}{F_2} (r) - \frac{f_1'}{F_1} (\phi(r)) + \frac{f_2'}{F_2} (r) - \frac{f_1'}{F_1} (\phi(r)).$$

Then, $\phi''(r) >_s H(r)$, where $>_s$ denotes that the LHS is strictly positive whenever the RHS is weakly positive.

The power of Theorem 3 is that $H$ does not depend on the behavior of $\phi$ after $r$. Because this proposition is at the heart of our results, and because the proof illustrates some novel points, we provide it here.

**Proof** From Theorem 1

$$\phi'(r) = \frac{S_1(\phi(r)) \frac{f_2}{F_2}(r)}{S_2(r) \frac{f_1}{F_1}(\phi(r))} = \frac{T}{B},$$

(6)
and so $\frac{\phi''}{\phi'} = (\log \phi')'$ is
\[
\phi'(r) \frac{S_1'(r)}{S_1(r)} - \frac{S_2'(r)}{S_2(r)} + \left( \log \frac{f_2}{F_2} \right)'(r) - \phi'(r) \left( \log \frac{f_1}{F_1} \right)'(r). \tag{7}
\]
By log-concavity of $F_2$, $\left( \log \frac{f_2}{F_2} \right)' > 0$, and so
\[
\left( \log \frac{f_2}{F_2} \right)'(r) \geq \phi'(r) \left( \log \frac{f_2}{F_2} \right)'(r). \tag{8}
\]
Note that $\left( \log \frac{f_2}{F_2} \right)'(r) = \frac{f_1}{F_2}(r) - 2 \frac{f_2}{F_2}(r)$ and similarly for $\left( \log \frac{f_1}{F_1} \right)'$. From (1), $S_1'(\phi(r)) = -F_2(r)$, and so
\[
\frac{S_1'(r)}{S_1(r)} = -\frac{F_2(r)}{S_1(r)} = -\frac{f_2}{F_2}(r) \quad \tag{9}
\]
and similarly
\[
\frac{S_2'(r)}{S_2(r)} = \frac{-\frac{f_1}{F_1}(\phi(r))}{B} = -\phi'(r) \frac{\frac{f_1}{F_1}(\phi(r))}{T}. \tag{10}
\]
Substitute (8), (9), and (10) into (7), collect terms, and cancel $\phi'(r) > 0$ to obtain
\[
\phi''(r) \geq s \left( \frac{1}{T} - 2 \right) \left( \frac{f_1}{F_1}(\phi(r)) - \frac{f_2}{F_2}(r) \right) + \left( \frac{f_2'}{f_2} - \frac{f_1'}{f_1} \right)(\phi(r)). \tag{11}
\]
Now,
\[
S_2(r) = \int_{\phi(r)}^{\epsilon_1 - A} \frac{F_1(s)}{\phi(s)} ds = \int_{\phi(r)}^{\phi(\epsilon_1 - A)} F_1(s) \psi'(s) ds < \frac{1}{\phi'(r)} \int_{\phi(r)}^{\epsilon_1} F_1(s) ds,
\]
where the strict inequality follows since $r$ is a global minimum of $\phi'$, with $\phi'(r) \leq 1$, since $\phi'$ is continuous, and since $\lim_{c \to \epsilon_1 - A} \phi'(c) > 1$. Multiplying both sides by $\phi'(r) \frac{f_1}{F_1}(\phi(r))$ yields $\phi'(r)B < W_f F_1(\phi(r))$. Since $T = \phi'(r)B$, we have
\[
T < W_f F_1(\phi(r)). \tag{12}
\]
Substituting (12) into (11) yields the result. ■

Central to the proof is (12). Term $T$ involves the surplus $S_1(\phi(r))$, which as a function of the entire equilibrium is inherently forbidding. The inequality (12) lets us replace this with a much simpler object, one which can be analyzed using the tools of $\rho$-concavity discussed above.

5 The Geometry of Allocations for Symmetric (or Shifted) Costs

In this section, we specialize our model to one that is simple but highly relevant. We consider the case $F_1 = F_2 = F$. One way of thinking of this is that the only remaining asymmetry in the model is in the preferences of the bidder, as expressed through $A$. But, recalling the isomorphism between auction forms discussed above, another setting to which this applies is when the buyer has no preferred supplier, but the costs of the two suppliers have distributions that are shifted by a constant.²³

So, for example, consider a buyer contracting for a bulk good from two suppliers who differ only in their distance from the buyer, or of on-site professional services again from suppliers at differing distances. Then, it is reasonable to think of costs before delivery as symmetrically distributed, but transportation costs as differing by a fixed known amount. The question then is to what degree the buyer should optimally subsidize the extra transportation costs of the more distant supplier. In settings like these, either first or second price mechanisms that treat the players differently are particularly easy to achieve. Rather than naming one bidder as “favored”, the mechanism can treat both players according to the same rules, but specify what part of transportation costs are the responsibility of the buyer.

As another example, consider a firm that needs to replace a key piece of software. One vendor may supply a product compatible with the existing capabilities of the firm, while another may provide a product that requires new hardware or training. By writing the rules of the auction to specify the scope of the project, the costs of the hardware or training can be reflected or not in how the two bidders compete. In this light, Cabral and Greenstein (1990) have analyzed federal procurement auctions for computers where incumbents are favored through a fixed handicap.²⁴ Wolfstetter

²³ We relax the assumption that costs are symmetric (or shifted) later.
²⁴ This handicap, generated by the General Accounting Office to reflect an estimate of the changeover costs of moving to a new system, is routinely felt by the department
and Lengweiler (2006) present additional examples of handicaps and bonuses implemented via scoring rules.

To begin our analysis, we begin with some basic properties of $\phi$.

**Lemma 3** Let $F_1 = F_2 = F$, and let $A \geq 0$. Then,

1. $\phi'(0) > 1$,
2. $\lim \inf_{c \to 1-A} \phi'(c) > 1$, and
3. $\phi(c) \in (c, c + A)$ for all $c \in (0, 1 - A)$.

The first two parts tell us that $\phi' > 1$ at each boundary. The third part states that the favored bidder takes part of his advantage in the form of a higher margin, thus undoing some of the distortion designed by the auctioneer, and part in the form of a higher probability of winning.\(^{25}\)

This in hand, we turn to our first application of Theorem 3. We show that if $f$ is weakly increasing, then the distortion induced by $FPHA$, $\phi(c) - c$, is monotonic. We relax the assumption that $f$ is increasing later.

**Proposition 4** Assume that $F_1 = F_2 = F$, and that $f$ is weakly increasing. Then, $\phi'(c_2) > 1$ for all $c_2$.

**Proof** By Lemma 3, if $\phi'(c_2) \leq 1$ anywhere, then $\phi'$ has an interior global minimum $r$ with $\phi'(r) \leq 1$. Since $f$ is increasing, by (5), $\frac{1}{W_{f,F}(r)} + 2 \geq 0$, while by Lemma 3 and log-concavity of $f$ and $\bar{F}$, $\frac{f}{F}(\phi(r)) - \frac{f}{F}(r)$ and $\frac{\bar{f}}{\bar{F}}(r) - \frac{\bar{f}}{\bar{F}}(\phi(r))$ are each non-negative. But then, $H(r)$ is non-negative, and so by Theorem 3, $\phi''(r) > 0$, contradicting that $\phi'$ is minimized at $r$. \(\blacksquare\)

As one illustration of the implications of this proposition, we have the following result.

**Proposition 5** Assume that $F_1 = F_2$. If $\phi' \geq 1$ everywhere, then $\beta_1$ and $\beta_2$ lie on either side of the symmetric equilibrium strategy $\beta_s$ of a standard first price auction. That is,

$$\beta_1 \geq \beta_s \geq \beta_2.$$  

Thus, 1 bids less aggressively than if he were not favored, and 2 more.

Figure 1 shows how $\beta_1$ and $\beta_2$ vary in $A$ for the uniform case. It is an interesting conjecture that $\beta_1$ and $\beta_2$ should move monotonically further apart as $A$ grows for general $f$.

\(^{25}\)The proof of this Lemma is subsumed by that of Lemmas 12 and 15 in the appendix.
Figure 1: The equilibrium bid functions for $A = 0$ (dotted), $A = .2$ (dashed) and $A = .4$ (solid) for uniformly distributed costs.

6 Ranking First and Second Price Handicap Auctions

We now turn to another implication of Proposition 4. We identify conditions under which we can compare the merits of first and second price mechanisms. We do this by comparing how each mechanism compares to the mechanism which is optimal subject to always allocating the job.

6.1 Second Price Mechanisms

In a second price bonus auction (SPBA) the auctioneer announces a bonus $A$, and requests sealed bids from 1 and 2. The low bidder wins.\(^{26}\) When 1 wins, he receives $\min(b_2 + A, c_1)$, while if 2 wins, he receives $\min(b_1, 1)$. Putting these maxima on payments guarantees that the bidders do not receive more than their highest possible cost. It is weakly dominant for 2 to set $\beta_2(c_2) = c_2$, and 1 to set $\beta_1(c_1) = c_1 - A$. Thus, 1 wins if and only if $A \geq c_1 - c_2$. Because $\beta_1(c_1) = c_1 - A$, 2 never wins when $c_2 > \bar{c}_1 - A$. So, without loss of generality, we restrict $A \leq \bar{c}_1$.

