Mathematical Finance, Vol. 18, No. 2 (April 2008), 199–238

OPTIMALITY AND STATE PRICING IN CONSTRAINED FINANCIAL MARKETS WITH RECURSIVE UTILITY UNDER CONTINUOUS AND DISCONTINUOUS INFORMATION

MARK SCHRODER

Department of Finance, Michigan State University

COSTIS SKIADAS

Department of Finance, Northwestern University

We study marginal pricing and optimality conditions for an agent maximizing generalized recursive utility in a financial market with information generated by Brownian motion and marked point processes. The setting allows for convex trading constraints, non-tradable income, and non-linear wealth dynamics. We show that the FBSDE system of the general optimality conditions reduces to a single BSDE under translation or scale invariance assumptions, and we identify tractable applications based on quadratic BSDEs. An appendix relates the main optimality conditions to duality.

KEY WORDS: optimal portfolio, recursive utility, marked point processes, BSDE, FBSDE

1. INTRODUCTION

In this paper, we characterize marginal pricing and optimal consumption/portfolio choice for an investor who trades in a financial market in which information is generated by Brownian motion and marked point processes, and therefore prices can evolve continuously but can also have totally inaccessible jumps. The investor maximizes recursive utility given a possibly non-tradable income stream, and faces convex trading constraints such as missing markets, short-sale constraints, margin requirements, and position limits. The investor's wealth dynamics allow for a non-linear drift term that can capture, for example, market impact, or differential borrowing and lending rates. The paper is essentially an extension of our earlier work on scale/translation invariant formulations (Schroder and Skiadas 2003, 2005) to include jumps, non-linear wealth dynamics, as well as a unified and simplified characterization of state price densities. In special cases, we investigate the applicability of quadratic BSDE methodology in the presence of jumps. For example, quadratic BSDEs and associated ODE-based solutions turn out to be applicable for a class of translation-invariant formulations with a proposed type of non-additive recursive utility that combines the advantages of exponential additive utility with quadratic

This paper replaces an older version titled "Quasi-arbitrage, Optimality, State Pricing and Duality in Constrained Financial Markets with Continuous and Discontinuous Information." In writing this paper, we have benefited from earlier discussions with Jakša Cvitanić, Darrell Duffie, and Thaleia Zariphopoulou. We also value the feedback we have received from Jerome Detemple. We are responsible for any errors.

Manuscript received August 2006; final revision received April 2007.

Address correspondence to Mark Schroder, Eli Broad Graduate School of Management, Dept. of Finance, Michigan State University, 323 Eppley Center, East Lansing, MI 48824; e-mail: schroder@bus.msu.edu.

representations of risk aversion. On the other hand, for scale-invariant (homothetic) formulations, the quadratic BSDE methodology is of more limited applicability once jumps are introduced.

Our use of recursive utility is mainly motivated by its well-known decision-theoretic advantages over its additive special case (see, for example, Epstein 1992). As explained in Skiadas (2008), any two agents with expected discounted utility that agree on their preferences over deterministic consumption plans must be equally risk averse. Recursive utility allows a partial separation of risk aversion from preferences over deterministic plans. Moreover, we will see that certain non-additive recursive utility specifications present tractability advantages in the presence of price jumps. The continuous-time limit of stochastic recursive utility was introduced by Duffie and Epstein (1992), who termed their utility "stochastic differential utility" (SDU). In an example, they computed the continuous-time limit of Epstein and Zin (1989) utility,[1] which provides a single-parameter extension of expected discounted power utility, and preserves homotheticity. While the power coefficient of the latter is determined by choice over deterministic plans, the new parameter of Epstein–Zin utility adjusts risk aversion without affecting preferences over deterministic plans. In a Brownian setting, Lazrak and Quenez (2003) proposed generalized SDU (GSDU) as a way of unifying SDU with various multiple-prior formulations appearing in the literature. Skiadas (2008) argues that GSDU includes interesting models of source-dependent risk aversion that are outside the SDU class. In this paper, we introduce a natural extension of GSDU to our filtration structure, and we refer to it simply as "recursive utility."

This paper contributes to a large literature of dynamic portfolio theory that is rooted in the seminal papers of Merton (1969, 1971), Karatzas et al. (1987), and Cox and Huang (1989). Immediately related to this work are the papers by[2] El Karoui et al. (2001) and Schroder and Skiadas (1999, 2003, 2005), who assumed Brownian information and recursive utility. The first two of these papers provided optimality conditions in complete markets as a FBSDE system. Schroder and Skiadas (1999) assumed linear wealth dynamics and SDU, and emphasized the solution and existence theory for Epstein–Zin utility. El Karoui et al. (2001) on the other hand developed optimality conditions with GSDU and wealth dynamics that can be non-linear, and they proved existence based on growth-Lipschitz conditions (that exclude Epstein–Zin utility). Assuming linear wealth dynamics, Schroder and Skiadas (2003, 2005) emphasized the role of scale and translation invariance, respectively, under recursive utility and convex trading constraints in incomplete markets, providing necessary and sufficient conditions for optimality, but leaving aside issues of existence. The scale-invariant formulation is based on homothetic utility (which includes the Epstein–Zin case), and tradable income. The translation-invariant formulation is based on quasi-linear utility (which includes the recursive extension of additive discounted exponential utility), and allows non-tradable income. In both cases, the FBSDE system of the optimality conditions uncouples, and reduces to a single BSDE.

[1] Epstein–Zin utility is a homothetic parametric example of Kreps and Porteus (1978) utility that is increasingly replacing expected discounted power utility as a benchmark dynamic utility form in economic theory. Based on a heuristic argument, Duffie and Epstein (1992) proposed a continuous-time version of Epstein–Zin utility as a solution to a certain BSDE, which violates the Lipschitz-growth assumptions required by the BSDE theory of Pardoux and Peng (1990) and Duffie and Epstein (1992). For the Brownian case, existence, uniqueness, and basic properties of the BSDE arising with continuous-time Epstein–Zin utility was shown by Schroder and Skiadas (1999). BSDE theory has been extended in other directions by Lepeltier and Martin (1997, 1998, 2002) and many others (see also El Karoui and Mazliak 1997).

[2] See also Tang and Li (1994) for a stochastic maximum principle in a jump-diffusion setting.

As noted earlier, the above papers are extended here by introducing dynamics that are driven by marked point processes, as well as Brownian motion, thus accommodating price jumps that are increasingly becoming central components in the modeling of stochastic volatility and risk premia (see, for example, the book by Cont and Tankov [2004]). Moreover, the paper provides a unified framework, extending El Karoui et al. (2001) by the inclusion of trading constraints, and our earlier papers by allowing wealth non-linearities,[3] as in Cuoco and Cvitanić (1998), and more general trading constraints. The paper's main results are a set of sufficient conditions for state pricing and optimality in the general formulation, followed by the simplification of these conditions and the proof of their necessity (under regularity) in the translation/scale invariant cases, which are the paper's main focus. Necessity in the general case can be dealt with easily if one allows unrestricted lump-sum consumption. As the scale/translation invariant cases illustrate, however, the form of allowable plans, on which the necessity argument is sensitive, is application-dependent. For this reason, the discussion of necessity in the general case is left out of the paper. Finally, we give a set of tractable specifications based on quadratic BSDEs and recursive utilities that are necessarily non-additive in the presence of jumps.

Questions of BSDE existence and computation will not be addressed. The definition of recursive utility with marked point processes, and the associated optimality conditions, require BSDE forms of the type that is discussed (for the case of Poisson measures) by Tang and Li (1994), Barles et al. (1997), Pardoux (1997), Pardoux et al. (1997), Becherer (2006), Royer (2006), and others, albeit, under Lipschitz restrictions on the BSDE driver that rule out interesting representations of risk aversion. Analogous comments apply with respect to numerical methods. The emphasis of this paper's analysis is on the formulation of broad conditions that uncouple the dependence of wealth, utility, and shadow price of wealth in the forward–backward system of the optimality conditions, resulting in a single BSDE, or corresponding PDE in a Markovian setting (see Ma et al. 1994). While this uncoupling plays a significant role in simplifying the numerical solution of the optimality conditions, we do not address computational issues, except for pointing out classes of incomplete-market models for which the BSDEs reduce to ODE systems of the Riccati type. The numerical solution of BSDEs has received considerable attention recently, with contributions by Douglas et al. (1996), Chevance (1997), Bally and Pages (2002), Ma et al. (2002), Zhang (2004), Bouchard and Touzi (2004), Bouchard and Elie (2005), Gobet et al. (2005), Lemor et al. (2006), and others. We hope the type of applications discussed in this paper helps motivate work on the theory and computations of BSDEs and FBSDEs in interesting directions.

Several authors have considered unconstrained optimal portfolio choice with jump-diffusion prices, additive utility, and linear wealth dynamics. Merton (1971) solved a portfolio problem with power utility and a single stock with Poisson jump risk and i.i.d. instantaneous returns. The Merton solution was extended by Framstad et al. (1998) to Lévy-type price dynamics. Aase (1984, 1986) considered a general stochastic investment opportunity set and logarithmic utility (implying myopic behavior). Solutions with a stochastic investment opportunity set that maximize expected power utility of terminal wealth have been obtained by Wu (2003), who assumed an Ornstein–Uhlenbeck instantaneous expected stock return, and Liu and Longstaff (2003), who assumed a jump-diffusion process for the return diffusion coefficient. The monograph by Øksendal and

[3] El Karoui et al. (2001) also allow the budget equation to be non-linear with respect to consumption. While this type of generality does not present difficulties, it detracts from our modeling focus and we therefore forgo it.

Sulem (2005) reviews the application of the Hamilton–Jacobi–Bellman (HJB) approach in jump-diffusion contexts. Existence of an optimal policy for an agent with time-additive utility in complete markets and jump-diffusion returns was studied in Jeanblanc-Picqué and Pontier (1990) and Bardhan and Chao (1995).

A number of papers, starting with He and Pearson (1991), Karatzas et al. (1991), and Cvitanić, and Karatzas (1992), use a duality approach to optimal consumption/portfolio choice and pricing under convex trading constraints in Brownian settings with additive utility. Notable examples with liquidity constraints include He and Pagés (1993), El Karoui and Jeanblanc-Picqué (1998), and Detemple and Serrat (2003). While duality is not our main focus, we relate our optimality conditions to duality in an Appendix. Another main approach to constrained dynamic portfolio theory relies on the HJB equation. Notable examples with Brownian information, additive utility, and i.i.d. instantaneous returns include Zariphopoulou (1994) and Vila and Zariphopoulou (1997), who deal with borrowing constraints, and Duffie et al. (1997), who study non-tradable income. Cuoco (1997) shows existence of an optimum with non-tradable income, convex trading constraints, and additive utility in a general Brownian setting. Kramkov and Schachermeyer (1999, 2003) study duality and existence issues with non-tradable income assuming expected utility for terminal wealth, and more general semi-martingale price processes.

More recently, Detemple and Rindisbacher (2005) provided a solution to a constrained portfolio problem with Brownian information, a stochastic-volatility Vasicek short-rate process, and power expected utility for terminal wealth. Their formulation is within the scope of the models examined in Schroder and Skiadas (2003), which is in turn within the scale-invariant formulation of this paper.[4] In particular, the optimal trading strategy expression of Example 7.3 of Schroder and Skiadas (2003) applies,[5] which used in conjunction with Proposition 31 of Schroder and Skiadas (2003) reduces the problem to a single BSDE, the computation of which we have not addressed. The Detemple–Rindisbacher analysis relies on backward equations involving Malliavin derivatives, including an explicitly solved parametric case, extending earlier contributions by Ocone and Karatzas (1991), and Detemple et al. (2003).

The remainder of this paper is organized in seven sections and two appendices. Section two sets up the problem primitives. Section three relates optimality to a notion of state pricing, without yet introducing the specific recursive utility form, and gives sufficient conditions for a given process to be a state price density. Section four begins the specialization of the theory to recursive utility, with the main result being sufficient optimality conditions as a FBSDE system. The remaining sections present applications based on translation or scale invariance, including necessity results for the state-pricing and optimality conditions introduced earlier as sufficient conditions, and applications based on quadratic BSDEs. Appendix A relates the main text's optimality conditions to a dual

[4] Detemple and Rindisbacher (2005), Section 6, suggest that our method does not uncouple the forward–backward equations of the optimality conditions. The confusion arises because they use the condition that value is proportional to wealth at the optimum without first normalizing the utility so that it is ordinally equivalent and homogeneous of degree one. In Schroder and Skiadas (2003) and this paper we use such a normalization. Alternatively, as in Merton (1971) and Schroder and Skiadas (1999), one can leave the utility as an expected power, but then the value function must be expressed as a power of wealth. In either case, the forward–backward equations of the optimality conditions uncouple due to the scale-invariant structure of the problem.

[5] While the trading constraints there are stated as being constant, the exact same expressions and arguments apply if they are allowed to be stochastic. The same will true of the modeling of trading constraints in this paper.

characterization of the agent's problem. Appendix B contains proofs omitted from the main text.

2. STOCHASTIC SETTING AND NOTATION

We begin by defining the stochastic and informational setting, and by setting up some notation. For background mathematical results we refer to Jacod and Shiryaev (2003), which is hereafter abbreviated to JS. The stochastic setting is similar to that of Björk et al. (1997). Vectors are always assumed to be column vectors, with a prime denoting transposition.

Given is a probability space $(\Omega, \mathcal{F}, P)$, with corresponding expectation operator $\mathbb{E}$, and a *mark space* $(E, \mathcal{B}(E))$, where E is either Euclidean space, with $\mathcal{B}(E)$ denoting the Borel σ-algebra, or a discrete space, with $\mathcal{B}(E)$ denoting the set of all subsets of E. Information is revealed by:[6]

(i) A d-dimensional standard Brownian motion, $B = (B^1, \ldots, B^d)'$.
(ii) A sequence of random times $\{T_n\}$, such that $T_{n+1} > T_n$ a.s. and $\lim_{n\to\infty} T_n = \infty$ a.s.
(iii) A sequence of E-valued random variables $\{J_n\}$.

The agent's problem will be formulated over a finite time horizon $[0, T]$. Associated with the sequence $\{(T_n, J_n)\}$ is the counting random measure $p : \Omega \times \mathcal{B}([0, T]) \otimes \mathcal{B}(E) \to \{1, 2, \ldots\}$, where

$$p([0, t], S) = \sum_{n=1}^{\infty} 1\{T_n \le t, J_n \in S\}, \quad t \le T, \quad S \in \mathcal{B}(E).$$

The random measure p is known as an E-marked point process, or E-valued multivariate point process (JS, definition II.1.23).

The underlying filtration, $\{\mathcal{F}_t : t \in [0, T]\}$, relative to which all processes are assumed to be adapted, is defined as the augmented filtration generated by (B, p), meaning that $\mathcal{F}_0$ is generated by the P-null events, and

$$\mathcal{F}_t = \mathcal{F}_0 \vee \sigma\{B(s), p([0, s], S) : s \le t, S \in \mathcal{B}(E)\}, \quad t \in [0, T].$$

We also assume that $\mathcal{F} = \mathcal{F}_T$ throughout. The optional and predictable σ-algebras relative to this filtration are denoted $\mathcal{O}$ and $\mathcal{P}$, respectively.

The compensator of $p(\omega, dt \times dz)$ is assumed throughout to be of the form $h(\omega, t, dz)\, dt$, for an *intensity kernel* h. The corresponding compensated random measure is defined as

$$\hat{p}(\omega, dt \times dz) = p(\omega, dt \times dz) - h(\omega, t, dz)\, dt.$$

The intensity assumption implies that the stopping times $\{T_n\}$ arrive as "surprises," a statement made precise in corollary II.1.19 of JS. The Brownian motion on the other hand reveals information continuously in time, meaning that Bayesian estimates conditionally on Brownian motion are updated continuously.

[6] While our setting includes a quite general class of jump processes, for notational simplicity, we assume that almost surely there is a finite number of jump events over finite time intervals. The extension to point processes violating this condition (using the stochastic integral for point processes) is straightforward, as long as the martingale representation property holds.

We think of the Brownian motions dB_t^i and the compensated point processes $\hat{p}(dt \times dz)$ as forming a set of (conditionally zero-mean) instantaneous linear factors. It will be convenient to notationally unify the labeling of all factors by introducing the factor index set

$$Z = \{z^1, \dots, z^d\} \cup E.$$

The symbols $z^1, \dots, z^d$, which are assumed not to be members of E, label the respective Brownian factors $dB_t^1, \dots, dB_t^d$. An element $z \in E$ labels the point-process factor corresponding to jumps of size z. A generic $z \in Z$ can therefore label either a Brownian or a point-process factor. The set Z is given a measurable structure by the σ-algebra $\mathcal{B}(Z)$ generated by $\mathcal{B}(E)$ and all subsets of $\{z^1, \dots, z^d\}$.

Extending the interpretation of an instantaneous linear factor structure, we now wish to define predictable instantaneous factor loadings for a local martingale. We define the set $\mathcal{V}$ of *volatility processes*[7] to consist of every Z-indexed process of the form σ: $\Omega \times [0, T] \times Z \to \mathbb{R}$ satisfying:

(i) σ is $\mathcal{P} \otimes \mathcal{B}(Z)$-measurable.
(ii) $\int_0^T (\sum_{i=1}^d \sigma(t, z^i)^2 + \int_E |\sigma(t, z)| h(t, dz))\, dt < \infty$ a.s.

A process M in this context is a local martingale if and only if[8] there exists some $\sigma \in \mathcal{V}$ such that

$$dM_t = \sum_{i=1}^d \sigma(t, z^i)\, dB_t^i + \int_E \sigma_t(t, z)\hat{p}(dt \times dz). \tag{2.1}$$

We interpret $\sigma(t, z)$ as the time-t loading of dM_t on factor $z \in Z$.

A key part of our analysis will be a notion of linear factor pricing. For this purpose, it will be notationally convenient to define the kernel $\nu : \Omega \times [0, T] \times \mathcal{B}(Z) \to \mathbb{R}_+$ by

$$\nu(\omega, t, \{z^i\}) = 1 \text{ for } i = 1, \dots, d, \quad \text{and} \quad \nu(\omega, t, S) = h(\omega, t, S) \text{ for } S \in \mathcal{B}(E).$$

Therefore, for any (suitably measurable) function of the form $b : \Omega \times [0, T] \times Z \to \mathbb{R}$,

$$\int_Z b(t, z)\nu(t, dz) = \sum_{i=1}^d b(t, z^i) + \int_E b(t, z)h(t, dz), \tag{2.2}$$

provided the last integral is well defined.

