Scale-invariant asset pricing and consumption/portfolio choice with general attitudes toward risk and uncertainty

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April 2013, forthcoming in *Mathematics and Financial Economics*

Abstract

Motivated by notions of aversion to Knightian uncertainty, this paper develops the theory of competitive asset pricing and consumption/portfolio choice with homothetic recursive preferences that allow essentially any homothetic uncertainty averse certainty-equivalent form. The market structure is scale invariant but otherwise general, allowing any trading constraints that scale with wealth. Technicalities are minimized by assuming a finite information tree. Pricing restrictions in terms of consumption growth and market returns are derived and a simple recursive method for solving the corresponding optimal consumption/portfolio choice problem is established.

*Kellogg School of Management, Department of Finance, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208. This working paper replaces an earlier version titled "Scale-Invariant Asset Pricing Theory: A General Discrete Framework with Ambiguity-Aversion Applications." I thank Snehal Banerjee, Soohun Kim, Peter Klibanoff, Ioan Mirciov, Dimitri Papanikolaou, Jonathan Parker, Mark Schroder and Viktor Todorov for helpful discussions or feedback. I am responsible for any errors.

1 Introduction

This paper shows how scale invariance of preferences and markets can be used on a finite information tree to establish a general and simple theory of asset pricing and consumption/portfolio choice, incorporating recursive utility with any risk/uncertainty-averse conditional certainty equivalent (CE). Scale invariance of preferences means homotheticity. Scale invariance of markets means that (possibly constrained) positions can be scaled with wealth. Equilibrium pricing restrictions are given in terms of consumption growth or market returns, based on the solution of a single backward recursion on the information tree. In the case of a constant but non-unit elasticity of intertemporal substitution (EIS), pricing restrictions are shown to further simplify, dispensing with the backward recursion altogether, just as in the familiar special case of Epstein and Zin (1991) and Weil (1989, 1990). A simple recursive procedure for computing optimal consumption/portfolio policies follows essentially as a corollary of the pricing theory.

In the asset-pricing literature, scale-invariant recursive utility appears mainly in the Epstein-Zin-Weil form, which is the homothetic, constant-EIS case of Kreps and Porteus (1978) utility (that is, recursive utility with an expected-utility CE). Chapter 6 of Skiadas (2009) reviews the motivation behind this type of preferences and presents a simplified version of the present paper, restricted to the case of Kreps-Porteus utility and linear markets. The contribution here relative to that chapter is to relax the assumption of an expected-utility CE and to allow for any type of scale-invariant trading constraints.

A major motivation for broadening the class of allowable CEs is ambiguity aversion in the sense of Knight (1921) and Ellsberg (1961). Ambiguity-averse representations include the maxmin expected utility of Gilboa and Schmeidler (1989) and its variational extensions by Maccheroni, Marinacci, and Rustichini (2006), Chateauneuf and Faro (2009), and Cerreira-Vioglio et. al. (2008) (which include the entropic variational preferences of Hansen and Sargent (2001) and Strzalecki (2011)). Skiadas (2013) shows that scale invariance superimposed on any of these preferences has strong parametric implications and provides a unified representation of homothetic ambiguity averse preferences, which can be directly imported to the present paper.[1] Epstein and Schneider (2010) survey the existing asset-pricing literature with ambiguity aversion.

This paper relates to a long line of research of asset pricing and portfolio choice with recursive utility. Following Epstein and Zin (1991) and Weil (1989), discrete-time pricing calculations with Epstein-Zin-Weil utility that rely on specific consumption dynamics have several precedents in the literature, for example, Bansal and Yaron (2004) and Hansen, Heaton, and Li (2008), although much of the earlier theory was developed in continuous-time settings. Duffie and Epstein (1992a,b) formulated and applied to asset pricing a continuous-time version of Kreps-Porteus utility. The variational or "utility gradient" approach for recursive preferences, adopted in this paper, was

[1] The homothetic case of second-order expected utility of Klibanoff, Marinacci, and Mukerji (2005) and Nau (2006) is also consistent with the present paper. Skiadas (2008b) argues that in high frequency with Brownian/Poisson information, recursive utility with smooth second order expected utility becomes quantitatively indistinguishable from expected utility, a fact that limits the appeal of this approach for asset pricing and portfolio choice.

introduced in Skiadas (1992), Duffie and Skiadas (1994) and Schroder and Skiadas (1999), and was further developed in El Karoui, Peng, and Quenez (2001) and Schroder and Skiadas (2003, 2005, 2008). The last three references are closely related to this paper, emphasizing the role of scale or translation invariance and allowing for trading constraints, but in continuous-time settings and more specialized CE specifications.

The variational approach employed in this paper is complementary to the dynamic-programming approach in the tradition of Merton (1990). While the Bellman equation is a highly flexible and intuitive tool, an advantage of the variational approach is that it separates the analysis of the market and the analysis of preferences. The set of all state-price densities (or Arrow-Debreu prices) that are consistent with no incremental arbitrage at the reference plan is characterized independently of preferences. This part of the argument relates to the traditional literature of arbitrage pricing, including the more recent literature of limits to arbitrage, surveyed by Gromb and Vayanos (2010). A separate argument characterizes the superdifferential of the utility function, without reference to the market. The two components are then put together to characterize optimality. Thus if one changes the market specification, the utility part of the analysis need not be repeated, and likewise if one changes the utility specification, the market side of the analysis need not be repeated.

The remainder of this paper is organized in seven sections and an appendix. Section 2 introduces the stochastic setting and some notation and convex analysis facts used throughout the paper. Section 3 introduces the market structure and characterizes all state-price densities that are consistent with no incremental arbitrage at a given reference plan. Sections 4, 5 and 6 develop the recursive utility structure. Section 7 presents the main pricing results, and Section 8 presents the optimal consumption/portfolio theory. The Appendix contains proofs omitted from the main text.

2 Preliminaries

We begin with the basic primitives defining the stochastic setting and a review of some notation and simple results from convex analysis to be used later on. This section can be skimmed over and referred to as needed. For additional background material we will occasionally refer to Skiadas (2009), which is henceforth abbreviated to S09.

2.1 Information and Processes

Taken as primitive are a finite state space Ω and a filtration $\{\mathcal{F}_t : t = 0, 1, \ldots, T\}$, where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_T$ is the set of all subsets of Ω. Each algebra $\mathcal{F}_t$ is generated by a partition of Ω, denoted $\mathcal{F}_t^0$. A *spot* is any pair (F, t) such that $F \in \mathcal{F}_t^0$, $t \in \{0, \ldots, T\}$. Spots can be visualized as nodes on an information tree. The *immediate successors* of spot $(F, t-1)$ are the spots $(F_1, t), \ldots, (F_n, t)$, where $F_1, \ldots, F_n$ are the elements $\mathcal{F}_t^0$ whose union is F. Spots of the form (F, T) are *terminal.* The set of all $\mathcal{F}_t$-measurable random variables is denoted $L(\mathcal{F}_t)$.

