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On the uniqueness of fully informative rational expectations equilibria★

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Summary. This paper analyzes two equivalent equilibrium notions under asymmetric information: risk neutral rational expectations equilibria (rn-REE), and common knowledge equilibria. We show that the set of fully informative rn-REE is a singleton, and we provide necessary and sufficient conditions for the existence of partially informative rn-REE. In a companion paper (DeMarzo and Skiadas (1996)) we show that equilibrium prices for the larger class of quasi-complete economies can be characterized as rn-REE. Examples of quasi-complete economies include the type of economies for which demand aggregation in the sense of Gorman is possible (with or without asymmetric information), the setting of the Milgrom and Stokey no-trade theorem, an economy giving rise to the CAPM with asymmetric information but no normality assumptions, the simple exponential-normal model of Grossman (1976), and a case of no aggregate endowment risk. In the common-knowledge context, we provide necessary and sufficient conditions for a common knowledge posterior estimate, given common priors, to coincide with the full communication posterior estimate.

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1 Introduction

In the literature of competitive rational expectations equilibria with asymmetrically informed agents there are several well known basic models in which a fully informative equilibrium is known to exist, but the possibility of partially informative equilibria is either not resolved, or not characterized.

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The following are three such familiar examples:

(a) When agents have "concordant" beliefs and rational expectations, and the initial allocation is ex-ante Pareto optimal, Milgrom and Stokey (1982) have shown that there is no trade by risk-averse agents no matter what their private information is. The associated prices can be fully informative, meaning that they are also equilibrium prices in an economy in which all agents make their private information public.[1] Partially informative prices are also a possibility, however, and a clear understanding of when exactly partial revelation is possible is lacking.

(b) Grossman (1976, 1978) has demonstrated a fully informative equilibrium in a simple model in which agents with constant absolute risk aversion trade risky assets and one risk-free asset, having observed private signals, and being initially endowed with some of these assets. Grossman's linear equilibrium relies on the assumption that payoffs and signals are jointly normally distributed. The possibility of partially informative equilibria has remained an open question, despite the simplicity of the model.

(c) Aumann (1976) has argued that if some communication process has caused agents with common priors to have common knowledge of their estimate of a random variable, then this estimate has to be common among all agents. The estimate need not coincide with what the agents would have agreed upon, however, had they pooled all their private information. A characterization of when the common posterior has to agree with the full communication posterior is lacking. While not directly involving trading, we will show that posterior estimates in Aumann's setting can be viewed as equilibrium prices in an economy with risk-neutral agents, and whether the posterior is the full-communication posterior corresponds exactly to whether these equilibrium prices are fully informative or not.

In a paper closely related to the present one (DeMarzo and Skiadas 1996), hereafter abbreviated to DS, we have shown that all of the above examples are instances of a type of economy with asymmetric information that we call quasi-complete. There are several other examples of quasi-complete economies of interest. For example, in DS we show that the class of economies that allow demand aggregation in the sense of Gorman (1953) under symmetric information (spanned endowments and linear risk tolerance being key ingredients) are quasi-complete even if agents possess heterogeneous information. As a consequence, Gorman aggregation in these economies is possible even under asymmetric information and partially informative prices. Grossman's (1976) model is a special case, as is an asymmetric information version of the CAPM of Sharp (1964) and Lintner (1965), with quadratic utilities, endowment spanning, but no normality assumptions.

[1] We use the term "fully informative" as opposed to "fully revealing" to emphasize the distinction that in a fully informative equilibrium agents need not in fact know other agents' private signal realizations, but given their information they would not alter their decisions even if they did observe the agents' pooled signals.

Another example of a quasi-complete economy discussed in DS is one with asymmetric information, common priors, and no aggregate endowment risk. Quasi-complete economies always have a fully informative equilibrium, and partially informative equilibria may or may not exist, depending on the underlying distribution of the asset payoffs and agent signals. In this paper we develop conditions that are necessary and sufficient for the existence of partially informative equilibria in quasi-complete economies.

A key result of DS that is of interest to us in this paper is that equilibrium prices of quasi-complete economies can always be characterized as equilibrium prices in an equivalent risk-neutral economy. With this fact in mind, in this paper we formulate equilibrium prices directly in terms of the notion of a risk-neutral rational expectations equilibrium (rn-REE). We will show that the set of fully informative rn-REE prices is a singleton, and we will provide necessary and sufficient conditions for the existence of partially informative rn-REE. In the context of the common-knowledge literature, these conditions can be interpreted as necessary and sufficient for a common-knowledge posterior estimate (under common priors) to coincide with the full-communication posterior. The same conditions can be applied to any instance of a quasi-complete economy. For example, as a corollary of this paper's results, we show in DS that the fully informative equilibrium in Grossman's (1976) model is in fact unique, but also that minor deviations from the assumption that asset payoffs and signals are jointly normally distributed lead to the existence of partially informative equilibria.

Examples of robust partially informative rational expectations equilibria have been constructed by Ausubel (1990), and Polemarchakis and Siconolfi (1993). The results of this paper also provide a way of constructing partially informative equilibria in the context of quasi-complete economies. Quasi-complete economies, however, always possess a fully informative equilibrium, and therefore partial revelation is always associated with indeterminacy.

The remainder of the paper is organized in five sections and an appendix. In Section 2 we define the equivalent notions of a risk-neutral rational expectations equilibrium (rn-REE) and a common-knowledge equilibrium (CKE). We show that the set of fully informative rn-REE is a singleton, while partially informative rn-REE can also exist. In Section 3 we develop the "separably oriented" (SO) condition, which is sufficient for uniqueness, and, in a sense, nearly necessary as well. For example, it is the SO condition that is used in DS to prove uniqueness in Grossman's model. In Section 4 we formulate the "overlapping diagonals" condition (OD), the violation of which is easy to confirm, and provides an easy way of showing examples of the existence of robust partially informative equilibria. We also present special cases in which the OD condition is sufficient for uniqueness. In Section 5, we develop the "approximately separately oriented" (ASO) condition, which is both necessary and sufficient for uniqueness, and we discuss its relationship with the more tractable SO and OD conditions. Section 6 concludes with a discussion of "randomized" versus "pure" prices, notions introduced in Section 2. The appendix develops a requisite mathematical duality result.

2 Risk-neutral REE and common knowledge

We begin with the formal notion of a risk-neutral rational expectations equilibrium (rn-REE), followed by an interpretation in the context of the common-knowledge literature.

Uncertainty is represented throughout by the probability space $(\Omega, \mathscr{F}, P)$, with E denoting the corresponding expectation operator. There are n agents. Agent i observes a private signal $S_i : \Omega \to A_i$, where $(A_i, \mathscr{A}_i)$ is a measurable space. Naturally, S_i is assumed $\mathscr{F}/\mathscr{A}_i$-measurable. We let $S = (S_1, \ldots, S_n)$, $A = A_1 \times \cdots \times A_n$, and $\mathscr{A} = \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n$ (the product σ-algebra). We also take as primitive an $\mathbb{R}^m$-valued integrable random variable $V = (V_1, \ldots, V_m)$, and a σ-algebra $\mathscr{R} \subseteq \mathscr{F}$. Given any random vector Z, the notation $\sigma(Z)$ denotes the smallest sub-σ-algebra of $\mathscr{F}$ with respect to which Z is measurable.

The following are assumed throughout the paper:

Standing Assumption 1. *The random vectors S and V are $\mathscr{R}$-measurable.*

Standing Assumption 2. *There exists a random variable with continuous cumulative distribution function that is independent of $\mathscr{R}$.*

For example, one can take $\mathscr{R} = \sigma(S, V)$, although in the context of quasi-complete economies one would allow $\mathscr{R}$ to include other "relevant" information. The second assumption stipulates a rich enough source of extraneous uncertainty, and it can always be satisfied after an appropriate enlargement of the underlying probability space.

2.1 Risk-neutral REE

The central equilibrium notion analyzed in this paper is that of a risk-neutral rational expectations equilibrium, defined as follows:

Definition 1. *A risk-neutral rational expectations equilibrium (rn-REE), p, is a $\mathbb{R}^m$-valued random vector with the following properties:*

(a) *For every $i \in \{1, \ldots, n\}$, $p = E[V|S_i, p]$.*

(b) *p is conditionally independent of $\mathscr{R}$ given S.*

A rn-REE, p, is fully informative if $p = E[V|S, p]$, and partially informative otherwise. A rn-REE, p, is unique if $p = q$ a.s. for every rn-REE q, and pure if it is $\sigma(S)$-measurable.