A *cash flow* is any optional process x such that $\mathbb{E}[\int_0^T x_t^2\, dt + x_T^2] < \infty$. We interpret x_t as a time-t payment rate, and x_t as a lump-sum terminal payment. The set of all cash flows is denoted $\mathcal{H}$, which we regard as a Hilbert space under the inner product

$$(x \mid y) = \mathbb{E}\left[\int_0^T x_t y_t\, dt + x_T y_T\right], \quad x, y \in \mathcal{H}.$$

We identify any two elements $x, \tilde{x} \in \mathcal{H}$ such that $(x - \tilde{x} \mid x - \tilde{x}) = 0$. A cash flow x is *strictly positive* if the process $1\{x_t \le 0\}$ is identified with the zero process in $\mathcal{H}$. The set of strictly positive cash flows is denoted $\mathcal{H}^{++}$. An element $\pi \in \mathcal{H}^{++}$ will also be interpreted

[7] The set $\mathcal{V}$ should not be confused with the set of processes of finite variation in JS.

[8] As explained in Björk et al. (1997), this claim follows from the Fundamental Representation Theorem (JS, chapter III, 4d, theorem 4.29 and corollary 4.31).

later on as a state price density, in which case $(\pi \mid x)$ is interpreted as a present value[9] of the cash flow x.

The set of predictable process is denoted $\mathcal{P}$ (same as the predictable σ-algebra), and

$$\mathcal{P}_1 = \left\{ x \in \mathcal{P} : \int_0^T |x(t)|\, dt < \infty \text{ a.s.} \right\}.$$

The set of all (real-valued) semi-martingales is denoted $\mathcal{S}$, while

$$\mathcal{S}_\alpha = \left\{ X \in \mathcal{S} : \mathbb{E}\left[\text{ess sup}_t |X_t|^\alpha\right] < \infty \right\}, \quad \alpha = 1, 2.$$

Given any *càdlàg* (right-continuous with left limits) process X, we write

$$X_{t-} = \lim_{s \uparrow t} X_s, \quad \Delta X_t = X_t - X_{t-}, \quad X_{0-} = X_0.$$

The semi-martingale X is *strictly positive* if $X_t > 0$ and $X_{t-} > 0$ a.s. for all t. The set of strictly positive semi-martingales is denoted $\mathcal{S}^{++}$, while $\mathcal{S}_\alpha^{++} = \mathcal{S}_\alpha \cap \mathcal{S}^{++}$. We recall that a semi-martingale X is defined to be *special* if it can be uniquely decomposed as $X = A + \hat{X}$, where A is of finite variation and predictable, and $\hat{X}$ is a local martingale with $\hat{X}_0 = 0$. Proposition 4.23 of JS shows that a semi-martingale X is special if $X - X_0 \in \mathcal{S}_1$.

3. MARKET, OPTIMALITY, AND STATE PRICING

We consider an agent who is endowed with some initial wealth and income over time, and can trade in a financial market, possibly under constraints. The formal primitives of this set-up are introduced below, followed by sufficient conditions for state pricing and optimality.

We assume that the agent's preferences are dynamically consistent, and we therefore formulate the optimization problem from the point of view of time zero (see, for example, Skiadas [2008] for related discussion). Formally, the agent is characterized by

(i) A convex cone $\mathcal{C} \subseteq \mathcal{H}$ whose elements are the *consumption plans*. Every element of $\mathcal{C}$ is assumed to be valued in a given open interval $I_c \subseteq \mathbb{R}$ (typically equal to $\mathbb{R}$ or $\mathbb{R}_{++}$).
(ii) A concave function $U_0 : \mathcal{C} \to \mathbb{R}$, representing the agent's (time-zero) *utility function.* We assume that U_0 is (strictly) *increasing*, meaning that, for every non-zero cash flow x taking non-negative values, $c, c + x \in \mathcal{C}$ implies $U(c + x) > U(c)$.
(iii) A pair $(w_0, e) \in \mathbb{R} \times \mathcal{C}$ representing the agent's *endowment*: an initial wealth w_0, followed by an endowed consumption plan e. For $t < T$, e_t represents an endowment rate, while e_T represents a lump-sum terminal endowment.

The financial market consists of trading in a default-free money-market account and m risky assets. The agent's financial wealth process will be denoted W, a *càdlàg* process, with W_t representing total time-t financial wealth, not including the present value of the future endowment. The agent's risky-asset positions will be represented by a predictable process $\phi = (\phi^1, \ldots, \phi^m)' \in \mathcal{P}^m$. The agent's *pre-time-$t$* market positions (that is, positions

[9] The results of the following two sections extend to a setting with cumulative consumption plans, in which case one must define $(\pi \mid X) = \mathbb{E}[\int_0^T \pi_{t-} dX_t + [\pi, X]_T]$, where X is a special semi-martingale, with $X_t - X_s$ representing a cumulative payment over $(s, t] \subseteq [0, T]$. In this paper, we assume we can write $dX_t = x_t dt$ for $t < T$.

just prior to time-t lump-sum payments) are specified by the $(1+m)$-dimensional random vector (W_{t-}, ϕ_t), where ϕ_t^i and $W_{t-} - \sum_{i=1}^m \phi_t^i$ are the pre-time-t balances in asset i and the money market, respectively. The time-t conditional expected instantaneous return of the agent's portfolio is $f(t, W_{t-}, \phi_t)\,dt$. The money-market cumulative returns are predictable, while the vector of the cumulative returns of the m assets is assumed to be a special semi-martingale, whose local martingale part is denoted $\hat{R}$. The vector (W_{t-}, ϕ_t) will be constrained to lie in a convex set $K \subseteq \mathbb{R}^{1+m}$ at all times.

Given the above interpretations, we now formally define the financial market by the following primitives:

(i) A closed convex set[10] $K \subseteq \mathbb{R}^{1+m}$.

(ii) A $\mathcal{P} \otimes \mathcal{B}(K)$-measurable function $f : \Omega \times [0, T] \times K \to \mathbb{R}$ such that, for every $(\omega, t) \in \Omega \times [0, T]$, $f(\omega, t, \cdot)$ is concave and satisfies the regularity condition

$$f(\omega, t, w, \alpha) \leq \bar{f}_w(\omega, t)w + \bar{f}_\phi(\omega, t)'\alpha, \quad (w, \alpha) \in K, \tag{3.1}$$

for some $\mathbb{R}_+$-valued $\bar{f}_w \in \mathcal{P}_1$ and some $\bar{f}_\phi \in \mathcal{P}_1^m$.

(iii) An m-dimensional local martingale $\hat{R}$ with predictable representation

$$d\hat{R}_t = \sum_{i=1}^d \sigma^R(t, z^i)\,dB_t^i + \int_E \sigma^R(t, z)\hat{p}(dt \times dz), \quad \sigma^R \in \mathcal{V}^m.$$

EXAMPLE 3.1 (linear budget equation). Suppose the money market's instantaneous return is $r_t\,dt$ for some $\mathbb{R}_+$-valued short-rate process $r \in \mathcal{P}_1$, and the risky assets' instantaneous excess returns relative to r are $dR_t = \mu_t^R dt + d\hat{R}_t$ for some $\mu^R \in \mathcal{P}_1^m$. In this case,

$$f(\omega, t, w, \alpha) = r(\omega, t)w + \mu^R(\omega, t)'\alpha, \quad (w, \alpha) \in K.$$

EXAMPLE 3.2 (decreasing marginal returns). Marginal expected returns that decrease with the size of the investment can be modeled by allowing μ^R in the last example to be a function of α, provided $\mu^R(\omega, t, \alpha)'\alpha$ is concave in α.

EXAMPLE 3.3 (different borrowing and lending rates). Extending Example 3.1, suppose that the money-market rate r_t applies only to positive balances, while borrowing through the money market occurs at the rate $r_t + b_t$, for some $b \in \mathcal{P}_1$ valued in $\mathbb{R}_+$. In this case,

$$f(\omega, t, w, \alpha) = r(\omega, t)w + \mu^R(\omega, t)'\alpha - b(\omega, t)\left(\mathbf{1}'\alpha - w\right)^+, \quad (w, \alpha) \in K.$$

For a related analysis see appendix B of Cvitani and Karatzas (1992).

A *trading plan* is any m-dimensional process $\phi \in \mathcal{P}^m$ such that $\phi_t'\bar{f}_\phi \in \mathcal{P}_1$ and $\phi'\sigma^R \in \mathcal{V}$. A wealth process is any semi-martingale W such that $\bar{f}_w W_- \in \mathcal{P}_1$ and $W^- \in \mathcal{S}_2$. The last restriction on $W_t^- = \max\{0, -W_t\}$ can be thought of as a form of credit constraint that rules out doubling-type strategies. A *plan* is a triple of a consumption plan c, a wealth

[10] The analysis goes through if K is assumed to be a mapping from $\Omega \times [0, T]$ to closed convex subsets of $\mathbb{R}^m$. We forgo this generality for simplicity of exposition.

process W, and a trading plan ϕ. The plan (c, W, ϕ) is *feasible* if it satisfies the trading constraint

$$(W_{t-}, \phi_t) \in K,$$

and the *budget equation*

$$W_t = w_0 + \int_0^t (f(s, W_{s-}, \phi_s) + e_s - c_s)\, ds + \int_0^t \phi_s' d\hat{R}_s, \quad c_T = W_T + e_T. \tag{3.2}$$

As noted earlier, for $t < T$, c_t and e_t represent consumption rates, while c_T and e_T represent terminal lump-sum consumption. The plan (c, W, ϕ) is *optimal* if it is *feasible* and $U(c) \geq U(\tilde{c})$ for any feasible plan $(\tilde{c}, \tilde{W}, \tilde{\phi})$. A consumption plan c is *feasible* (resp. *optimal*) if there exist a wealth process W and a trading plan ϕ such that (c, W, ϕ) is a feasible (resp. optimal) plan. Clearly, a consumption plan c is optimal if and only if it is feasible and $U(c) \geq U(\tilde{c})$ for every feasible consumption plan $\tilde{c}$. Finally, a trading plan is *feasible* (resp. *optimal*) if it is part of a feasible (resp. optimal) plan.

Given the feasible consumption plan c, the set of *feasible incremental cash flows relative to c* is defined as

$$\mathcal{X}(c) = \{x \in \mathcal{H} : c + x \text{ is a feasible consumption plan}\}.$$

The process $\pi \in \mathcal{H}$ is a *state-price density at c* if

$$(\pi \mid x) \leq 0 \quad \text{for every } x \in \mathcal{X}(c).$$

Note that the state-price density property depends only on the market and reference plan, and not on the agent's preferences. The characterization of state-price densities is of independent interest since they represent admissible pricing rules that are consistent with the lack of incremental arbitrage opportunities at the reference plan.

The process $\pi \in \mathcal{H}$ is a *utility super-gradient density of U_0 at c* if

$$U_0(c + x) \leq U_0(c) + (\pi \mid x) \quad \text{for all } x \text{ such that } c + x \in \mathcal{C}. \tag{3.3}$$

The process $\pi \in \mathcal{S}_2$ is a *utility gradient density of U_0 at c* if, for any x such that $c + \alpha x \in \mathcal{C}$ for some $\alpha > 0$,

$$(\pi \mid x) = \lim_{\alpha \downarrow 0} \frac{U_0(c + \alpha x) - U_0(c)}{\alpha}.$$

The following characterization of optimality in terms of state prices and a utility (super-)gradient density is essentially the same as in Schroder and Skiadas (2003, 2005), and its simple proof is therefore omitted.

PROPOSITION 3.1. *Suppose (c, W, ϕ) is a feasible plan. If $\pi \in \mathcal{H}$ is both a super-gradient density of U_0 at c and a state-price density at c, then the plan (c, W, ϕ) is optimal. Conversely, if the plan (c, W, ϕ) is optimal and $\pi \in \mathcal{H}$ is a utility gradient density of U_0 at c, then π is a state-price density at c.*

In the remainder of this section, we fix a reference feasible plan (c, W, ϕ) and a semimartingale $\pi \in \mathcal{H}^{++}$ with the predictable representation

$$\frac{d\pi_t}{\pi_{t-}} = -\zeta(t)\, dt - \sum_{i=1}^d \eta(t, z^i)\, dB_t^i - \int_E \eta(t, z)\hat{p}(dt \times dz), \quad \zeta \in \mathcal{P}_1, \quad \eta \in \mathcal{V}. \tag{3.4}$$

Below, we formulate conditions on the coefficients (ζ, η) that are sufficient for π to be a state-price density at c. In the following section, we will combine these conditions with the dynamics of a utility super-gradient density, resulting in sufficient optimality conditions as a consequence of Proposition 3.1. The necessity of the state pricing conditions that follow will be shown in later sections (under regularity assumptions) for more special translation/scale invariant formulations.

The process η, which can be thought of as a market-price-of-risk process, will enter our conditions below only through the quantity

$$\Theta(t) = \int_Z \sigma^R(t,z)\eta(t,z)\nu(t,dz) = -\operatorname{Cov}_{t-}\left(d\hat{R}_t, \frac{d\pi_t}{\pi_{t-}}\right). \tag{3.5}$$

The key to formulating a state pricing condition is the following duality result on the present value of feasible incremental cash flows. (Part (b) will be of use in a later section.)

LEMMA 3.1. *Suppose the plan $(c+x, W+V, \phi+\delta)$ is feasible, $\pi \in \mathcal{S}_2^{++}$, and the process*

$$D_t = f(t, W_t + V_t, \phi_t + \delta_t) - f(t, W_t, \phi_t) - \zeta_t V_t - \Theta_t' \delta_t \tag{3.6}$$

satisfies $\mathbb{E}[\int_0^T \pi_t D_t^+ dt] < \infty$.

(a) If $\pi W \in \mathcal{S}_1$, then $(\pi \mid x) \le \mathbb{E}[\int_0^T \pi_t D_t \, dt]$.
(b) If $\pi V \in \mathcal{S}_1$, then $(\pi \mid x) = \mathbb{E}[\int_0^T \pi_t D_t \, dt]$.

Proof. See Appendix B. □

We define and denote the *super-differential of $f(\omega, t, \cdot) : K \to \mathbb{R}$ at $(\bar{w}, \bar{\alpha}) \in K$* by

$$\partial f(\omega, t, \bar{w}, \bar{\alpha}) = \left\{ (d_w, d_\phi) \in \mathbb{R}^{1+m} : \begin{array}{l} f(\omega, t, w, \alpha) \le f(\omega, t, \bar{w}, \bar{\alpha}) \\ \qquad + d_w(w - \bar{w}) + d_\phi'(\alpha - \bar{\alpha}), \text{ all } (w, \alpha) \in K \end{array} \right\}.$$

We write $(\zeta, \Theta) \in \partial f(W_-, \phi)$ a.e. to mean that $(\zeta(\omega, t), \Theta(\omega, t)) \in \partial f(\omega, t, W(\omega, t-), \phi(\omega, t))$ for almost every (ω, t) relative to $dP \times dt$. It should be emphasized that, although notationally suppressed, the constraint set K is an integral part of the definition of ∂f, as illustrated in the examples below.

PROPOSITION 3.2. *Suppose that (c, W, ϕ) is a feasible plan, $\pi \in \mathcal{S}_2^{++}$ has the predictable representation (3.4) for some $(\zeta, \eta) \in \mathcal{P} \times \mathcal{V}$, Θ is well-defined by (3.5), and $\pi W \in \mathcal{S}_1$. If*

$$(\zeta, \Theta) \in \partial f(W_-, \phi) \quad \text{a.e.}, \tag{3.7}$$

then π is a state-price density at c.

Proof. Suppose $(c+x, W+V, \phi+\delta)$ is a feasible plan and D is defined in (3.6). Condition (3.7) implies that $D \le 0$ a.e. Applying part (a) of the last lemma gives $(\pi \mid x) \le 0$. □

EXAMPLE 3.4 (linear budget equation and no trading contraints). Consider the setting of Example 3.1 with $K = \mathbb{R}^{1+m}$. Then condition (3.7) is equivalent to $(\zeta, \Theta) = (r, \mu^R)$ a.e.

EXAMPLE 3.5 (collateral constraint). Suppose that there is a single risky asset ($m = 1$), and, as in Example 3.1, $f(\omega, t, w, \alpha) = r(\omega, t)w + \mu^R(\omega, t)\alpha$. We consider an agent who faces the collateral constraint:

$$K = \{(w, \alpha) \in \mathbb{R}^2 : w \geq \varrho|\alpha|\},$$

for some $\varrho \in (0, 1)$. Then condition (3.7) is equivalent to the a.e. validity of the following restrictions:

$$\delta_t = \zeta_t - r_t \geq 0, \quad \varepsilon_t = \mu_t^R - \Theta_t \in [-\varrho\delta_t, \varrho\delta_t],$$

$$(\phi_t > 0 \implies \varepsilon_t = \varrho\delta_t), \quad (\phi_t < 0 \implies \varepsilon_t = -\varrho\delta_t), \quad (W_t > \varrho|\phi_t| \implies \delta_t = 0).$$

Papers analyzing collateral constraints in a Brownian setting and additive utility include Cuoco and Liu (2000) and Liu and Longstaff (2004).

The reader can work out examples of the form of condition (3.7) for cases of short-sale constraints, incomplete markets, differential borrowing and lending rates, and so on.

4. OPTIMALITY UNDER RECURSIVE UTILITY

In the remainder of this paper we explore optimality for specifications with recursive utility. We begin in this section with a formulation of sufficient optimality conditions as a FBSDE system. The setting will be specialized in subsequent sections by imposing translation or scale invariance assumptions that nest familiar formulations with exponential or power additive utility. For these cases, we will see that the FBSDE system of the optimality system reduces to a single BSDE, which, under regularity assumptions, is also necessary for optimality.

We henceforth assume that there exist a finite measure h^* on $\mathcal{B}(E)$ and some constant $M > 0$ such that

$$h(\omega, t, S) \leq Mh^*(S) \quad \text{for every } (\omega, t, S) \in \Omega \times [0, T] \times \mathcal{B}(E). \tag{4.1}$$

$\mathcal{Z}$ denotes the set of every $\mathcal{B}(Z)$-measurable function $\upsilon : Z \to \mathbb{R}$ such that $\int_E |\upsilon(z)|h^*(dz) < \infty$. We endow $\mathcal{Z}$ with the norm $\|\upsilon\|_{\mathcal{Z}} = \sum_{i=1}^d \upsilon(z^i)^2 + \int_E |\upsilon(z)|h^*(dz)$, relative to which the set $\mathcal{B}(\mathcal{Z})$ of Borel subsets of $\mathcal{Z}$ is defined.