A (stochastic) process $x : \Omega \times \{0, 1, \ldots, T\} \to \mathbb{R}$ is *adapted* if $x_t = x(\cdot, t) \in L(\mathcal{F}_t)$ for every time t (and therefore x_0 is constant). If x is an adapted process and $F \in \mathcal{F}_t^0$, then $x(F, t)$ denotes the common value of x_t on F, which is referred to as the value x takes at spot (F,t). A process x is *predictable* if x_0 is constant and $x_t \in L(\mathcal{F}_{t-1})$ for every time $t>0$. If x is a predictable process and $F \in \mathcal{F}^0_{t-1}$, then $x(F,t)$ denotes the common value of x_t on F. *Period* t is the time interval starting at time $t-1$ and ending at time t. The time-t value of a predictable (resp. adapted) process is known at the beginning (resp. end) of period t. The sets of all adapted processes and all predictable processes are denoted $\mathcal{L}$ and $\mathcal{P}$, respectively, with $\mathcal{P}_0 = \{x \in \mathcal{P} : x_0 = 0\}$.

If x is either a random variable or a process, the notation $x \geq 0$ means that x is valued in $[0,\infty)$, while we say that x is *strictly positive* to indicate that x is valued in $(0,\infty)$. For any set S of random variables or processes, $S_+ = \{x \in S : x \geq 0\}$ and $S_{++} = \{x \in S : x \text{ is strictly positive}\}$. For any function of the form $f : S \to \mathbb{R}$, we say that f is *increasing* if $x \geq y \neq x \implies f(x) > f(y)$; and that f is *decreasing* if $-f$ is increasing.

The symbol $\mathbf{1}$ can represent any vector (for example, a random variable or stochastic process) whose entries are all one. For any $F \subseteq \Omega$, 1_F denotes the random variable that takes the value one on F and zero on $\Omega \setminus F$.

2.2 Probabilities

Fixed throughout the paper is a reference probability P on $\mathcal{F}_T = 2^\Omega$ that assigns a positive value to every nonempty event. The corresponding expectation operator is denoted $\mathbb{E}$, while $\mathbb{E}_t$ is short for the conditional expectation operator $\mathbb{E}[\,\cdot \mid \mathcal{F}_t]$.

$\mathcal{Q}$ denotes the set of every probability on $\mathcal{F}_T$ that, like P, assigns a positive value to every nonempty event. Given any $Q \in \mathcal{Q}$, the corresponding expectation operator is denoted $\mathbb{E}^Q$, with the usual abbreviation $\mathbb{E}^Q_t$ for the conditional expectation operator $\mathbb{E}^Q[\,\cdot \mid \mathcal{F}_t]$. The *density* of Q (with respect to P) is the random variable, denoted dQ/dP, whose value at state ω is the ratio $Q(\{\omega\})/P(\{\omega\})$. The *conditional density process* of Q (with respect to P) is the process

$$\xi^Q_t = \mathbb{E}_t\left[\frac{dQ}{dP}\right], \quad t = 0,1,\ldots,T. \tag{1}$$

For any spot (F,t), $\xi^Q(F,t) = Q(F)/P(F)$, which is the ratio of the probabilities assigned by Q and P to the path on the information tree that leads from time zero to spot (F,t). The process ξ^Q is a strictly positive martingales of unit mean. Conversely, any such martingale ξ defines an element Q of $\mathcal{Q}$ by letting $Q(F) = \mathbb{E}[\xi_T 1_F]$, $F \in \mathcal{F}$, in which case, $\xi^Q = \xi$. The following change-of-measure formula will be useful (see, for example, in Lemma B.48 in S09):

$$\mathbb{E}^Q_t[x] = \mathbb{E}_t\left[\frac{\xi^Q_{t+1}}{\xi^Q_t} x\right]. \tag{2}$$

Given any $\pi \in \mathcal{L}_{++}$, let the processes $\zeta \in \mathcal{P}_0$ and $\xi \in \mathcal{L}_{++}$ be defined so that

$$\frac{\pi_t}{\pi_{t-1}} = \frac{1}{1+\zeta_t}\frac{\xi_t}{\xi_{t-1}}, \quad \text{where} \quad \frac{1}{1+\zeta_t} = \mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}}\right] \quad \text{and} \quad \xi_0 = 1. \tag{3}$$

Setting $\zeta_0 = 0$, the second equation in (3) defines the predictable process ζ. Given ζ, the first equation in (3) recursively defines ξ, which is a strictly positive unit-mean martingale. Let Q be the unique probability such that $\xi^Q = \xi$. We call the pair (Q, ζ) the *risk-neutral representation* of π. Given any pair $(Q, \zeta) \in \mathcal{Q} \times \mathcal{P}_0$, a process $\pi \in \mathcal{L}_{++}$ that has (Q, ζ) as its risk-neutral representation must satisfy recursion (3) with $\xi = \xi^Q$, and is therefore uniquely determined up to positive scaling.

2.3 Certainty Equivalents

We use the term *certainty equivalent (CE) on* $(0, \infty)^n$ to mean an increasing continuous function of the form $\nu : (0, \infty)^n \to (0, \infty)$ such that $\nu(s\mathbf{1}) = s$ for all $s \in (0, \infty)$. Let us fix any inner product[2] $(\cdot \mid \cdot)$ on $(0, \infty)^n$. The *superdifferential*[3] of a concave CE $\nu : (0, \infty)^n \to (0, \infty)$ at $x \in (0, \infty)^n$ is the set

$$\partial \nu(x) = \{p \in \mathbb{R}^n : \nu(y) \leq \nu(x) + (p \mid y - x) \text{ for all } y \in (0, \infty)^n\}.$$

Since ν is concave and its domain is an open set, $\partial \nu(x) \neq \emptyset$ (see, for example, Proposition A.31 of S09). Since ν is increasing, $\partial \nu(x) \subseteq \mathbb{R}^n_{++}$.

The following duality is key for our analysis.

Lemma 1 *Suppose $\nu : (0, \infty)^n \to (0, \infty)$ is a concave CE and $X \subseteq \mathbb{R}^n$ is a convex set containing the origin. For any $z \in (0, \infty)^n$, the following two conditions are equivalent:*

1. $\nu(z) = \max\{\nu(z + x) : x \in X \text{ and } z + x \in (0, \infty)^n\}$.
2. *There exists some $p \in \partial \nu(z)$ such that $(p \mid x) \leq 0$ for all $x \in X$.*

Proof. $(2 \implies 1)$ Suppose $p \in \partial \nu(z)$ is such that $(p \mid x) \leq 0$ for all $x \in X$. For any $x \in X$ such that $z + x \in (0, \infty)^n$, $\nu(z + x) \leq \nu(z) + (p \mid x) \leq \nu(z)$.

$(1 \implies 2)$ See Appendix. ■

The CE ν is *scale invariant (SI)* if $\nu(sx) = s\nu(x)$ for every $s \in (0, \infty)$ and $x \in (0, \infty)^n$. The following implications of scale invariance will be of repeated use.

Lemma 2 *If ν is as an SI concave CE, then for every $x \in (0, \infty)^n$,*

$$[\ s \in (0, \infty) \implies \partial \nu(sx) = \partial \nu(x)\ ] \quad \textit{and} \quad [\ p \in \partial \nu(x) \implies \nu(x) = (p \mid x)\ ]. \tag{4}$$

Proof. The first implication is immediate from the definitions. For the second one, given any $p \in \partial \nu(x)$, note that $2\nu(x) = \nu(x + x) \leq \nu(x) + (p \mid x)$ and therefore $\nu(x) \leq (p \mid x)$. Similarly, $\nu(x) = \nu(2x - x) \leq 2\nu(x) - (p \mid x)$ and therefore $\nu(x) \geq (p \mid x)$. ■

[2] This is the same as saying that we fix a positive definite matrix $\Pi \in \mathbb{R}^{n \times n}$ and write $(x \mid y) = x'\Pi y$ for all column vectors $x, y \in (0, \infty)^n$.