In its simplest interpretation, a rn-REE represents a set of competitive equilibrium prices in an economy consisting of n risk-neutral agents with common prior P, trading m risky assets with payoffs V, and one risk-free asset with both payoff and price normalized to be equal to one. Agent i observes the private signal S_i as well as the information revealed by prices.

Since agents are risk-neutral, they expect zero profit in equilibrium, and their specific trades can be anything as long as they clear the market (for example, we can always assume there is no trade). Alternatively, one can think of the m risky assets as representing forward contracts, with p being the corresponding equilibrium forward prices.

In DS we explain that rn-REE arise as a characterization of equilibrium prices in the much broader class of quasi-complete economies, in which risk-neutrality is no longer assumed, and equilibrium trades need not be indeterminate. Under this interpretation, the underlying probability P does not represent prior beliefs, but rather an "equivalent martingale measure" (EMM). The construction of an EMM in each of the examples briefly referred to in the Introduction is spelled out in DS, to which we refer for further details.

A rn-REE is fully informative if it remains a rn-REE after all agents have observed the pooled signals S. Part (b) of Definition 1 states that prices cannot reveal more relevant information (as defined by $\mathscr{R}$) than contained in the pooled signals S. On the other hand, we do allow for the possibility that prices are randomized, that is, they contain information not observed by any of the agents, which somehow has entered prices through some unmodeled price formation process. This type of randomized prices is further discussed in DS, and Dutta and Morris[2] (1997). While randomized prices provide the cleanest form of our results, in Section 6 we will also provide versions involving only pure prices. Of course, proving uniqueness in the broader class of randomized prices considered here always implies uniqueness in the narrower class of pure prices.

The following simple proposition shows that the set of fully informative rn-REE is a singleton. The possibility of partially informative rn-REE is illustrated in the example following this result.

Proposition 1. *The random vector p is a fully informative rn-REE if and only if* $p = E[V|S]$.

Proof. By the law of iterated expectations, we have

$$E[V|S_i, E[V|S]] = E[E[V|S]|S_i, E[V|S]] = E[V|S] \ .$$

This shows that $E[V|S]$ is a rn-REE. The same argument with S in place of S_i shows that p is fully informative. The converse is immediate from the definitions. □

Example 1. Suppose that $m = 1$, $n = 2$, $V = S_1 S_2$, and that S_1 and S_2 are stochastically independent and have zero mean. Then $E[V|S_i] = 0$ for $i \in \{1, 2\}$, and therefore $p = 0$ is a partially informative (pure) rn-REE. Of course, $p = V$ is also a rn-REE. This completes Example 1.

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[2] Dutta and Morris developed their examples concurrently and independently of this paper. Besides the connection of their common beliefs equilibrium and our definition of (randomized) equilibrium prices, there are similarities between some of their examples, and our example in Section 3.2.

The main objective of subsequent sections is to provide necessary and sufficient conditions for rn-REE uniqueness.

2.2 Common knowledge equilibria

Another interpretation of rn-REE is obtained in the context of the common-knowledge literature surveyed by Geanakoplos (1994). To show this connection, we introduce the notion of a common knowledge equilibrium (CKE), a slight extension of the notion of common-knowledge posterior estimates discussed by Aumann (1976) and many others.

A CKE is a collection of information sets, one for each agent, that can be thought of as the outcome of some communication process that somehow has caused all agents' posterior estimates of V to become common knowledge. The relevant (as defined by $\mathscr{R}$) information communicated cannot exceed what is collectively known by all the agents. We do allow, however, for the possibility of (endogenous) noise in the communication process, in the sense that an agent may obtain information that is irrelevant given the agents' pooled signals. It is this last possibility which makes our definition more inclusive than, say, Aumann's (1976).

The formal definition of a CKE is as follows:

Definition 2. *A collection, $(\mathscr{F}_1, \ldots, \mathscr{F}_n)$, of sub-$\sigma$-algebras of $\mathscr{F}$ is a common-knowledge equilibrium (CKE) if, for every i,*

(a) *S_i is $\mathscr{F}_i$-measurable.*

(b) *$E[V|\mathscr{F}_i]$ is $\bigcap_{j=1}^n \mathscr{F}_j$-measurable (common knowledge).*

(c) *$\mathscr{F}_i$ is conditionally independent of $\mathscr{R}$ given S.*

The CKE is fully informative if $E[V|\mathscr{F}_i] = E[V|S]$ for all i.

By the results of McKelvey and Page (1986) (simplified by Nielsen, Brandenburger, Geanakoplos, McKelvey, and Page, 1990), part (b) of Definition 2 can be replaced by the apparently weaker requirement that a stochastically monotone aggregate of the posterior estimates (as defined by McKelvey and Page) is common knowledge.

Aumann's (1976) fundamental observation is that if all the conditional estimates, $E[V|\mathscr{F}_i]$, are common knowledge, then they are equal to each other and to $E\left[V|\bigcap_{i=1}^n \mathscr{F}_i\right]$. It is well known, however, that in a CKE agents do not have to agree with the estimate, $E[V|S]$, that would have prevailed had all agents made their private information public. In other words, not every CKE is fully informative. For a simple concrete example, just take $\mathscr{F}_i = \sigma(S_i)$ in Example 1.

The following result shows that the notions of rn-REE and CKE are equivalent. While straightforward, this conclusion has the important implication that all our results about rn-REE have direct counterparts for CKE.

Proposition 2. (a) *Suppose that p is a rn-REE. Then $(\sigma(S_1, p), \ldots, \sigma(S_n, p))$ is a CKE, and it is fully informative if and only if p is fully informative.*

(b) *Suppose that* $(\mathscr{F}_1, \ldots, \mathscr{F}_n)$ *is a CKE. Then* $E\left[V|\bigcap_{i=1}^n \mathscr{F}_i\right]$ *is a rn-REE, and it is fully informative if and only if* $(\mathscr{F}_1, \ldots, \mathscr{F}_n)$ *is fully informative.*

Proof. Part (a) is immediate from the definitions. To show part (b), suppose that $(\mathscr{F}_1, \ldots, \mathscr{F}_n)$ is a CKE, and let $p = E[V|\mathscr{F}_0]$, where $\mathscr{F}_0 = \bigcap_{i=1}^n \mathscr{F}_i$. Using the fact that $E[V|\mathscr{F}_i]$ is $\mathscr{F}_0$-measurable, and the law of iterated expectations, we have

$$E[V|\mathscr{F}_i] = E[E[V|\mathscr{F}_i]|\mathscr{F}_0] = p \ .$$

(The last line is essentially Aumann's (1976) argument.) Since, p is $\mathscr{F}_i$-measurable, we also have $E[V|\mathscr{F}_i, p] = p$. Applying the operator $E[\cdot|S_i, p]$, we conclude that $E[V|S_i, p] = p$. Clearly, p, being $\mathscr{F}_0$-measurable, is conditionally independent of $\mathscr{R}$ given S, and therefore p is a rn-REE. By Proposition 1, p is fully informative if and only if $p = E[V|S]$, and since $p = E[V|\mathscr{F}_i]$ for every i, p is fully informative if and only if $(\mathscr{F}_1, \ldots, \mathscr{F}_n)$ is fully informative. □

One can also easily extend the above equivalence to the notions of pure rn-REE and pure CKE, where the latter is defined by strengthening part (c) of Definition 2, requiring that each $\mathscr{F}_i$ is $\sigma(S)$-measurable. Given Proposition 2, Proposition 1 and Example 1 have the obvious counterparts. Similarly, most subsequent results in this paper can be interpreted as necessary and/or sufficient conditions for the existence of partially informative CKE.

A process of consecutive communications of posterior estimates that leads to common knowledge has been discussed by Geanakoplos and Polemarchakis (1982), and in greater generality by Washburn and Teneketzis (1984) (who also give further related references). These papers do not deal with extraneous information or noise. One can easily modify the above formulations, however, to allow for the public announcement of additional noisy information at every stage, leading to information sets corresponding to the CKE notion of this section. For example, if p is a partially informative rn-REE, the public announcement of p causes the communication process to converge in one step to a partially informative CKE.

3 The "separably oriented" condition

The key to determining rn-REE uniqueness lies in properties of the function, $f : A \to \mathbb{R}^m$, giving the fully informative rn-REE as a function of the signals $f(S) = E[V|S]$. In this section we introduce a condition on f that is always sufficient for rn-REE uniqueness, and in a certain sense almost necessary as well. At the cost of mathematical complexity, a true necessary and sufficient condition for rn-REE uniqueness is given in Section 5.