Recursive utility is defined in terms of the following primitives:

- (i) A set $\mathcal{U} \subseteq \mathcal{S}_2$ of *utility processes*. Every utility process is assumed to be valued in the interval $I_U \subseteq \mathbb{R}$ (typically, $I_U = I_c$).
- (ii) A function $F : \Omega \times [0, T] \times I_c \times I_U \times \mathcal{Z} \to \mathbb{R}$, called the *aggregator*, such that $F(\omega, T, c, U, \Sigma)$ does not depend on the arguments $(U, \Sigma) \in I_U \times \mathcal{Z}$, and is therefore denoted $F(\omega, T, c)$. The aggregator F is assumed to be $\mathcal{O} \otimes \mathcal{B}(I_c) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{Z})$-measurable.

We will be using (c, U, Σ) to denote both a triple of processes and a dummy variable in $I_c \times I_U \times \mathcal{Z}$, with the meaning being clear from the context.

The utility specification is based on the following condition, assumed throughout the rest of this paper. Equation (4.2) below is the natural extension of the BSDE notion of Pardoux and Peng (1990) and Pardoux (1997) to our filtration. Intuitively, we think of a BSDE as a general backward recursion on the information tree. A heuristic decision-theoretic motivation for defining utility as a BSDE solution can be found in Skiadas (2008). (While the latter refers to a Brownian filtration, the discussion extends to any setting with a locally linear factor structure, such as the present one.) General existence results for our applications do not yet exist, although several partial results are available in the BSDE literature.[11]

CONDITION 4.1 (Standing Assumption).

(*a*) *Given any* $c \in \mathcal{C}$, *there exists a unique process pair* $(U, \Sigma) \in \mathcal{U} \times \mathcal{V}$ *such that* $\Sigma(\omega, t, \cdot) \in \mathcal{Z}$ *for a.e.* (ω, t) *and*

$$dU_t = -F(t, c_t, U_t, \Sigma_t)\, dt + \sum_{i=1}^{d} \Sigma_t(z^i)\, dB_t^i + \int_E \Sigma_t(z)\hat{p}(dt \times dz), \qquad U_T = F(T, c_T), \tag{4.2}$$

where $F(T, c_T)$ *denotes the terminal utility from lump-sum consumption* c_T. *We write* $(U(c), \Sigma(c))$ *when the dependence on* c *is to be emphasized. The utility function* $U_0 : \mathcal{C} \to \mathbb{R}$ *is specified by letting* $U_0(c)$ *be the initial value of the* utility process $U(c)$.

(*b*) *The aggregator* $F(\omega, t, c, U, \Sigma)$ *is concave in* (c, U, Σ) *and differentiable in* c. *For any* $(\omega, t, U, \Sigma) \in \Omega \times [0, T] \times I_U \times \mathcal{Z}$, *the function* $F_c(\omega, t, \cdot, U, \Sigma)$ *maps* I_c *onto* $(0, \infty)$, *where* F_c *denotes the partial derivative of* F *with respect to* c.

REMARK 4.1. If $I_c = \mathbb{R}_{++}$, the assumption on F_c of part (b) guarantees that a non-negativity constraint on consumption is non-binding.

For any $(\omega, t) \in \Omega \times [0, T]$, the *super-differential* of $F(\omega, t, \cdot)$ at $(c, U, \Sigma) \in I_c \times I_U \times \mathcal{Z}$, denoted $\partial F(\omega, t, c, U, \Sigma)$, is the set of all $(d_c, d_U, d_\Sigma) \in (0, \infty) \times \mathbb{R} \times \mathcal{Z}$ satisfying

$$F(\omega, t, c + \alpha, U + \beta, \Sigma + \gamma) \le F(\omega, t, c, U, \Sigma) + d_c\alpha + d_U\beta + \int_Z d_\Sigma(z)\gamma(z)\nu(\omega, t, dz),$$

for all $(\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R} \times \mathcal{Z}$ such that $(c + \alpha) \in I_c$, $U + \beta \in I_U$, and γ is bounded. Since F is assumed to be differentiable in the consumption argument, it will be convenient to define the super-differential with respect to the arguments (U, Σ) :

$$\partial_{U,\Sigma} F(\omega, t, c, U, \Sigma) = \{(d_U, d_\Sigma) : (F_c(\omega, t, c, U, \Sigma), d_U, d_\Sigma) \in \partial F(\omega, t, c, U, \Sigma)\}.$$

EXAMPLE 4.1 (n-variate point process). Suppose there is no Brownian component in the filtration, and the mark space is finite; that is,

$$d = 0 \quad \text{and} \quad E = \{1, \dots, n\}.$$

In this case, $\mathcal{Z} = \mathbb{R}^n$. If $F(\omega, t, c, U, \Sigma)$ is differentiable in (c, U, Σ) for every (ω, t), and the kernel h is strictly positive, then the super-differential $\partial F_{U,\Sigma}(\omega, t, c, U, \Sigma)$ consists of the single element (d_U, d_Σ), where

$$d_U = F_U(\omega, t, c, U, \Sigma) \quad \text{and} \quad d_\Sigma(z) = \frac{F_{\Sigma(z)}(\omega, t, c, U, \Sigma)}{h\,(\omega, t, \{z\})}, \qquad z = 1, \dots, n,$$

[11] For example, the argument of appendix A of Duffie and Epstein (1992) applies in a general filtration but requires that there is an ordinally equivalent version in which the aggregator is not a function of the volatility terms. Pardoux (1997) showed an existence result for the case of Poisson random measures and Lipschitz restrictions on the aggregator.

where, in this context, F_U and $F_{\Sigma(z)}$ represent partial derivatives with respect to the obvious arguments. This completes the example.

In the remainder of this section, we fix a reference plan (c, W, ϕ), and we formulate sufficient conditions for its optimality. The processes (U, Σ, λ) are defined by

$$U_t = U_t(c), \quad \Sigma_t = \Sigma_t(c), \quad \text{and} \quad \lambda_t = F_c\left(t, c_t, U_t, \Sigma_t\right).$$

Recalling part (b) of Condition 4.1, we can invert the equation defining λ to express the consumption plan c as

$$c_t = \mathcal{I}\left(t, \lambda_t, U_t, \Sigma_t\right), \tag{4.3}$$

where the function $\mathcal{I} : \Omega \times [0, T] \times (0, \infty) \times I_U \times \mathcal{Z} \to I_c$ is defined implicitly through the equation

$$F_c\left(t, \mathcal{I}\left(t, y, U, \Sigma\right), U, \Sigma\right) = y, \quad y \in (0, \infty).$$

We derive optimality conditions by computing a utility super-gradient density at c as a stochastically discounted version of the process λ.

PROPOSITION 4.1. *Suppose that* $(F_U, F_\Sigma) \in \mathcal{O} \times \mathcal{V}$ *is such that*

$$(F_U, F_\Sigma) \in \partial_{U,\Sigma} F\left(c, U, \Sigma\right) \ a.e. \tag{4.4}$$

and the process $\mathcal{E} \in \mathcal{S}_2^{++}$ *solves the SDE*

$$\frac{d\mathcal{E}_t}{\mathcal{E}_{t-}} = F_U(t)\, dt + \sum_{i=1}^{d} F_\Sigma(t)(z^i)\, dB_t^i + \int_E F_\Sigma(z)\hat{p}(dt \times dz), \quad \mathcal{E}_0 = 1. \tag{4.5}$$

Assuming it is an element of $\mathcal{H}$, *the process* $\pi = \mathcal{E}\lambda$ *is a super-gradient density of* U_0 *at* c.

Proof. See Appendix B. □

To derive optimality conditions, suppose that λ follows the dynamics

$$\frac{d\lambda_t}{\lambda_{t-}} = \mu_t^\lambda dt + \sum_{i=1}^{d} \sigma_t^\lambda(z^i)\, dB_t^i + \int_E \sigma_t^\lambda(z)\hat{p}(dt \times dz). \tag{4.6}$$

Applying integration by parts (JS I4.45,52) to $\pi = \mathcal{E}\lambda$, with $\mathcal{E}$ defined in (4.5), we find

$$\begin{aligned}\frac{d\pi}{\pi_-} &= \left(F_U + \mu^\lambda + \int_Z \sigma^\lambda(z)F_\Sigma(z)\nu(dz)\right) dt \\ &\quad + \sum_{i=1}^{d}\left(F_\Sigma(z^i) + \sigma^\lambda(z^i)\right) dB^i + \int_E \left(F_\Sigma(z) + \sigma^\lambda(z) + F_\Sigma(z)\sigma^\lambda(z)\right)\hat{p}(dt \times dz).\end{aligned}$$

Matching coefficients with the predictable representation (3.4), we obtain

$$\begin{aligned}-\zeta &= F_U + \mu^\lambda + \int_Z \sigma^\lambda(z)F_\Sigma(z)\nu(dz),\\ -\eta(z) &= F_\Sigma(z) + \sigma^\lambda(z) + F_\Sigma(z)\sigma^\lambda(z)1_{\{z\in E\}}.\end{aligned}$$

Combining these restrictions with the utility BSDE (4.2), the definition of $(\mu^\lambda, \sigma^\lambda)$, the budget equation, the state-pricing restriction of Proposition 3.1, and the definition of

(F_U, F_Σ), we obtain the following optimality conditions in the form of a constrained FBSDE:

CONDITION 4.2 *(optimality conditions). Suppose that the processes*

$$\big(U, \Sigma, \lambda, \sigma^\lambda, W\big) \in \mathcal{U} \times \mathcal{V} \times \mathcal{S}^{++} \times \mathcal{V} \times \mathcal{S}$$

with the trading strategy ϕ solve the system:

$$dU = -F(\mathcal{I}(\lambda, U, \Sigma), U, \Sigma)\, dt + \sum_{i=1}^{d} \Sigma(z^i)\, dB^i + \int_E \Sigma(z)\hat{p}(dt \times dz),$$

$$\frac{d\lambda}{\lambda_-} = -\left(\zeta + F_U + \int_Z \sigma^\lambda(z) F_\Sigma(z)\nu(dz)\right) dt + \sum_{i=1}^{d} \sigma^\lambda(z^i)\, dB^i + \int_E \sigma^\lambda(z)\hat{p}(dt \times dz),$$

$$U_T = F(T, W_T + e_T), \quad \lambda_T = F_c(T, W_T + e_T),$$

$$dW = (f(W, \phi) + e - \mathcal{I}(\lambda, U, \Sigma))\, dt + \phi' d\hat{R}, \quad W_0 = w_0,$$

$$\Theta = -\int_Z \sigma^R(z)\big(F_\Sigma(z) + \sigma^\lambda(z) + F_\Sigma(z)\sigma^\lambda(z)1_{\{z \in E\}}\big)\nu(dz),$$

$$(\zeta, \Theta) \in \partial f(W, \phi), \quad (F_U, F_\Sigma) \in \partial_{U,\Sigma} F(\mathcal{I}(\lambda, U, \Sigma), U, \Sigma), \quad (\phi, W) \in K.$$

The sufficiency of the above conditions for optimality, given some integrability assumptions, is shown in the following result, as a straightforward consequence of our earlier arguments.

PROPOSITION 4.2. *Suppose that Condition 4.2 holds, equation (4.3) defines a consumption plan $c \in \mathcal{C}$, $\mathcal{E} \in \mathcal{S}_2^{++}$ solves SDE* (4.5), *$\pi = \lambda\mathcal{E} \in \mathcal{S}_2$, and $\pi W \in \mathcal{S}_1$. Then (c, ϕ, W) is an optimal plan.*

Proof. By construction, c is a feasible plan. By Proposition 3.2 and the above calculations, π is a state-price density at c. By Proposition 4.1, π is also a utility super-gradient density at c. The optimality of c then follows from Proposition 3.1. □

In subsequent sections, we simplify the above optimality conditions based on translation or scale invariance assumptions that allow us to write simple expressions relating the backward components (U, λ) to the forward component W, effectively reducing the above FBSDE system to a single BSDE. In those cases, Condition 4.2 will be shown (under some regularity) to also be necessary for optimality.

5. TRANSLATION-INVARIANT FORMULATION

We consider a first class of problems for which the FBSDE of the optimality conditions uncouples, based on a notion of translation invariance (or quasi-linearity), which includes familiar additive discounted exponential utility formulations. This section generalizes Schroder and Skiadas (2005) by allowing jumps, a non-linear budget equation, and more general trading constraints.

Throughout this section and the next one, we specialize last section's setting by imposing the following restrictions on the consumption set, the trading constraint set $K \subseteq \mathbb{R}^{1+m}$, and the function $f : \Omega \times [0, T] \times K \to \mathbb{R}$ defining the budget equation (3.2).

CONDITION 5.1.

(a) Consumption can take any value: $I_c = \mathbb{R}$.

(b) For any $c \in \mathcal{C}$ *and bounded* $b \in \mathcal{H}$, $c + b \in \mathcal{C}$.

(c) The vector[12] $\kappa \in \mathbb{R}^m$ *and process* $\mu^\kappa \in \mathcal{P}_1$ *are such that*

$$(w, \alpha) \in K \implies (w + v, \alpha + v\kappa) \in K \text{ for all } v \in \mathbb{R}, \tag{5.1}$$

$$(w, \alpha) \in K \implies f(\omega, t, w + v, \alpha + v\kappa) = f(\omega, t, w, \alpha) + v\mu^\kappa(\omega, t) \text{ for all } v \in \mathbb{R}. \tag{5.2}$$

(d) The bounded consumption plan $\gamma \in \mathcal{H}$ *and the semi-martingale* $\Gamma \in \mathcal{S}_2^{++}$ *are related by*

$$d\Gamma_t = \left(\Gamma_t \mu_t^\kappa - \gamma_t\right) dt + \Gamma_{t-}\kappa' d\hat{R}_t, \quad \Gamma_T = \gamma_T. \tag{5.3}$$

We interpret $\kappa = (\kappa^1, \ldots, \kappa^m)$ as a portfolio allocation, where κ^i represents the value proportion allocated to asset $i \in \{1, \ldots, m\}$, with the remaining proportion allocated to the money market. Conditions (5.1) and (5.2) state that, relative to the position $(w, \alpha) \in K$, the agent can invest any incremental amount of wealth v to a portfolio with value weights κ whose instantaneous return μ^κ does not depend on the agent's market positions. In this interpretation, $v\kappa_i$ is the amount invested in asset $i \in \{1, \ldots, m\}$, and $v(1 - \sum_{i=1}^m \kappa_i)$ is the amount invested in the money market.

EXAMPLE 5.1. Assume, as in Example 3.1, that $f(w, \alpha) = wr + \alpha'\mu^R$. Then equation (5.2) is satisfied with $\mu^\kappa = r + \kappa'\mu^R$.

We interpret the consumption plan γ as a dividend process generated by a fund, that we call the *γ-fund*, whose value allocation is κ at all times, and whose time-t value is Γ_t. The terminal value Γ_T is paid off as a liquidating lump-sum dividend γ_T.

EXAMPLE 5.2. Let $r \in \mathcal{P}_1$ represent the rate process of the money market account, which is assumed to not depend on the agent's trading plan. Suppose that both γ and r are deterministic processes. In this case the γ-fund is implemented by trading in the money market alone; that is, equation (5.3) is satisfied with

$$\kappa = 0, \quad \mu^\kappa = r, \quad \text{and} \quad \Gamma_t = \int_t^T e^{-\int_t^s r_u du}\gamma_s \, ds + e^{-\int_t^T r_u \, du}\gamma_T.$$

This completes the example.

Let the set $K^0 \subseteq \mathbb{R}^m$ and the function $f^0 : \Omega \times [0, T] \times K^0 \to \mathbb{R}$ be defined by

$$K^0 = \{\alpha : (0, \alpha) \in K\} \quad \text{and} \quad f^0(\omega, t, \alpha) = f(\omega, t, 0, \alpha).$$

[12] In Schroder and Skiadas (2005) we assumed, for simplicity, that $\varrho = \Gamma_{-}\kappa$ is constant. Here the exposition is simpler taking κ to be constant instead. The analysis in both papers applies essentially with no change if κ and ϱ are allowed to be stochastic processes, although the resulting generality is not substantial, since one can repackage the assets to make κ or ϱ constant.

Condition 5.1(c) is equivalent to the representations

$$K = \{(w, \alpha) \in \mathbb{R}^{1+m} : \alpha - w\kappa \in K^0\} \quad \text{and}$$

$$f(\omega, t, w, \alpha) = f^0(\omega, t, \alpha - w\kappa) + w\mu^\kappa(\omega, t).$$

Defining the super-differential notation ∂f^0 analogously to ∂f, it is straightforward to check that, under Condition 5.1, the key state-pricing restriction $(\zeta, \Theta) \in \partial f(W, \phi)$ a.e. is equivalent to

$$\mu^\kappa = \zeta + \kappa'\Theta \quad \text{and} \quad \Theta \in \partial f^0(\phi - W\kappa), \quad \text{a.e.} \tag{5.4}$$

Based on this observation, in the following theorem we characterize the dynamics of a state price density under Condition 5.1. The sufficiency proof, which specializes Proposition 3.2, only makes use of part (c) of Condition 5.1. The necessity proof on the other hand requires some additional regularity, and uses all parts of Condition 5.1. We say that return jumps are *bounded above* if σ^R has an upper bound on $\Omega \times [0, T] \times E$, and *bounded away from zero* if there exists $\varepsilon > 0$ such that $\sigma^R(\omega, t, z) \geq \varepsilon$ for all $(\omega, t, z) \in \Omega \times [0, T] \times E$.

THEOREM 5.1. *Suppose Condition 5.1 holds, (c, W, ϕ) is a feasible plan, $\pi \in \mathcal{S}_2^{++}$ has the predictable representation (3.4) for some $(\zeta, \eta) \in \mathcal{P} \times \mathcal{V}$, and Θ is well-defined by (3.5).*

- *(a) (Sufficiency) Suppose $\pi W \in \mathcal{S}_1$. If (5.4) holds, then π is a state-price density at c.*
- *(b) (Necessity) Suppose returns jumps are bounded above and away from zero. If π is a state-price density at c, then (5.4) holds.*

Proof. See Appendix B. □

Turning our attention to the preference side of the problem, we specialize the assumed recursive utility Condition 4.1 by imposing a special aggregator structure:

CONDITION 5.2. *$I_U = \mathbb{R}$ and the aggregator F is of the form*

$$F(\omega, t, c, U, \Sigma) = G\left(\omega, t, \frac{c}{\gamma(\omega, t)} - U, \Sigma\right), \quad F(\omega, T, c) = \frac{c}{\gamma(\omega, T)},$$

for some strictly positive bounded cash flow γ, and some function $G : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ that we call an absolute aggregator.