[3] The term subdifferential is often used in place of superdifferential, following Rockafellar (1970), who nevertheless suggests on p. 308 that the term "superdifferential" might be more appropriate. Our use of the term corresponds to Clarke's notion of generalized gradient (see, for example, Clarke (1990)) as well as the way the term is used in the literature of viscosity solutions of Hamilton-Jacobi-Bellman equations (see, for example, Bardi and Capuzzo-Dolcetta (2008)). The terms Fréchet superdifferential (and D-superdifferential for the finite-dimensional case) also map to what we call superdifferential, given concavity (see Clarke, Ledyaev, Stern, and Wolenski (1998), p. 142).

3 Market, Optimality and State Pricing

We consider an agent who consumes a positive amount of a single good at every spot. The agent is endowed with some initial (financial) wealth that can be exchanged in a market for a feasible consumption plan. In this section we define the market and feasibility, as well as the associated notion of a state-price density relative to a given feasible consumption plan. We then proceed to characterize all state-price densities and to related them to optimality under an abstract utility function on the set of all consumption plans. The remainder of the paper is about implications of the structure of this utility function.

At each nonterminal spot $(F, t-1)$, the agent selects a *portfolio allocation* out of a nonempty subset $A(F,t)$ of a Euclidean space $\mathbb{R}^J$. Setting $A_0 = \{0\}$, one can view A as a set-valued predictable process assigning the set $A(F,t)$ to spot $(F, t-1)$. A *trading policy* is any column vector $\psi = (\psi^1, \ldots, \psi^J)'$, where $\psi^j \in \mathcal{P}_0$ for every j, such that

$$\psi_t \in A_t, \quad t = 1, \ldots, T, \tag{5}$$

meaning that $\psi(F,t) \in A(F,t)$ for every nonterminal spot $(F, t-1)$.

Let $(F_1, t), \ldots, (F_n, t)$ be the immediate successor spots to spot $(F, t-1)$. The selection of the portfolio allocation $\alpha \in A(F,t)$ generates a period-t gross return $R^i_{F,t}(\alpha) \in \mathbb{R}$ at spot (F_i, t). Letting

$$R_{F,t}(\alpha) = \sum_{i=1}^{n} R^i_{F,t}(\alpha)\, 1_{F_i},$$

we regard $R_{F,t}$ as a function from $A(F,t)$ to $L(\mathcal{F}_t)$ with the property $R_{F,t}(\alpha) = R_{F,t}(\alpha)\, 1_F$. The *period-t return* corresponding to a trading policy ψ is the $\mathcal{F}_t$-measurable random variable $R_t[\psi_t]$, where

$$R_t[\psi_t]\, 1_F = R_{F,t}(\psi(F,t)), \quad \text{for every } F \in \mathcal{F}^0_{t-1}.$$

Example 3 (Linear Returns) *Suppose the agent can trade in* $1 + J$ *assets, indexed* $0, \ldots, J$. *Asset zero implements single-period default-free borrowing and lending with corresponding* short-rate process $r \in \mathcal{P}_0$; *a dollar invested in asset zero at time* $t-1$ *generates* $1+r_t$ *dollars at time* t. *The* excess-return processes *of the remaining* J *assets are listed in the column vector* $\tilde{r} = (\tilde{r}^1, \ldots, \tilde{r}^J)'$ *of adapted processes, with* $\tilde{r}_0 = 0$. *A dollar invested in asset* j *at time* $t-1$ *generates* $1 + r_t + \tilde{r}^j_t$ *dollars at time* t, *and therefore*

$$R_t[\psi_t] = 1 + r_t + \psi_t' \tilde{r}_t, \quad \psi_t \in A_t.$$

The specification of A_t *reflects trading constraints. For example, letting* $A_t = \mathbb{R}^J_+$ *means that the risky assets cannot be sold short.*

The following restrictions on A and R are assumed throughout.

Condition 4 *For every nonterminal spot* $(F, t-1)$, $A(F,t)$ *is closed and convex, the function* $R_{F,t} : A(F,t) \to L(\mathcal{F}_t)$ *is continuous and concave, and the restriction of* $R_{F,t}(\alpha)$ *on* F *is strictly positive for every* $\alpha \in A(F,t)$.

The agent is endowed with an initial wealth $w \in (0, \infty)$. A *wealth process* is any strictly positive adapted process W such that $W_0 = w$. A *consumption policy* is any adapted process ϱ such that $\varrho_t \in (0,1)$ for $t < T$ and $\varrho_T = 1$. The interpretation of ϱ_t is as the fraction of time-t wealth consumed; in particular, all terminal wealth is assumed to be consumed. An *allocation policy* is a pair of a consumption policy ϱ and a trading policy ψ. The allocation policy (ϱ, ψ) *generates* the wealth process W determined recursively by the *budget equation*

$$W_0 = w \quad \text{and} \quad W_t = W_{t-1} \left(1 - \varrho_{t-1}\right) R_t[\psi_t], \quad t = 1, \ldots, T. \tag{6}$$

A *consumption plan* is any process c in $\mathcal{L}_{++}$, with $c(F,t)$ representing contingent consumption at spot (F,t). The allocation policy (ϱ, ψ) is said to *finance* the consumption plan $c = \varrho W$. A consumption plan c is *feasible* if it can be financed by some allocation policy.

Let us now fix a reference feasible consumption plan c. Relative to c, we define the set of feasible incremental cash flows:

$$X(c) = \{x : c + x \text{ is a feasible consumption plan}\}.$$

We regard the set $\mathcal{L}$ of all adapted processes as an inner product space, where the inner product of the processes x and y is defined by

$$(x \mid y) = \mathbb{E}\left[\sum\nolimits_{t=0}^{T} x_t y_t\right]. \tag{7}$$

A *state-price density (SPD) at* c is any adapted process π with the property

$$x \in X(c) \quad \Longrightarrow \quad (\pi \mid x) \leq 0.$$

For convenience, throughout this paper, we use the term *utility function* to mean an increasing continuous function of the form $U_0 : \mathcal{L}_{++} \to \mathbb{R}$ such that $U_0(s\mathbf{1}) = s$ for every $s \in (0, \infty)$. The last restriction means that utility will always be measured in terms of equivalent annuities and entails no loss of generality, since any increasing continuous function $\tilde{U} : \mathcal{L}_{++} \to \mathbb{R}$ is ordinally equivalent to the utility function $U_0 = f^{-1} \circ \tilde{U}$, where $f(s) = \tilde{U}(s\mathbf{1})$. Note that $\mathcal{L}_{++}$ can be identified with $(0,\infty)^n$, where n is the total number of spots, and therefore every utility function is a CE on $(0,\infty)^n$ in the sense discussed in Section 2.3.

We henceforth take as given a concave utility function U_0, representing the given agent's preferences over consumption plans from the perspective of time zero. The feasible consumption plan c is *optimal* if there exists no feasible consumption plan $\tilde{c}$ such that $U_0(\tilde{c}) > U_0(c)$. Clearly, c is *optimal* if and only if $U_0(c + x) \leq U_0(c)$ for every $x \in X(c)$. While the preceding definition of optimality is from the perspective of time zero, the recursive dynamic utility to be introduced implies dynamic consistency, and therefore time-zero optimality is equivalent to optimality at every spot. (Chapter 6 of S09 gives a detailed explanation of this point.)