The condition that will concern us in this section is defined as follows:

Definition 3. *The function* $f : A \to \mathbb{R}^m$ *is separably oriented* (*SO*) *if there exist product-measurable and bounded functions,* $g_i : A_i \times \mathbb{R}^m \to \mathbb{R}^m$, *such that*

$$f(s) \neq r \Rightarrow (f(s) - r) \cdot \sum_{i=1}^{n} g_i(s_i, r) > 0, \quad (s, r) \in A \times \mathbb{R}^m \ . \tag{1}$$

The term "separably oriented" is motivated by the geometric interpretation of the condition: for any given reference point r, the vector $f(s) - r$ forms an acute angle with the vector $\sum_i g_i(s_i, r)$.

The SO condition is useful in applications for proving equilibrium uniqueness. For example, in DS we utilize this condition to prove that the fully informative (linear) equilibrium in Grossman's (1976) model is unique, thus ruling out partially informative (nonlinear) equilibria.

Some functional forms of f that imply the SO condition are summarized in the following example. The proofs of all claims in the example are straightforward, and are left to the reader.

Example 2. In this example we assume that f is real-valued ($m = 1$), and we provide some functional forms that deliver the SO condition. For $m > 1$, f is SO if each one its components is SO, and therefore the following functional forms are useful in this case, too.

The function $f : A \to \mathbb{R}$ is *ordinally additive* if $f(s) = k\left(\sum_{i=1}^{n} h_i(s_i)\right)$, $s \in A$, for some measurable and bounded functions $h_i : A_i \to \mathbb{R}$ and strictly increasing $k : \mathbb{R} \to \mathbb{R}$. An ordinally additive f is necessarily SO, and arises, for example, in the uniqueness proof in Grossman's (1976) economy.[3]

A weaker condition than ordinal additivity is the following. The function $f : A \to \mathbb{R}$ is *implicitly additive* if there exist bounded product-measurable functions $h_i : A_i \times \mathbb{R} \to \mathbb{R}$ such that $f(s) = \sum_{i=1}^{n} h_i(s_i, f(s))$ for all $s \in A$, and $\left(\sum_i h_i(s_i, r)\right) - r$ is strictly decreasing in $r \in \mathbb{R}$, for every $s \in A$. (The latter monotonicity condition implies that f is uniquely determined by the h_i.) If f is implicitly additive, then f is SO. Conversely, if $f : A \to \mathbb{R}$ is SO, with the additional restriction on the g_i that $f(s) = r \Rightarrow \sum_i g_i(s_i, r) = 0$, then f is implicitly additive. This completes Example 2.

3.1 The sufficiency and near necessity of the SO condition

The sufficiency of the SO condition for rn-REE uniqueness amounts to a simple "adding-up" argument:

Proposition 3. *Suppose that $f : A \to \mathbb{R}^m$ satisfies $f(S) = E[V|S]$ and is SO. Then $f(S)$ is the unique rn-REE.*

Proof. Suppose that the assumptions of the proposition hold, and p is a rn-REE. Using the law of iterated expectations, and the fact that p is conditionally independent of V given S, we have, for every i,

$$p = E[V|S_i, p] = E[E[V|S, p]|S_i, p] = E[f(S)|S_i, p] \ .$$

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[3] Strictly speaking, in Grossman's example some of the functions assumed bounded above are not bounded, but all the arguments that lead to uniqueness go through. We assume boundedness here for convenience of exposition.

Therefore, $E[f(S)-p|S_i,p]=0$ for every i. Letting the g_i be as in Definition 3, we have $E[(f(S)-p)\cdot g_i(S_i,p)]=0$ for all i. Adding up,

$$E\left((f(S)-p)\cdot\sum_{i=1}^{n} g_i(S_i,p)\right)=0\ .$$

In view of (1), this equation implies that $p=f(S)$ a.s. □

While the SO condition is not in general necessary for rn-REE uniqueness (a counterexample will be given in the appendix), it becomes necessary under some regularity conditions:

Proposition 4. *Suppose that A is finite, and $P[S=s]>0$ for all $s\in A$. Suppose further that every function $f: A\rightarrow \mathbb{R}^m$ satisfying $f(S)=E[V|S]$ is not SO. Then there exists a partially informative rn-REE.*

The proof of this result is considerably more elaborate than Proposition 3, and follows from the more general results of Section 5. The underlying intuition is fairly straightforward, however, and will be illustrated through a simple example in the following subsection.

3.2 An example

We now consider a special case that illustrates the arguments leading to Proposition 4. We assume that $n=2, m=1, \mathscr{R}=\sigma(S)$, and $A_1=A_2=\{0,1\}$, with $P[S=s]>0$ for every $s\in A$. In particular, $V=f(S)$ for a function $f: A\rightarrow \mathbb{R}$. We assume that f is not SO, and using this fact we are going to demonstrate the existence of a partially informative rn-REE.

Because A is finite, the technical restrictions on the g_i of boundedness and measurability in Definition 3 can be safely ignored (this is part of the proof of Proposition 9). Since f is not SO, there exists $r\in\mathbb{R}$ such that $f\neq r$ and there is no choice of the $g_i(s_i,r)$ for which (1) holds. Without loss of generality, we assume that this is the case for $r=0$ (otherwise, we would apply the same arguments with $f-r$ in place of f). In matrix form, we are assuming that there is no $g\in\mathbb{R}^4$ such that $Fg\gg 0$, where

$$F=\begin{bmatrix} f(0,0) & 0 & f(0,0) & 0\\ 0 & f(1,0) & f(1,0) & 0\\ f(0,1) & 0 & 0 & f(0,1)\\ 0 & f(1,1) & 0 & f(1,1)\end{bmatrix}\quad\text{and}\quad g=\begin{bmatrix} g_1(0,0)\\ g_1(1,0)\\ g_2(0,0)\\ g_2(1,0)\end{bmatrix}\ .$$

Equivalently, the range of F does not intersect the interior of the positive cone in $\mathbb{R}^4$. By the separating hyperplane theorem, there exists some nonzero row vector $\psi=[\psi(0,0),\psi(1,0),\psi(0,1),\psi(1,1)]$ such that $\psi F=0$ and $0\neq\psi\geq 0$. By proper scaling, we assume ψ is a probability distribution. Moreover, we can always choose ψ so that $f(s)\psi(s)\neq 0$ for some s. (To see that, simply remove from F the rows corresponding to the values of f that vanish, and apply the same argument in a lower dimensional space.) Let the probability Q be defined so that dQ/dP is $\sigma(S)$-measurable, and $Q[S = s] = \psi(s)$ for all s. It follows easily that

$$E^Q[f(S)|S_i] = 0, \quad i \in \{1, \ldots, n\} \ , \tag{2}$$

where E^Q denotes the expectation operator relative to Q. Therefore, zero price is a partially informative rn-REE when the underlying probability P is replaced by Q.

Our next task is to take the partially informative rn-REE under Q, and randomize it appropriately in order to create a partially informative rn-REE under the original probability P. Imagine four coins, one for every $s \in A$, such that coin s will land heads with probability $k\psi(s)/P[S = s]$, for some constant $k \in (0, 1]$. (The constant k can be chosen arbitrarily in the interval $(0, K]$, where $K = \min_{s \in A} P[S = s]/\psi(s)$.) The formal justification for the availability of such coins can be provided using Assumption 2. We define a price vector, p, as follows. On the event $\{S = s\}$, we assign the value zero to p if coin s turns out heads, and the value $f(s)$ if the coin turns out tails. By construction, $P[p = 0|S = s] = k\psi(s)/P[S = s]$. Therefore $k = P[p = 0]$, and, by Bayes' rule, $\psi(s) = P[S = s|p = 0]$. Using the equation $\psi F = 0$, it follows that p is a partially informative rn-REE. On the event $\{p \neq 0\}$, p is fully informative, but on the event $\{p = 0\}$ both agents expect a payoff of zero, while f is not identically zero. This proves Proposition 4 for this example.

The example also illustrates the indeterminacy that results when the SO condition is violated. In the specific context, if f is not SO, then there exists a manifold (with boundary) of partially informative rn-REE, whose dimension is at least one. One source of indeterminacy is the value of the parameter k. Another source of indeterminacy is the fact that the probability Q solving the system of equations (2) is typically non-unique. To illustrate, consider the extension of the above example in which each signal space consists of $N \geq 3$ possible values, and there are $n \geq 2$ agents. Then the vector ψ lives in a space of dimension N^n. On the other hand, the number of constraints in the condition $\psi F = 0$ is only Nn. Of those, $n - 1$ are redundant, because the columns corresponding to each agent all add up to the same vector. (This corresponds to the fact that if $E^Q[V|S_i] = 0$ then $E^Q[V] = 0$.) Adding the constraint that ψ is a probability, we are left with a set of probabilities, Ψ, that generically is the intersection of an $(N^n - Nn + n - 2)$-dimensional linear manifold and the positive cone of an N^n-dimensional Euclidean space. For some (robust) choice of the primitives, it is possible that the manifold of pure partially informative REE has dimension $N^n - Nn + n - 2$. Moreover, any element of Ψ produces a continuum of partially revealing rn-REE (parameterized by k).