The above condition implies that U is quasi-linear with respect to γ :

$$U(c + k\gamma) = U(c) + k \quad \text{for all } k \in \mathbb{R} \text{ and } c \in \mathcal{C}.$$

EXAMPLE 5.3 (expected discounted exponential utility). Suppose that β is a given optional process, and the utility process $V(c)$ of the consumption plan c is well-defined by

$$V_t = \mathbb{E}\left[\int_t^T - \exp\left(-\int_t^s \beta_u\, du - \frac{1}{\gamma_s} c_s\right) ds - \exp\left(-\int_t^T \beta_u\, du - \frac{1}{\gamma_T} c_T\right) \middle|\, \mathcal{F}_t\right]. \tag{5.5}$$

The ordinally equivalent utility $U_t(c) = -\log(-V_t(c))$ satisfies Condition 5.2 with

$$G(\omega, t, x, \Sigma) = \beta(\omega, t) - \exp(-x) - \frac{1}{2}\sum_{i=1}^{d} \Sigma(z^i)^2$$

$$- \int_E (\exp(-\Sigma(z)) - 1 + \Sigma(z))h(\omega, t, dz).$$

To show this claim, multiply (5.5) by $\exp(-\int_0^t \beta_u\, du)$, and subtract $\int_0^t \exp(-\int_0^t \beta_u du - c_s/\gamma_s)\, ds$ from both sides, resulting in a martingale on the right-hand side. Then compute the expansion of $\mathrm{d}V$ using integration by parts and a martingale representation of the right-hand-side, and finally expand $U = -\log(-V)$ using Itô's lemma for semimartingales (JS, theorem 4.57).

In the context of the agent's optimization problem, we think of the process γ as part of the preference specification. The argument that follows requires that there exists a trading strategy that generates γ as a dividend stream. Example 5.2 provides a simple case in which γ is generated by trading in the money-market account alone. If either the instantaneous returns of the money market or γ is stochastic, then generally one must trade in risky assets to generate γ. A trading strategy that generates γ can be packaged as a single synthetic security, and be named, say, asset one in our formal setting. In this case, the necessary financing condition (5.3) is satisfied with $\kappa = (1, 0, \ldots, 0)$, provided that trading in asset one is unrestricted, and its instantaneous returns are independent of the agent's positions.

Fixing a candidate optimal plan (c, W, ϕ), let $U = U(c)$, $\Sigma = \Sigma(c)$, and $\lambda = F_c(c, U, \Sigma)$. This section's conditions together imply that, at an optimum, and additional pre-time-t dollar of financial wealth can be optimally invested in the γ-fund, followed by consumption of the resulting incremental dividend stream. This observation and the quasi-linearity of the utility function with respect to γ suggest the following relationships at an optimum, whose validity will be verified formally later on:[13]

$$\phi_t = \phi_t^0 + W_{t-}\kappa, \quad U_t = \frac{1}{\Gamma_t}(Y_t + W_t), \quad \lambda_t = \frac{1}{\Gamma_t}. \tag{5.6}$$

The pair (Y, ϕ^0) is determined by a BSDE, given below, that is independent of financial wealth. The portfolio ϕ_t^0 is optimal at time t given zero time-t financial wealth. Relative to this portfolio, ϕ_t is computed by investing all financial wealth in the γ-fund. Finally, λ_t equals the coefficient of W_t in the expression relating optimal utility and wealth, and represents the time-t shadow price of wealth.

We now put together the above insights to simplify the optimality conditions, starting with some convenient notation. The dividend-yield process of the γ-fund is

$$\delta = \frac{\gamma}{\Gamma_-}. \tag{5.7}$$

The function $J : \Omega \times [0, T] \times Z \to (0, \infty)$ is defined by

$$J(\omega, t, z) = 1 + 1_{\{z \in E\}}\kappa' \sigma^R(\omega, t, z), \quad (\omega, t, z) \in \Omega \times [0, T] \times Z. \tag{5.8}$$

The functions $\mathcal{X}, G^* : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ are defined by

$$G_x(\omega, t, \mathcal{X}(\omega, t, y, \Sigma), \Sigma) = y, \tag{5.9}$$

[13] The definition of ϕ^0 here differs from than in Schroder and Skiadas (2005), where ϕ^0 corresponds to the optimal strategy that results in zero utility.

$$G^{*}(\omega, t, y, \Sigma) = \sup_{x\in\mathbb{R}}\{G(\omega, t, x, \Sigma) - yx\} \tag{5.10}$$
$$= G(\omega, t, \mathcal{X}(\omega, t, y, \Sigma), \Sigma) - y\mathcal{X}(\omega, t, y, \Sigma).$$

The super-differential of G with respect to the last argument is denoted $\partial_{\Sigma} G$, and is defined analogously to $\partial_{U,\Sigma} F$. Specializing the super-gradient density calculation in this context, and using the conjectured relationships (5.6), one arrives to the following optimality conditions:

CONDITION 5.3 *(optimality conditions under translation-invariance with respect to γ). The processes $(Y, \sigma^{Y}, \Sigma, G_{\Sigma}, \phi^{0}) \in \mathcal{O} \times \mathcal{V} \times \mathcal{V} \times \mathcal{V} \times \mathcal{P}^{m}$ satisfy*

$$dY = -\left(e - Y\mu^{\kappa} + f^{0}(\phi^{0}) + \Gamma G^{*}(\delta, \Sigma) - \Gamma \int_{Z} \kappa' \sigma^{R}(z)\Sigma(z)\nu(dz)\right) dt$$
$$+ \sum_{i=1}^{d} \sigma^{Y}(z^{i})\, dB_{t}^{i} + \int_{E} \sigma^{Y}(z)\hat{p}(dt \times dz), \qquad Y_{T} = e_{T},$$
$$\Sigma = \frac{\sigma^{Y} + (\phi^{0} - \kappa Y_{-})'\sigma^{R}}{J\Gamma_{-}}, \qquad -\int_{Z} \frac{\sigma^{R}(z)}{J(z)}\big(G_{\Sigma}(z) - \kappa'\sigma^{R}(z)\big)\nu(dz) \in \partial f^{0}(\phi^{0}),$$
$$G_{\Sigma} \in \partial_{\Sigma} G(\mathcal{X}(\delta, \Sigma), \Sigma), \qquad \phi^{0} \in K^{0}.$$

The sufficiency of Condition 5.3 for optimality, given integrability assumptions, is shown in the following result as an application of Proposition 4.2. The necessity of Condition 5.3 for optimality is also shown for the case of a smooth aggregator and integrability assumptions.

THEOREM 5.2. *Suppose that Conditions 5.1 and 5.2 hold.*

(a) (Sufficiency) Suppose Condition 5.3 is satisfied, the wealth process W solves the SDE

$$dW = (e - \delta Y + f^{0}(\phi^{0}) + W_{-}(\mu^{\kappa} - \delta) - \gamma \mathcal{X}(\delta, \Sigma))\, dt \tag{5.11}$$
$$+ (\phi^{0} + W_{-}\kappa)' d\hat{R}, \qquad W_{0} = w_{0},$$

and the process $\mathcal{E} \in \mathcal{S}_{2}^{++}$ solves the SDE

$$\frac{d\mathcal{E}_{t}}{\mathcal{E}_{t-}} = -\delta dt + \sum_{i=1}^{d} G_{\Sigma}(t)(z^{i})\, dB_{t}^{i} + \int_{E} G_{\Sigma}(z)\hat{p}(dt \times dz), \qquad \mathcal{E}_{0} = 1.$$

With ϕ, U, and λ defined in (5.6), assume the regularity restrictions: $U \in \mathcal{U}$, $\lambda\mathcal{E} \in \mathcal{S}_{2}$, and $\lambda\mathcal{E} W \in \mathcal{S}_{1}$. Finally, suppose the consumption plan c is specified by

$$c_{t} = \gamma_{t}\left(U_{t} + \mathcal{X}\left(t, \delta, \Sigma_{t}\right)\right), \quad t < T, \quad \text{and} \quad c_{T} = W_{T} + e_{T}.$$

Then (c, ϕ, W) is an optimal plan, $U(c) = U$, $\Sigma(c) = \Sigma$, and $F_{c}(c, U, \Sigma) = \lambda$.

(b) (Necessity) Suppose the plan (c, W, ϕ) is optimal, $G(\omega, t, \cdot)$ is differentiable for every (ω, t), and return jumps are bounded above and away from zero. Let $U = U(c)$, $\Sigma = \Sigma(c)$, $x = c/\gamma - U$, $G_{x} = G_{x}(x, \Sigma)$, $G_{\Sigma} = G_{\Sigma}(x, \Sigma)$, and let Y and ϕ^{0} be defined by (5.6). Suppose $\mathcal{E} \in \mathcal{S}_{2}^{++}$ solves the SDE

$$\frac{d\mathcal{E}_t}{\mathcal{E}_{t-}} = -G_x(t)\,dt + \sum_{i=1}^{d} G_\Sigma(t)(z^i)\,dB_t^i + \int_E G_\Sigma(z)\hat{p}(dt \times dz), \quad \mathcal{E}_0 = 1,$$

and $\lambda, \lambda\mathcal{E} \in \mathcal{S}_2$. *Then Condition 5.3 is satisfied.*

Proof. See Appendix B. □

6. SOLUTIONS BASED ON QUADRATIC BSDEs

In this section, we identify a subclass of the translation-invariant formulation for which the BSDE characterizing optimality is quadratic. Further, we impose restrictions under which the quadratic BSDE can be reduced to an ODE system of the Riccati form. The technique is familiar in finance application involving linear BSDEs, as in affine term-structure and credit-risk models (see, for example, Duffie 2005, Duffie et al. 2003, and Piazzesi 2005). Here we extend the quadratic BSDE analysis of Schroder and Skiadas (2005) by including jumps.

Throughout this section we adopt last section's setting (Conditions 5.1 and 5.2), specialized by the following restrictions on the market and preferences:

(i) No trading constraints: $K = \mathbb{R}^{1+m}$.
(ii) Linear wealth dynamics: $f(W, \phi) = rW + \phi'\mu^R$ and $\mu^\kappa = r + \kappa'\mu^R$, for some $\mu^R \in \mathcal{P}^m$.
(iii) The absolute aggregator takes the quasi-quadratic form

(6.1)

$$G(\omega, t, x, \Sigma) = g(\omega, t, x) - \int_Z \left(q(\omega, t, z)\Sigma(z) + \frac{1}{2}Q(\omega, t, z)\Sigma(z)^2\right)\nu(\omega, t, dz),$$

for some predictable functions $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $q : \Omega \times [0, T] \times Z \to \mathbb{R}$, and $Q : \Omega \times [0, T] \times Z \to \mathbb{R}_{++}$. For simplicity, we assume that q and Q are bounded.

Even though there are no trading constraints, the market can be incomplete, and in particular the endowment e may not be tradable. The quasi-quadratic absolute aggregator assumption imposes a preference structure that is inconsistent with time-additivity in the presence of jumps, yet provides a more tractable class of problems than more conventional additive utilities do.

We specialize the optimality conditions in this case after introducing some notation. With J given in (5.8), we define

$$g^*(\omega, t, y) = \sup_{x \in \mathbb{R}} (g(\omega, t, x) - yx) \quad \text{and} \quad \tilde{\mu}^R = \mu^R - \int_Z \frac{\sigma^R(z)}{J(z)} \left(q(z) + \kappa'\sigma^R(z)\right)\nu(dz).$$

Also, for any $\sigma_1 \in \mathcal{V}^{l\times k}$ and $\sigma_2 \in \mathcal{V}^{k\times n}$, we use the notation

$$(\sigma_1, \sigma_2) = \int_Z \frac{Q(z)}{\Gamma_- J(z)^2}\sigma_1(z)\sigma_2(z)\nu(dz),$$

and we assume throughout that $(\sigma^R, \sigma^{R'})$ is a.e. invertible.

In this context, the BSDE satisfied by Y in optimality Condition 5.3 takes the quadratic form

$$dY = -\left(\alpha - \beta Y - (b, \sigma^Y) - \frac{1}{2}(\sigma^Y, \sigma^Y) + \frac{1}{2}(\sigma^Y, \sigma^{R\prime})A^{-1}(\sigma^R, \sigma^Y)\right) dt \tag{6.2}$$
$$+ \sum_{i=1}^{d} \sigma^Y(z^i)(dB^i + q(z^i)\, dt) + \int_E \sigma^Y(z)\hat{p}(dt \times dz), \qquad Y_T = e_T,$$

with coefficients

$$\alpha = e + \Gamma g^*(\delta) + \frac{1}{2}\tilde{\mu}^{R\prime}(\sigma^R, \sigma^{R\prime})^{-1}\tilde{\mu}^R, \qquad \beta = r,$$
$$A = (\sigma^R, \sigma^{R\prime}), \quad b(z) = \frac{\Gamma_- J(z)}{Q(z)}\big(q(z)1_{\{z\in E\}} + \kappa'\sigma^R(z)\big) + \tilde{\mu}^{R\prime}(\sigma^R, \sigma^{R\prime})^{-1}\sigma^R(z).$$

(The apparent redundancy in the above notation will be of use in the last section, where the coefficients of the same BSDE receive new definitions.) The corresponding optimal strategy is given as in Theorem 5.2, with

$$\phi^0 - \kappa Y_- = (\sigma^R, \sigma^{R\prime})^{-1}(\tilde{\mu}^R - (\sigma^R, \sigma^Y)).$$

Next, we identify a class of problems for which the quadratic BSDE (6.2) can be solved in terms of a solution to an ODE system. We introduce an n-dimensional state process X with dynamics

$$dX_t = \mu_t^X dt + \sum_{i=1}^{d} \sigma_t^X(z^i)\, dB_t^i + \int_E \sigma_t^X(z)\hat{p}(dt \times dz), \tag{6.3}$$

where $\mu^X \in \mathcal{P}_1^n$ and $\sigma^X \in \mathcal{V}^n$, and we seek a solution to BSDE (6.2) of the form

$$Y_t = Y_t^0 + Y_t^{1\prime} X_t, \tag{6.4}$$

for deterministic processes $Y^0 : [0, T] \to \mathbb{R}$ and $Y^1 : [0, T] \to \mathbb{R}^n$. This type of solution is made possible by the following restrictions:

CONDITION 6.1. *There exist deterministic processes* $\alpha^0 : [0, T] \to \mathbb{R}; \alpha^1, C^0, D^1[i, j] : [0, T] \to \mathbb{R}^n$; *and* $C^1, D^0 : [0, T] \to \mathbb{R}^{n\times n}$, *such that*

$$\alpha = \alpha^0 + \alpha^{1\prime} X_t, \quad \mu^X - (b, \sigma^X) = C^0 + C^{1\prime} X,$$
$$\big(\sigma_i^X, \sigma^{R\prime}\big) A^{-1}\big(\sigma^R, \sigma_j^X\big) + a\big(\sigma_i^X, \sigma_j^X\big) = D_{ij}^0 + D^1[i, j]' X, \quad i, j = 1, \ldots, n.$$

Finally, β *is deterministic, and* $Y_T = \bar{Y}^0 + \bar{Y}^{1\prime} X_T$, *where* $\bar{Y}^0 \in \mathbb{R}$ *and* $\bar{Y}^1 \in \mathbb{R}^n$.

REMARK 6.1. For each $(i, j) \in \{1, \ldots, n\}^2$, D_{ij}^0 is the (i,j) entry of the $n \times n$ matrix D^0, while $D^1[i, j]$ is an n-dimensional column vector.

Suppose Condition 6.1 holds and the deterministic processes (Y^0, Y^1) solve the ODE system (where the dots on the left-hand-side represent time-derivatives):

$$\left.\begin{aligned} -\dot{Y}_t^0 &= \alpha_t^0 - \beta_t Y_t^0 + C_t^{0\prime} Y_t^1 + \frac{1}{2} Y_t^{1\prime} D_t^0 Y_t^1, & Y_T^0 &= \bar{Y}^0,\\ -\dot{Y}_t^1 &= \alpha_t^1 - \big(\beta_t - C_t^1\big) Y_t^1 + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} D_t^1[i, j] Y_i^1(t) Y_j^1(t), & Y_T^1 &= \bar{Y}^1. \end{aligned}\right\} \tag{6.5}$$

(Note that the equation in Y^1 does not depend on Y^0.) Direct computation using Itô 's lemma shows that if (Y^0, Y^1) solves (6.5), then equation (6.4) defines a solution to BSDE (6.2), provided that the drift and diffusion terms of (6.2) are suitably integrable so that the respective integrals are well-defined.

EXAMPLE 6.1. We assume that the aggregator parameters g and Q, the jump-rate intensity kernel h, the short-rate process r, the asset volatility process σ^R, and the cash flow γ are all deterministic (that is, not dependent of the state variable $\omega \in \Omega$), and therefore $\kappa = 0$ and the processes Γ and $g^*(\gamma/\Gamma)$ are also deterministic. We postulate $N \geq m$ types of shocks represented by the partition $Z = Z_1 \cup \cdots \cup Z_N$ (where $Z_i \cap Z_j = \emptyset$ for $i \neq j$). We assume that the instantaneous returns of asset i are driven only by shocks in Z_i; that is, $z \notin Z_i \implies \sigma_i^R(z) = 0$. Further, we assume

$$\mu^X = \mu^0 + \mu^1 X_t, \quad e = e^0 + e^{1\prime} X, \quad \mu_i^R = s_i\sqrt{\upsilon_i^0 + \upsilon_i^{1\prime} X}, \quad i = 1, \ldots, m,$$

$$\sigma_t^X(z) = \bar{\sigma}(z)\sum_{i=1}^N 1_{\{z\in Z_i\}}\sqrt{\upsilon_i^0 + \upsilon_i^{1\prime} X}, \quad q(z) = \bar{q}(z)\sum_{i=1}^N 1_{\{z\in Z_i\}}\sqrt{\upsilon_i^0 + \upsilon_i^{1\prime} X}, \quad z \in Z,$$

where $s_i, \upsilon_i^0, e^0 \in \mathbb{R}; \upsilon_i^1, \mu^0, e^1 \in \mathbb{R}^n; \mu^1 \in \mathbb{R}^{n\times n}$; $\bar{\sigma} : Z \to \mathbb{R}^n$, and $\bar{q} : Z \to \mathbb{R}$. Direct computation shows that Condition 6.1 is satisfied, implying a solution of the form (6.4), provided the coefficients (Y^0, Y^1) solve the ODE system (6.5) with $\bar{Y}^0 = e^0$ and $\bar{Y}^1 = e^1$.