\(^{26}\)As before, ties are inessential.
An SPBA can always be replicated by an open mechanism with bonuses. To see this, imagine an open descending price mechanism where bidders choose when to drop out and the last active bidder wins, where if 2 wins, he receives the prevailing price, while if 1 wins, he receives the prevailing price plus a bonus $A$, and where we start the clock at $c_1 - A$. It is weakly dominant for 2 to drop out at $c_2$ and for 1 to drop out at $c_1 - A$. Thus all of our results about SPBAs can be directly translated into statements about open auctions.

6.2 Optimal Mechanisms

Let

$$\omega_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$$

be the virtual cost of $i$. By construction $\omega_i' = 1 + \rho F_i$, giving another simple link to $\rho$-concavity. Because $F_i$ is log-concave, $\omega_i' \geq 1$.\(^{27}\) Consider any deterministic mechanism in which the buyer always buys. From incentive compatibility, any such mechanism is characterized by an increasing function $\eta$ such that 1 wins if and only if $c_1 < \eta(c_2)$. Adapting Myerson (1981) or Riley and Samuelson (1981) in the obvious manner, the buyer’s expected surplus from mechanism $\eta$ is

$$BS(\eta) = v_2 - 1 + \int \int I_{(c_1 < \eta(c_2))} (\Delta - \omega_1(c_1) + \omega_2(c)) f_1(c_1) f_2(c_2) dc_1 dc_2. \quad (13)$$

This is intuitive. Always buying from 2 gives the buyer surplus $v_2 - 1$, since 2 must receive 1 if he is to sell for all $c_2$. The integral represents the change in buyer surplus from buying from 1 according to $\eta$.

So, among mechanisms that always buy, 1 optimally wins if $\Delta > \omega_1(c_1) - \omega_2(c_2)$ and 2 wins otherwise.\(^{28}\) Let $\eta_M$ given by

$$\Delta = \omega_1(\eta_M(c_2)) - \omega_2(c_2) \quad (14)$$

be this mechanism.

The connection between the geometry of $\eta_M$ and $\rho$-concavity is again strong.

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\(^{27}\)While log concavity or equivalently $\rho F_i > 0$ is the conventional assumption in a large part of the auction literature (partly on the basis that it is interpretable), the implementability condition that $\omega_i' > 0$ is precisely $\rho F_i > -1$, i.e., that $\frac{1}{\rho F_i}$ is convex.

\(^{28}\)Whether or not exclusion is optimal, the constrained mechanism serves its purpose in helping us compare the FPHA and SPBA.
Lemma 4 If $F_1 = F_2 = F$ and $\rho_F$ is increasing, then $\eta_M' \leq 1$. If $\rho_F$ is decreasing, then $\eta_M' \geq 1$.

To see this, note from (14) that since $\omega$ is increasing $\eta_M(c_2) \geq c_2$, and that

$$
\eta_M'(c_2) = \frac{1 + \rho_F(c_2)}{1 + \rho_F(\eta_M(c_2))}
$$

yielding the result.$^{29}$

6.3 A Ranking Result

Consider mechanisms $\eta_1$ and $\eta_2$. Say that $\eta_1$ dominates $\eta_2$ if $\eta_1$ and $\eta_M$ generate the same allocation on a superset of the set over which $\eta_2$ and $\eta_M$ do. From (13), $\eta_1$ gives the buyer higher surplus, but this notion is much stronger, as it requires that the allocation is realization by realization closer to the optimal one.

Despite the strength of our dominance criterion, our results allow us to show simple conditions under which, for any FPHA, a well chosen second price mechanism dominates it.

Theorem 4 Assume that $F_1 = F_2 = F$, that $f$ is increasing, and that $\rho_F$ is increasing. Then, for any $A$, there is $\hat{A}$ such that $\text{SPBA}_{\hat{A}}$ dominates $\text{FPHA}_A$.

To see the proof, consider Figure 2. From Lemma 4, $\eta_M' \leq 1$ everywhere, while from (14), $\eta_M$ lies above the diagonal. By Proposition 4, $\phi' > 1$ everywhere, and $\phi(0) = 0$. Hence, either $\phi$ and $\eta_M$ cross as illustrated, or $\phi$ lies everywhere below $\eta_M$. In the first case, choose $\hat{A}$ so that $\text{SPBA}_{\hat{A}}$ generates the allocation given by the dotted line in Figure 2. This agrees with the optimal allocation strictly more often than does $\text{FPHA}_A$. In the second, the same is true of any $\hat{A}$ giving an allocation that lies above $\phi$ and below $\eta_M$.

Under the conditions of Proposition 4, for any given first price handicap mechanism, there is a superior second price mechanism. Our conditions are sufficient for this result, but far from necessary. In Mares and Swinkels (2010b), we show how to extend the result to a setting in which there are several players drawing according to each of two distributions. One can

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$^{29}$So, if (and only if) $\rho_F$ is a constant, then $\eta_M' = 1$, and so the optimal mechanism can be implemented by an appropriate $\text{SPBA}$. This is true for the power distributions, $F(c) = c^\alpha$. Gabaix (2009) argues that such distributions are common in economic settings.
also numerically calculate equilibria of two player asymmetric first price auctions, and show that the dominance result continues to hold in a wide variety of other examples. In particular, we have no counterexample to \( \phi' > 1 \). In Mares and Swinkels (2010a) we show how bounds on \( \rho_F \) translate into recommendations for second price handicaps that do not depend on further details of \( F \), but guarantee remarkably good performance. Those bounds are strongest in precisely the case (concave virtual costs) where our dominance results do not apply (because \( \eta_M \) has slope greater than 1, and so our crossing argument fails).

### 6.4 On the Relationship to Maskin and Riley

Consider a setting with shifted costs, in which player 1 has costs distributed on \([-s, 1-s]\) according to \( F_1(c) = F_2(c + s) \). Then, a direct translation of Maskin and Riley (2000a) shows that a symmetric first price mechanism gives higher expected surplus to the buyer than a symmetric second price mechanism. Setting \( A = \Delta = s \) transforms this into an auction with symmetrically distributed costs but a \( \Delta \) value advantage and handicap for 1. So, Maskin and Riley’s result implies that a second price mechanism with handicap \( A = \Delta \) is worse in expected surplus terms than a first price mech-
anism with handicap $A = \Delta$. Hence, for example, a first price RFP process is better than a second price (or open) one.

Maskin and Riley assume that $f$ is increasing, and thus we are, if virtual costs are convex, in the world of Theorem 5, where we show that second price mechanisms are better than first. So, our results, at first glance, seem contradictory! This sense of contradiction is strengthened when one considers the recent work of Kirkegaard (2010b), which extends and generalizes Maskin and Riley to a larger class of settings and in particular shows that the result is robust to the addition of a reserve price. The key to the resolution is to note that in both Maskin and Riley and in Kirkegaard, what is effectively being compared is the first and second price auction for $A = \Delta$. While natural, this turns out to be a pretty bad choice for the auctioneer, especially in the second price case, which for $A = \Delta$ universally distorts too far in favor of 1.

In contrast, we show that for any given value advantage $\Delta$ and $F_{PHA_A}$ there is a bonus $\tilde{A}$ such that the $SPBA_{\tilde{A}}$ does better (in the strong sense of ex-post dominance). So, while Maskin and Riley and Kirkegaard show that between a focal pair of asymmetric auctions one prefers the first price mechanism, we show that if one can choose which handicap to offer, one will prefer a second price mechanism.

7 Asymmetric Costs

We now turn to a situation in which costs are asymmetric, with the goal both of arguing that Theorem 4 is fairly robust, and to illustrate on a technical level what is involved.\footnote{Several different extensions in this direction are feasible.} We begin with the case of asymmetric increasing densities. In the next section, we relax the monotonicity assumption.

To relate asymmetric cost distributions, it is useful, as we did in Proposition 2, to define $\gamma$ implicitly by

$$F_1(\gamma(c_2)) = F_2(c_2).$$

Define $\lambda_A$ by $\lambda_A(c_2) = c_2 + A$, so that $SPBA_A$ implements precisely $\lambda_A$. Of particular interest to us will be $\lambda_\tau$, where $\tau = \tilde{c}_1 - 1$. This is the line of slope 1 that passes through $(\tilde{c}_1, 1)$. See Figure 3.

Our first assumption is that $F_1$ is, in a particular sense, a stretch and convexification of $F_2$.\footnotemark
Assumption 1 $\gamma' \geq 1$ and $\gamma'' \geq 0$.\footnote{That is, $F_1$ dominates $F_2$ in the dispersive order. See Shaked and Shanthikumar (1994). For application of the dispersive order to auction theory, see Gauza and Penalva (2009).}

This is, of course, automatic when $F_1 = F_2$. An implication of $A1$ is that $\gamma \leq \lambda_\tau$. In Example 2 in Maskin and Riley (2000a), a “strong” buyer has values which are a linear stretch around 0 of those of a “weak” buyer, and the first price auction is superior. Our class of asymmetric auctions consists of those where $F_1$ can be viewed as a convexification and stretch of $F_2$, and so includes this case.\footnote{Here again, we will have the “contradictory” result that when handicaps can be chosen, a second price mechanism is preferred. The general “stretch” used by Maskin and Riley is somewhat different than ours, but coincides for this example.}

We remain in the case where virtual costs are convex.