We combine the SPD notion with that of the utility superdifferential to characterize optimality. Recall that U_0 can be regarded as a CE on $\mathcal{L}_{++}$. Given the inner product (7), the superdifferential of U_0 at c is therefore the (nonempty) set

$$\partial U_0(c) = \{\pi \in \mathcal{L}_{++} : U_0(c+y) \leq U_0(c) + (\pi \mid y) \text{ if } c + y \in \mathcal{L}_{++}\}.$$

The following key observation, which is essentially a restatement of Lemma 1, motivates our interest in the preceding notion of an SPD at c.

Lemma 5 *Suppose the utility function $U_0 : \mathcal{L}_{++} \to \mathbb{R}$ is concave and c is any feasible consumption plan. Then c is optimal if and only if there exists some $\pi \in \partial U_0(c)$ that is an SPD at c.*

Our next task it to characterize all state-price densities at the feasible consumption plan c, without any reference to preferences. Given $\pi \in \mathcal{L}_{++}$, it will be helpful to define the process

$$V_t = \mathbb{E}_t\left[\sum_{s=t}^{T} \frac{\pi_s}{\pi_t} c_s\right], \quad t = 0, \ldots, T. \tag{8}$$

In a hypothetical complete (linear) market in which π is a state-price density, V_t represents the cum-dividend time-t value of a traded contract that generates c as a dividend process. (See Chapter 5 of S09 for details.)

The paper's central result on state pricing follows.

Theorem 6 *Suppose the consumption plan c is financed by the allocation policy (ϱ, ψ), which generates the wealth process W. The following are true for any $\pi \in \mathcal{L}_{++}$.*

(a) Let V be defined by (8). Then $V = W$ if and only if

$$1 = \mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}} R_t[\psi_t]\right], \quad t = 1, \ldots, T. \tag{9}$$

(b) The process π is an SPD at c if and only if

$$1 = \mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}} R_t[\psi_t]\right] = \max_{\alpha_t \in A_t} \mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}} R_t[\alpha_t]\right], \quad t = 1, \ldots, T. \tag{10}$$

Remark 7 *Condition (9) is the same as (10) if $A_t = \{\psi_t\}$. This proves that $V = W$ if and only if π is an SPD at c relative to the market generated by varying ϱ but not ψ (that is, with $X(c)$ defined as the set of every x such that $c + x$ is a consumption plan that is financed by some consumption policy and the given trading policy ψ).*

Remark 8 *The conclusions of Theorem 6 can be equivalently stated in terms of the risk-neutral representation (Q, ζ) of π :*

(a) $V = W$ if and only if $1 + \zeta_t = \mathbb{E}^Q_{t-1} R_t[\psi_t]$ for $t = 1, \ldots, T$.

(b) π is an SPD at c if and only if

$$1 + \zeta_t = \mathbb{E}^Q_{t-1} R_t[\psi_t] = \max_{\alpha_t \in A_t} \mathbb{E}^Q_{t-1} R_t[\alpha_t], \quad t = 1, \ldots, T. \tag{11}$$

We close this section with an example, which is a discrete version of corresponding results derived by Schroder and Skiadas (2003) in a continuous-time setting with Brownian information.

Example 9 (Linear Returns and Trading Constraints) *Consider the setting of Example 3 and fix* $\pi \in \mathcal{L}_{++}$*, with risk-neutral representation* (Q, ζ). *In a hypothetical complete market for which* π *is an SPD,* ζ *is the short-rate process (in the credit-risk-free sense of Example 3) and* Q *is the unique equivalent martingale measure (see Chapter 5 of S09). If the risky assets were all traded in this hypothetical complete market, then it should be the case that* $\mathbb{E}_{t-1}\tilde{r}_t = -\ cov_{t-1}[\xi_t^Q/\xi_{t-1}^Q, \tilde{r}_t]$. *Since* A *can imply trading constraints, the last equality is not necessary for* π *to be an SPD at* c. *The difference between the conditional expected excess returns in the actual market and those in the hypothetical complete market implied by* π *is*

$$\varepsilon_t = \mathbb{E}_{t-1}\tilde{r}_t + \ cov_{t-1}\left[\frac{\xi_t^Q}{\xi_{t-1}^Q}, \tilde{r}_t\right] = \mathbb{E}_{t-1}\left[\frac{\xi_t^Q}{\xi_{t-1}^Q}\tilde{r}_t\right] = \mathbb{E}_{t-1}^Q\tilde{r}_t.$$

Letting $\varepsilon_0 = 0$*, this defines a process* $\varepsilon \in \mathcal{P}_0$. *In an unconstrained market,* π *is an SPD at* c *if and only if* $\zeta = r$ *and* $\varepsilon = 0$. *We now generalize this statement to account for trading constraints.*

Given any nonempty set $S \subseteq \mathbb{R}^J$*, the support function* $\delta_S : \mathbb{R}^J \to (-\infty, \infty]$ *is defined by*

$$\delta_S(y) = \sup\left\{x'y : x \in S\right\}.$$

We define the process $\delta_A[\varepsilon] \in \mathcal{P}_0$ *by letting* $\delta_A[\varepsilon](F, t) = \delta_{A(F,t)}(\varepsilon(F, t))$*, for every nonterminal spot* (F, t).

Suppose the consumption plan c *is financed by the allocation policy* (ϱ, ψ)*, which generates the wealth process* W*. Specializing Theorem 6 (in the form of Remark 8) to the current setting, we conclude:*

(a) $W = V$ *if and only if* $\zeta = r + \delta_A[\varepsilon]$.

(b) The process π *is an SPD at* c *if and only if*

$$\zeta = r + \delta_A[\varepsilon] \quad \text{and} \quad \psi'\varepsilon = \delta_A[\varepsilon]. \tag{12}$$

If a set $S \subseteq \mathbb{R}^J$ *is a cone (meaning that* $x \in S$ *and* $s \in (0, \infty) \implies sx \in S$*) then* $\delta_S(y) = 0$ *if* $x'y \leq 0$ *for all* $x \in S$*, and* $\delta_S(y) = \infty$ *otherwise. Therefore, if* A *is a cone at every spot, then condition* (12) *is equivalent to*

$$\zeta = r \quad \text{and} \quad \psi_t'\varepsilon_t = \max\left\{\alpha_t'\varepsilon_t : \alpha_t \in A_t\right\}.$$

For example, if there is a single risky asset $(J = 1)$ *that cannot be sold short* $(A_t = \mathbb{R}_+)$*, then condition* (12) *becomes* $\zeta = r$ *and* $\varepsilon \leq 0$*, with* $\varepsilon < 0$ *only if* $\psi = 0$.

4 Recursive Utility

We saw in Lemma 5 that optimality of a feasible consumption plan c is equivalent to the existence of an element of the utility superdifferential at c that is also an SPD at c. The SPD property was characterized in Theorem 6 in terms of a recursive market structure. We now shift our attention to the utility side, postulating a recursive utility structure and characterizing the utility superdifferential.