In our second example of a problem class satisfying Condition 6.1, we obtain a BSDE solution in which Y is a quadratic of a state process, with deterministic coefficients. We embed this quadratic formulation in the above affine setting by a suitable expansion of the state variables.[14] Consider a new k-dimensional state process x with drift $\mu^x \in \mathcal{P}_1^k$ and volatility $\sigma^x \in \mathcal{V}^k$, meaning that equation (6.3) holds with x in place of X. We fix a non-negative integer $l \leq k$, and we use the matrix block notation

$$x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \quad \text{where} \quad x^1(\omega, t) \in \mathbb{R}^l \quad \text{and} \quad x^2(\omega, t) \in \mathbb{R}^{k-l}. \tag{6.6}$$

We are interested in solutions to BSDE (6.2) of the form:

$$Y_t = y_t^0 + y_t^{1\prime} x_t + x_t^{1\prime} y_t^{\mathrm{q}} x_t^1 \tag{6.7}$$

where the deterministic coefficients $y^0 : [0, T] \to \mathbb{R}$, $y^1 : [0, T] \to \mathbb{R}^k$, and $y^{\mathrm{q}} : [0, T] \to \mathbb{R}^{l\times l}$ solve an ODE system. Note that the linear term involves the entire process x, while the quadratic term is on x^1 only. We assume that y_t^{q} is a symmetric matrix for all $t \in [0, T]$.

For any symmetric matrix M, we let vec(M) denote the column vector that lists all the lower triangular elements of M (the diagonal included) in some fixed order, say column-by-column. We define the expanded state process

$$X = \begin{bmatrix} x \\ \mathrm{vec}(x^1 x^{1\prime}) \end{bmatrix}. \tag{6.8}$$

Equations (6.4) and (6.7) are then equivalent if the deterministic coefficients (Y^0, Y^1) and $(y^0, y^1, y^{\mathrm{q}})$ are related by

$$Y^0 = y^0 \quad \text{and} \quad Y^1 = \begin{bmatrix} y^1 \\ \mathrm{vec}\,(y^{\mathrm{q}}) \end{bmatrix},$$

[14] A similar reduction of a quadratic model to an affine one is given by Cheng and Scaillet (2005).

in which case the ODE system for (Y^0, Y^1) is equivalent to an ODE system for $(y^0, y^1, y^{\mathrm{q}})$. Given a solution to the latter and Condition 6.1, a solution to BSDE (6.2) of the form (6.7) is obtained. An example of this type follows:

EXAMPLE 6.2. As in the last example, we assume that g, Q, h, r, σ^R, and γ are deterministic. Given the state process x, with drift μ^x and volatility σ^x, we use the block notation (6.6), and analogously $\sigma^x = [\sigma^{1'}, \sigma^{2'}]'$, where σ^1 is ℓ-dimensional. We further impose the restrictions:

$$\mu^x = \mu^0 + \begin{pmatrix} \mu^{11} & 0 \\ \mu^{21} & \mu^{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix},$$

$$\sigma^1 \text{ is deterministic}, \quad \sigma^2(z) = \sigma^{20}(z) + \sigma^{21}(z)x^1, \quad z \in Z,$$

$$\mu^R = \upsilon^0 + \upsilon^1 x^1, \quad e = e^0 + e^{1\prime} x + \frac{1}{2} x^{1\prime} e^{\mathrm{q}} x^1, \quad q(z) = q^0(z) + q^1(z)' x^1, \quad z \in Z,$$

where $\mu^0 \in \mathbb{R}^k$, $\mu^{11} \in \mathbb{R}^{l\times l}$, $\mu^{21} \in \mathbb{R}^{(k-l)\times l}$, $\mu^{22} \in \mathbb{R}^{(k-l)\times(k-l)}$, $\upsilon^0 \in \mathbb{R}^m$, $\upsilon^1 \in \mathbb{R}^{m\times l}$, $e^0 \in \mathbb{R}$, $e^1 \in \mathbb{R}^k$, $e^{\mathrm{q}} \in \mathbb{R}^{l\times l}$; $q^0 : Z \to \mathbb{R}$, $q^1 : Z \to \mathbb{R}^l$, $\sigma^{20} : Z \to \mathbb{R}^{k-l}$, $\sigma^{21} : Z \to \mathbb{R}^{(k-l)\times l}$. Defining the expanded state process X as above, Condition 6.1 can be verified by straight calculation, implying a solution of the form (6.7), provided the coefficients solve an ODE system.

7. SCALE-INVARIANT FORMULATION

In the previous two sections we studied simplifications of the optimality conditions based on translation invariance assumptions. In this section, and the following section, we provide analogous arguments based on scale invariance with respect to wealth. The scale-invariant formulation is consistent with constant relative risk aversion, and Epstein–Zin utility (including additive power or logarithmic utility). On the other hand, scale invariance precludes the possibility of a non-traded income stream, and we therefore set $e = 0$ below. The analysis that follows extends Schroder and Skiadas (2003) by allowing non-linear wealth dynamics and jumps.

The following condition is assumed throughout the rest of the main part of this paper.

CONDITION 7.1.

- *(a) Consumption is strictly positive:* $I_c = (0, \infty)$.
- *(b) For all* $c \in \mathcal{C}$, $\tilde{c} \in \mathcal{H}^{++}$ *and* $\tilde{c} \le c$ *implies* $\tilde{c} \in \mathcal{C}$.
- *(c) A* wealth process *W is further restricted by requiring that* $W_{t-} > 0$ *a.s. for all t.*
- *(d) The agent's endowment consists of a positive initial financial wealth only:* $e = 0$ *and* $w_0 > 0$.
- *(e) The constraint set K is a cone (with origin at zero) and* $f(\omega, t, \cdot)$ *is homogeneous of degree one.*

Part (e) of the condition is equivalent to the representations

$$K = \{(w, w\alpha) : w \in (0, \infty), \alpha \in K^1\} \quad \text{and}$$

$$f(\omega, t, w, \alpha) = w f^1(\omega, t, \alpha/w), \quad (w, \alpha) \in K,$$

where $K^1 \subseteq \mathbb{R}^m$ and $f^1 : \Omega \times [0, T] \times K^1 \to \mathbb{R}$ are defined by

$$K^1 = \{\alpha \in \mathbb{R}^m : (1, \alpha) \in K\} \quad \text{and} \quad f^1(\omega, t, \alpha) = f(\omega, t, 1, \alpha).$$

The above condition implies that trading constraints and wealth dynamics depend on trading plans only through the corresponding wealth allocations. The term "strategy" will refer to a plan expressed in terms of proportions of pre-jump wealth. The strategy corresponding to the plan (c, W, ϕ) is therefore the pair (ρ, ψ), where

$$\rho_t = \frac{c_t}{W_{t-}} \quad \text{and} \quad \psi_t = \frac{\phi_t}{W_{t-}}.$$

Formally, a *consumption strategy* is any process $\rho \in \mathcal{P}_1$ that is valued in $(0, \infty)$, and a *trading strategy* is any process $\psi \in \mathcal{P}^m$ such that the stochastic integral $\int \psi' d\hat{R}$ is well-defined. A *strategy* is a pair (ρ, ψ) of a consumption strategy and a trading strategy. A strategy (ρ, ψ) *generates* a wealth process W through the budget equation, which in this context takes the form

$$W_0 = w_0, \quad \frac{dW_t}{W_{t-}} = (f^1(t, \psi_t) - \rho_t)\, dt + \psi_t' d\hat{R}_t. \tag{7.1}$$

The strategy (ρ, ψ) *finances* the consumption plan c if it generates a wealth process W such that $c_t = \rho_t W_{t-}$ for $t < T$ and $c_T = W_T$. The strategy (ρ, ψ) is *feasible* (respectively *optimal*) if it finances a feasible (respectively optimal) consumption plan. We note that the strategy (ρ, ψ) is feasible if and only if it finances a consumption plan and $\psi \in K^1$ a.e.

Under Condition 7.1, the central condition $(\zeta_t, \Theta_t) \in \partial f(t, W_t, \phi_t)$ a.e., used to characterize state-price dynamics, is equivalent to the restrictions

$$\zeta_t = f^1(t, \psi_t) - \psi_t' \Theta_t \quad \text{and} \quad \Theta_t \in \partial f^1(t, \psi_t) \quad \text{a.e.} \tag{7.2}$$

(A proof of this claim is given in lemma 19 of Schroder and Skiadas, 2003.)

EXAMPLE 7.1. Suppose that $f^1(\omega, t, \psi) = r(\omega, t) + \psi' \mu^R(\omega, t)$, and define the support function $\delta_{K^1} : \mathbb{R}^m \to (-\infty, \infty]$ by

$$\delta_{K^1}(y) = \sup\{\psi' y : \psi \in K^1\}.$$

Then condition (7.2) is equivalent to

$$\zeta = r + \delta_{K^1}(\varepsilon), \quad \psi' \varepsilon = \delta_{K^1}(\varepsilon), \quad \varepsilon = \mu^R - \Theta, \qquad \text{a.e.}$$

If K^1 is a cone, then $\delta_{K^1}(\varepsilon) = 0$. For example, setting $K = \mathbb{R}_+^m$ corresponds to a short-sale constraint on all assets except the money market account. In this case, the above condition is equivalent to the following restrictions (in an a.e. sense):

$$\zeta = r, \quad \mu^R \leq \Theta, \quad \text{and} \quad \mu_t^{Ri} = \Theta_t^i \text{ on } \{\psi_t^i > 0\}, \quad i = 1, \ldots, m.$$

The following characterization of state-price dynamics extends Proposition 3.2 with a necessity argument for a scale-invariant market.

THEOREM 7.1. *Suppose that Condition 7.1 holds, and* (ρ, ψ) *is a feasible strategy that generates the wealth process* W *and finances the consumption plan* c. *Suppose also that* $\pi \in \mathcal{S}_2^{++}$ *has the predictable representation (3.4) for some* $(\zeta, \eta) \in \mathcal{P} \times \mathcal{V}$, Θ *is well-defined by (3.5), and* $\pi W \in \mathcal{S}_1$.

(a) (*Sufficiency*) *If* (ζ, Θ) *satisfies (7.2), then* π *is a state-price density at c.*
(b) (*Necessity*) *If* π *is a state-price density at c, return jumps are bounded, and* ρ *is* càdlàg, *then* (ζ, Θ) *satisfies (7.2).*

Proof. See Appendix B. □

So far, we have imposed a scale-invariant structure on the market, resulting in the state-pricing characterization (7.2). On the preference side, we further specialize the recursive utility structure of Condition 4.1 by imposing the following homogeneity restriction on the aggregator:

CONDITION 7.2. $I_U = (0, \infty)$ *and the aggregator F is of the form*

$$F(\omega, t, c, U, \Sigma) = UG\left(\omega, t, \frac{c}{U}, \frac{\Sigma}{U}\right), \quad F(T, c) = c,$$

for some function $G : \Omega \times [0, T] \times (0, \infty) \times \mathcal{Z} \to \mathbb{R}$ *that we call a* proportional aggregator.

The above restriction implies that the utility function is homogeneous of degree one:

$$U(kc) = kU(c) \quad \text{for all } k \in \mathbb{R} \text{ and } c \in \mathcal{C}.$$

Making the change of variables

$$x_t = \frac{c_t}{U_{t-}} \quad \text{and} \quad \sigma_t = \frac{\Sigma_t}{U_{t-}}, \tag{7.3}$$

the BSDE for the utility process $U = U(c)$ can be written as

$$\frac{dU_t}{U_{t-}} = -G(t, x_t, \sigma_t)\, dt + \sum_{i=1}^{d} \sigma_t(z^i)\, dB_t^i + \int_E \sigma_t(z)\hat{p}(dt \times dz), \quad U_T = c_T. \tag{7.4}$$

The following example shows that the familiar additive discounted power or logarithm utility is included as a special case.

EXAMPLE 7.2. Given the bounded deterministic processes $b : [0, T] \to \mathbb{R}$ and scalars $\gamma, D > 0$, suppose that the utility process $V(c)$ corresponding to the consumption plan c is specified by

$$V_t(c) = \mathbb{E}\left[\int_t^T e^{-\int_t^s b_u du}\, \frac{c_s^{1-\gamma} - 1}{1-\gamma} ds + e^{-\int_t^T b_u du}\, D\frac{c_T^{1-\gamma} - 1}{1-\gamma} \,\middle|\, \mathcal{F}_t\right], \tag{7.5}$$

where we assume that c is sufficiently integrable for $V(c)$ to be well defined. For $\gamma = 1$, we interpret the function $(x^{1-\gamma} - 1)/(1-\gamma)$ as $\log x$ (which is the limit as $\gamma \to 1$). The dynamic utility V is ordinally equivalent to a recursive utility U specified by BSDE (7.4) for a proportional aggregator of the form

(7.6)

$$G(t, x, \sigma) = \alpha_t + \beta_t \frac{x^{1-\gamma} - 1}{1-\gamma} - \frac{\gamma}{2}\sum_{i=1}^{d} \sigma(z^i)^2 - \int_E \left[\sigma(z) - \frac{(1+\sigma(z))^{1-\gamma} - 1}{1-\gamma}\right] h(t, dz),$$

for some $\alpha : [0, T] \to \mathbb{R}$ and $\beta : [0, T] \to (0, \infty)$. (Incidentally, this expression makes clear the problem with the utility specification (7.5). The parameter γ, which is used to model risk aversion, is entirely determined by $G(t, x, 0)$; that is, by the utility values of deterministic plans.) To transform V into a recursive utility U with aggregator (7.6), we can set

$$\frac{U_t^{1-\gamma}-1}{1-\gamma}=\beta_t V_t, \quad \alpha_t=0, \quad \frac{1}{\beta_t}=\int_t^T e^{-\int_t^s b_u du} ds+e^{-\int_t^T b_u du}\, D. \tag{7.7}$$

Alternatively, provided $\gamma \neq 1$, we can set

$$\frac{U_t^{1-\gamma}-1}{1-\gamma}=\beta_t V_t+\int_t^T \alpha_s e^{-\int_t^s b_u du} ds, \quad \alpha_t=\frac{\beta_t-b_t}{1-\gamma}, \quad \beta_t=\frac{1}{D}. \tag{7.8}$$

The expressions for α and β in transformations (7.7) and (7.8) become identical if and only if b is constant and $D=1/b\,(=\int_0^\infty e^{-bt}dt)$, in which case $\alpha=0$ and $\beta=b$, and (7.8) is valid even if $\gamma=1$ (with the interpretation $\alpha=0/0=0$). On the other hand, if $D\neq 1/b$, transformation (7.7) results in a time-dependent aggregator, and transformation (7.8) results in a time-independent one, but requires that $\gamma\neq 1$. Intuitively, if $\gamma\neq 1$, a change in the unit of account for terminal consumption allows an arbitrary terminal utility weight in (7.5), and hence the embedding of the utility in an infinite-horizon version with stationary aggregator. With $\gamma=1$, however, $\log(\lambda_T c_T)=\log(\lambda_T)+\log(c_T)$ for every strictly positive λ_T, and therefore changing units for terminal consumption has no effect on the corresponding weight of terminal utility. This completes the example.

The scale-invariance assumptions on the market and the homogeneity of the utility function suggest that the optimal utility process U, the optimal wealth process W, and the corresponding shadow-price-of-wealth process λ are related by

$$U_t=\lambda_t W_t. \tag{7.9}$$

This simple relationship is used below to uncouple the FBSDE of the optimality conditions, reducing it to a single constrained BSDE for λ.

We let G_x denote the partial derivative of $G(\omega, t, x, \sigma)$ with respect to x, while the super-differential $\partial_\sigma G$ is defined analogously to $\partial_{U,\Sigma}F$. Under the above aggregator restriction, and with the notation (7.3), lemma 19 of Schroder and Skiadas (2003) shows that the condition $(F_U, F_\Sigma)\in\partial_{U,\Sigma}F(c, U, \Sigma)$ is equivalent to (omitting time indices):

(7.10)

$$F_\Sigma=G_\sigma\in\partial_\sigma G(x,\sigma), \quad F_U(c,U,\Sigma)=G(x,\sigma)-G_x(x,\sigma)\,c-\int_Z G_\sigma(z)\sigma(z)\nu(dz).$$

The functions $\mathcal{X}, G^*:\Omega\times[0,T]\times(0,\infty)\times\mathcal{Z}\to\mathbb{R}$ are defined by (5.9) and (5.10), respectively, but with the domain and the interpretation of the variables being different than in the last section. (Here $x\in(0,\infty)$ is a possible value of c/U_-, $y\in(0,\infty)$ is a possible value of λ, and the role of Σ is played by $\sigma=\Sigma/U_-$.) Using equation (7.9), the optimality conditions can be reduced to

CONDITION 7.3 (optimality conditions under scale invariance relative to wealth). $(\lambda, \sigma^{\lambda}, \sigma, G_{\sigma}, \psi) \in \mathcal{S}^{++} \times \mathcal{V} \times \mathcal{V} \times \mathcal{V} \times \mathcal{P}^m$ *solve the constrained BSDE:*

$$\frac{d\lambda}{\lambda_-} = -\left(G^*(\lambda, \sigma) + f^1(\psi) + \int_Z \psi' \sigma^R(z) \sigma^{\lambda}(z) \nu(dz)\right) dt$$

$$+ \sum_{i=1}^{d} \sigma^{\lambda}(z^i)\, dB^i + \int_E \sigma^{\lambda}(z) \hat{p}(dt \times dz), \qquad \lambda_T = 1,$$

$$\sigma(z) = \sigma^{\lambda}(z) + \psi' \sigma^R(z)(1 + \sigma^{\lambda}(z) 1_{\{z \in E\}}), \qquad G_{\sigma} \in \partial_{\sigma} G(\lambda, \sigma), \qquad \psi \in K^1,$$

$$-\int_Z \sigma^R(z) \left(G_{\sigma}(z) + \sigma^{\lambda}(z) + G_{\sigma}(z) \sigma^{\lambda}(z) 1_{\{z \in E\}}\right) \nu(dz) \in \partial f^1(\psi).$$

THEOREM 7.2. *Suppose Conditions 7.1 and 7.2 hold. In each part below, we assume that $\mathcal{E}$ solves SDE (4.5), where (F_U, F_{Σ}) is computed by equation (7.10).*

(a) (Sufficiency) Suppose Condition 7.3 holds, the strategy (ρ, ψ), where $\rho = \lambda \mathcal{X}(\lambda, \sigma)$, generates the wealth process W, and finances the consumption plan $c \in \mathcal{C}$. Suppose also that $\lambda W \in \mathcal{U}$, and $\mathcal{E}, \lambda\mathcal{E} \in \mathcal{S}_2^{++}$. Then the strategy (ρ, ψ) is optimal and $U(c) = \lambda W$.