4 The "overlapping diagonals" condition

In applications, such as with quasi-complete economies, we are interested in demonstrating that a given specification of the primitives of an economy leads to the existence of partially informative equilibria. In the context of

Proposition 4, that would require us to prove that a function f satisfying $f(S) = E[V|S]$ is not SO. We now provide a new condition, called the overlapping diagonals (OD) condition, that is necessary for f to be SO if $m = 1$, and whose violation, when it occurs, is robust and easy to confirm. We will also encounter special cases in which the OD condition is equivalent to the SO condition. Moreover, under regularity assumptions and with $m = 1$, the OD condition will be shown to be necessary for rn-REE uniqueness even when the SO condition is violated. Finally, in a special setting of a two agent economy, the OD condition delivers uniqueness of a rational expectations equilibrium under risk-aversion and heterogeneous beliefs, and without assuming quasi-completeness in the sense of DS.

To state the OD condition, we introduce, for any $I \subseteq \{1, \ldots, n\}$, the notation (x_I, y_{-I}) to denote the vector z with $z_i = x_i$ for $i \in I$, and $z_i = y_i$ for $i \notin I$.

Definition 4. *The function $f : A \to \mathbb{R}$ has overlapping diagonals (OD) if, for all $I \subseteq \{1, \ldots, n\}$ and $x, y \in A$,*

$$\max\{f(x), f(y)\} \geq \min\{f(x_I, y_{-I}), f(y_I, x_{-I})\} \ . \tag{3}$$

The term "overlapping diagonals" is suggestive of the following geometric picture: Imagine the rectangle formed by the four points x, y, (x_I, y_{-I}), and (y_I, x_{-I}), and for each diagonal consider the interval defined by the values of f at the end points of the diagonal. The two intervals overlap for any choice of x and y in A, if and only if f has OD. For example, the OD condition is violated in Example 1.

Example 3. As in Section 3.2, suppose that $m = 1$, $n = 2$, $A_1 = A_2 = \{0, 1\}$, and $\mathscr{R} = \sigma(S)$, implying that $V = f(S)$ for some function $f : \{0, 1\}^2 \to \mathbb{R}$. In other words, all relevant uncertainty is represented by a two by two matrix, where the one agent knows which row contains the true state, and the other agent knows which column contains the true state. The elements of the matrix indicate the possible values of V, and the fully informative rn-REE is of course V itself. In such a simple context one can easily confirm that the OD condition is necessary and sufficient for V to be the unique rn-REE. Surprisingly, the pattern that arises in this simplest interesting instance of the problem, turns out to be key in quite more general settings described below.

4.1 The necessity of the OD condition

With $m = 1$, the OD condition follows from the SO condition, and is therefore necessary for rn-REE uniqueness under the assumptions of Proposition 4.

Proposition 5. *If $f : A \to \mathbb{R}$ is SO, then f has OD.*

Proof. Suppose that (3) is violated for some $x, y \in A$ and $I \subseteq \{1, \ldots, n\}$. We can then find some level $r \in \mathbb{R}$ such that

$$\max\{f(x), f(y)\} < r < \min\{f(x_I, y_{-I}), f(y_I, x_{-I})\} \ . \tag{4}$$

If f is SO, there exist functions $g_i : A_i \times \mathbb{R} \to \mathbb{R}$, such that $f(s) > r \Rightarrow \sum_i g_i(s_i, r) > 0$, and $f(s) < r \Rightarrow \sum_i g_i(s_i, r) < 0$, for all $s \in A$. Combining these implications with (4), one obtains a contradiction. □

The OD condition is also necessary for rn-REE uniqueness in some infinite settings in which Proposition 4 does not apply. We now put together two useful, from an applications standpoint, cases in which the violation of the OD condition leads to the existence of partially informative rn-REE. For this purpose, we call a measure, μ, on a Euclidean space *diffuse* if every set of μ-measure zero has Lebesgue measure zero.

Proposition 6. *Suppose that* $m = 1$, *and either one of the following regularity conditions are valid*:
(*a*) *A is finite and* $P[S = s] > 0$ *for all* $s \in S$.
(*b*) *A_i is a Euclidean space, for all i, f is continuous, and the distribution* of *S is diffuse*.
If the function $f : A \to \mathbb{R}$ *satisfying* $f(S) = E[V|S]$ *does not have OD, then there exists a partially informative rn-REE.*

The first part of Proposition 6 is a corollary of the last two propositions. The second part is a corollary of the more general results of Section 5. It is of some interest to note that, at least for the case of a finite A, the violation of the OD condition is a robust phenomenon: small perturbations of the function f also violate the OD condition. (A similar statement can be made when A is compact and f is continuous.) Therefore, the existence of partially informative equilibria as a consequence of the violation of the OD condition is likewise robust to small perturbations of the function f.

4.2 Some special cases

While f having OD is not in general sufficient for f to be SO, the two notions are equivalent in the following special case, which includes Examples 1 and 3, as well as the setting of Section 3.2.

Proposition 7. *Suppose that* $n = 2$, *and at least one of* A_1 *or* A_2 *is countable* (*or finite*). *Then* $f : A \to \mathbb{R}$ *is SO if and only if it has OD.*

Proof. The "only if" part is a special case of Proposition 5. We now prove the converse, assuming that A_2 is countable. Given any $s = (s_1, s_2) \in A$ and $r \in \mathbb{R}$, we define the sets

$$G(s_1, r) = \{y \in A_2 : f(s_1, y) > r\} \ ,$$
$$H(s_2, r) = \{y \in A_2 : f(x, s_2) < r \text{ and } f(x, y) > r \text{ for some } (x, y) \in A\} \ .$$

These sets have the properties:

$$f(s) < r \quad \Rightarrow \quad G(s_1, r) \subseteq H(s_2, r) \,,$$
$$f(s) > r \quad \Rightarrow \quad H(s_2, r) \subseteq G(s_1, r) \backslash \{s_2\} \text{ and } s_2 \in G(s_1, r) \ .$$

The first implication is immediate from the definitions, while the second one follows from the OD condition. Let now ν be any finite measure on the subsets of A_2, with $\nu(s_2) \equiv \nu(\{s_2\}) > 0$ for all $s_2 \in A_2$. Defining, for every $s \in A$ and $r \in \mathbb{R}$,

$$g_1(s_1, r) = \nu(G(s_1, r)) \quad \text{and} \quad -g_2(s_2, r) = \nu(H(s_2, r)) + \frac{\nu(s_2)}{2} \ ,$$

condition (1) follows. This shows that f is SO. □

With two agents, one risky asset, one risk-free asset, and a finite signal space that describes all relevant uncertainty, the OD condition turns out to be sufficient for equilibrium uniqueness, even when the equilibrium notion is extended to allow for risk aversion and heterogeneous priors (without the assumption of quasi-completeness). We now formalize and prove this result.

Suppose that $m = 1$, $n = 2$, A is finite, and $V = f(S)$ for some $f : A \to \mathbb{R}$. Deviating from our usual setup, suppose that agent i has prior P_i, a probability that is absolutely continuous with respect to P, and maximizes a smooth concave von Neumann-Morgenstern utility given the private signal S_i and the observed equilibrium prices. The two agents trade one risky asset with payoff V, and a risk-free asset with both its payoff and price normalized to be equal to one in every state. Let p represent an equilibrium price of the risky asset. (Alternatively, one can think of the agents entering into a single forward contract with payoff V and forward price p). A vector of *Arrow-Debreu state price densities* (*s.p.d.*), $\pi = (\pi_1, \pi_2)$, is any strictly positive random vector valued in $\mathbb{R}^2$. In an equilibrium, the first order conditions of optimality for the two agents take the form:

$$E_i[\pi_i(V - p)|S_i, p] = 0, \quad i \in \{1, 2\} \ , \tag{5}$$

where π is a s.p.d. vector, and E_i denotes the expectation operator with respect to P_i. For more details on the equilibrium notion alluded to above, one can refer to DS. Mathematically, the first order conditions (5) provide sufficient restrictions to uniquely determine the equilibrium price of the risky asset if f has OD. For example, it is sufficient that f be monotone in at least one of its two arguments.