Recursive utility is defined in terms of a conditional certainty equivalent, which specifies, at every nonterminal spot, a rule for collapsing a one-period-ahead contingent payoff to an equivalent certain payment at the given spot.

Definition 10 *A* conditional certainty equivalent (conditional CE) *is a mapping υ that assigns to each time $t \in \{0, \ldots, T-1\}$ a continuous function of the form $\upsilon_t : L(\mathcal{F}_{t+1})_{++} \to L(\mathcal{F}_t)_{++}$ such that $\upsilon_t(s\mathbf{1}) = s$ for every $s \in (0, \infty)$, and the following conditions hold for every nonterminal spot (F, t) and $x, y \in L(\mathcal{F}_{t+1})_{++}$:*

- *$x = y$ on F implies $\upsilon_t(x) = \upsilon_t(y)$ on F.*
- *$x \geq y \neq x$ on F implies $\upsilon_t(x) > \upsilon_t(y)$ on F.*

Consider any nonterminal spot (F, t), and let $(F_1, t+1), \ldots, (F_n, t+1)$ be its immediate successor spots. The two bullet conditions of Definition 10 state that the value of $\upsilon_t(x)$ on F, which we denote $\upsilon_{F,t}(x)$, is an increasing function of the restriction of x on F. Given this fact, we write $\upsilon_{F,t}(x1_F) = \upsilon_{F,t}(x)$ (even though $x1_F$ is not valued in $(0, \infty)$). We will occasionally use the notation

$$\bar{\upsilon}_{F,t}(\alpha) = \upsilon_{F,t}\left(\sum\nolimits_{i=1}^{n} \alpha_i 1_{F_i}\right), \quad \alpha \in (0, \infty)^n, \tag{13}$$

which defines a CE $\bar{\upsilon}_{F,t}$ on $(0, \infty)^n$ (in the sense discussed in Section 2.3). Conversely, given a CE $\bar{\upsilon}_{F,t}$ at each nonterminal spot (F, t), equations (13) determine an entire conditional CE υ on the information tree. We call the conditional CE υ *concave, differentiable,* or *scale invariant* if $\bar{\upsilon}_{F,t}$ has the respective property for every choice of a nonterminal spot (F, t).

The sense in which we use the term recursive utility can now be rigorously defined as follows.

Definition 11 *An* aggregator *is a function $f : \{0, \ldots, T-1\} \times (0, \infty)^2 \to (0, \infty)$ such that, for every nonterminal time t, the section $f(t, \cdot)$ is increasing and continuous and satisfies $f(t, 1, 1) = 1$. The utility function $U_0 : \mathcal{L}_{++} \to (0, \infty)$ is* recursive utility *if there are a conditional CE υ and an aggregator f such that for any given $c \in \mathcal{L}_{++}$, $U_0(c)$ is the initial value of the process U in $\mathcal{L}_{++}$ specified by the backward recursion*

$$U_t = f(t, c_t, \upsilon_t(U_{t+1})) \quad t < T; \quad U_T = c_T. \tag{14}$$

In this case, we refer to U as the utility process *of c.*

An aggregator f is said to be *concave, differentiable,* or *scale invariant (SI)* if $f(t, \cdot)$ has the respective property for every time $t < T$. Mathematically speaking, $f(t, \cdot)$ is a CE on $(0, \infty)^2$, and therefore $f(t, \cdot)$ is SI if and only if it is homogeneous of degree one.

Chapter 6 of S09 provides ordinal axioms that characterize preferences with a recursive utility representation and establishes some basic properties of recursive utility. For example, the aggregator f determines and is determined by preferences over deterministic consumption plans, while making the conditional CE υ more risk averse, with f fixed, makes the utility function U_0 more risk averse. Concavity (resp. scale invariance) of U_0 corresponds to concavity (resp. scale invariance) of both the aggregator and the conditional CE. Every recursive utility in this paper will be further restricted by the ordinal properties of scale invariance and quasiconcavity, properties that imply concavity. Postponing discussion of scale invariance until the next section, we close this section with an extension of Proposition 6.15 of S09 that gives a key recursive formula for the superdifferential of concave recursive utility, assuming a smooth aggregator but a potentially nonsmooth concave conditional CE. The latter generality is essential in accommodating some of the most common representations of ambiguity aversion.

Consider any concave conditional CE υ. For each time $t < T$ and random variable $z \in L(\mathcal{F}_{t+1})_{++}$, we define the *superdifferential of* υ_t at z :

$$\partial \upsilon_t (z) = \left\{ \kappa_{t+1} \in L(\mathcal{F}_{t+1})_{++} : \upsilon_t (y) \leq \upsilon_t (z) + \mathbb{E}_t [\kappa_{t+1} (y - z)] \text{ for all } y \in L(\mathcal{F}_{t+1})_{++} \right\}. \quad (15)$$

To make this definition more concrete, given any nonterminal spot (F, t), let n be the number of its immediate successor spots and consider the function $\bar{\upsilon}_{F,t}$ defined in (13). It is then clear that $\kappa_{t+1} \in \partial \upsilon_t (z)$ if and only if for every choice of $F \in \mathcal{F}_t^0$, we have

$$p \in \partial \bar{\upsilon}_{F,t} (\alpha), \quad \text{where} \quad z 1_F = \sum\nolimits_{i=1}^{n} \alpha_i 1_{F_i} \quad \text{and} \quad p_i = \kappa (F_i, t+1) P [F_i \mid F].$$

To anchor the discussion to something familiar, we review the standard case of a smooth expected-utility CE. Specifications that are motivated by ambiguity aversion are discussed in the following section.

Example 12 (Expected-Utility CE) *Suppose* $\upsilon_{t-1} = u^{-1} \mathbb{E}_{t-1}^{Q} u$ *for some* $Q \in \mathcal{Q}$ *and differentiable increasing concave function* $u : (0, \infty) \to \mathbb{R}$. *Then*

$$\partial \upsilon_{t-1} (U_t) = \left\{ \frac{u' (U_t)}{u' (\upsilon_{t-1} (U_t))} \frac{\xi_t^Q}{\xi_{t-1}^Q} \right\}, \quad \textit{for every } U_t \in L(\mathcal{F}_t)_{++}. \quad (16)$$

The recursive characterization of the superdifferential of any concave recursive utility with differentiable aggregator f is given below, with f_c and f_υ denoting the partial derivatives of f with respect to its consumption and CE arguments, respectively.

Lemma 13 *Suppose* U_0 *is recursive utility with a concave conditional CE* υ *and a concave, differentiable aggregator* f. *Given any* $c \in \mathcal{L}_{++}$, *let* U *be the corresponding utility process (defined by recursion* 14*) and let* q *and* λ *be the processes defined by*

$$q_t = \frac{f_\upsilon (t, c_t, \upsilon_t (U_{t+1}))}{f_c (t, c_t, \upsilon_t (U_{t+1}))} \quad \textit{and} \quad \lambda_t = f_c (t, c_t, \upsilon_t (U_{t+1})), \quad t < T; \quad q_T = 0, \quad \lambda_T = 1. \quad (17)$$

Then $\pi \in \partial U_0 (c)$ *if and only if* π *solves a recursion of the form*

$$\pi_0 = \lambda_0 \quad \textit{and} \quad \frac{\pi_t}{\pi_{t-1}} = q_{t-1} \kappa_t \lambda_t, \quad \kappa_t \in \partial \upsilon_{t-1} (U_t). \quad (18)$$

Given a reference consumption plan c, we refer to the process λ defined in (17) as the *shadow-price-of-wealth process.* (The reason for this term is explained in Remark 6.16 of S09.) Later sections explain how the factors q_{t-1}, κ_t and λ_t of recursion (18) can be computed in terms of consumption growth and market returns, assuming U_0 is scale invariant.