(b) (Necessity) Suppose (ρ, ψ) is an optimal strategy that generates the wealth process W, and finances the consumption plan $c \in \mathcal{C}$. Let $(U, \Sigma) = (U(c), \Sigma(c))$ and $\lambda = G_x(x, \sigma)$, with (x, σ) defined in (7.3). Suppose further that $G(\omega, t, \cdot)$ is differentiable for all (ω, t), ρ is càdlàg*, return jumps are bounded, and $\mathcal{E}, \lambda\mathcal{E} \in \mathcal{S}_2^{++}$. Then Condition 7.3 is satisfied and $U = \lambda W$.*

Proof. See Appendix B. □

EXAMPLE 7.3 (a robustly optimal consumption strategy). Suppose that the proportional aggregator is given as $G(\omega, t, x, \sigma) = \beta(\omega, t) \log(x) + G(\omega, t, 1, \sigma)$, for some process β. In the context of part (a) of Theorem 7.2, we have $\lambda = \beta/x = \beta U_-/c$ and $U = \lambda W$. Therefore the optimal consumption strategy is $\rho = \beta$ for any specification of the trading constraints and price dynamics.

EXAMPLE 7.4 (myopic optimal trading strategy). This example generalizes the solution of Aase (1984, 1986), who considered the case of an unconstrained agent maximizing expected utility of terminal wealth. Suppose the proportional aggregator takes the form

$$G(\omega, t, x, \sigma) = g(\omega, t, x) - \sum_{i=1}^{d} \left(q(\omega, t, z^i) \sigma(z^i) + \frac{1}{2} \sigma(z^i)^2\right)$$

$$- \int_E [\sigma(z) - \kappa(\omega, t, z) \log(1 + \sigma(z))] h(\omega, t, dz),$$

for some predictable function $\kappa : \Omega \times [0, T] \times Z \to \mathbb{R}_{++}$. (As shown in Example 7.2, expected discounted logarithmic utility is the special case.) Putting technical integrability requirements aside, the trading strategy component, ψ, of the optimality conditions in this case is only restricted by the requirement that $\Theta \in \partial f^1(\psi)$ and $\psi \in K^1$, where

$$\Theta = \sum_{i=1}^{d} \sigma^R(z^i) q(z^i) + \int_E \sigma^R(z) \left(1 - \frac{\kappa(z)}{1 + \psi' \sigma^R(z)}\right) h(dz) + \left(\sum_{i=1}^{d} \sigma^R(z^i) \sigma^R(z^i)'\right) \psi.$$

The computation of the optimal strategy ψ is therefore myopic, and does not require the solution of the BSDE for λ .

8. QUASI-QUADRATIC PROPORTIONAL AGGREGATOR

Continuing in last section's setting, but without trading constraints (although the market can be incomplete), we explore simplifications to the optimality conditions resulting from the assumption of a quasi-quadratic form (6.1) for the proportional aggregator. Example 7.2 shows that in the case of a Brownian filtration this section's setting includes the additive discounted power or logarithmic case (as well as Epstein–Zin utility, as discussed in Schroder and Skiadas 2003, Skiadas 2008). In the presence of jumps, however, the quasi-quadratic proportional aggregator specification is inconsistent with time additivity, yet preserves the homotheticity assumption, and offers clear tractability advantages.

The following conditions are assumed throughout:

(i) Condition 7.1 is satisfied with $K = \mathbb{R}^{1+m}$ and the linear wealth dynamics of Example (7.1). Therefore, $K^1 = \mathbb{R}^m$ and $f^1(\omega, t, \psi) = r_t + \psi' \mu_t^R$.
(ii) Condition 7.2 is satisfied for a proportional aggregator of the quasi-quadratic form (6.1), where q and Q are assumed bounded.

We use the notation

$$g^*(\omega, t, y) = \sup_{x\in\mathbb{R}}(g(\omega, t, x) - yx), \quad y \in (0, \infty),$$

$$A(\sigma) = \int_Z Q(z)\sigma^R(z)\sigma^R(z)'(1 + \sigma(z)1_{\{z\in E\}})^2 \nu(dz), \quad \sigma \in \mathcal{V}.$$

$A(\sigma)$ depends on σ only through its jump components $\sigma(z),\ z \in E$. Provided $A(\sigma^\lambda)$ is a.e. invertible, one can easily check that the optimality Condition 7.3 reduces to the following BSDE satisfied by λ, which includes an expression for the optimal trading strategy ψ:

$$\frac{d\lambda}{\lambda_-} = -\left(r + g^*(\lambda) + \frac{1}{2}\psi' A(\sigma^\lambda)\psi - \int_Z \left(q(z)\sigma^\lambda(z) + \frac{1}{2} Q(z)\sigma^\lambda(z)^2\right)\nu(dz)\right)$$
$$+ \sum_{i=1}^{d} \sigma^\lambda(z^i)\, dB^i + \int_E \sigma^\lambda(z)\hat{p}(dt \times dz), \quad \lambda_T = 1,$$

$$\psi = A(\sigma^\lambda)^{-1}\left(\mu^R + \int_Z \sigma^R(z)\left\{\sigma^\lambda(z) - \left[q(z) + Q(z)\sigma^\lambda(z)\right]\left[1 + \sigma^\lambda(z)1_{\{z\in E\}}\right]\right\}\nu(dz)\right).$$

EXAMPLE 8.1 (mean-variance efficient portfolio). Suppose that g is state independent, $q = 0,\ Q_t(z) = \theta_t$ for all $z \in Z$, where θ is deterministic and strictly positive, and the jump-rate intensity kernel h is also deterministic. Suppose further that one of the following two conditions hold:

(i) r, μ^R, and σ^R are deterministic processes.
(ii) $\theta = 1$, and either $\sigma_t^R(z) = 0$ for $z \in E$ (no return jumps) or (r, μ^R, σ^R) is adapted to the natural filtration generated by the Brownian motion B.

Then the above expression for the optimal trading strategy ψ simplifies to the instantaneously mean-variance efficient trading strategy

$$\psi = \left(\theta \int_Z \sigma^R(z)\sigma^R(z)'\nu(dz)\right)^{-1} \mu^R.$$

As in Schroder and Skiadas (2003) for the Brownian case, in order to exploit a quadratic BSDE structure, a special function form is required of the component g of the quasi-quadratic expression of the proportional aggregator G. For the rest of this section, we assume that[15]

$$g(\omega, t, c) = \delta(\omega, t) + \beta(t) + \beta(t)\log\left(\frac{c}{\beta(t)}\right),$$

for some $\delta \in \mathcal{P}$ and deterministic process β.

Making the change of variables

$$Y_t = \log(\lambda_t),$$

we note that $g^*(t, \lambda_t) = \delta_t - \beta_t Y_t$. To state the BSDE satisfied by Y we introduce the notation:

$$(\sigma_1, \sigma_2) = \sum_{i=1}^{d}(Q(z^i) - 1)\sigma_1(z^i)\sigma_2(z^i), \quad \sigma_1 \in \mathcal{V}^{l\times k},\ \sigma_2 \in \mathcal{V}^{k\times n}.$$

The above BSDE for λ in this context is equivalent to BSDE (6.2) for Y, with $Y_T = 0$, and the following new definition of the coefficients:

$$A = \int_Z Q(z)\sigma^R(z)\sigma^R(z)' \exp[2\sigma^Y(z)1_{\{z\in E\}}]\nu(dz),$$

$$b(z^i) = \tilde{\mu}^R A^{-1}\sigma^R(z^i), \quad i = 1, \ldots, d,$$

$$\alpha = r + \delta + \frac{1}{2}\tilde{\mu}^{R\prime} A^{-1}\tilde{\mu}^R - \int_E \left\{\sigma^Y(z) + [q(z) - 1](\exp[\sigma^Y(z)] - 1)\right\} h(dz)$$
$$- \frac{1}{2}\int_E Q(z)(\exp[\sigma^Y(z)] - 1)^2 h(dz),$$

$$\tilde{\mu}^R = \mu^R - \int_Z \sigma^R(z)q(z)\nu(dz)$$
$$+ \int_E \sigma^R(z)\{1 - Q(z)\exp[\sigma^Y(z)]\}\{\exp[\sigma^Y(z)] - 1\}h(dz).$$

The dependence of the coefficients on $\sigma^Y(z)$ for $z \in E$ means that BSDE (6.2) is no longer quadratic in the jump component of volatility. In order to recover a quadratic structure, our applications will be limited to cases with deterministic volatility.

In Section 6, we saw that if Condition 6.1 is satisfied and (Y^0, Y^1) solve the ODE system (6.5), then equation (6.4) defines a solution to BSDE (6.2), provided that the drift and diffusion terms of (6.2) are suitably integrable so that the respective integrals are well-defined. We conclude with two examples in the current context for which Condition 6.1 is satisfied and the optimality conditions reduce to an ODE system.

[15] The case of no intermediate consumption can be formally embedded to this setting by letting $\beta = \beta\log(c/\beta) = 0$.

EXAMPLE 8.2. We assume that the risk-aversion process Q, the jump-rate intensity kernel h, and the asset volatility process σ^R are all deterministic (not dependent on the state variable $\omega \in \Omega$). We postulate $N \geq m$ types of shocks represented by the partition $Z = Z_1 \cup \cdots \cup Z_N$ (where $Z_i \cap Z_j = \emptyset$ for $i \neq j$). We assume that the instantaneous returns of asset i are driven only by shocks in Z_i; that is, $z \notin Z_i \implies \sigma_i^R(z) = 0$. We also assume that, for some $\kappa^0, s_i, \upsilon_i^0 \in \mathbb{R}$ and $\kappa^1, \upsilon_i^1 \in \mathbb{R}^n$,

$$r + \delta = \kappa^0 + \kappa^{1\prime} X \quad \text{and} \quad \mu_i^R = s_i \sqrt{\upsilon_i^0 + \upsilon_i^{1\prime} X}, \quad i = 1, \ldots, m,$$

and $\upsilon_i^1 \neq 0$ implies $\sigma_i^R(z) 1_{\{z \in E\}} = 0$; in other words, if μ_i^R is stochastic for risky asset i, then the volatility of asset i has no jump component. Finally, we assume that, for some $\mu^0 \in \mathbb{R}^n$, $\mu^1 \in \mathbb{R}^{n \times n}$, $\bar{\sigma} : Z \to \mathbb{R}^n$, and $\bar{q} : Z \to \mathbb{R}$,

$$\mu^X = \mu^0 + \mu^1 X,$$

$$\sigma^X(z^i) = \bar{\sigma}(z^i) \sum_{j=1}^{N} 1_{\{z^i \in Z_j\}} \sqrt{\upsilon_j^0 + \upsilon_j^{1\prime} X}, \quad i = 1, \ldots, d,$$

$$q(z^i) = \bar{q}(z^i) \sum_{j=1}^{N} 1_{\{z^i \in Z_j\}} \sqrt{\upsilon_j^0 + \upsilon_j^{1\prime} X}, \quad i = 1, \ldots, d,$$

$$z \in E \implies \sigma^X(z) \text{ and } q(z) \text{ are deterministic.}$$

Direct computation shows that in this context Condition 6.1 is satisfied, implying a solution of the form (6.4), provided the coefficients (Y^0, Y^1) solve the ODE system (6.5) with $\bar{Y}^0 = 0$ and $\bar{Y}^1 = 0$. The ODE for Y^1 is independent of $\sigma^Y(z)$ for $z \in E$ and is therefore of the Riccati form. Once Y^1 is solved, the deterministic value of σ^Y can be substituted into the ODE for Y^0, which is then also of the Riccati form.

EXAMPLE 8.3. We assume that Q, h, and σ^R are deterministic. Given the state process x, with drift μ^x and volatility σ^x, we use the block notation (6.6), and analogously $\sigma^x = [\sigma^{1\prime}, \sigma^{2\prime}]'$, where σ^1 is ℓ-dimensional. We further impose the restrictions:

$$\mu^x = \mu^0 + \begin{pmatrix} \mu^{11} & 0 \\ \mu^{21} & \mu^{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix},$$

$$\sigma^1 \text{ is deterministic, and } \sigma^1(z) = 0 \quad z \in E,$$

$$\sigma^2(z) \text{ is deterministic for all } z \in E, \text{ and } \sigma^2(z^i) = \sigma^{20}(z^i) + \sigma^{21}(z^i) x^1, \quad i = 1, \ldots, d,$$

$$\mu^R = \upsilon^0 + \upsilon^1 x^1, \quad r + \delta = \kappa^0 + \kappa^{1\prime} x + \tfrac{1}{2} x^{1\prime} \kappa^{\mathrm{q}} x^1, \quad q(z) = q^0(z) + q^1(z)' x^1, \quad z \in Z,$$

where $\mu^0 \in \mathbb{R}^k$, $\mu^{11} \in \mathbb{R}^{l \times l}$, $\mu^{21} \in \mathbb{R}^{(k-l) \times l}$, $\mu^{22} \in \mathbb{R}^{(k-l) \times (k-l)}$, $\upsilon^0 \in \mathbb{R}^m$, $\upsilon^1 \in \mathbb{R}^{m \times l}$, $\kappa^0 \in \mathbb{R}$, $\kappa^1 \in \mathbb{R}^k$, $\kappa^{\mathrm{q}} \in \mathbb{R}^{l \times l}$; $q^0 : Z \to \mathbb{R}$, $q^1 : Z \to \mathbb{R}^l$, $\sigma^{20} : Z \to \mathbb{R}^{k-l}$, $\sigma^{21} : Z \to \mathbb{R}^{(k-l) \times l}$. Defining the expanded state process X as in Section 6, Condition 6.1 can be verified by straight calculation, implying a solution of the quadratic form (6.7), provided the coefficients solve an ODE system. Using the notation $y^1 = [y^{11}, y^{12}]'$, were y^1 is ℓ-dimensional, the assumption $\sigma^1(z) = 0$ for $z \in E$ implies $\sigma^Y(z) = y^{12\prime} \sigma^2(z)$ for $z \in Z$. It can be shown that the ODE for y^{12} simplifies to a non-homogeneous linear system. Given a solution to the latter, the ODE system for $(y^0, y^{11}, y_t^{\mathrm{q}})$ is of the Riccati form.

APPENDIX A: DUALITY

A number of papers, such as Karatzas et al. (1991), Cvitani and Karatzas (1992), and Cuoco and Cvitanić (1998), focus on duality formulations under additive utility and Brownian information. In this appendix, we extend the duality notion of these papers to our setting, and we argue that (under some regularity) the sufficient optimality conditions we formulated for the primal problem are also sufficient for optimality in the dual problem. These same conditions were shown, under regularity assumptions, to be necessary for optimality in the scale or translation invariant problem classes. Duality is defined for an arbitrary time-zero utility function, $U_0 : \mathcal{C} \to \mathbb{R}$, which is not necessarily a recursive utility.[16]

We consider the general market model of Section 3. For simplicity, we assume that $\mathcal{C} = \mathcal{H}$. We define the convex dual of the function f appearing in the wealth dynamics by

$$f^*(\omega, t, z, \theta) = \sup\{f(\omega, t, w, \alpha) - wz - \alpha'\theta : (w, \alpha) \in K\},$$

noting that the constraint set K is an important part of this definition. We let $\mathcal{D}$ denote the set of all $(\zeta, \Theta) \in \mathcal{P}_1^{1+m}$ such that $f^*(\zeta, \Theta) \in \mathcal{P}_1$. The *primal problem* is the agent's problem as defined in Section 3. For any $(\zeta, \Theta) \in \mathcal{D}$, we define the (ζ, Θ)-*problem* by making the following changes to the *primal* problem:

(i) Asset returns are not dependent of the agent's market positions, there are no trading constraints, and there is free disposal (in a sense formalized below).
(ii) The short-rate process corresponding to the money market is ζ, and the excess return dynamics of assets $1, \ldots, m$ are $dR = \Theta dt + d\hat{R}$ (where R is an m-vector of cumulative excess returns).
(iii) The agent's endowment is $e + f^*(\zeta, \Theta)$.

The plan (c, W, ϕ) is (ζ, Θ)-*feasible* if there exists some non-negative-valued process p (representing free disposal) such that

$$dW = \big(W\zeta + \phi'\Theta + e + f^*(\zeta, \Theta) - c - p\big)\, dt + \phi' d\hat{R}, \qquad W_0 = w_0. \tag{A.1}$$

A consumption plan is (ζ, Θ)-*feasible* if it is part of a (ζ, Θ)-feasible plan. We let

$$V(\zeta, \Theta) = \sup\{U_0(c) : c \text{ is } (\zeta, \Theta)\text{-feasible}\}.$$

The plan (c, ϕ, W) is (ζ, Θ)-*optimal* if it is (ζ, Θ)-feasible and $V(\zeta, \Theta) = U_0(c)$. The *dual problem* is that of finding a $(\zeta, \Theta) \in \mathcal{D}$ that minimizes $V(\zeta, \Theta)$.

Our first observation is that the optimal value of the dual problem is at least as large as the optimal value of the primal problem:

LEMMA A.1. $\inf\{V(\zeta, \Theta) : (\zeta, \Theta) \in \mathcal{D}\} \geq \sup\{U_0(c) : c \text{ feasible}\}$.