Proposition 8. *Suppose that $n = 2$, $m = 1$, A is finite,*[4] *and $V = f(S)$ for some function $f : A \to \mathbb{R}$ with overlapping diagonals. If the random variable p satisfies* (5) *for some vector of Arrow-Debreu state price densities π, then $P[p = f(S)] = 1$.*

Proof. By Proposition 7, since f has OD it is SO. Let g_i, $i \in \{1, 2\}$, be the corresponding functions of Definition 3. In particular, $f(s) > r \Rightarrow g_1(s_1, r) + g_2(s_2, r) > 0$, and $f(s) < r \Rightarrow g_1(s_1, r) + g_2(s_2, r) < 0$.

Let ν be the distribution of p on $\mathbb{R}$, and let $\psi : A \times \mathbb{R} \to (0, \infty)$ be the conditional density of S given p, in the sense that

[4] The proof shows that, except for minor summability conditions, it is sufficient to take A_1 finite, and A_2 countable. For expositional simplicity we only consider the finite case here.

$$P[S \in B, p \in C] = \sum_{s \in B} \int_{r \in C} \psi(s,r) \nu(dr)$$

for all $B \subseteq A$ and any Borel set C. We also define the strictly positive functions $h_i : A \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,2\}$, to satisfy $h_i(S,p) = E[(dP_i/dP)\pi_i | S,p]$.

By a change of measure and the law of iterated expectations, (5) implies that $E\left[h_i(S,p)(f(S)-p)1_{\{S_i=s_i\}}|p\right] = 0$ for all $s_i \in A_i$, and all $i \in \{1,2\}$. Therefore, for some set $C \subseteq \mathbb{R}$ with $\nu(C) = 1$, and all $r \in C$, we have

$$\begin{aligned} \sum_{s_2 \in A_2} \psi(s,r) h_1(s,r)(f(s)-r) = 0, \quad s_1 \in A_1; \\ \sum_{s_1 \in A_1} \psi(s,r) h_2(s,r)(f(s)-r) = 0, \quad s_2 \in A_2 \ . \end{aligned} \tag{6}$$

Next we show that (6) in conjunction with the existence of the functions g_i above imply that for all $r \in C$ and $s \in A$ such that $\psi(s,r) > 0$, we have $f(s) = r$. This of course implies that $P[p = f(S)] = 1$.

To confirm our claim, fix any $r \in C$, and let $A_r = \{s \in A : \psi(s,r) > 0\}$. Suppose that, contrary to our claim, there exists $(s_1^0, s_2^0) \in A_r$ such that $f(s_1^0, s_2^0) < r$, and therefore $g_1(s_1^0, r) + g_2(s_2^0, r) < 0$. (The case $f(s^0) > r$ is treated symmetrically.) By (6), it follows that there must then exist some $(s_1^1, s_2^0) \in A_r$ such that $f(s_1^1, s_2^0) > r$. The latter inequality implies that $g_1(s_1^1, r) + g_2(s_2^0, r) > 0$, and therefore $g_1(s_1^1, r) > g_1(s_1^0, r)$. Similarly, (6) also implies that there exists some $(s_1^1, s_2^1) \in A_r$ such that $f(s_1^1, s_2^1) < r$. Repeating these steps, we obtain a sequence $\{(s_1^k, s_2^k) : k = 0, 1, 2, \ldots\}$ such that $g_1(s_1^k, r) < g_1(s_1^{k+1}, r)$ for all k. But this contradicts the assumption that A_1 is finite, and completes the proof. □

There are deeper formal connections between the OD and SO conditions. For $m = 1$, the OD and SO conditions are closely related to the independence condition and additive representation, respectively, discussed by Vind (1991), who was motivated by completely unrelated economic questions. (This can be seen by taking the set Q considered by Vind to be a level set of the form $\{f > \alpha\}$ or $\{f < \alpha\}$.) For $m = 1$ and $n > 3$, Vind's results can be used to formulate the equivalence of the SO and OD conditions under further regularity and continuity properties.[5] The details are beyond the scope of this paper.

[5] In interpreting Vind's results, the reader should be aware that all topological properties in Vind (1991) are with respect to an underlying order topology that differs from the Euclidean topology. For example, Vind's Theorem 6 is not valid relative to the Euclidean topology: it can easily be proved that the set $Y = \{x \in \mathbb{R}^n : \lceil x_1 \rceil + \cdots + \lceil x_n \rceil \leq 0\}$ (where $\lceil x \rceil$ is the smallest integer greater than or equal to x) is closed (in the Euclidean topology), essential, and independent, but it has no representation of the form $x \in Y \Leftrightarrow \sum_i f_i(x) \geq 0$ for f_i that are continuous in the Euclidean topology.

5 A necessary and sufficient condition for uniqueness

As pointed out earlier, the SO condition is sufficient but only "nearly necessary" for rn-REE uniqueness. In this final section we formulate a true necessary and sufficient condition, which is an approximate version of the SO condition (ASO). We also discuss how this new ASO condition is related to the SO and OD conditions. Some of the mathematical machinery needed in the proofs of this section is developed in the appendix.

5.1 The ASO condition

We begin with some terminology and notation that will be used throughout this section. We define $\mathscr{D}$ as the set of every probability distribution on $A \times \mathbb{R}^m$ whose marginal distribution on A is the distribution of the pooled signals S. More formally, letting $\mathscr{B}_m$ denote the Borel subsets of $\mathbb{R}^m$, $\mathscr{D}$ contains every countably additive function of the form $\pi : \mathscr{A} \otimes \mathscr{B}_m \to [0,1]$ such that $\pi(B \times \mathbb{R}^m) = P[S \in B]$ for all $B \in \mathscr{A}$. We now introduce the ASO condition in terms of the set $\mathscr{D}$.

Definition 5. *The function* $f : A \to \mathbb{R}^m$ is *approximately separably oriented with respect to* $\pi \in \mathscr{D}$, *or* π*-ASO for short, if there exist sequences of product-measurable functions* $\{h^k : A \times \mathbb{R}^m \to \mathbb{R}^m\}_{k=1,2,\ldots}$ *and* $\{g_i^k : A_i \times \mathbb{R}^m \to \mathbb{R}^m\}_{k=1,2,\ldots}$, $i \in \{1,\ldots,n\}$, *and a product-measurable function* $h : A \times \mathbb{R}^m \to \mathbb{R}^m$, *such that*

(a) *For every* i *and* k, g_i^k *is bounded, and* h^k *is* π*-integrable.*
(b) $(f(s) - r) \cdot \sum_{i=1}^n g_i^k(s_i, r) \geq h^k(s,r)$, *or all* $(s,r) \in A \times \mathbb{R}^m$.
(c) $\int |h^k - h| d\pi \to 0$ *as* $k \to \infty$.
(d) $f(s) \neq r \;\Rightarrow\; h(s,r) > 0$ *for all* $(s,r) \in A \times \mathbb{R}^m$.

The function f *is ASO if it is* π*-ASO for every* $\pi \in \mathscr{D}$.

The following are some elementary relationships between the SO and ASO conditions:

Proposition 9. (a) *If* f *is SO, then* f *is ASO.*
(b) *If* A *is finite,* $P[S = s] > 0$ *for all* $s \in A$, *and* f *is ASO, then* f *is SO.*

Proof. Part (a) is immediate from the definitions. We now show part (b). Given any $r \in \mathbb{R}^m$, consider the condition of Definition 5 corresponding to the fact that f is π-ASO when π is the distribution of the random vector (S, r). Since A is finite, and $P[S = s] > 0$ for all $s \in A$, for a sufficiently large integer $k(r)$, we have

$$f(s) \neq r \;\Rightarrow\; (f(s) - r) \cdot \sum_{i=1}^n g_i^{k(r)}(s_i, r) > 0, \quad s \in A \;.$$

For each i, define the function $g_i : A_i \times \mathbb{R}^m \to \mathbb{R}^m$ by letting $g_i(s_i, r) = g_i^{k(r)}(s_i, r)$. Then (1) is satisfied by construction, but we do not know whether the functions g_i are measurable and bounded. We now define new functions

$\hat{g}_i : A_i \times \mathbb{R}^m \to \mathbb{R}^m$ that, in addition to satisfying (1), are measurable and bounded by construction.