Assumption 2 $\rho_{F_1}$ is increasing

Theorem 5 Assume $A1$ and $A2$, that $f_1$ is increasing, and that $\Delta > \omega_1(\tilde{c}_1) - \omega_2(1)$. Then, for any $A$, there is $\bar{A}$ such that $SPBA_{\bar{A}}$ dominates $FPHA_A$.

The assumption $\Delta > \omega_1(\tilde{c}_1) - \omega_2(1)$ says that $\Delta$ is large enough that when both players have their worst cost types, the optimal mechanism gives the job to 1. This is automatic for $\Delta > 0$ when $F_1 = F_2$, and otherwise requires that $\Delta$ be sufficiently large compared to the asymmetry between $F_1$ and $F_2$.

As before, the proof of Theorem 5 is essentially geometric. See Figure 3 which generalizes Figure 2. First, we show that because $\Delta > \omega_1(\tilde{c}_1) - \omega_2(1)$, $\eta_M$ lies above $\lambda_\tau$ and has slope at most 1. But then, for any $A \leq \tau$, $FPHA_A$ is dominated by $SPHA_\tau$. The proof (see the appendix) uses Theorem 3 to show that for $A > 7$, $\phi'(c_2) \geq 1$ over all relevant ranges. But then, a SPBA which implements a line of slope 1 through the unique crossing point of $\eta_M(c_2)$ and $\phi'(c_2)$ dominates $FPHA_A$. This is $\lambda_{\bar{A}}$ as illustrated in Figure 3.

8 Other Densities

Now let us turn to the case of densities that are not increasing. Given log-concavity, this includes decreasing densities, and densities which are hump shaped. We will restrict attention to densities for which $f_1(\tilde{c}_1) = 0$\footnote{Our Assumption 3 below fails automatically if $f_1$ neither increases nor goes to zero. It is an open question to what extent our results can be extended to this case.}. In
exchange for the assumption that \( f_1 \) is increasing, we need two assumptions. Each is true when \( f_1 \) is increasing, but much more generally as well.

Our first assumption is analogous to the assumption of convex virtual costs, but applied to the reverse cumulative instead of the cumulative.

**Assumption 3** \( \rho_{F_1} \) is decreasing.

By Corollary 2 (see Appendix 10.2), this is automatic if \( f_1 \) is increasing. It can never be satisfied if \( f_1 \) decreases near \( \bar{c}_1 \) but \( f_1 (\bar{c}_1) > 0 \). For densities where \( f_1 (\bar{c}_1) = 0 \), the condition is neither vacuous nor difficult to satisfy. Weyl and Fabinger (2009) require an analogous monotonicity condition for their results. They show that the condition is satisfied by all standard distributions used in econometric estimation.

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34 If \( f \) is decreasing on \((\bar{c}, \tilde{c}_1)\), then \( W_{F'} > 0 \) on \((\bar{c}, \tilde{c}_1)\). If \( f (\tilde{c}_1) > 0 \), but \( f' (\tilde{c}_1) \) is finite, then, since \( \frac{\tilde{c}_1}{f_1} (\tilde{c}_1) = 0 \), \( W_{F_1} (\tilde{c}_1) = 0 \).

35 Choose \( g \) increasing and log-concave. Then, \( W_G \) is increasing (Corollary 1). Setting \( f_1 = G \), \( W_{f_1} = W_G \) is increasing, and thus so is \( W_{F_1} \).

If \( g \) is decreasing with \( g (1) > 0 \), then \( W_G (1) = \frac{1}{2} \), while (Corollary 1), \( W_G (c) > \frac{1}{2} \) for \( c < 1 \). So, (excepting infinitely many sign changes in \( W_G \)), \( W_G \) will be decreasing over some interval near 1. On this interval, \( W_{F_1} = W_{f_1} \) is also decreasing.

36 That \( A2 \) and \( A3 \) are consistent is easily shown by example. One reason why the two
Our second assumption requires that in a sense the concavity of \( \log f \) versus that of \( \log F \) is at its smallest at \( c_1 \).

**Assumption 4** \( \frac{(\log f)''}{(\log F_1)''} \) is minimized at \( c_1 \).  

This is satisfied broadly but not universally in examples we have checked. It is a simple exercise that \( A2 - A4 \) are satisfied for \( F_1(c) = F_2(c) = 1 - (1 - c)^\alpha \).

**Lemma 5** Let \( G \) satisfy \( A3 \) and \( A4 \), and let \( F_1 \) be given by \( F_1 = G^n \), for some \( n \in (0,\infty) \). Then, \( F_1 \) satisfies \( A3 \) and \( A4 \) as well.

When \( n \) is an integer greater than one, \( F_1 \) represents the most favorable from \( n \) draws from \( G \), while when \( \frac{1}{n} \) is an integer, \( G \) represents the most favorable from \( n \) draws from \( F_1 \).

Lemma 5 provides a ready source of both hump-shaped and decreasing examples. Choose \( g \) increasing and log-concave with \( W_g(c_1) \) finite. Then, \( W_G \) is negative and increasing (Corollary 2), so that \( A2 \) is satisfied, and \( A4 \) is automatic. It is easily checked that for \( n \) near enough 1, \( f_1 \) will thus be hump-shaped, while for \( n \) large enough, \( f_1 \) will be decreasing.

In the same spirit, starting from \( g \) increasing and log-concave, consider \( f_1 = G \). Then, by Corollary 2, \( W_{f_1} = W_G \) is increasing, and thus so is \( W_{F_1} \). This generates a class of decreasing densities which satisfy \( A3 \). It is easy to check that densities constructed this way satisfy \( A4 \) as well. Finally, by judicious choice of \( \Delta \), one can start from symmetric distributions that satisfy the conditions, and distort one or the other of them via \( \gamma \) in such a way as to continue to satisfy them.

Another issue we must face when \( f_1(c_1) = 0 \) is that over a range near 1, the slope of the optimal allocation function, \( \eta_M' \), will no longer be less than 1.

**Lemma 6** Assume \( A1 \), that \( f_1(c_1) = 0 \), and that \( \gamma \) is not the identity. Then, for any finite \( \Delta, \eta_M(1) = c_1 \). Further, \( \eta_M(c) \) crosses \( \gamma \) at some unique \( c_\Delta \in (0,1) \). As \( \Delta \) increases, so does \( c_\Delta \), and for \( \Delta \to \infty, c_\Delta \to 1 \). On some interval near 1, \( \eta_M'(c) > 1 \).

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\(^{37}\) The expression is well-defined at \( c_1 \). See Lemma 16 in Appendix.

\(^{38}\) By log-concavity \( \frac{(\log g)''}{(\log G)''} \) is weakly positive by. Since \( g \) is increasing, \( W_G(c_1) = 0 \) and thus by Lemma 16 the expression converges to 0 at \( c_1 \).
The problem is that virtual costs diverge as costs increase. Since near 1, \( \gamma' > 1 \), (as \( \gamma \) is not the identity) they diverge faster for player 1, eventually swamping any given \( \Delta \).

We thus have the possibility illustrated in Figure 4. For clarity, we have zoomed in on the top right corner of the picture, starting at some cost \( K \) for 2 and \( \eta_M(K) \) for 1. Here, the SPBA through the intersection of \( \eta_M \) and \( \phi \) does not dominate \( \phi \), because on the shaded triangular region, \( \phi \) is getting things right while \( \lambda_A \) is not.\(^{39}\) The key is that \( \lambda_r \) no longer lies everywhere above \( \eta_M \). It can be shown, however, that as either \( \Delta \) grows or \( F_1 \) and \( F_2 \) become close to each other, the region of concern becomes arbitrarily small.

To attain a result in this setting, we relax our notion of one mechanism being better than another from dominance to simple ex-ante superiority. We will also need a different form of our assumption that \( \Delta \) is not too small relative to the asymmetry between \( F_1 \) and \( F_2 \). To do this, define \( c_\tau \) as the (unique) point at which \( \eta_M \) crosses \( \lambda_r \), and consider the line \( \delta \) given by

\[
\delta(c) = \begin{cases} 
\lambda_r(c) & \text{for all } c \leq c_\tau \\
\eta_M(c) & \text{for all } c \geq c_\tau 
\end{cases}
\]

It is illustrated in Figure 5.

\(^{39}\)Since virtual costs diverge in this case, some exclusion is optimal. We leave the question of the effects of such exclusion for future research.
Assumption 5 There exists \( \hat{A} \) such that \( BS(\lambda_{\hat{A}}) \geq BS(\delta) \).

That is, there is a second price auction that does at least as well, in ex-ante terms, as \( \delta \). This assumption can again be interpreted as saying that \( \Delta \) is not too small relative to the asymmetry between \( F_1 \) and \( F_2 \).\(^{40}\)

Theorem 6 Assume A1-A5. Then, the ex-ante optimal SPBA gives the buyer higher expected surplus than any FPHA.

The proof relies on many of the same ideas as that of Theorem 5. As before, \( A \leq \tau \) do not make sense. We then show that for \( A > \tau \), \( \phi' > 1 \). Then, depending on whether \( \phi \) crosses \( \eta_M \), we either show a dominating second price auction as before, or we show that \( \phi \) is in fact dominated by \( \delta \). But then, by A5, we are again done.