Finally, it is worth noting that recursion (18) can be equivalently stated in terms of the risk-neutral representation (Q, ζ) of π, as

$$\frac{1}{1+\zeta_t} = q_{t-1}\mathbb{E}_{t-1}\left[\kappa_t \lambda_t\right] \quad \text{and} \quad \frac{\xi_t^Q}{\xi_{t-1}^Q} = \frac{\kappa_t \lambda_t}{\mathbb{E}_{t-1}\left[\kappa_t \lambda_t\right]}, \quad \xi_0 = 1. \tag{19}$$

5 SI Conditional CEs

A utility function $U_0 : \mathcal{L}_{++} \to \mathbb{R}$ is defined to be *scale invariant (SI)* if for every $a, b \in \mathcal{L}_{++}$ and $s \in (0,\infty)$, $U_0(a) > U_0(b)$ implies $U_0(sa) > U_0(sb)$. Recall that every utility function U_0 in this paper is, by definition, increasing continuous and normalized in the sense that $U_0(sc) = sU_0(c)$ if $s \in (0,\infty)$. Therefore, U_0 is SI if and only if it is homogeneous of degree one. The focus of the remainder of this paper is on a recursive SI utility function U_0 that is quasiconcave and therefore concave (see, for example, Proposition 3.33 of S09). These properties of U_0 are easily shown to be equivalent to the requirement that the corresponding conditional CE and aggregator are SI and concave. This section discusses SI concave conditional CEs and the following section discusses SI concave aggregators.

The definition of scale invariance of a conditional CE υ can be restated as the requirement that

$$\upsilon_t\left(s_t U_{t+1}\right) = s_t \upsilon_t\left(U_{t+1}\right) \quad \text{for every } s, U \in \mathcal{L}_{++}. \tag{20}$$

Lemma 2, applied at each nonterminal spot, implies the following important properties of a concave SI conditional CE υ :

$$\partial\upsilon_t\left(s_t U_{t+1}\right) = \partial\upsilon_t\left(U_{t+1}\right) \quad \text{for every } s, U \in \mathcal{L}_{++}. \tag{21}$$

$$\kappa_t \in \partial\upsilon_{t-1}\left(U_t\right) \implies \upsilon_{t-1}\left(U_t\right) = \mathbb{E}_{t-1}\left[\kappa_t U_t\right] \quad \text{for every } U \in \mathcal{L}_{++}. \tag{22}$$

Some examples of conditional CEs of interest follow. For any coefficient $\gamma \in \mathbb{R}$, we use the notation u_γ for the function on $(0,\infty)$ defined by[4]

$$u_\gamma(z) = \begin{cases} \log(z), & \text{if } \gamma = 1; \\ z^{1-\gamma}/(1-\gamma), & \text{if } \gamma \neq 1. \end{cases} \tag{23}$$

Example 14 (SI Expected-Utility CE) *As is well-known (and follows from Theorem 3.37 of S09), the expected-utility conditional CE of Example 12 is SI if and only if there is a* coefficient of relative risk aversion (CRRA) $\gamma \in [0,\infty)$ *such that*

$$\upsilon_{t-1}\left(U_t\right) = u_\gamma^{-1}\mathbb{E}_{t-1}^Q u_\gamma\left(U_t\right).$$

[4] This definition is consistent with the use of u_γ in Skiadas (2013), where it is important that the range of u_γ is either $\pm(0,\infty)$ or $\mathbb{R}$. Nothing would change in the present paper, however, if we were to redefine $u_\gamma(z) = \left(z^{1-\gamma} - 1\right)/(1-\gamma)$, which is more consistent in the sense that $u_1 = \lim_{\gamma\to 1} u_\gamma$.

In this case, the superdifferential expression (16) *specializes to*

$$\partial \upsilon_{t-1}(U_t) = \left\{ \left(\frac{U_t}{\upsilon_{t-1}(U_t)} \right)^{-\gamma} \frac{\xi_t^Q}{\xi_{t-1}^Q} \right\}. \tag{24}$$

By developing the theory in terms of a general SI conditional CE, we can accommodate any of the SI ambiguity-averse specifications in Skiadas (2013), with the associated computation of the CE superdifferential being a routine exercise. If the CE is smooth, as in the case[5] of source-dependent constant relative risk aversion, then the superdifferential is obtained by computing the partial derivatives of $\bar{\upsilon}_{F,t}$, for each nonterminal spot (F,t). More generally, given the unifying multiple-prior representation of Theorem 5 in Skiadas (2013), each element of the CE superdifferential can be obtained by fixing a corresponding minimizing prior and differentiating as if the minimizing prior were exogenous. Rather than presenting the notationally burdensome general case, we illustrate with a simple parametric example,[6] which is within the broader divergence class of Maccheroni, Marinacci, and Rustichini (2006). In this example a single parameter provides a continuum of CE specifications, ordered by ambiguity aversion, which connects the familiar entropic and quadratic cases. The entropic case defines a conditional CE that is of the SI expected utility form with a CRRA greater than one, while the quadratic case defines a mean-variance criterion.

Example 15 (A Parametric Divergence CE Class) *We assume that for some positive scalar parameters* p *and* χ,

$$\upsilon_{t-1}(U_t) = \inf_{Q \in \mathcal{Q}} \exp\left(\mathbb{E}_{t-1}^Q \log(U_t) + \frac{1}{\chi} \mathbb{E}_{t-1} \varphi_p \left(\frac{\xi_t^Q}{\xi_{t-1}^Q} \right) \right), \tag{25}$$

where

$$\varphi_p(y) \equiv \frac{y^p - y}{p-1} - y + 1.$$

Note that φ_p *is strictly convex, smooth and satisfies* $\varphi_p(1) = \varphi_p'(1) = 0$ *and* $\varphi_p'(\infty) = \infty$. *Note also that* $\varphi_p'(0+) = -\infty$ *if and only if* $p \in (0,1]$. *Some examples of* φ_p *are plotted in Figure 1. The higher the value of* p, *the larger* φ_p *(pointwise) and therefore the higher values the conditional CE* υ *takes, corresponding to lower uncertainty aversion. Assuming that the infimum in* (25) *is achieved as a minimum, Corollary 2 in Section 2.8 of Clarke (1990) implies*[7] *that*

$$\partial \upsilon_{t-1}(U_t) = \left\{ \left(\frac{U_t}{\upsilon_{t-1}(U_t)} \right)^{-1} \frac{\xi_t^Q}{\xi_{t-1}^Q} : Q \textit{ is a minimizing probability in } \mathcal{Q} \right\}. \tag{26}$$

[5] See Skiadas (2012) for an axiomatic characterization of recursive utility with a constant EIS, constant rate of impatience and constant source-dependent CRRA.