Proof. Suppose (c, ϕ, W) is a feasible plan in the primal problem and $(\zeta, \Theta) \in \mathcal{D}$. By the definition of f^*, the process

$$p_t = f^*(t, \zeta_t, \Theta_t) - f(t, W_t, \phi_t) + W_{t-}\zeta_t + \phi_t'\Theta_t\, dt$$

[16] Convex duality can be extended with regard to the concave aggregator of a recursive utility representation, as discussed by El Karoui et al. (2001). Although not developed here, an analysis analogous to that of this appendix applies with regard to this type of duality as well.

takes non-negative values. The budget equation (3.2) for (c, ϕ, W) in the primal problem and the definition of p imply the budget equation (A.1) in the (ζ, Θ)-problem, and therefore the (ζ, Θ)-feasibility of (c, ϕ, W). This proves that $V(\zeta, \Theta) \geq U_0(c)$. □

The above lemma allows for the possibility of a duality gap. We argue that if a feasible plan (c, W, ϕ) satisfies the sufficient optimality conditions of the main part of this paper, then (c, W, ϕ) closes the duality gap. Ignoring for now integrability conditions, we recall that the sufficient optimality conditions require the existence of some (ζ, η) such that, with Θ defined in (3.5), $(\zeta, \Theta) \in \partial\mu(\phi, W)$ and equation (3.4) defines a utility super-gradient density π of U_0 at c. By Proposition 3.2, $(\zeta, \Theta) \in \partial\mu(\phi, W)$ implies (under some regularity) that π is a state-price density at c, and therefore $(\pi \mid x) \leq 0$ for any feasible consumption plan $c + x$. The super-gradient density property of π implies $U(c + x) \leq U(c) + (\pi \mid x)$, and therefore $U(c + x) \leq U(c)$, verifying the optimality of c. In the following result, we confirm that the same optimality conditions lead to the (ζ, Θ)-optimality of c, a fact that implies there is no duality gap.

PROPOSITION A.1. *Suppose that (c, W, ϕ) is a feasible plan, $\pi \in \mathcal{S}_2^{++}$ has the predictable representation (3.4), Θ is defined in* (3.5) $(\zeta, \Theta) \in \mathcal{D}$, *and $\pi W \in \mathcal{S}_1$. If $(\zeta, \Theta) \in \partial f(\phi, W)$ a.e. and π is a super-gradient density of U_0 at c, then (c, ϕ, W) is both optimal and (ζ, Θ)-optimal, and*

$$U_0(c) = \max\{U_0(\tilde{c}) : \tilde{c} \text{ feasible}\} = \min\{V(\tilde{\zeta}, \tilde{\Theta}) : (\tilde{\zeta}, \tilde{\Theta}) \in \mathcal{D}\}.$$

Proof. By Proposition 3.2, π is a state price density at c. Applying Proposition 3.1 shows that the plan (c, W, ϕ) is optimal (in the primal problem). We verify (ζ, Θ)-optimality next. The assumption $(\zeta, \Theta) \in \partial f(W, \phi)$ implies that $f^*(\zeta, \Theta) = f(W, \phi) - W\zeta - \phi'\Theta$. The last equation and the budget equation (3.2) for (c, ϕ, W) in the primal problem imply the budget equation (A.1) in the (ζ, Θ)-problem with $p = 0$ (no disposal). While we did not allow for free disposal in Proposition 3.2, the same argument goes through in the context of the (ζ, Θ)-problem to show that π is a state price density at c relative to (ζ, Θ)-feasibility. If $c + x$ is any (ζ, Θ)-feasible consumption plan, it follows that $(\pi \mid x) \leq 0$, and therefore $U(c + x) \leq U(c) + (\pi \mid x) \leq U(c)$, where we used the assumption that π is a utility super-gradient density of U_0 at c. This confirms the (ζ, Θ)-optimality of the plan (c, ϕ, W). The proposition's last claim follows from Lemma A.1. □

APPENDIX B: PROOFS

This appendix contains proofs omitted from the main text. For simplicity, we omit in most places the obvious a.s. or a.e. qualifications.

B.1. Proof of Lemma 3.1

Given any feasible plan (c, W, ϕ), integration by parts, the dynamics of π, and the budget equation imply

$$d(\pi_t W_t) = \pi_{t-}\left(f(t, W_{t-}, \phi_t) + e_t - c_t - \zeta_t W_t - \phi_t'\Theta_t\right) dt + dM_t,$$

where M is a local martingale. Subtracting this equation from its version obtained by putting $(c + x, W + V, \phi + \delta)$ in place of (c, W, ϕ) results in $d(\pi_t V_t) = \pi_t(D_t - x_t)\, dt +$

dN_t, where N is another local martingale. Letting $\{\tau_n\}$ be a localizing stopping-time sequence for N, integrating from 0 to τ_n, and taking expectations results in

$$\mathbb{E}\left[\pi_{\tau_n} V_{\tau_n}\right] = \mathbb{E}\left[\int_0^{\tau_n} \pi_t \left(D_t - x_t\right) dt\right]. \tag{B.1}$$

By the definition of a wealth process, $(W + V)^- \in \mathcal{S}_2$. Since $\pi \in \mathcal{S}_2^{++}$, we can apply Fatou's lemma as follows

$$\liminf_{n\to\infty} \mathbb{E}[\pi_{\tau_n}(W_{\tau_n} + V_{\tau_n})] \geq \mathbb{E}[\pi_T(W_T + V_T)] = \mathbb{E}[\pi_T(W_T + x_T)].$$

On the other hand, if $\pi W \in \mathcal{S}_1$, $\mathbb{E}[\pi_{\tau_n} W_{\tau_n}]$ converges to $\mathbb{E}[\pi_T W_T]$. The proof of part (a) is completed by taking the limit inferior on both sides of equation (B.1) as $n \to \infty$. Part (b) follows similarly.

B.2. Proof of Proposition 4.1

Given the consumption plans $c, c + x \in \mathcal{C}$, let

$$(U, \Sigma) = (U(c), \Sigma(c)) \quad \text{and} \quad (U + Y, \Sigma + \Delta) = (U(c + x), \Sigma(c + x)).$$

Integration by parts, the dynamics of π, and the utility dynamics imply

$$d(\mathcal{E}_t U_t) = -\mathcal{E}_t \left(F(t, c_t, U_t, \Sigma_t) - F_U(t)U_t - \int_Z F_\Sigma(t, z)\Sigma(t, z)\nu(t, dz)\right) dt + dM_t,$$

where M is a local martingale. We subtract the above equation from the same equation with $(c + x, U + Y, \Sigma + \Delta)$ in place of (c, U, Σ) to find

$$d\left(\mathcal{E}_t Y_t\right) = -\mathcal{E}_t D_t\, dt - \pi_t x_t\, dt + dN_t, \tag{B.2}$$

where N is another local martingale, and

$$\begin{aligned} D_t = {} & F(t, c_t + x_t, U_t + Y_t, \Sigma_t + \Delta_t) - F(t, c_t, U_t, \Sigma_t) \\ & - F_c(t, c_t, U_t, \Sigma_t)x_t - F_U(t)Y_t - \int_Z F_\Sigma(t, z)\Delta(t, z)\nu(\omega, t, dz) \leq 0 \text{ a.s.} \end{aligned}$$

(The last inequality follows from (4.4).) Integrating (B.2) from 0 to any stopping time τ gives

$$Y_0 \leq \int_0^{\tau} \pi_t x_t\, dt + \mathcal{E}_\tau Y_\tau - (N_\tau - N_0).$$

Let $\{\tau_n : n = 1, 2, \ldots\}$ be an increasing sequence of stopping times that converges to T and such that N stopped at τ_n is a martingale. Then

$$Y_0 \leq \mathbb{E}\left[\int_0^{\tau_n} \pi_t x_t\, dt + \mathcal{E}_{\tau_n} Y_{\tau_n}\right], \quad n = 1, 2, \ldots$$

We let $n \to \infty$, and apply dominated convergence. Since $\pi, x \in \mathcal{H}$, the first term of the right-hand side converges to $\mathbb{E}[\int_0^T \pi_t x_t\, dt]$. Since $\mathcal{E} \in \mathcal{S}_2$ and $U, U + Y \in \mathcal{U} \subseteq \mathcal{S}_2$, it follows that $\mathbb{E}[\mathcal{E}_{\tau_n} Y_{\tau_n}] = \mathbb{E}[\mathcal{E}_{\tau_n}(U_{\tau_n} + Y_{\tau_n})] - \mathbb{E}[\mathcal{E}_{\tau_n} U_{\tau_n}]$ converges to $\mathbb{E}[\mathcal{E}_T Y_T]$. The proof is completed by applying the gradient inequality to $F(T, \cdot)$ to conclude that $Y_T \leq F_c(T, c_T)x_T = \lambda_T x_T$.

B.3. Proof of Theorem 5.1

Sufficiency: Suppose Condition 5.1 holds. Given any $(\omega, t, w, \alpha) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m$, one can easily check that $\partial f(\omega, t, w, \alpha)$ is equal to the set of all $(z, \theta) \in \mathbb{R} \times \mathbb{R}^m$ such that $\mu^\kappa(\omega, t) = z + \kappa'\theta$ and $\theta \in \partial f^0(\omega, t, \alpha - w\kappa)$. Given this observation, sufficiency follows by Proposition 3.2.

Necessity: Suppose $\pi \in \mathcal{S}_2^{++}$ is a state-price density at c, and returns jumps are bounded. For simplicity, we omit the a.e. qualifier throughout.

Step 1 We show that

$$\pi_s \Gamma_s = \mathbb{E}\left[\int_s^T \pi_t \gamma_t \, dt + \pi_T \Gamma_T \,\middle|\, \mathcal{F}_s \right], \quad s \in [0, T], \tag{B.3}$$

reflecting the fact that trading in the γ-fund is unrestricted, and therefore the cash flow γ is priced correctly by any state price density. We fix any (deterministic) time $s \in [0, T)$, event $F \in \mathcal{F}_s$, and time length $\varepsilon \in (0, T - s)$, and we define the processes $\dot{g}$ and g by

$$\dot{g}(\omega, t) = \frac{1}{\varepsilon} 1_{\{\omega \in F,\ s < t \le s + \varepsilon\}} \quad \text{and} \quad g(\omega, t) = \int_0^t \dot{g}(\omega, u)\, du, \quad (\omega, t) \in \Omega \times [0, T].$$

Consider the cash flow

$$x_t = g_t \gamma_t - \dot{g}_t \Gamma_t, \quad t \in [0, T), \quad x_T = g_T \Gamma_T.$$

(Since $\Gamma \in \mathcal{S}_2$ and γ is assumed bounded, one can easily confirm that $x \in \mathcal{H}$.) We show that $x \in \mathcal{X}(c)$ by verifying the feasibility of the plan $(c + x, W + g\Gamma, \phi + g\Gamma_-\kappa)$. Integration by parts and the dynamics of Γ in (5.3) imply that

$$d(g\Gamma) = g\, d\Gamma + \dot{g}\Gamma\, dt = (g\Gamma\mu^\kappa - x)\, dt + g\Gamma_-\kappa' d\hat{R}.$$

The budget equation for (c, W, ϕ) in this context is

$$dW = (W\mu^\kappa + f^0(\phi - W\kappa) + e - c)\, dt + \phi' d\hat{R}.$$

Combining the last two equations gives the budget equation for the plan $(c + x, W + g\Gamma, \phi + g\Gamma_-\kappa)$:

$$\begin{aligned} &d(W + g\Gamma) \\ &\quad = \big((W + g\Gamma)\mu^\kappa + f^0(\phi + g\Gamma_-\kappa - (W + g\Gamma)\kappa) + e - (c + x)\big)\, dt + (\phi + g\Gamma_-\kappa)'\, d\hat{R}. \end{aligned}$$

By Condition 5.1, $(W, \phi) \in K$ implies $(W + g\Gamma, \phi + g\Gamma_-\kappa) \in K$, completing the proof that $x \in \mathcal{X}(c)$. Similarly, $-x \in \mathcal{X}(c)$, and therefore $(\pi \mid x) = 0$, an equation that can be expanded to

$$\mathbb{E}\left[1_F \left(\int_s^{s+\varepsilon} \pi_t g_t \gamma_t \, dt - \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \pi_t \Gamma_t \, dt + \int_{s+\varepsilon}^T \pi_t \gamma_t \, dt + \pi_T \Gamma_T \right) \right] = 0.$$

Letting $\varepsilon \downarrow 0$ and using the right continuity of π and Γ, the boundedness of γ, and the assumption $\pi \in \mathcal{S}_2^{++}$, we find

$$\mathbb{E}\left[1_F \left(-\pi_s \Gamma_s + \int_s^T \pi_t \gamma_t \, dt + \pi_T \Gamma_T \right) \right] = 0.$$

Since the last equation holds for all $F \in \mathcal{F}_s$, the proof of (B.3) is complete.

Step 2 We show that

$$\mu^\kappa = \zeta + \kappa'\Theta. \tag{B.4}$$

Consider the martingale

$$M_t = \pi_t\Gamma_t + \int_0^t \pi_s\gamma_s\,ds = \mathbb{E}\left[\int_0^T \pi_s\gamma_s\,ds + \pi_T\Gamma_T \,\middle|\, \mathcal{F}_t\right],$$

where the last equation follows from (B.3). Integration by parts and the dynamics of Γ imply

$$dM_t = \pi_{t-}\Gamma_t\left(\mu_t^\kappa - \zeta_t - \int_Z \kappa_t'\sigma^R(t,z)\eta(t,z)\nu(t,dz)\right)dt + dN_t,$$

for a local martingale N. Setting the drift term to zero results in (B.4).

Step 3 We show that

$$\Theta \in \partial f^0(\phi^0), \quad \text{where } \phi^0 = \phi - W\kappa.$$

Given the form of the function f and equation (B.4), for any the feasible plan $(c + x, W + V, \phi + \Delta)$, the process D in Lemma 3.1 is equal to $D(\hat{\Delta})$, where $\hat{\Delta} = \Delta - V\kappa$, and the function $D : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$D(\hat{\alpha}) = f^0(\phi^0 + \hat{\alpha}) - f^0(\hat{\alpha}) - \hat{\alpha}'\Theta.$$

For any positive integer N let

$$S^N(\omega, t) = \left\{\hat{\alpha} \in \mathbb{R}^m : \phi^0(\omega, t) + \hat{\alpha} \in K^0,\ \|\hat{\alpha}\| \leq N\right\},$$

$$D^N(\omega, t) = \max\left\{D(\omega, t, \hat{\alpha}) : \hat{\alpha} \in S^N\right\}.$$

Since $D(\omega, t, 0) = 0$, we know that $D^N(\omega, t) \geq 0$. We will show that $D^N = 0$ a.e.

Applying the Measurable Maximum Theorem (see, for example, theorem 17.18 in Aliprantis and Border [1999]), we select a predictable process $\hat{\Delta}$ such that

$$D^N(\omega, t) = D\left(\omega, t, \hat{\Delta}(\omega, t)\right) \quad \text{and} \quad \hat{\Delta}(\omega, t) \in S^N.$$

Fixing any positive integer N, we also define the stopping time and stopped process

$$\tau^N = \inf\{t \in [0, T] : |V_t| \geq N \text{ or } \Gamma_t \leq 1/N\}, \quad \hat{\Delta}_t^N = \hat{\Delta}_t 1_{\{t \leq \tau^N\}},$$

where V is the incremental wealth process in the feasible plan $(\tilde{c}, \tilde{W}, \tilde{\phi}) = (c + x, W + V, \phi + \Delta^N)$ that we now construct. Let $\Delta^N = \hat{\Delta}^N + V\kappa$ and $x = k\gamma$ for predictable k . The budget equations are therefore

$$dW = \left(W\mu^\kappa + f^0(\phi^0) + e - c\right)dt + \phi' d\hat{R}$$

$$d\tilde{W} = \left(\tilde{W}\mu^\kappa + f^0(\phi^0 + \hat{\Delta}^N) + e - c - k\gamma\right)dt + \tilde{\phi}' d\hat{R}.$$

Given the first equation, the second is satisfied if

$$dV = (V\mu^\kappa + f^0(\phi^0 + \hat{\Delta}^N) - f^0(\phi^0) - k\gamma)\,dt + (\hat{\Delta}^N + V\kappa)' d\hat{R}.$$

Letting

$$k_t = \begin{cases} 1 & \text{if } t \leq \tau^N \\ V_{\tau^N}/\Gamma_{\tau^N} & \text{if } t > \tau^N \end{cases},$$

then $V_t = \Gamma_t V_{\tau^N}/\Gamma_{\tau^N}$ for $t > \tau^N$. Given the assumption that return jumps are bounded above and away from zero, $V_t 1_{\{t \le \tau^N\}}$ and $V_{\tau^N}/\Gamma_{\tau^N}$ are bounded. Therefore, using the assumptions $\pi, \Gamma \in \mathcal{S}_2$, is follows that $\pi V \in \mathcal{S}_1$. Lemma 3.1 implies $0 \ge \mathbb{E}[\int_0^T \pi_t D_t^N dt]$, and therefore $D^N = 0$ a.e.

B.4. Proof of Theorem 5.2

Sufficiency: Let $x = \mathcal{X}(\delta, \Sigma)$. Using the identities (5.6) and $c = \gamma(x + U)$, and the quasi-linearity of the budget equation in Condition 5.1, the wealth dynamics (5.11) imply the budget equation (3.2). Applying integration by parts to $Y = U\Gamma - W$ and substituting the resulting volatility expression of Condition 5.3, as well as $G^*(\delta, \Sigma) = G(x, \Sigma) - \delta x$, it follows that U satisfies the utility BSDE. Therefore (c, W, ϕ) is a feasible plan and $U = U(c)$. Optimality is proven by showing that the process $\pi = \mathcal{E}/\Gamma$ is both a utility super-gradient and a state-price density at c, and applying Proposition 3.1.

By assumption $\lambda = 1/\Gamma$. The aggregator form implies that $\lambda = \gamma^{-1} G_x(t, x, \Sigma) = F_c(c, U, \Sigma)$. Letting $F_U = -G_x(x, \Sigma) (= -\delta)$ and $F_\Sigma = G_\Sigma(x, \Sigma)$, it follows that $(F_U, F_\Sigma) \in (\partial_{U,\Sigma} F)(c, U, \Sigma)$. By Proposition 4.1, π is a super-gradient density of U_0 at c.