Finally, when costs are symmetric (but densities are not necessarily increasing), we can return to our stronger notion of dominance.

Corollary 1 Let \( F_1 = F_2 = F \) and \( \Delta > 0 \). Assume that \( F \) satisfies A2-A4. Then the optimal FPHA is dominated by an appropriately chosen SPBA.

\(^{40}\)In particular, it can be shown that for any \( \varepsilon > 0 \), there is \( \Delta \) large enough that \( BS(\lambda_{\tau+\varepsilon}) > BS(\delta_\tau) \).
9 Conclusion

Our results suggest an underexploited connection between $\rho$-concavity and the study of auctions and mechanism design more generally. That the slope of the bid function is one to one with the local $\rho$-concavity of the associated surplus function and that this in turn is tightly tied to the $\rho$-concavity properties of the underlying distributions certainly suggests that the connection is worth exploration. For example, it would be interesting to see what these tools have to say about first-price auctions with more than 2 players.

We show that for a class of auctions, second-price mechanisms are ex-post superior to first price mechanisms. These results should be interesting to an economic theorist, but also to firms that engage in procurement.

The derivation of the result that $\phi' > 1$ uses techniques that are new, and that seem likely have wider applicability. The degree to which one can generate bounds on the surplus available to each player, and use that to partially characterize equilibrium bid functions seems intriguing. An obvious topic for further research is to get a better understanding of the examples suggesting that $\phi' > 1$ holds more widely than under our conditions.

Our ranking results are primarily in terms of dominance. Further exploration using expected buyer surplus is merited, as is auction design when quality is affected by pre-auction effort.\footnote{See Arozamena and Cantillion (2004) for interesting work in this direction.} Other simple auction forms, such as percentage-bonus auctions, deserve more consideration, especially because of their wide-spread use in practice.\footnote{Marion (2006) and (2007) looks at federal highway contracts with minority preferences which are governed by a proportional mechanism. In a recent paper, Kirkegaard (2010a) argues that the competition for Canadian government research grants is best described by a combination of proportional and fixed handicaps. See also related experimental results by Schotter and Weigelt (1992).} The question of exclusion is also interesting. Extending our analysis to these environments is a question for further research.

10 Appendix

10.1 Proof of Theorem 1

That $\phi(0) = c_1$ follows since a bid of $\beta_2(0) + A$ by 1 wins with probability one and thus dominates any lower bid, and similarly for a bid of $\beta_1(c_1) - A$ by 2. Similarly, $\phi(1 - A) = c_1$. 


If $1$ with type $c$ bids as if his type is $\tilde{c}$, his surplus is $S_1(\tilde{c}; c) = \tilde{F}_2(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. By the envelope theorem,

$$S'_1(c) = \frac{\partial}{\partial c} S_1(\tilde{c}; c) \bigg|_{\tilde{c}=c} = -\tilde{F}_2(\psi(c)). \quad (15)$$

Given that $b_1$ is restricted to be at most $\tilde{c}_1$, $S_1(\tilde{c}_1) = 0$, yielding the expression for $S_1$. Similarly,

$$S'_2(c) = -\tilde{F}_1(\phi(c)), \quad (16)$$

and for $c_2 > \tilde{c}_1 - A$ no $b_2 > c_2$ ever wins, and so $S_2(\tilde{c}_1 - A) = 0$, yielding (1).

Since $\tilde{F}_2(\psi(c))(\beta_1(c) - c) = S_1(c)$, and by (15), $\beta_1(c) = c + \frac{S_1(c)}{S'_1(c)}$.

But then, wherever $\psi$ is differentiable,

$$\beta'_1(c) = 1 + \frac{S'_1(c) (-S'_1(c)) + S_1(c) S''_1(c)}{(S'_1(c))^2} = W_{S_1}(c),$$

and similarly for $\beta'_2$, giving (2).

As strictly increasing functions, $\beta_1$ and $\beta_2$ are differentiable almost everywhere. And, as $\beta_1(\phi(c)) = \beta_2(c) + A$, $\phi$ is continuous, and where $\beta'_1$ and $\beta'_2$ exist,

$$\phi'(c) = \frac{\beta'_2(c)}{\beta'_1(\phi(c))} > 0. \quad (17)$$

Substituting (2) into (17) gives

$$\phi'(c) = \frac{W_{S_2}(c)}{W_{S_1}(\phi(c))} = \frac{S_2(c) f_1(\phi(c))}{S_1(\phi(c))} \frac{\phi'(c)}{f_2(\phi(c))}, \quad (18)$$

using (15) and (16) to evaluate $W_{S_2}$ and $W_{S_1}$. Canceling $\phi'(c) > 0$ and rearranging yields (3).

Now, let us show that $\phi \in C^{k+1}[0, \tilde{c}_1 - A]$. We show the result on $[0,a]$, $a < \tilde{c}_1 - A$. Since $a$ is arbitrary, the result follows. We have that $\phi$ is $C^0[0,a]$. Assume that $\phi \in C^{\tilde{k}}[0,a]$ where $0 \leq \tilde{k} \leq k$. Then, $f_1(\phi)$ and $\tilde{F}_1(\phi)$ belong to $C^{\tilde{k}}[0,a]$. Since $S_2 = \int_{c_1-A}^{\phi(s)} ds$ it follows from the fundamental theorem of calculus that $S_2 \in C^{\tilde{k}+1}[0,a]$ and similarly for $S_1$. But, as a bounded, continuous function on a compact interval, $\phi$ is absolutely continuous and so

$$\phi(c) = \phi(0) + \int_0^c \phi'(t) \, dt = c_1 + \int_0^c \frac{S_1(\phi(t))}{S_2(t)} f_2(\phi(t)) dt.$$
As each part of the integrand is positive and belongs to $C^k[0,a]$, it follows that $\phi \in C^{k+1}[0,a]$. By induction, $\phi \in C^{k+1}[0,a]$. It follows that $\beta_1 \in C^{k+1}[0,c_1]$ and $\beta_2 \in C^{k+1}[0,c_1 - A]$.

### 10.2 Proofs for Sections 3 and 5

We use a series of Lemmas.

**Lemma 7** If $f$ is a log-concave density then $F$ and $\int f$ are themselves log-concave. If $f$ is a (strictly) log-concave density then $f F$ is (strictly) decreasing and $f F$ is (strictly) increasing.

For a proof see, e.g., Prekopa (1971).

**Lemma 8** Let $g$ be positive and log-concave on some interval near 1, with $g(1) = 0$. Then, $W_G(1) = \frac{1}{2 - W_g(1)}$.

**Proof** Let $c^*$ be such that $g$ is log-concave on $[c^*, 1]$ and $g'(c^*) < 0$. By log-concavity, $g'(s) < 0$ for all $s \in [c^*, 1)$. Thus, l'Hôpital's rule applies to give

$$W_G(1) = \lim_{s \to 1} \left( -\frac{g'(s)G(s)}{g''(s)} \right) = -\lim_{s \to 1} \frac{g''(s)G(s) - g'(s)g(s)}{2g'(s)g(s)}$$

$$= -\lim_{s \to 1} \frac{g''(s)G(s)}{2g'(s)g(s)} + \frac{1}{2} = \frac{1}{2} \lim_{s \to 1} \left( -\frac{g''(s)g(s)g'(s)G(s)}{(g'(s))^2} \right) + \frac{1}{2}$$

$$= \frac{1}{2} \lim_{s \to 1} (W_G(s)W_g(s)) + \frac{1}{2}$$

$$= \frac{1}{2} (W_G(1)W_g(1) + 1),$$

where the last step is valid noting that by Lemma 7 $G$ is log-concave on $[c^*, 1]$ and that since $g$ is decreasing near zero, $W_G(1) \geq 0$, and so $W_G(1) \leq 1$ is finite. Rearranging yields the result. □

The following lemma shows why the assumption that $W_G(1)$ is finite is mild.

**Lemma 9** Assume that $g$ is $C^\infty [0,1]$ and has $g(1) = 0$ and $g^{(k)}(1) < \infty$ for all $k$. Then, $W_g(1) \in [0, 1)$. In particular, if we let $n$ be such that $g^{(n)}(1) \neq 0$ while $g^{(k)}(1) = 0$ for all $k \in \{0, 1, \ldots, n-1\}$, then $W_g(1) = \frac{n}{n+1}$.

Note that $n$ must be finite, otherwise by Taylor's expansion $f \equiv 0$. 43
Proof Since $g^{(n)}(1) \neq 0$ while $g^{(n-1)}(1) = 0$,

$$W_{g^{(n-1)}}(1) = \frac{g^{(n-1)}(1)g^{(n+1)}(1)}{(g^{(n)}(1))^2} = 0.$$ 

Assume by induction that $W_{g^{(n-k)}}(1) = \frac{k-1}{k}$ for some $k \in \{1, 2, \cdots, n\}$. Then, since $g^{n-k}(1) = 0$, Lemma 8 applies to $g^{(n-k)}$ (which, since $W_{g^{(n-k)}}(1) < 1$, is log-concave on some interval near 1) to yield

$$W_{g^{(n-k+1)}}(1) = \frac{1}{2 - W_{g^{(n-k)}}(1)} = \frac{k}{k+1}. \qed$$

Proof of Theorem 2 Define $\bar{W}_g(c) = 1 - \rho_g(c)$ and $\bar{W}_g(c) = 1 - \bar{\rho}_g(c)$. An equivalent expression to (4) is

$$2 - \bar{W}_g(c) \leq \frac{1}{\bar{W}_G(c)} \leq 2 - W_g(c). \quad (19)$$

To see this, make the substitutions and manipulate, noting that since $\rho_g > -1$, $\bar{W}_g(c) \leq \bar{W}_g(c) < 2$, while $W_G > 0$ when $g' < 0$, and so the cross multiplications are valid.