[6] Note that the presentation in Skiadas (2013) is based on a filtration that is generated by two sources of uncertainty, an ambiguous one (horse-race uncertainty) and an unambiguous one (roulette risk). One could take the filtration $\{\mathcal{F}_t\}$ of the present paper to coincide with that in Skiadas (2013) (as, for example, would be required if one were to adopt the source-dependent CRRA specification). In Example 15, we take the alternative view that roulette risk is nontradeable and this paper's filtration $\{\mathcal{F}_t\}$ is identified with the horse-race filtration in Skiadas (2013).

[7] Clarke's result can be applied at the CE $\nu = \bar{\upsilon}_{F,t}$ at each nonterminal spot (F,t). A quick way to show the superdifferential expression is to use an envelope theorem for the directional derivatives of ν at w, which characterize $\partial\nu(w)$ via Theorem 23.2 of Rockafellar (1970).

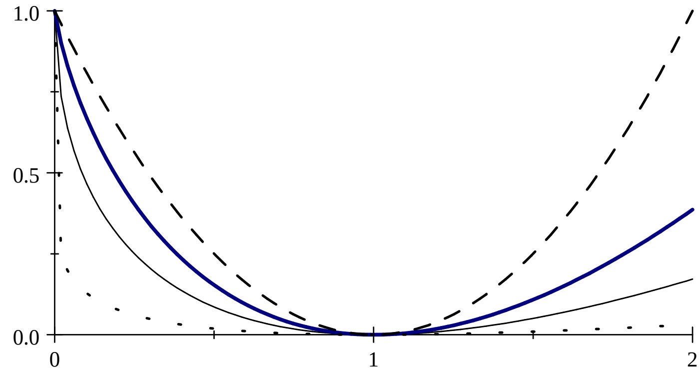

Figure 1: Examples of the graph of the function φ_p. From top to bottom the lines correspond to p taking the values $2, 1, 0.5$ and 0.1.

Letting $\varphi = \varphi_p/\chi$ in Lemma 8 of Skiadas (2008b) leads to the following conclusions.

Case of $p \in (0,1)$. *In this case, the infimum in* (25) *is a minimum, for any $U \in \mathcal{L}_{++}$. The unique minimizing $Q \in \mathcal{Q}$ can be computed recursively by*

$$\frac{\xi_t^Q}{\xi_{t-1}^Q} = \left(1 + \frac{p-1}{p}\chi \log\left(\frac{k_{t-1}}{U_t}\right)\right)^{1/(p-1)}, \quad \xi_0^Q = 1, \tag{27}$$

where k_{t-1} is the unique element of $L(\mathcal{F}_{t-1})_{++}$ such that $\mathbb{E}_{t-1}\xi_t^Q/\xi_{t-1}^Q = 1$.

Case of $p = 1$. *This case corresponds to the entropic specification $\varphi_1(y) = y\log y - y + 1$, resulting in the well-known identity $\upsilon_{t-1} = u_\gamma^{-1}\mathbb{E}_{t-1}u_\gamma$, with $\gamma = 1 + \chi$.*

Case of $p \in (1,\infty)$. *In this case, for any $U \in \mathcal{L}_{++}$, a minimum in* (25) *exists if and only if*

$$\mathbb{E}_{t-1}\left[\left(\frac{p-1}{p}\chi \log\left(\frac{U_t^*}{U_t}\right)\right)^{1/(p-1)}\right] < 1, \quad t = 1, \ldots, T, \tag{28}$$

where U^* is the least predictable process such that $U^* \geq U$ and $U_0^* = U_0$. Therefore, equation (25) *defines a conditional CE only on the cone of adapted processes satisfying condition* (28). *For any $U \in \mathcal{L}_{++}$ satisfying* (28), *the minimizing $Q \in \mathcal{Q}$ is again computed by* (27). *An application with this type of conditional CE requires either a further restriction of the consumption set (so that all corresponding utility processes satisfy* (28)*) or a suitable extension of the conditional CE (meaning that the preceding representation is only valid locally at a reference consumption plan). The issue is illustrated by setting $p = 2$, corresponding to the quadratic divergence specification $\varphi_2(y) = (y-1)^2$, resulting in the conditional CE*

$$\upsilon_{t-1}(U_t) = \exp\left(\mathbb{E}_t \log(U_t) - \frac{\chi}{4} Var_{t-1}[\log(U_t)]\right), \qquad (\textit{assuming } p = 2) \tag{29}$$

for any adapted process U such that $\mathbb{E}_{t-1}\log(U_t^*/U_t) < 2/\chi$. Further details on the quadratic case can be found in Maccheroni, Marinacci, and Rustichini (2006) and Maccheroni, Marinacci, Rustichini, and Taboga (2009).

6 Proportional Aggregators

An SI recursive utility is specified by a pair of an SI conditional CE, discussed in the last section, and an SI aggregator, which is the topic of this section.

An aggregator f is said to be *scale invariant (SI)* if the section $f(t,\cdot):(0,\infty)^2\to(0,\infty)$ is homogeneous of degree one for every $t\in\{0,\ldots,T-1\}$. In this case,

$$f_t(c,\upsilon)=\upsilon g_t\left(\frac{c}{\upsilon}\right),\quad t=0,\ldots,T, \tag{30}$$

where $g_t(x)=f_t(x,1)$. We will refer to the function g as a proportional aggregator, a term defined more formally as follows.

Definition 16 *A* proportional aggregator *is a mapping g that assigns to each nonterminal time t a continuous and increasing function $g_t:(0,\infty)\to(0,\infty)$ such that $g_t(1)=1$ and the mapping $x\mapsto g_t(x)/x$ is decreasing. The proportional aggregator g is* concave *or* differentiable *if g_t has the respective property for every time $t<T$.*

If g is any proportional aggregator, then (30) defines an SI aggregator f. If f and g are so related, concavity of f is equivalent to concavity of g_t for all $t<T$.

For simplicity, every proportional aggregator in this paper is assumed to be strictly concave and differentiable, with a derivative that is unbounded near zero and goes to zero at infinity. These restrictions are adopted in the interest of expositional economy and are peripheral to our main focus, which is the structure of the conditional CE. More formally, the following condition is henceforth assumed.

Condition 17 *$U_0:\mathcal{L}_{++}\to(0,\infty)$ is recursive utility with a concave and SI conditional CE υ, and a strictly concave and differentiable proportional aggregator g. The derivative of the proportional aggregator satisfies*

$$\lim_{x\downarrow 0}g_t'(x)=\infty\quad\textit{and}\quad\lim_{x\to\infty}g_t'(x)=0. \tag{31}$$

Several transformations of the proportional aggregator g appear in the sequel. These are all introduced here for easy reference.

Since the derivative g_t' is decreasing and continuous, condition (31) implies the existence of the inverse function $\mathcal{I}_t:(0,\infty)\to(0,\infty)$, defined by the requirement that

$$g_t'(\mathcal{I}_t(\lambda))=\lambda.$$

The convex dual of g_t is the function

$$g_t^*(\lambda)=\max_{x\in(0,\infty)}\{g_t(x)-\lambda x\}=g_t(\mathcal{I}_t(\lambda))-\lambda\mathcal{I}_t(\lambda),\quad\lambda\in(0,\infty). \tag{32}$$

The *elasticity* of g_t is the function

$$h_t(x)=\frac{d\log g_t(x)}{d\log x}=\frac{xg_t'(x)}{g_t(x)},\quad x\in(0,\infty). \tag{33}$$

Since x, $g_t(x)$ and $g_t'(x)$ are all positive, so is $h_t(x)$. The assumption that $g_t(x)/x$ is decreasing is equivalent to the requirement that

$$h_t(x) \in (0,1), \quad x \in (0,\infty).$$

Since $g_t(1) = 1$, it follows in particular that $g_t'(1) = h_t(1) \in (0,1)$.