Finally, we verify that $\pi = \mathcal{E}\lambda$ is a state-price density at c. Applying Itô's lemma to $\lambda = 1/\Gamma$ and using the Γ dynamics (5.3), we find

$$\frac{d\lambda}{\lambda_-} = \left(-\mu^\kappa + \delta + \int_Z \frac{\sigma^\lambda(z)^2}{1 + 1_{\{z \in E\}} \sigma^\lambda(z)} \nu(dz) \right) dt + \sum_{i=1}^d \sigma^\lambda(z^i)\, dB_t^i + \int_E \sigma^\lambda(z) \hat{p}(dt \times dz),$$

$$\sigma^\lambda = -\frac{\kappa' \sigma^R}{J}.$$

The above equations and integration by parts applied to $\pi = \mathcal{E}\lambda$ imply the state-price dynamics (3.4) with

$$\zeta = \mu^\kappa + \int_Z \frac{\kappa' \sigma^R(z)}{J(z)} \left[G_\Sigma(z) - \kappa' \sigma^R(z) \right] \nu(t, dz) \quad \text{and} \quad \eta = G_\Sigma + \sigma^\lambda + G_\Sigma \sigma^\lambda 1_{\{z \in E\}}.$$

By Theorem 5.1, π is a state-price density at c.

Necessity: That $\Gamma = 1/\lambda$ follows by the same argument used in the proof of theorem 9(b) in Schroder and Skiadas (2005). Since $\lambda = G_x(x, \Sigma)/\gamma$, it follows that $x = \mathcal{X}(\delta, \Sigma)$ a.e. Defining $Y = U\Gamma - W$ and $\phi^0 = \phi - W\kappa$, substituting into the budget equation (3.2), and using the quasi-linearity in wealth from Condition 5.1, we obtain the wealth dynamics (5.11). Applying Itô's lemma to $Y = U\Gamma - W$, and using the dynamics of Γ, U and W, and the definition of G^* confirms the formula for Σ in Condition 5.3 and the BSDE for Y. The rest of Condition 5.3 follows from the necessity parts of Proposition 9 and Theorem 5.1.

B.5. Proof of Theorem 7.1

Sufficiency: Follows from Proposition 3.2.

Necessity: The proof is similar to the proof of theorem 7(b) in Schroder and Skiadas (2003). Their Lemma A.1 still applies (with *càdlàg* instead of continuous ϱ, and using the fact that jumps are bounded), showing that the state-price density π correctly prices any feasible consumption plan:

$$\pi_t W_t = \mathbb{E}\left[\int_t^T \pi_s c_s\, ds + \pi_T c_T \,\middle|\, \mathcal{F}_t\right], \quad t \in [0, T]. \tag{B.5}$$

Applying integration by parts to the martingale

$$M_t = \pi_t W_t + \int_0^t \pi_s c_s\, ds,$$

we get

$$dM_t = \pi_{t-} W_{t-} \left\{ f^1(t, \psi_t) - \zeta(t) + \psi_t' \int_Z \sigma^R(t, z)\eta(t, z)\nu(t, dz) \right\} dt + dN_t,$$

for a local martingale N. Setting the drift term to zero results in

$$\zeta_t = f^1(t, \psi_t) - \psi_t' \Theta_t.$$

Consider any feasible strategy $(\tilde{\psi}, \tilde{\rho})$ with corresponding wealth process $\tilde{W}$. Then D_t of Lemma 3.1 is given by

$$D_t = \tilde{W}_t \left\{ f^1(t, \tilde{\psi}_t) - f^1(t, \psi_t) - \Theta_t(\tilde{\psi}_t - \psi_t) \right\}.$$

Given the fact that π is a state price density at c, if $\mathbb{E}[\int_0^T \pi_t D_t^+ dt] < \infty$ and $\pi(\tilde{W}\tilde{\psi} - W\psi) \in \mathcal{S}_1$, then

$$0 \geq (\pi \mid x) = \mathbb{E}\left[\int_0^T \pi_t D_t\, dt\right]. \tag{B.6}$$

For any positive integer N, let $S^N(\omega, t)$ be the set of all $\hat{\psi} \in \mathbb{R}^m$ such that

$$|\hat{\psi} - \psi(\omega, t)| \leq N \quad \text{and} \quad f^1(\omega, t, \hat{\psi}) - f^1(\omega, t, \psi(\omega, t)) - \Theta(\omega, t)(\hat{\psi} - \psi(\omega, t)) \leq N.$$

Applying a measurable selection theorem (see, for example, theorem 17.13 in Aliprantis and Border [1999]), let $\tilde{\psi}$ be a predictable selection from S^N, and let $\tilde{W}$ be the trading strategy generated by $(\tilde{\psi}, \rho)$. We also define the stopping time

$$\tau^N = \inf\left\{ t \in [0, T] : |\tilde{W}_t\tilde{\psi}_t - W_t\psi_t| \geq N \ \text{ or} \, \tilde{W}_t / W_t \geq N \right\}.$$

Let

$$\tilde{\psi}_t^N = \tilde{\psi} 1_{\{t < \tau^N\}} + \psi 1_{\{t \geq \tau^N\}},$$

and let $\tilde{W}^N$ be the wealth process generated by $(\tilde{\psi}^N, \rho)$. By the construction of the stopping time and the assumed bound on return jumps, $\rho \tilde{W}^N \leq b\rho W$ for some scalar $b > 0$, and therefore $\rho \tilde{W}^N \in \mathcal{C}$. Inequality (B.6) applied to $(\tilde{\psi}^N, \rho)$ and $\tilde{W}^N$ implies $f^1(\tilde{\psi}^N) - f^1(\psi) - \Theta_t(\tilde{\psi}^N - \psi) \leq 0$ a.s. Taking the union over all N finishes the proof.

B.6. Proof of Theorem 7.2

Sufficiency: Proposition 4.1 implies that $\pi = \lambda\mathcal{E}$ is a utility super-gradient density at c, and Proposition 3.2 implies that π is also a state-price density ($\pi W \in \mathcal{S}_1$ follows from $\pi W = \lambda\mathcal{E} W = U\mathcal{E}$, and $\mathcal{E} \in \mathcal{S}_2$). Optimality follows from Proposition 3.1.

Necessity: Having established the pricing equation (B.5), the proof that $U = \lambda W$ is the same as in the necessity proof of theorem 23 in Schroder and Skiadas (2003). Proposition 4.1 implies that $\pi = \mathcal{E}\lambda$ is a utility gradient density at c. Because c is optimal, Proposition 3.1 implies that π is also a state-price density. It follows from Theorem 7.1 that condition (7.2) holds, and therefore Condition 7.3 is satisfied.

REFERENCES

AASE, K. K. (1984): Optimum Portfolio Diversification in a General Continuous-Time Model, *Stochast. Process. Appl.* 18, 81–98.

AASE, K. K. (1986): Ruin Problems and Myopic Portfolio Optimization in Continuous Trading, *Stochast. Process. Appl.* 21, 213–227.

ALIPRANTIS, C. D., and K. C. BORDER (1999): *Infinite Dimensional Analysis: A Hitchhikers Guide*, 2nd end. Berlin: Springer-Verlag.

BALLY, V., and G. PAGES (2002): A Quantization Algorithm for Solving Discrete Time Multidimensional Optimal Stopping Problems, *Bernoulli* 9, 1003–1049.

BARDHAN, I., and X. CHAO (1995): Martingale Analysis for Assets with Discontinuous Returns, *Math. Oper. Res.* 20, 243–256.

BARLES, G., R. BUCKDAHN, and E. PARDOUX (1997): Backward Stochastic Differential Equations and Integro-Partial Differential Equations, *Stochast. Stochast. Rep.* 60, 57–83.

BECHERER, D. (2006): Bounded Solutions to Backward SDEs with Jumps for Utility Optimization and Indifference Hedging, *Ann. Appl. Probab.* 16, 2027–2054.

BJÖRK, T., Y. KABANOV, and W. RUNGGALDIER (1997): Bond Market Structure in the Presence of Marked Point Processes, *Math. Finance* 7, 211–223.

BOUCHARD, B., and R. ELIE (2005): Discrete Time Approximation of Decoupled Forward–Backward SDE with Jumps. Working Paper, LPMA, CNRS, UMR 7599, Université Paris 6 and CREST.

BOUCHARD, B., and N. TOUZI (2004): Discrete-Time Approximation and Monte-Carlo Simulation of Backward Stochastic Differential Equations, *Stochast. Process. Appl.* 111, 175–206.

CHENG, P., and O. SCAILLET (2005): Linear-Quadratic Jump-Diffusion Modeling with Application to Stochastic Volatility. Working Paper, HEC Geneva, Switzerland.

CHEVANCE, D. (1997): Numerical Methods for Backward Stochastic Differential Equations, in *Numerical Methods in Finance*, L. Rogers, and D. Talay, eds. Cambridge: Cambridge University Press, 232–244.

CONT, R., and P. TANKOV (2004): *Financial Modelling with Jump Processes*. London: Chapman & Hall/CRC.

COX, J., and C.-F. HUANG (1989): Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process, *J. Econ. Theory* 49, 33–83.

CUOCO, D. (1997): Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income, *J. Econ. Theory* 72, 33–73.

CUOCO, D., and J. CVITANI (1998): Optimal Consumption Choices for a Large Investor, *J. Econ. Dyn. Contl.* 22, 401–436.

CUOCO, D., and H. LIU (2000): A Martingale Characterization of Consumption Choices and Hedging Costs with Margin Requirements, *Math. Finance* 10, 355–385.

CVITANI , J., and I. KARATZAS (1992): Convex Duality in Constrained Portfolio Optimization, *Ann. Appl. Probab.* 2, 767–818.

DETEMPLE, J., and M. RINDISBACHER (2005): Closed-Form Solutions for Optimal Portfolio Selection with Stochastic Interest Rate and Investment Constraints, *Math. Finance* 15, 539–568.

DETEMPLE, J., and A. SERRAT (2003): Dynamic Equilibrium with Liquidity Constraints, *Rev. Financial Stud.* 16, 597–629.

DETEMPLE, J. B., R. GARCIA, and M. RINDISBACHER (2003): A Monte Carlo Method for Optimal Portfolios, *J. Finance* 58, 401–446.

DOUGLAS Jr., J., J. MA, and P. PROTTER (1996): Numerical Methods for Forward-Backward Stochastic Differential Equations, *Ann. Appl. Probab.* 6, 940–968.

DUFFIE, D. (2005): Credit Risk Modeling with Affine Processes, *J. Banking Finance* 29, 2751–2802.

DUFFIE, D., and L. EPSTEIN (1992): Stochastic Differential Utility, *Econometrica* 60, 353–394.

DUFFIE, D., D. FILIPOVIC, and W. SCHACHERMAYER (2003): Affine Processes and Applications in Finance, *Ann. Appl. Probab.* 13, 984–1053.

DUFFIE, D., V. FLEMING, M. SONER, and T. ZARIPHOPOULOU (1997): Hedging in Incomplete Markets with HARA Utility, *J. Econ. Dyn. Contl.* 21, 753–782.

EL KAROUI, N., and M. JEANBLANC-PICQUÈ (1998): Optimization of Consumption with Labor Income, *Finance Stochast.* 2, 409–440.

EL KAROUI, N., and L. MAZLIAK (eds.) (1997): *Backward Stochastic Differential Equations*. England: Addison Wesley Longman, Essex.

EL KAROUI, N., S. PENG, and M.-C. QUENEZ (2001): A Dynamic Maximum Principle for the Optimization of Recursive Utilities under Constraints, *Ann. Appl. Probab.* 11, 664–693.

EPSTEIN, L. (1992): Behavior under Risk: Recent Developments in Theory and Applications, in *Advances in Economic Theory*, J.-J. Laffont ed. Cambridge: Cambridge University Press.

EPSTEIN, L., and S. ZIN (1989): Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework, *Econometrica* 57, 937–969.

FRAMSTAD, N. C., B. ØKSENDAL, and A. SULEM (1998): Optimal Consumption and Portfolio in a Jump Diffusion Market, in *Mathematical Finance* A. Shiryaev and A. Sulem, eds. Paris: INRIA, 8–20.

GOBET, E., J. LEMOR, and X. WARIN (2005): A Regression-Based Monte Carlo Method to Solve Backward Stochastic Differential Equations, *Ann. Appl. Probab.* 15, 2172–2202.

HAMADENE, S. (1996): Équations Différentielles Stochastiques Rétrogrades: Le Cas Localement Lipschitzien, *Ann. Inst. Henri Poincaré* 32, 645–659.

HE, H., and H. PAGÈS (1993): Labor Income, Borrowing Constraints and Equilibrium Asset Prices: A Duality Approach, *Econ. Theory* 3, 663–696.

HE, H., and N. PEARSON (1991): Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constrains: The Infinite Dimensional Case, *J. Econ. Theory* 54, 259–304.

JACOD, J., and A. N. SHIRYAEV (2003): *Limit Theorems for Stochastic Processes*, 2nd edn. Berlin Heidelberg: Springer-Verlag.

JEANBLANC-PICQUÉ, M., and M. PONTIER (1990): Optimal Porfolio for a Small Investor in a Market Model with Discontinuous Prices, *Appl. Math. Opt.* 22, 287–310.

KARATZAS, I., J. LEHOCZKY, and S. SHREVE (1987): Optimal Portfolio and Consumption Decisions for a Small Investor on a Finite Horizon, *SIAM J. Contl. Opt.* 25, 1557–1586.

KARATZAS, I., J. LEHOCZKY, S. SHREVE, and G. XU (1991): Martingale and Duality Methods for Utility Maximization in an Incomplete Market, *SIAM J. Contl. Opt.* 29, 702–730.

KOBYLANSKI, M. (2000): Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth, *Ann. Probab.* 28, 558–602.

KRAMKOV, D., and W. SCHACHERMAYER (1999): The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets, *Ann. Appl. Probab.* 9, 904–950.

KRAMKOV, D., and W. SCHACHERMAYER (2003): Necessary and Sufficient Conditions in the Problem of Optimal Investment in Incomplete Markets, *Ann. Appl. Probab.* 13, 1504–1516.

KREPS, D., and E. PORTEUS (1978): Temporal Resolution of Uncertainty and Dynamic Choice Theory, *Econometrica* 46, 185–200.

LAZRAK, A., and M. C. QUENEZ (2003): A Generalized Stochastic Differential Utility, *Math. Oper. Res.* 28, 154–180.

LEMOR, J., E. GOBET, and X. WARIN (2006): Rate of Convergence of an Empirical Regression Method for Solving Generalized Backward Stochastic Differential Equations, *Bernoulli* 12, 889–916.

LEPELTIER, J.-P., and J. SAN MARTÍN (1997): Backward Stochastic Differential Equations with Continuous Coefficient, *Stat. Probab. Lett.* 32, 425–430.

LEPELTIER, J.-P., and J. SAN MARTÍN (1998): Existence for BSDE with Superlinear-Quadratic Coefficient, *Stochast. Stochast. Rep.* 63, 227–240.

LEPELTIER, J.-P., and J. SAN MARTÍN (2002): On the Existence or Non-Existence of Solutions for Certain Backward Stochastic Differential Equations, *Bernoulli* 8, 123–137.

LIU, J., and F. A. LONGSTAFF (2003): Dynamic Asset Allocation with Event Risk, *J. Finance* 58, 231–259.

LIU, J., and F. A. LONGSTAFF (2004): Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities, *Rev. Financial Stud.* 17(3), 611–641.

MA, J., P. PROTTER, J. SAN MARTIN, and S. TORRES (2002): Numerical Method for Backward Stochastic Differential Equations, *Ann. Appl. Probab.* 12, 302–316.

MA, J., P. PROTTER, and J. YONG (1994): Solving Forward-Backward Stochastic Differential Equations Explicitly. A Four Step Scheme, *Probab. Theory Relat. Fields* 98, 339–359.

MERTON, R. (1969): Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case, *Rev. Econ. Stat.* 51, 247–257.

MERTON, R. (1971): Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *J. Econ. Theory* 3, 373–413, Erratum 6 (1973): 213–214.

OCONE, D., and I. KARATZAS (1991): A Generalized Clark Representation Formula, with Application to Optimal Portfolios, *Stochast. Stochast. Rep.* 34, 187–220.

ØKSENDAL, B., and A. SULEM (2005): *Applied Stochastic Control of Jump Diffusions*. New York: Springer.

PARDOUX, E. (1997): Generalized Discontinuous Backward Stochastic Differential Equations, in *Backward Stochastic Differential Equations*, N. El Karoui and L. Mazliak, eds. England: Addison Wesley Longman, Essex. 207–219.

PARDOUX, E., and S. PENG (1990): Adapted Solution of a Backward Stochastic Differential Equation, *Syst. Contl. Lett.* 14, 55–61.

PARDOUX, E., F. PRADEILLES, and Z. RAO (1997): Probabilistic Interpretation of a System of Semi-Linear Parabolic Partial Differential Equations, *Annales de l'institut Henri Poincaré Henri Poincaré* 33, 467–490.

PIAZZESI, M. (2005): Affine Term Structure Models, in *Handbook of Financial Econometrics*. Amsterdam, New York: Elsevier, North Holland.

ROYER, M. (2006): Backward Stochastic Differential Equations with Jumps and Related Non-Linear Expectations, *Stochast. Process. Appl.* 116, 1357–1376.

SCHRODER, M., and C. SKIADAS (1999): Optimal Consumption and Portfolio Selection with Stochastic Differential Utility, *J. Econ. Theory* 89, 68–126.

SCHRODER, M., and C. SKIADAS (2003): Optimal Lifetime Consumption-Portfolio Strategies under Trading Constraints and Generalized Recursive Preferences, *Stochast. Process. Appl.* 108, 155–202.

SCHRODER, M., and C. SKIADAS (2005): Lifetime Consumption-Portfolio Choice under Trading Constraints and Nontradable Income, *Stochast. Process. Appl.* 115, 1–30.

SKIADAS, C. (2008): Dynamic Portfolio Choice and Risk Aversion, in *Handbooks in Operation Research and Management Science*, Vol. 15, J.R. Birge and V. Lenetsky, eds. Elsevier B.V., 789–843.

TANG, S., and X. LI (1994): Necessary Conditions for Optimal Control of Stochastic Systems with Random Jumps, *SIAM J. Contl. Opt.* 32, 1447–1475.

VILA, J.-L., and T. ZARIPHOPOULOU (1997): Optimal Consumption and Portfolio Choice with Borrowing Constraints, *J. Econ. Theory* 77, 402–431.

WU, L. (2003): Jumps and Dynamic Asset Allocation, *Rev. Quant. Finance Account.* 20, 207–243.

ZARIPHOPOULOU, T. (1994): Consumption-Invenstment Models with Constraints, *SIAM J. Contl. Opt.* 32, 59–85.

ZHANG, J. (2004): A Numerical Scheme for BSDEs, *Ann. Appl. Probab.* 14, 459–488.