Let

$$\bar{J}(c) = \frac{1}{\bar{W}_G(c)} + 2 - \bar{W}_g(c),$$

$$J(c) = \frac{1}{W_G(c)} + 2 - W_g(c),$$

and

$$\bar{J}(c) = \frac{1}{W_G(c)} + 2 - \bar{W}_g(c),$$

so that $\bar{J}(c) \leq J(c) \leq \bar{J}(c)$. Note that

$$W_G'(c) = W_G(c) \left( -\frac{g(c)}{G(c)} - \frac{2g'(c)}{g(c)} + \frac{g''(c)}{g'(c)} \right)$$

$$= W_G(c) \left( -\frac{g'(c)}{g(c)} \right) \left( \frac{-1}{W_G(c)} + 2 - W_g(c) \right)$$

and that

$$W_G(c) \left( -\frac{g'(c)}{g(c)} \right) = \frac{G(c)}{g(c)} \left( \frac{g'(c)}{g(c)} \right)^2 > 0,$$
and so
\[ W'_G(c) =_s J(c), \tag{20} \]
where \(_s =_s \) denotes that LHS and RHS have the same sign. Assume that at some \( c \), \( \frac{1}{W'_G(c)} > 2 - W_g(c) \), or, equivalently, \( J(c) < 0 \). Then, \( J(c) \leq \bar{J}(c) < 0 \), and so \( W'_G(c) < 0 \), and since \( W_g(c) \) is weakly increasing,
\[ J'(c) = \frac{W'_G(c)}{(W_G(c))^2} - W'_g(c) < 0 \]
as well. Thus, if \( J(c) < 0 \), \( J'(c) < 0 \), and it follows that \( \bar{J}(s) < 0 \) for all \( s \in [c, 1] \), and so \( \bar{J}(1) < 0 \). But, as \( W_g \) is continuous and \( W_g(1) \) is finite,
\[ W_g(1) = W_g(1) = \bar{W}_g(1), \]
and so from Lemma 8
\[ 2 - \bar{W}_g(1) = \frac{1}{W_G(1)} = 2 - W_g(1), \tag{21} \]
and hence \( \bar{J}(1) = 0 \), a contradiction. The case \( J(c) < 0 \) is analogous. 

What has been proved here is \( \bar{J}(c) \geq 0 \), and \( \bar{J}(c) \leq 0 \). Together with (20), we thus have the following result.

**Corollary 2** If on some interval \([\bar{c}, 1]\), \( g \) is decreasing and \( g(1) = 0 \) while \( \rho_g \) is increasing, then \( \rho_G \) is decreasing on \([\bar{c}, 1]\). If \( g \) is decreasing, while \( \rho_g \) is increasing on \([\bar{c}, 1]\), then \( \rho_G \) is increasing on \([\bar{c}, 1]\). If \( g \) is log-concave and increasing at \( c \) then \( \rho_G \) is increasing at \( c \).

This is immediate, since if \( W_g \) is increasing then \( W_g(c) = W_g(c) \), and so \( \bar{J}(c) = \bar{J}(c) \geq 0 \), and similarly if \( W_g \) is decreasing. The second part follows from the observation that if \( g \) at \( c \) is increasing and log-concave then \( g' \) is positive and decreasing, as is \( \frac{G'}{g} \). Thus, \( -W_G = \frac{G'}{g^2} \) is decreasing.

**Proof of Lemma 1** Assume \( W_f \tilde{F} \) is increasing and \( \gamma \) is concave. Then,
\[ W_f F_{\circ \gamma}(c) = \frac{\gamma'(c) f(\gamma(c)) \int_c^1 \tilde{F}(\gamma(s)) \, ds}{(F(\gamma(c)))^2} \]
and so if
\[ \gamma'(c) \int_c^1 \tilde{F}(\gamma(s)) \, ds \geq \int_{\gamma(c)}^1 \tilde{F}(s) \, ds \tag{22} \]
then
\[ W_f \bar{F}_\gamma(c) \geq W_f \bar{F}(\gamma(c)) \geq W_f \bar{F}(c) \]
since \( W_f \bar{F} \) is increasing and \( \gamma(c) \geq c \).

To prove (22) define
\[ Q(c) = \gamma'(c) \int_c^1 \bar{F}(\gamma(s)) \, ds - \int_{\gamma(c)}^1 \bar{F}(s) \, ds. \]

Since \( \gamma \) is concave, \( \lim_{c \to 1} \gamma'(c) < \infty \), and so \( Q(1) = 0 \). Also,
\[
Q'(c) = \gamma''(c) \int_c^1 \bar{F}(\gamma(s)) \, ds - \gamma'(c) \bar{F}(\gamma(c)) + \gamma'(c) \bar{F}(\gamma(c)) \\
= \gamma''(c) \int_c^1 \bar{F}(\gamma(s)) \, ds \leq 0
\]
again since \( \gamma \) is concave. Since \( Q' \leq 0 \) and \( Q(1) = 0 \), \( Q(c) \geq 0 \) or equivalently (22). The other case is similar.

**Proof of Lemma 2** Define
\[ Q(c) = \alpha \int_c^1 g^\alpha(s) \, ds - g^{\alpha-1}(c) \int_c^1 g(s) \, ds. \]

Note that \( Q(1) = 0 \) and that
\[
Q'(c) = -\alpha g^\alpha(c) + g^\alpha(c) - (\alpha - 1) g'(c) g^{\alpha-2}(c) \int_c^1 g(s) \, ds \\
= (1 - \alpha) g^\alpha(c) \left( 1 + \frac{g'(c) \int_c^1 g(s) \, ds}{g^2(c)} \right) \\
= (1 - \alpha) g^\alpha(c) \rho_f g(c) \leq 0
\]
since by assumption \( \rho_g \geq 0 \) and so by Theorem 2 \( \rho_f g(c) \geq 0 \). Thus \( Q(c) \geq 0 \).

But,
\[
W_f g^\alpha(c) - W_f g(c) = -\frac{g'(c)}{g^{\alpha+1}(c)(c)} \left( \alpha \int_c^1 g^\alpha(s) \, ds - g^{\alpha-1}(c) \int_c^1 g(s) \, ds \right) \\
= s Q(c) \geq 0
\]
since \( g' \leq 0 \).
Proof of Proposition 5 Since $\phi'(c) \geq 1$, and $\bar{F}$ is log-concave,

$$
\left( \ln \frac{\bar{F}(\phi(c))}{F(c)} \right)' = \phi'(c) \left( \ln \bar{F}'(\phi(c)) \right) - (\ln \bar{F})'(c) \\
\leq (\ln \bar{F})'(\phi(c)) - (\ln \bar{F})'(c) \leq 0
$$

using $\phi(c) \geq c$. Thus, $\frac{\bar{F}(\phi(c))}{F(c)}$ is decreasing, and so

$$
\frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\int_c^1 F(s)ds} \leq \bar{F}(\phi(c)) \cdot \\
\frac{F'(c)}{F(c)}.
$$

But then,

$$
\beta_s(c) = c + \frac{\int_c^1 \bar{F}(s)ds}{F(c)} = c + \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))} = \beta_2(c),
$$

and analogously, $\beta_1(c) \geq \beta_s(c)$. ■

10.3 Proof of Theorem 5

We will prove Theorem 5 (and 6) closely following the geometric arguments behind our previous ranking result, establishing that $\eta_M < 1$ in the relevant domain, and, using Theorem 3, that $\phi' > 1$. Most of the development in this section is also needed for the proof of Theorem 6. So, in what follows, we will at many places replace the assumption that $f$ is increasing by $A3$ (which is weaker by Corollary 2).

10.3.1 Geometry of $\gamma$ and $\eta_M$

We will start with some geometric properties that link $\gamma$ to $\eta_M$. Define $\gamma_F$ by

$$
\frac{\gamma_F(c_2)}{F_1(c_2)} = \frac{\gamma_F(c_2)}{F_2(c_2)}.
$$

Lemma 10 Under $A1$ and $A3$, $\gamma_F \geq \gamma$ and $(\gamma_F)' \geq 1$.

---

Footnote 15

As in Footnote 15, since the objects are monotone, there is no ambiguity in defining $\gamma_F(c_2) = \bar{c}_1$ if $\frac{\phi_1}{\phi_2}(c_1) < \frac{\phi_2}{\phi_1}(c_2)$ for all $c_1$, and analogously if the inequality is reversed.
**Proof** Differentiating \( F_1 (\gamma (c)) = F_2 (c) \) we have \( f_1 (\gamma (c)) \gamma' (c) = f_2 (c) \), and so
\[
\frac{f_2}{F_2} (c) = \gamma' (c) \frac{f_1}{F_1} (\gamma (c)).
\]
Thus, since \( \gamma' \geq 1 \),
\[
\frac{f_1}{F_1} (\gamma (c)) - \frac{f_2}{F_2} (c) = (1 - \gamma' (c)) \frac{f_1}{F_1} (\gamma (c)) \leq 0.
\]
As \( \frac{f_1}{F_1} \) is increasing, it follows that \( \gamma_{\bar{F}} \geq \gamma \), and in particular, \( \gamma_{\bar{F}} (0) \geq \gamma (0) \).