Let $a, b \in \mathbb{R}^m$ be such so that $a_j < f(s)_j < b_j$ for all $s \in A$ and $j \in \{1, \ldots, m\}$. For every $s \in A$, we define $\hat{g}_i(s_i, r) = 1$ if $r \leq a$, and $\hat{g}_i(s_i, r) = -1$ if $r \geq b$. This guarantees that (1) holds (with $g_i = \hat{g}_i$) for all $r \notin [a, b]$. Consider now any $r \in [a, b]$, and let $N(r)$ be a small enough open interval of r so that

$$(f(s) - r') \cdot \sum_{i=1}^{n} g_i(s_i, r) > 0 \quad \text{for all } r' \in N(r) \text{ and } s \in A \ .$$

Since $[a, b]$ is compact, there is a finite set $\{r_1, \ldots, r_K\}$, such that $[a, b] \subseteq \bigcup_{k=1}^{K} N(r_k)$. Define recursively: $I_0 = \emptyset$, and $I_k = ([a, b] \cap N(r_k)) \setminus \bigcup_{j<k} I_j$, so that $\{I_k\}$ forms a finite partition of $[a, b]$, while $I_k \subseteq N(r_k)$ for every k. Defining $\hat{g}_i(s, r) = g_i(s, r_k)$ whenever $r \in I_k$, it follows easily that the $\hat{g}_i$ have the desired properties. $\square$

5.2 The main theorem

Recall that $f(S) = E[V|S]$ is always a (fully informative) rn-REE. The question is whether other (partially informative) rn-REE exist. This question is answered in the following central theorem, which generalizes the results of Section 3.1:

Theorem 1. *Suppose that $E[V|S] = f(S)$ for some function $f : A \to \mathbb{R}^m$. Then $f(S)$ is the unique rn-REE if and only if f is ASO.*

In the remainder of this subsection we give a proof of this result, utilizing Theorem A of the appendix. We begin with some preliminary notation, terminology, and a technical lemma.

Given any random variable Z (valued in any measurable space), we let $Z^{\sim}$ denote its distribution (meaning that $Z^{\sim}(B) = P[Z \in B]$ for any measurable set B). An *admissible price vector* is any random vector valued in $\mathbb{R}^m$ that is conditionally independent of $\mathcal{R}$ given S.

Lemma 1. *Suppose that $\pi \in \mathcal{D}$. Then there exist an admissible price vector p, and a random variable U, with the following properties*:

(a) *$P[U \leq \alpha] = \alpha$ for all $\alpha \in [0, 1]$.*

(b) *U is independent of $\mathcal{R} \vee \sigma(p)$.*

(c) *$\pi = (S, p)^{\sim}$.*

Proof. First, we notice that our standing Assumption 2 implies the existence of a $[0, 1]$-uniform random variable that is independent of $\mathcal{R}$. Indeed, if X is any random variable independent of $\mathcal{R}$, with continuous cumulative distribution function F_X, then $F_X(X)$ is uniformly distributed and independent of $\mathcal{R}$.

A second fact we will use is the following: Suppose that U_0 is a $[0, 1]$-uniform random variable that is independent of $\mathcal{R}$, and suppose that v is a probability measure on $\mathscr{B}_k$, the Borel σ-algebra on $\mathbb{R}^k$, for any integer k. Then there exists a measurable function $F : [0,1] \to \mathbb{R}^k$ such that $F(U_0)$ has distribution v. Briefly, this follows from the well-known fact that there exists a Borel isomorphism (see, for example, Dudley (1989), Chapter 13) $H : [0,1] \to \mathbb{R}^k$. Letting $G : [0,1] \to [0,1]$ be defined by $G(\alpha) = \inf\{x : \alpha \leq v(\{H(u) : u \leq x\})\}$, it follows easily that $G(U_0)$ has distribution vH, and therefore $H(G(U_0))$ has distribution v. We can therefore simply define $F = H \circ G$.

A corollary of the above observations is that there exist $[0,1]$-uniform random variables, U_1 and U_2, that are mutually independent, and jointly independent of $\mathscr{R}$. Letting $U = U_1$, we now use U_2 and the discussion of the last paragraph to construct a (U_2, S)-measurable price vector p so that U and p satisfy (a) through (c).

We fix any $\pi \in \mathscr{D}$, and we define μ to be the distribution of S. Let $\eta : \mathscr{B}_m \times A \to [0,1]$ be a regular conditional distribution[6] of π given S, meaning that $\eta(\cdot, s)$ is a probability measure for all $s \in A$, $\eta(B, \cdot)$ is $\mathscr{A}$-measurable for all $B \in \mathscr{B}_m$, and

$$\pi(C,B) = \int_C \int_B \eta(dr,s)\mu(ds), \quad C \in \mathscr{A}, \ B \in \mathscr{B}_m \ .$$

By our earlier discussion, there exists, for every $s \in A$, a function $F_s : [0,1] \to \mathbb{R}^m$ such that $F_s(U_2)$ has distribution $\eta(\cdot, s)$. Letting $F(\alpha, s) = F_s(\alpha)$, our explicit construction of F_s above shows that F can be chosen to be measurable. Finally, the Lemma is proved by taking $U = U_1$ and $p = F(U_2, S)$. □

A corollary of Lemma 1 is the following useful characterization of the set $\mathscr{D}$:

$$\mathscr{D} = \{(S,p)^{\sim} : p \text{ is an admissible price vector}\} \ .$$

Proof of Theorem 1. If f is ASO, then uniqueness follows by Theorem A of the appendix, applied with $X = f(S) - p$ and $\mathscr{F}_i = \sigma(S_i, p)$, where p is any $\mathbb{R}^m$-valued random vector conditionally independent of $\mathscr{R}$ given S.

Conversely, suppose that f is not π-ASO for some $\pi \in \mathscr{D}$. Let p and U be as in Lemma 1. We will use p and U to construct a partially revealing rn-REE. Arguing by Theorem A as above, it follows that there exists probability Q, absolutely continuous with respect to P, such that dQ/dP is bounded, and $E^Q[f(S) - p|S_i, p] = 0$ for all i, and $Q[f(S) \neq p] > 0$. Let $Z = kE[dQ/dP|S, p]$, where $k > 0$ is any scalar such that Z is valued in $[0,1]$. By the law of iterated expectations, we have

$$E[(f(S) - p)Z|S_i, p] = kE\left[(f(S) - p)\frac{dQ}{dP}\,\middle|\, S_i, p\right] = 0 \ .$$

[6] For a discussion of regular conditionals see, for example, Dudley (1989, Chapter 10), or Theorem 8.1 of the appendix of Ethier and Kurtz (1986), which is in a form that applies more directly to the present setting.

Let $F = \{U \leq Z\}$, with 1_F denoting the random variable that is equal to one on F, and zero otherwise. Clearly, F is conditionally independent of $\mathscr{R}$ given S, and $P[F \mid S, p] = Z$. Again by the law of iterated expectations,

$$\begin{aligned} E[\,(f(S) - p)1_F | S_i, p] &= E[\,(f(S) - p)E[\,1_F | S, p]|S_i, p] \\ &= E[\,(f(S) - p)Z | S_i, p] = 0 \ . \end{aligned}$$

We now define $q = p1_F + f(S)1_{\Omega \setminus F}$. Since $f(S) - q = (f(S) - p)1_F$, we have that $E[f(S) - q | S_i, p] = 0$ for all i. Since $p = q$ on F and $f(S) - q = 0$ on $\Omega \setminus F$, we also have that $E[f(S) - q | S_i, q] = 0$ for all i. By construction, q is admissible. Arguing as in the first part of the proof of Proposition 3 in Section 3.1, it follows that q is a rn-REE.

Finally, we show that q is not fully-informative. Let $\|\cdot\|$ denote the Euclidean norm. Using the definition of F, and the fact that $Q[f(S) \neq p] > 0$, we have

$$\begin{aligned} E[\,\|f(S) - q\| 1_F\,] &= E[\,\|f(S) - p\| E[\,1_F \mid S, p]\,] \\ &= kE\left[\,\|f(S) - p\| \frac{dQ}{dP}\right] = kE^Q[\,\|f(S) - p\|\,] > 0 \ . \end{aligned}$$

This shows that $f(S)$ is not a.s. equal to q. □

5.3 A connection between the ASO and OD conditions

Sufficient conditions for f to be ASO useful in applications are the same conditions, discussed in Example 2, that make f SO. We will now show that, under regularity assumptions, if $f : A \to \mathbb{R}$ is ASO, then f has OD. This is a useful result when f does not have OD, because it provides an easy way of proving that f is not ASO (in a robust way), and therefore the existence of partially informative rn-REE. For simplicity, we assume that signals are valued in Euclidean spaces. (There are straightforward extensions to settings in which each A_i is a Polish space.)