Also since
\[
f_1 (\gamma (c)) \left( \gamma' (c) \right)^2 + f_1 (\gamma (c)) \gamma'' (c) = f_2' (c),
\]
we have
\[
W_{\bar{F}_2} (c) = \frac{F_1 (\gamma (c))}{f_1 (\gamma (c))} \left( -f_1 (\gamma (c)) \left( \gamma' (c) \right)^2 - f_1 (\gamma (c)) \gamma'' (c) \right)
\]
\[
= W_{\bar{F}_1} (\gamma (c)) - \frac{F_1 (\gamma (c))}{f_1 (\gamma (c))} \frac{\gamma'' (c)}{\left( \gamma' (c) \right)^2}
\]
\[
\leq W_{\bar{F}_1} (\gamma (c)), \quad (23)
\]
since \( \gamma'' \geq 0 \). But then, since \( W_{\bar{F}_1} \) is increasing, and since \( \gamma_{\bar{F}} (c) \geq \gamma (c) \),
\[
W_{\bar{F}_1} \left( \gamma_{\bar{F}} (c) \right) \geq W_{\bar{F}_2} (c). \quad (24)
\]
Finally, differentiating \( \frac{F_2'}{F_1} \left( \gamma_{\bar{F}} (c_2) \right) = \frac{F_2'}{F_2} (c_2) \), noting that \( W_{\bar{F}_1} \leq 1 \) by log-concavity, and using (24) we have
\[
\gamma_{\bar{F}}' (c) = \frac{-1 + W_{\bar{F}_2} (c)}{-1 + W_{\bar{F}_1} \left( \gamma_{\bar{F}} (c) \right)} \geq 1. \quad \blacksquare
\]

**Lemma 11** Under \( A1 \) and \( A2 \), \( \eta_M \) crosses \( \gamma \) at most once on \([0, 1]\), and, if it crosses, does so from above. Anywhere that \( \eta_M (c) \geq \gamma (c) \), \( \eta''_M (c) \leq 1 \).

**Proof** Differentiating (14) we have
\[
\eta'_M (c_2) = \frac{1 + \rho_{\bar{F}_2} (c_2)}{1 + \rho_{\bar{F}_1} (\eta_M (c_2))}, \quad (25)
\]

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By computations analogous to (23) we have

\[ W_{F_1}(\gamma(c)) + \left( \frac{F_1(\gamma(c))}{f_1(\gamma(c))} \right) \frac{\gamma''(c_2)}{\gamma'(c_2)^2} = W_{F_2}(c_2) \]

and so \( \rho_{F_1}(\gamma(c_2)) \geq \rho_{F_2}(c_2) \). So, since \( \rho_{F_1} \) is increasing by \( A2 \), for any \( c_1 \geq \gamma(c_2) \), \( \rho_{F_1}(c_1) \geq \rho_{F_2}(c_2) \). In particular at a point where \( \eta_M(c_2) \geq \gamma(c_2) \)
we thus have by (25) that

\[ \eta'_M(c_2) \leq 1 \leq \gamma'(c_2). \]

**10.3.2 Geometry of \( \gamma \) and \( \phi \)**

Now, we examine the relationship of \( \gamma \) and \( \phi \).

**Lemma 12** Anywhere that \( \phi(c_2) \leq \min \left( \gamma_{F_2} \left( c_2 \right), c_2 + A \right) \), \( \phi'(c_2) \geq 1 \). Anywhere that \( \phi(c_2) \geq \max \left( \gamma_{F_2} \left( c_2 \right), c_2 + A \right) \), \( \phi'(c_2) \leq 1 \). Each inequality is strict unless \( \phi(c_2) = \gamma_{F_2} \left( c_2 \right) = c_2 + A \).

**Proof** From Theorem 1, for any \( c_2 > 0 \) such that \( c_1 = \phi(c_2) \) we have

\[
\phi'(c_2) = \frac{S_1(c_1) f_2(c_2)}{F_2(c_2)} - \frac{S_2(c_2) f_1(c_1)}{F_1(c_1)} = \frac{\beta_1(c_1) - c_1 f_2(c_2)}{\beta_2(c_2) - c_2 f_1(c_1)} = (1 + \frac{A - (c_1 - c_2)}{\beta_2(c_2) - c_2}) \left( \frac{f_2(c_2)}{f_1(c_1)} \right),
\]

(26)

since \( \beta_1(c_1) = \beta_2(c_2) + A \) by definition of \( \phi \). When \( \phi(c_2) \leq \min \left( \gamma_{F_2} \left( c_2 \right), c_2 + A \right) \),
each term in (26) at least 1, while when \( \phi(c_2) \geq \max \left( \gamma_{F_2} \left( c_2 \right), c_2 + A \right) \), each term is at most 1. ■

For the case \( f \) increasing, define \( c_M \) as the first point at which \( \eta_M(c_2) = \bar{c}_1 \). Since \( \Delta - \omega(\bar{c}_1) + \omega(1) > 0 \), then \( c_M < 1 \).

**Lemma 13** Under the conditions of Theorem 5, \( \eta_M \geq \lambda_{c_1 - c_M} \).
This follows since \( M(c_M) = c_1 > \gamma(c_M) \) (since \( c_M < 1 \)) and thus from Lemma 11, \( \eta_M \) is everywhere above \( \gamma \) and has \( \eta_M' \leq 1 \). On the other hand \( \eta_M(c_M) = \lambda_{c_1-c_M}(c_M) \) and \( \lambda'_{c_1-c_M} = 1 \) and so \( \eta_M \geq \lambda_{c_1-c_M} \) before \( c_M \).

These results in hand, we can begin by showing that first price mechanisms with \( A \leq \tau \) do not make sense.

**Lemma 14** Under the conditions of Theorem 5 if \( A \leq \tau \), then \( \text{FPBA}_A \) is dominated by \( \text{SPBA}_{\bar{c}_1-c_M} \).

**Proof** Since \( c_1 > c_M > 1 = \tau, \) and by Lemma 13, we have \( \eta_M \geq \lambda_{c_1-c_M} > \lambda_\tau \). It is thus enough to show that since \( \phi \) starts below \( \lambda_\tau \), it cannot get above it. But, from Lemma 10 \( \phi(0) = \gamma(0) \leq \gamma_{\text{no}}(0) \). So, since \( \gamma'(c_2) \geq 1 \) and \( \lambda'(c_2) = 1 \) for all \( c_2 \), if \( \phi(c_2) > \max(\gamma_{\text{no}}(c_2), \lambda_A(c_2)) \) anywhere, then there is a \( \hat{c}_2 \) where \( \phi(\hat{c}_2) > 1 \), and where \( \phi(\hat{c}_2) \geq \max(\gamma_{\text{no}}(\hat{c}_2), \lambda_A(\hat{c}_2)) \).

This contradicts Lemma 12. So, \( \phi \leq \max(\gamma_{\text{no}}, \lambda_A) < \lambda_\tau \). \( \blacksquare \)

### 10.3.3 Boundary Conditions for \( \phi' \)

Given Lemma 14, we can restrict attention to \( A > \tau \). For such \( A \), we aim to show that \( \phi' > 1 \) everywhere on \( (0, \bar{c}_1 - A) \). In this section, we show that this is true at the two boundaries of the domain of \( \phi' \), i.e., at 0 and at \( \bar{c}_1 - A \).

**Lemma 15** Under A1 and A3, if \( A > \tau \) then \( \phi'(0) > 1 \), \( \limsup_{c \to \bar{c}_1 - A} \phi'(c) = \infty \), and \( \liminf_{c \to \bar{c}_1 - A} \phi'(c) > 1 \).

So, (modulo a discontinuity of the second type in \( \phi' \) at \( \bar{c}_1 - A \)), \( \phi' \) becomes arbitrarily large as \( c \to \bar{c}_1 - A \). The key driver is that since \( \bar{c}_1 - A < 1 \), the behavior of \( S_1 \) at \( \bar{c}_1 \) and \( S_2 \) at \( \bar{c}_1 - A \) are very different, with the surplus of player 1 changing much more quickly. Unlike much of the development to date, this result depends crucially on \( A > \tau \).

**Proof** We present a proof based on the assumption that each of \( \beta_1', \beta_2' \) and \( \phi' \) has a well defined limit in the extended real line as \( c \to \bar{c}_1 - A \). A (surprisingly involved) complete proof is available in the working paper.

Note first that since \( \bar{c}_1 - 1 = \tau, \bar{c}_1 - A < 1 \). Since \( \gamma' \geq 1, \gamma(1) - \gamma(0) \geq 1 \), and so \( \bar{c}_1 - 1 \geq \gamma(0) \), or

\[
A > \bar{c}_1 - 1 \geq \gamma(0) = c_1 - 0.
\]
Lemma 12 thus applies at 0 since $\phi(0) = \gamma(0) \leq \gamma_f(0)$ by Lemma 10 and thus $\phi'(0) > 1$.