Theorem 2. *Suppose that each A_i is an open subset of some Euclidean space (depending on i), with $\mathscr{A}_i$ consisting of the usual Borel sets. Suppose also that for every $B \in \mathscr{A}$, $P[S \in B] = 0$ implies that B has Lebesgue measure zero. Then every continuous ASO function of the form $f : A \to \mathbb{R}$ has OD.*

Proof. Let μ be the distribution of S. Given any $\varepsilon > 0$ and $r \in \mathbb{R}$, we define $f : A \to \mathbb{R}$ to be (ε, r)-SO if there exist measurable sets $g_i : A_i \to \mathbb{R}$ such that

$$(f(s) \neq r \text{ and } s \notin N) \ \Rightarrow \ (f(s) - r) \sum_{i=1}^{n} g_i(s_i) > 0 \ ,$$

for some $N \in \mathscr{A}$ such that $\mu(N) < \varepsilon$. We first prove that if $f : A \to \mathbb{R}$ is ASO, then f is (ε, r)-SO for every $\varepsilon > 0$ and $r \in \mathbb{R}$. Fix any such ε and r, and consider the condition of Definition 5 corresponding to the fact that f is π-ASO for $\pi = (S, r)^{\sim}$. Let $\delta > 0$ be small enough so that $h(s, r) > \delta$ for all

$s \notin N_1$ for some $N_1 \in \mathscr{A}$ of μ-measure less than $\varepsilon/2$. Subsequently, choose k large enough so that $\left|h^k(s,r) - h(s,r)\right| < \delta$ for all $s \notin N_2$ for some $N_2 \in \mathscr{A}$ also of μ-measure less than $\varepsilon/2$. Letting $g_i = g_i^k(\cdot, r)$ and $N = N_1 \cup N_2$, it follows that f is (ε, r)-SO.

Suppose now that $f : A \to \mathbb{R}$ is continuous, ASO, and does not have OD. We will show a contradiction. Since f does not have OD, there exist $r \in \mathbb{R}$, nonempty $I \subseteq \{1, \ldots, n\}$, and some $x, x+y \in A$ such that

$$\max\{f(x), f(x+y)\} < r < \min\{f(x_I, (x+y)_{-I}), f((x+y)_I, x_{-I})\} \ . \tag{7}$$

Let us fix y, and define

$$y_0 = 0, \quad y_1 = y, \quad y_2 = (0_I, y_{-I}), \quad y_3 = (y_I, 0_{-I}) \ ,$$

so that $x + y_i$, $i = 0, \ldots, 3$, are the four points involved in (7). By continuity of f, there exists an open ball, B_0, such that (7) holds for all $x \in B_0$, while at the same time $\bigcap_i B_i = \emptyset$, where $B_i = B_0 + y_i$. Let now B be a ball large enough so that it contains $\bigcup_i B_i$, and let λ represent Lebesgue measure on A. Since $\lambda \ll \mu$, it is a standard result[7] that we can find $\varepsilon > 0$ small enough so that

$$\mu(B \cap C) < \varepsilon \Rightarrow \lambda(B \cap C) < \lambda(B_0), \quad C \in \mathscr{B}_m \ .$$

Let us fix such an ε for the remainder of the proof. Since f is ASO, f is (ε, r)-SO. We also fix the set $N \in \mathscr{A}$ that appears in the above definition of the (ε, r)-SO condition. Notice that by the choice of ε, we have

$$\lambda(B \cap N) < \lambda(B_0) \ . \tag{8}$$

If we can find some $x \in B_0$ so that $x + y_i \notin N$ for all i, then we can derive a contradiction from (7) and the (ε, r)-SO condition just as in the proof of Proposition 5. We therefore conclude this proof by showing that such an x exists.

Suppose to the contrary that for each $x \in B_0$, $x + y_i \in N$ for some $i \in \{0, 1, 2, 3\}$. Intuitively, it should be clear that (8) must then be violated. Formally, define recursively

$$N_i = (B_i \cap N) \setminus \left(\bigcup_{j<i} N_j + y_i - y_j \right), \qquad i = 0, 1, 2, 3 \ .$$

Then the sets N_i are pairwise disjoint, and $\{N_i - y_i : i = 0, 1, 2, 3\}$ forms a partition of B_0. Therefore, $\lambda(\bigcup_i N_i) = \sum_i \lambda(N_i) = \sum_i \lambda(N_i - y_i) = \lambda(B_0)$. But $\bigcup_i N_i \subseteq B \cap N$, contradicting (8). This proves that for some $x \in B_0$, $x + y_i \notin B_0$ for all i, and the proof the theorem is then completed by the same argument used in Proposition 5. $\square$

[7] If π and ρ are finite measures such that π is absolutely continuous with respect to ρ, then, for any given $\delta > 0$, there exists an $\varepsilon > 0$ such that $\rho(C) < \varepsilon$ implies $\pi(C) < \delta$, for all C. This is proved, for example, in Section 32 of Billingsley (1995). Here we apply this result with $\pi(C) = \mu(B \cap C)$ and $\rho(C) = \lambda(B \cap C)$.

6 The case of pure prices

We conclude with a brief discussion on how our results should be modified when all equilibrium prices are required to be pure (that is, S-measurable). The basic idea is illustrated by the example of Section 3.2. There we saw that if f is not SO (or does not have OD), then we can construct a partially informative pure rn-REE, not under the original probability P, but under some new probability Q absolutely continuous with respect to P. This situation generalizes.

Departing from our standard setup, in this section we assume that $\mathscr{R} = \mathscr{F}$, that is, all uncertainty is "relevant." By necessity, our standing Assumption 2 can no longer be valid, and is therefore dropped. To say that a price vector p is admissible (conditionally independent of $\mathscr{R}$ given S) is now the same as saying that p is $\sigma(S)$-measurable, that is, pure. Given any probability Q, we define p to be a rn-REE *under* Q, if it is a rn-REE in the sense of Definition 1, but with the underlying probability P replaced by the probability Q. We let E^Q denote the expectation operator with respect to Q.

In this setting we have the following version of Theorem 1:

Theorem 3. *In the context of this section, suppose that* $E[V|S] = f(S)$ *for some function* $f : A \to \mathbb{R}^m$. *Then the following are equivalent*:

(a) f *is ASO.*

(b) *For every probability* Q, *absolutely continuous with respect to* P, *and with* dQ/dP *bounded*, $E^Q[V|S]$ *is the unique rn-REE under* Q.

The result follows immediately from Theorem A of the appendix. Similarly, one can easily state a version of Proposition 4 corresponding to the above result. In either case, the condition of f not being ASO leads to the existence of a partially informative pure rn-REE under some probability Q. When these results are applied to quasi-complete economies, we can interpret the choice of the probability Q as corresponding to a choice of the priors held by the agents in the economy. Details of this interpretation can be found in DS.

Appendix: An auxiliary mathematical result

In this appendix we develop the basic duality theorem used in our treatment of the uniqueness of risk-neutral rational expectations equilibria. The presentation is self-contained.

We fix throughout a probability space $(\Omega, \mathscr{F}, P)$, and n sub-σ-algebras of $\mathscr{F}$, denoted $\mathscr{F}_1, \ldots, \mathscr{F}_n$. As in the main text, E denotes the expectation operator relative to P, and "a.s." means "with P-probability one." A random variable, Z, is *integrable* if $E(|Z|) < \infty$, and *bounded* if $P[Z > K] = 0$ for some constant K (depending on Z). Given any σ-algebra $\mathscr{G} \subseteq \mathscr{F}$, we let $L_1(\mathscr{G})$ be the set of all integrable $\mathscr{G}$-measurable random variables, and we let $L_\infty(\mathscr{G})$ be the set of all bounded $\mathscr{G}$-measurable random variables. For $p = 1$ or ∞, we write $L_p^m(\mathscr{G})$ to denote the Cartesian product of m copies of $L_p(\mathscr{G})$.

We now fix an integer m, and an $\mathbb{R}^m$-valued random variable $X \in L_1^m(\mathscr{F})$. We are interested in the validity of the following condition:

A1. *For every probability*[8] $Q \ll P$, *with* dQ/dP *bounded*, $E^Q[X|\mathscr{F}_i] = 0$ *a.s. for all* $i \in \{1, \ldots, n\}$ *implies* $Q[X = 0] = 1$.