Next, let us show that $\beta'_1(\bar{c}_1) = \beta'_2(\bar{c}_1 - A) = 0$. To see this, note first that for any $c < \bar{c}_1$, $\beta_1(c)$ earns $\bar{F}_2(\psi(c)) (\beta_1(c) - c)$, while a bid of $\bar{c}_1$ earns at least $\bar{F}_2(\bar{c}_1 - A) (\bar{c}_1 - c)$, and so

$$\bar{F}_2(\psi(c)) (\beta_1(c) - c) \geq \bar{F}_2(\bar{c}_1 - A) (\bar{c}_1 - c),$$

from which

$$\frac{\beta_1(c) - c}{\bar{c}_1 - c} \geq \frac{\bar{F}_2(\bar{c}_1 - A)}{\bar{F}_2(\psi(c))},$$

and so, since $\psi(\bar{c}_1) = \bar{c}_1 - A$,

$$\lim_{c \to \bar{c}_1} \frac{\beta_1(c) - c}{\bar{c}_1 - c} \geq 1. \tag{27}$$

But, by definition,

$$\lim_{c \to \bar{c}_1} \frac{\beta_1(\bar{c}_1) - \beta_1(c)}{\bar{c}_1 - c} = \beta'_1(\bar{c}_1). \tag{28}$$

Adding (27) and (28),

$$\lim_{c \to \bar{c}_1} \frac{\beta_1(\bar{c}_1) - c}{\bar{c}_1 - c} \geq 1 + \beta'_1(\bar{c}_1).$$

Since $\beta_1(\bar{c}_1) = \bar{c}_1$, it follows that $\beta'_1(\bar{c}_1) = 0$.

The first order condition for player 1’s profit at $c < \bar{c}_1$ is

$$\bar{F}_2(\psi(c)) \beta'_1(c) = f_2(\psi(c)) \psi'(c) (\beta_1(c) - c).$$

But, $\psi'(c) = \frac{\beta'_1(c)}{\beta_2(\psi(c))}$, and so

$$\bar{F}_2(\psi(c)) \beta'_1(c) = f_2(\psi(c)) \frac{\beta'_1(c)}{\beta_2(\psi(c))} (\beta_1(c) - c).$$

Cancelling $\beta'_1(c) > 0$, and rearranging,

$$0 < \beta'_2(\psi(c)) = \frac{f_2(\psi(c))}{\bar{F}_2(\psi(c))} (\beta_1(c) - c) < \frac{f_2(\bar{c}_1 - A)}{\bar{F}_2(\bar{c}_1 - A)} (\bar{c}_1 - c).$$

Thus, as $c \to \bar{c}_1$, $\beta'_2(\psi(c)) \to 0$, i.e., $\beta'_2(\bar{c}_1 - A) = 0$. 

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Now, let us show that \( \lim \phi'(c) \in \{0, \infty\} \). To see this, note that

\[
\lim_{c \to \tilde{c}_1 - A} \frac{\beta_1(\phi(c)) - \phi(c)}{\beta_2(c) - c} = \lim_{c \to \tilde{c}_1 - A} \frac{\beta_1'(\phi(c)) - 1}{\beta_2'(c) - 1} \phi'(c) = \lim_{c \to \tilde{c}_1 - A} \phi'(c),
\]

by l'Hôpital’s rule and the previous step. Assume that \( \lim \phi'(c) = \alpha \in (0, \infty) \). By Theorem 1

\[
\phi'(c) = \frac{\beta_1(\phi(c)) - \phi(c)}{\beta_2(c) - c} \frac{f_2(c)}{F_2(c)} \frac{f_1(\phi(c))}{F_1(\phi(c))},
\]

and so by (29), we have

\[
\alpha = \alpha \lim_{c \to \tilde{c}_1 - A} \frac{f_2(c)}{F_2(c)} \frac{f_1(\phi(c))}{F_1(\phi(c))} = 0,
\]

since \( \phi(c) \to \tilde{c}_1 \), while \( \tilde{c}_1 - A < \tilde{c}_2 \), a contradiction.

So, assume that \( \lim_{c \to \tilde{c}_1 - A} \phi'(c) = 0 \). Then, for any small \( t \), there is a last \( c(t) \) at which \( \phi'(c) = t \) (this is well defined since \( \phi \) is continuously differentiable and \([0, \tilde{c}_1 - A]\) is compact). But, by a change of variables,

\[
S_1(\phi(c(t))) = \int_{\phi(c(t))}^{c_1} F_2(\psi(s)) \, ds
\]

\[
= \int_{c(t)}^{\tilde{c}_1 - A} F_2(s) \phi'(s) \, ds,
\]

\[
< t (\tilde{c}_1 - A - c(t)),
\]

since \( \phi'(s) < t \), and \( F_2 < 1 \) for \( s > c(t) \).

Thus, since \( \frac{f_2}{F_2} \) is increasing

\[
S_1(\phi(c(t))) \frac{f_2}{F_2}(c(t)) < t (\tilde{c}_1 - A - c(t)) \frac{f_2}{F_2}(\tilde{c}_1 - A).
\]

The RHS converges to 0 as \( t \to 0 \) and \( c(t) \to \tilde{c}_1 - A \). But then the term \( \frac{1}{t} - 2 \) in (11) diverges for \( r = c(t) \) for small \( t \), the term \( \frac{f_2(\phi(c(t))) - f_2(c(t))}{F_2(\phi(c(t)))} \) diverges as well (noting that \( c(t) < \tilde{c}_1 - A < \tilde{c}_2 \)), and, by log-concavity of \( f \), the remaining term does not go to \(-\infty\). Hence, \( \phi''(c(t)) > 0 \), contradicting the construction of \( c(t) \).\(^{45}\)

\(^{45}\)At the point that (11) is derived, we have used only that \( \phi'(r) < 1 \), which holds for \( r = c(t) \) for small \( t \) by definition, and none of the other properties assumed in the statement of Theorem 3.
Since we have ruled out \( \lim_{c \to c_1 - A} \phi'(c) \in [0, \infty) \), we have that \( \lim_{c \to c_1 - A} \phi'(c) = \infty. \)

#### 10.3.4 Interior Minima and the Completion of the proof of Theorem 5

Given Lemma 15, if \( \phi' \leq 1 \) anywhere on \([0, c_1 - A]\), then \( \phi' \) achieves a global minimum at some \( r \in (0, c_1 - A) \) with \( \phi'(r) \leq 1 \). Given that \( \phi \) is \( C^2 \), and that \( r \) is interior, \( \phi''(r) = 0 \). Observe first that \( \phi(r) \leq r + A \). To see this, note that since \( \lambda_A \) lies weakly above \( \gamma \) (recall that \( \gamma \leq \lambda_r \) and \( A > \tau \)) it follows from Lemma 12 that if \( \phi(c_2) \geq \lambda_A(c_2) \), then \( \phi'(c_2) \leq 1 \). As \( \lambda_A' = 1 \), and since \( \phi(0) = \gamma(0) \leq \lambda_A(0) \), \( \phi \) cannot get above \( \lambda_A \). But, from (26), since by the previous observation, \( 1 + \frac{A - (\phi(r) - r)}{\beta_2(r) - r} \geq 1 \), and so, as \( \phi'(r) \leq 1 \), it must be that \( \frac{\partial}{\partial G} (r) \leq \frac{\partial}{\partial G} (\phi(r)). \)

But then, all requirements of Theorem 3 are met and so, since \( f_1 \) is increasing, we reach a contradiction as in Proposition 4. Hence, \( \phi' > 1 \) everywhere, and so, since \( h_M' \leq 1 \) by A2, a SPBA defined by their unique crossing dominates \( \phi \).

#### 10.4 Proofs for Section 8

**Proof of Lemma 5** Note that

\[
f_1 = -F_1' = nG^{n-1}g,
\]

and so

\[
\frac{f_1}{F_1} = \frac{\frac{\partial}{\partial G}}{G} \quad \text{and} \quad \frac{f_1'}{f_1} = -\left( n - 1 \right) \frac{\frac{\partial}{\partial G}}{G} + \frac{\frac{\partial}{\partial g}}{g}.
\]

Thus,

\[
W_{F_1} = \frac{F_1 - f_1'}{f_1} = \frac{1}{n} \left( n - 1 \right) \frac{\frac{\partial}{\partial G}}{G} - \frac{\frac{\partial}{\partial g}}{g} = \frac{n - 1}{n} + \frac{1}{n} W_G.
\]

So, \( W_{F_1} \) has the same monotonicity as \( W_G \).