We will show that A1 is equivalent to the following condition:

A2. *There exist a sequence* $\{Z^k : k = 1, 2, \ldots\}$ *in* $L_1(\mathscr{F})$, *and, for each* i, *a sequence* $\{Y_i^k : k = 1, 2, \ldots\}$ *in* $L_\infty^m(\mathscr{F}_i)$, *such that*

(a) $X \cdot \left(\sum_{i=1}^n Y_i^k\right) \geq Z^k$ *a.s. for all* k.

(b) *For some* $Z \in L_1(\mathscr{F})$, $\lim_{k\to\infty} E(|Z^k - Z|) = 0$ *and* $P[X \neq 0 \Rightarrow Z > 0] = 1$.

Remark A. The random variables Z^k and Z can always be chosen to be zero on $\{X = 0\}$, making Z non-negative.

Theorem A. *Conditions* A1 *and* A2 *are equivalent.*

Proof. We first show that A2 implies A1. Suppose that A2 holds, and let the sequences $\{Y_i^k\}$ and $\{Z^k\}$, and the random variable Z be as in A2. We can and do assume that Z is non-negative (see Remark A). Let now Q be a probability such that $Q \ll P$, dQ/dP is bounded, and $E^Q[X|\mathscr{F}_i] = 0$ for all i. We then have $E^Q(X \cdot Y_i^k) = 0$ for all i and k, and therefore,

$$0 = E^Q\left(X \cdot \sum_{i=1}^n Y_i^k\right) \geq E^Q(Z^k) \ ,$$

for all k. Letting $k \to \infty$, we find that $E^Q(Z) = 0$. But since Z is non-negative, and almost surely strictly positive on $\{X \neq 0\}$, it follows that $Q[X = 0] = 1$. This proves A1.

Conversely, suppose that A2 is false. We provide $L_1(\mathscr{F})$ and $L_\infty(\mathscr{F})$ with their usual norms, making the latter the topological dual of the former. Consider the following convex cones in $L_1(\mathscr{F})$:

$$A = \left\{X \cdot \sum_{i=1}^n Y_i \,:\, Y_i \in L_\infty^m(\mathscr{F}_i),\ i \in \{1, \ldots, n\}\right\}$$

$$B = \{W \in L_1(\mathscr{F}) : W \geq 0 \text{ a.s. and } P[X \neq 0 \Rightarrow W > 0] = 1\} \ .$$

One can easily check that $B \cap \overline{(A - B)} \neq \emptyset$ implies A2 (where the bar over $A - B$ denotes the closure operator in $L_1(\mathscr{F})$). Therefore, $B \cap \overline{(A - B)} = \emptyset$. Since $1_{\{X\neq 0\}} \in B$, we can then find a neighborhood N of $1_{\{X\neq 0\}}$ such that $N \cap (A - B) = \emptyset$. Let C be the convex cone generated by $B + N$ (not including the origin):

$$C = \{\lambda Z : \lambda > 0 \text{ and } Z \in B + N\} \ .$$

It then follows that $A \cap C = \emptyset$.

---

[8] The notation $Q \ll P$ means $P(F) = 0 \Rightarrow Q(F) = 0$, and dQ/dP denotes the Radon-Nikodym derivative of Q with respect to P.

By the Hahn-Banach separation theorem, there exists nonzero $W \in L_\infty(\mathscr{F})$ such that
(a) $E(WZ) \leq 0$ for all $Z \in A$, and
(b) $E(WZ) \geq 0$ for all $Z \in C$.

Condition (b) implies that $W \geq 0$ a.s. and that $E(W1_{\{X \neq 0\}}) > 0$ (since $1_{\{X \neq 0\}}$ is in the interior of C). Defining the probability Q by $dQ/dP = W/E(W)$, if follows that $Q \ll P$, dQ/dP is bounded, and $Q[X \neq 0] > 0$. On the other hand, since A is a linear subspace, condition (a) implies that $E(WZ) = E^Q(Z) = 0$ for all $Z \in A$, or, equivalently, $E^Q(X \cdot Y_i) = 0$ for all $Y_i \in L^m_\infty(\mathscr{F}_i)$ and $i \in \{1, \ldots, n\}$. Therefore, $E^Q[X|\mathscr{F}_i] = 0$ for all i. We have shown the existence of a measure Q that violates A1. This proves that if A2 is false, then so is A1. $\square$

In the case of a finite Ω, the following simpler condition is equivalent to A2, and therefore to A1, too:

A3. *For some* $Y_i \in L^m_\infty(\mathscr{F}_i)$, $i \in \{1, \ldots, n\}$,

$$P\left[X \neq 0 \ \Rightarrow \ X \cdot \sum_{i=1}^n Y_i > 0\right] = 1 \ .$$

Condition A3 corresponds to the SO condition of Section 3, which, for a finite signal space, is essentially necessary and sufficient for rn-REE uniqueness. It is therefore important to know if A3 is in fact equivalent to A2 (corresponding to the ASO condition of Section 5) in the infinite case as well. The following counterexample shows that it is not. The example also provides an instance of a function that is ASO but not SO.

Example A. We take $\Omega = [0, 1] \times [0, 1]$, with $\mathscr{F}$ being the usual Borel σ-algebra. The diagonal of Ω is denoted $D = \{\omega \in \Omega : \omega_1 = \omega_2\}$. The probability P is defined so as to place half its mass uniformly on $\Omega \setminus D$, and the other half uniformly on D. We also let $\Pi_i : \Omega \to [0, 1]$ be the projection function for the i^{th} coordinate, defined by $\Pi_i(\omega) = \omega_i$. We let $n = 2$, $\mathscr{F}_i = \sigma(\Pi_i)$, and

$$X = \begin{cases} +1 & \text{if } \omega_1 \leq \omega_2, \\ -1 & \text{if } \omega_1 > \omega_2 \ . \end{cases}$$

We first prove directly that A1 holds. Suppose that Q is a probability such that $Q \ll P$ and $E^Q[X|\Pi_i] = 0$ a.s. for $i \in \{1, 2\}$. Then $E^Q[X(\Pi_2 - \Pi_1)] = 0$, implying that $Q(D) = 1$. But we also have $E^Q(X) = E^Q(X1_D) = 0$, and since X is one on the diagonal, Q cannot be a probability. This proves A1, and by Theorem A, A2 also holds. Alternatively, the reader can confirm A2 directly, by taking $Y_1^k = (1/2) - k\Pi_1$ and $Y_2^k = k\Pi_2$, for $k \in \{1, 2, \ldots\}$, and $Z = 1/2$. (A plot of $X(Y_1^k + Y_2^k)$ along a diagonal perpendicular to D makes the validity of A2 clear.)

Finally, we show that A3 cannot hold.[9] If A3 held, then there would exist bounded measurable functions $g_i : [0,1] \to \mathbb{R}$, $i \in \{1,2\}$, and a set $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$,

$$\begin{aligned} \omega_1 \leq \omega_2 \ &\Rightarrow\ g_1(\omega_1) + g_2(\omega_2) > 0, \text{ and} \\ \omega_1 > \omega_2 \ &\Rightarrow\ g_1(\omega_1) + g_2(\omega_2) < 0 \ . \end{aligned} \tag{9}$$

Assuming the existence of a such a g_1 and g_2, we now derive a contradiction. The reader is advised to represent the following arguments diagrammatically on a unit square.

Let λ represent Lebesgue measure on $[0,1]$, and define

$$\Lambda(\delta) = \lambda(\{t \in [0, 1-\delta] : (t+\delta, t) \in \Omega_0\}), \quad \delta \in (0,1) \ .$$

By the definition of P, we have

$$\frac{1}{4} = P(\Omega_0 \cap \{\omega : \omega_1 > \omega_2\}) = \frac{1}{2}\int_0^1 \Lambda(\delta)\, d\delta \ .$$

It follows that the set $\Delta = \{\delta \in (0,1) : \Lambda(\delta) = 1 - \delta\}$ has Lebesgue measure one. In particular, there exists a sequence $\{\delta_n\} \subseteq \Delta$ converging to zero.

Using (9), it follows easily that

$$\begin{aligned} 0 &> \int_0^{1-\delta_n} g_1(t+\delta_n) + g_2(t)\, dt \\ &= \int_0^1 g_1(t) + g_2(t)\, dt - \left(\int_0^{\delta_n} g_1(t)\, dt + \int_{1-\delta_n}^1 g_2(t)\, dt \right) \ . \end{aligned}$$

Letting $n \to \infty$, the term in parenthesis vanishes, giving $\int_0^1 g_1(t) + g_2(t)\, dt \leq 0$. But since P places half its mass uniformly on D, $P(\Omega_0 \cap D) = 1/2$, and $\Omega_0 \cap D$ has positive Lebesgue measure. Therefore, the last inequality contradicts (9). □

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