Using (30), note that

\[
\frac{(\log f_1)''}{(\log F_1)''} = -\left( \frac{\frac{\partial}{\partial G}}{F_1} \right)' = \left( \frac{\frac{\partial}{\partial G}}{f_1} \right)' = \frac{(n - 1) \left( \frac{\partial}{\partial G} \right)' - \left( \frac{\partial}{\partial g} \right)'}{n} = \frac{n - 1}{n} + \frac{1}{n} \left( \frac{\partial}{\partial g} \right)'.
\]

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So, \( \frac{(\log f_1)''}{f_1''} \) and \( \frac{(\log g)''}{G''} \) reach their minima at the same point. □

**Proof of Lemma 6** By definition \( \Delta + \omega_2 (c) = \omega_1 (\eta_M(c)) \). Since \( f_1 (\bar{c}_1) = f_2 (1) = 0 \) and since \( \omega_i (c) = c + \frac{\xi_i}{f_i} (c) \), the left-hand side diverges as \( c_2 \) approaches 1 and thus \( \eta_M (c) \rightarrow \bar{c}_1 \). Imagine there exists \( \hat{c} \in (0, 1) \) such that \( \eta_M(c) > \gamma(c) \) for all \( c \in (\hat{c}, 1) \). Thus for all \( c \in (\hat{c}, 1) \)

\[
\Delta > \omega_1 (\gamma(c)) - \omega_2 (c)
\]

and since \( F_1 (\gamma(c)) = F_2 (c) \) and \( \gamma'(c)f_1 (\gamma(c)) = f_2 (c) \), (31) becomes

\[
\Delta > (\gamma(c) - c) + \frac{F_1 (\gamma(c))}{f_1 (\gamma(c))} \left( 1 - \frac{1}{\gamma'(c)} \right).
\]

But, since \( \gamma'(0) \geq 1 \), \( \gamma' \) is monotone and \( \gamma \) is not the identity, \( \gamma'(c) > 1 \) near 1, and so the RHS expression diverges, a contradiction. Thus there exists some \( \tilde{c} \) for which \( \eta_M(\tilde{c}) < \gamma(\tilde{c}) \). Since \( \gamma'(c) \geq 1 \), and since \( \eta_M(1) = \gamma(1) \), it must be that for some \( c^* \in (\hat{c}, 1) \), \( \eta_M(c^*) > 1 \). □

### 10.4.1 Proof of Theorem 6

We begin with two lemmas.

**Lemma 16** At any point in \( [\underline{c}_1, \bar{c}_1] \) where \( \frac{f_2}{f_1} \neq 0 \),

\[
\frac{(\log f_1)''}{(\log f_1)'} = \frac{W_2^2 (1 - W f_1)}{(1 - W f_1)}.
\]

If \( f_1 (\bar{c}_1) = 0 \), then \( \frac{(\log f_1)''}{(\log f_1)'} (\bar{c}_1) \) is well defined, and

\[
\frac{(\log f_1)''}{(\log f_1)'} (\bar{c}_1) = W_1 f_1 (\bar{c}_1).
\]

**Proof** Note that

\[
\frac{(\log f_1)''}{(\log f_1)'} = \frac{-\frac{f_1''}{f_1'}}{f_1'} = \frac{-\frac{f_1''}{f_1'} + \left( \frac{f_1'}{f_1} \right)^2}{f_1'}.
\]

(32)
On \([0, \tilde{c}_1]\), \(\frac{f_1}{F_1}\) and \(\frac{f_1}{f_1}\) are well-defined. So, where \(\frac{f_1}{f_1} \neq 0\), we have

\[
\frac{(\log f_1)''}{(\log F_1)''} = -\left(\frac{f_1}{f_1}\right)^2 \left(\frac{f''_1 f_1}{(f'_1)^2} - 1\right) = \frac{W^2_{\tilde{F}_1} (1 - W_{f_1})}{(1 - W_{F_1})},
\]

proving the first claim.

Assume that \(f_1(\tilde{c}_1) = 0\). Then, on some interval \((\tilde{c}, \tilde{c}_1)\), \(f'_1 < 0\) (since, by log-concavity, \(f'_1\) crosses 0 on at most one point or interval). Hence, on this interval, (33) holds. By Lemma 9, \(W_f(\tilde{c}_1) \in [0, 1]\), and, because \(f_1(\tilde{c}_1) = 0\), \(W_{\tilde{F}_1}(\tilde{c}_1) \in (\frac{1}{2}, 1)\). But then, the RHS of (33) is continuous on \((\tilde{c}, \tilde{c}_1)\), and \(\frac{(\log f_1)''}{(\log F_1)''}(\tilde{c}_1)\) is well defined. From Lemma 8 with \(g = f_1\), we have, with a little rearrangement,

\[
1 - W_{f_1}(\tilde{c}_1) = \frac{1 - W_{\tilde{F}_1}(\tilde{c}_1)}{W_{\tilde{F}_1}(\tilde{c}_1)},
\]

and so

\[
\frac{W^2_{\tilde{F}_1}(\tilde{c}_1) (1 - W_{f_1}(\tilde{c}_1))}{1 - W_{\tilde{F}_1}(\tilde{c}_1)} = W_{\tilde{F}_1}(\tilde{c}_1). \quad \blacksquare
\]

Next, we have the obvious analog to 14.

Lemma 17. Under the conditions of Theorem 6, if \(A \leq \tau\), then \(FPHA_A\) is dominated by \(\delta\).

The proof is similar to 14, noting first that because \(\phi\) starts below \(\lambda_r\), \(\phi\) cannot get above \(\lambda_r\). Subject to never being above \(\lambda_r\), \(\delta\) is the best allocation available.

Completion of the Proof of Theorem 6

Given Lemma 17, we can assume \(A > \tau\). Let us show first that \(\phi' > 1\). As before, if \(\phi' \leq 1\) anywhere then by Lemma 15, there is an interior minimum \(r\), and by Theorem 3, we have a contradiction if \(H(r) \geq 0\). Define

\[
K(t) = \left(1 - \frac{1}{W_{f \tilde{F}_1}(\phi(r))}\right) - 2 \left(\frac{f_1}{F_1} (t) - \frac{f_2}{F_2} (r)\right) + \left(\frac{f'_2}{f_2} (r) - \frac{f'_1}{f_1} (t)\right)
\]

so that

\[
H(r) = K(\phi(r)) = K(\gamma_r (r)) + \int^{\phi(r)}_{\gamma_r (r)} K'(t)dt.
\]

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By the same argument as in the proof of Theorem 5, \( \phi(r) \geq \gamma_\ell \ell(r) \). Thus, it is enough to show that \( K(\gamma_\ell \ell(r)) \geq 0 \) and that \( K'(t) \geq 0 \).

By definition, \( \frac{F}{F_1} \left( \gamma_\ell \ell \right) - \frac{F_2}{F_2}(r) = 0 \), and thus

\[
K(\gamma_\ell \ell(r)) = \frac{f_2'}{f_2}(r) - \frac{f_1'}{f_1}(\gamma_\ell \ell(r)).
\]

Divide \( \frac{F_2}{F_2}(r) \) by \( \frac{F_1}{F_1}(\gamma_\ell \ell(r)) \) by \( \frac{F_2}{F_1} \left( \gamma_\ell \ell \right) \) (the denominators are equal by definition), and apply (24) to conclude that \( K(\gamma_\ell \ell(r)) \geq 0 \).

Now,

\[
K'(t) = \left( \frac{1}{W_f \bar{F}_1(\phi(r)) - 2} \right) \left( \frac{f_1}{f_1} \right)'(t) + \left( -\frac{f_1'}{f_1} \right)'(t)
\]

\[
= s \left( \frac{1}{W_f \bar{F}_1(\phi(r)) - 2} \right) - \left( \frac{f_1'}{f_1} \right)'(t)
\]

\[
\geq \left( \frac{1}{W_f \bar{F}_1(\bar{c}_1) - 2} \right) - \left( \frac{f_1'}{f_1} \right)'(\bar{c}_1)
\]

\[= 0.\]

where the inequality follows by A4 and since \( W_f \bar{F}_1 \) is increasing, while the last line follows by Lemmas 2 and 16.

Hence, as before, we have \( \phi' > 1 \). Consider the first crossing point \( c_2^* \) of \( \phi \) and \( \eta_M \), and let \( \tau^* = \eta_M (c_2^* - c_2) \). Consider first the case \( c_2^* \leq c_\tau \), illustrated in Figure 6. Note that \( \tau^* \geq \tau \), and so, to the right of \( c_2^* \), \( \lambda_\tau^* \) lies above \( \eta_M \) (since between \( c_2^* \) and \( c_\tau \), \( \eta_M' \leq 1 \), while after \( c_\tau \), \( \eta_M \leq \lambda_\tau \)). To the left of \( c_2^* \), \( \eta_M' \leq 1 \). So, the SPBA implementing \( \lambda_\tau^* \) dominates FPHA\( A \).

If \( c_2^* > c_\tau \), then everywhere before \( c_\tau \), \( \phi < \lambda_\tau \leq \eta_M \). Since \( \delta \) coincides with \( \eta_M \) to the right of \( c_\tau \), it thus follows that \( \delta \) dominates \( \phi \). We are thus done since by A5, there exists \( \hat{A} \) such that \( BS(\lambda_{\hat{A}}) \geq BS(\delta) \).

**Proof of Corollary 1** In this case, A1 is clearly satisfied and \( \eta_M \) lies strictly above \( \lambda_\tau = \gamma \) (the main diagonal) on \([0, 1)\). Thus any FPHA\( A \) with \( A < 0 \) is dominated by FPHA\( 0 \). FPHA\( 0 \) cannot be optimal since small \( A > 0 \)

\[\text{Note that this is the only place in the development were we resort to the weaker notion of ex-ante surplus comparisons versus dominance.}\]

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distort away from the optimum on a small set of high costs, but improve the allocation on a substantially bigger set. And, since \( \eta_M \) lies above \( \gamma \), \( \eta_M' \leq 1 \), and so a SPBA implementing a line through the intersection of \( \phi \) and \( \eta_M \) is dominating as it was for Theorem 5.

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