The superdifferential calculation of Lemma 13 motivates the definition of the function

$$q_t(x) = \left(\frac{1}{h_t(x)} - 1\right)x, \quad x \in (0,\infty). \tag{34}$$

Fix any reference consumption plan c, and let U be the corresponding utility process. A simple calculation shows that the superdifferential characterization of Proposition 13 holds with

$$q_t = q_t(x_t) \quad \text{and} \quad \lambda_t = g_t'(x_t), \quad \text{where} \quad x_t = \frac{c_t}{\upsilon_t(U_{t+1})}, \qquad x_T = \lambda_T = 1. \tag{35}$$

While q_t denotes both the function (34) and the random variable $q_t(x_t)$, the meaning is generally clear from the context. In the following section the superdifferential is expressed in more interesting ways in terms of consumption growth and market returns.

The processes λ_t and $h_t(x_t)$ play a key role in our analysis by virtue of the following straightforward implications of the assumed scale invariance of the utility function.

Lemma 18 *Suppose $\pi \in \partial U_0(c)$ and V is defined in terms of c and π by (8). Then*

$$U_t = \lambda_t V_t \quad \textit{and} \quad \frac{c_t}{V_t} = h_t(x_t), \quad t = 0,\ldots,T,$$

with the convention $h_T \equiv 1$.

The elasticity h_t of g_t should not be confused with the *elasticity of intertemporal substitution (EIS),* which is defined as the reciprocal of the elasticity of the function q_t :

$$EIS_t(x) = \frac{d\log x}{d\log q_t(x)} = -\frac{d\log(c/\upsilon)}{d\log\left(f_c(t,c,\upsilon)/f_\upsilon(t,c,\upsilon)\right)}.$$

Equivalently, $(1/EIS_t) - 1$ is the elasticity of $(1/h_t) - 1$.

Example 19 (Constant EIS) *Imposing a constant EIS results in the proportional aggregator*

$$g_t(x) = \begin{cases} \left((1-\beta)x^{1-\delta} + \beta\right)^{1/(1-\delta)}, & \textit{if } \delta \neq 1; \\ x^{1-\beta}, & \textit{if } \delta = 1. \end{cases}$$

In this case,

$$\beta = 1 - g_t'(1), \quad h_t(x) = \frac{(1-\beta)x^{1-\delta}}{(1-\beta)x^{1-\delta} + \beta}, \quad q_t(x) = \frac{\beta x^\delta}{1-\beta} \quad \textit{and} \quad \mathrm{EIS}_t = \frac{1}{\delta}. \tag{36}$$

A constant-EIS proportional aggregator can be combined with any conditional CE specification. The most familiar case is reviewed in the following example.

Example 20 (Epstein-Zin-Weil Utility) *Combining the constant EIS proportional aggregator of the last example with the expected-utility SI conditional CE of Example 14 results in Epstein-Zin-Weil utility. Finally, we recall the well-known fact that Epstein-Zin-Weil utility is time additive if and only if* $\gamma = \delta$*, in which case*

$$\frac{1}{1-\beta}\frac{U_0(c)^{1-\gamma}-1}{1-\gamma} = \mathbb{E}^Q\left[\sum_{t=0}^{T-1}\beta^t\frac{c_t^{1-\gamma}-1}{1-\gamma} + \frac{\beta^T}{1-\beta}\frac{c_T^{1-\gamma}-1}{1-\gamma}\right].$$

7 Pricing with SI Recursive Utility

Having established the general SI recursive utility form of interest, in this section we turn our attention to pricing restrictions relative to a given optimal consumption plan. More precisely, we derive restrictions on the utility superdifferential, which when coupled with the assumption of optimality imply pricing restrictions in terms of consumption growth and/or market returns by virtue of the analysis of Section 3.

Recall that Condition 17 is a standing assumption. Fixed throughout this section is a reference consumption plan c. In a homogeneous-agent economy, c can be taken to be the aggregate consumption and U_0 the representative agent's utility function (see Section 3.4 of S09). Alternatively, c can represent the consumption plan of an individual agent with utility function U_0. We let U denoted the utility process associated with c, defined through the backward recursion

$$U_t = \upsilon_t(U_{t+1})\, g_t\left(\frac{c_t}{\upsilon_t(U_{t+1})}\right) \quad t < T; \quad U_T = c_T. \tag{37}$$

The associated process x, q and λ are defined in (35).

7.1 Pricing in Terms of Consumption Growth

We begin with a recursive method for computing $\partial U_0(c)$ as a function of the consumption growth process associated with the reference consumption plan c. Assuming optimality of c, which is to say that an element of $\partial U_0(c)$ must be an SPD at c, this computation together with the SPD characterization of Theorem 6 leads to joint restrictions on market returns and consumption growth.

Theorem 21 *The consumption-to-CE process* x *is determined by the backward recursion*

$$x_{t-1} = \upsilon_{t-1}\left(\frac{g_t(x_t)}{x_t}\frac{c_t}{c_{t-1}}\right)^{-1}, \quad t = 1, \ldots, T; \quad x_T = 1. \tag{38}$$

The process $\pi \in \mathcal{L}_{++}$ *is an element of* $\partial U_0(c)$ *if and only if* $\pi_0 = g_0'(x_0)$ *and*

$$\frac{\pi_t}{\pi_{t-1}} = q_{t-1}(x_{t-1})\, g_t'(x_t)\, \kappa_t \quad \text{for some } \kappa_t \in \partial\upsilon_{t-1}\left(\frac{g_t(x_t)}{x_t}\frac{c_t}{c_{t-1}}\right).$$

Proof. The definition $x_t = c_t/\upsilon_t(U_{t+1})$ and the utility recursion (37) give

$$U_t = \upsilon_t(U_{t+1})\, g_t(x_t) = c_{t-1}\frac{g_t(x_t)}{x_t}\frac{c_t}{c_{t-1}}.$$

Substituting the last expression for U_t in $x_{t-1} = c_{t-1}/\upsilon_{t-1}(U_t)$ and using the homogeneity of υ_{t-1} (equation 20) to cancel out c_{t-1} results in (38). The same expression for U_t and the fact that $\partial\upsilon_{t-1}$ is homogeneous of degree zero (equation 21) result in the identity

$$\partial\upsilon_{t-1}(U_t) = \partial\upsilon_{t-1}\left(\frac{g_t(x_t)}{x_t}\frac{c_t}{c_{t-1}}\right). \tag{39}$$

Lemma 13 completes the proof. ■

Example 22 (Unit EIS) *Assume the unit-EIS proportional aggregator* $g_t(x) = x^{1-\beta}$ *for some* $\beta \in (0,1)$. *In this case, the recursion for* x *is*

$$x_{t-1} = \upsilon_{t-1}\left(x_t^{-\beta}\frac{c_t}{c_{t-1}}\right)^{-1}, \qquad x_T = 1.$$

Given x, $\pi \in \partial U_0(c)$ *if and only if*

$$\frac{\pi_t}{\pi_{t-1}} = \beta x_{t-1}x_t^{-\beta}\kappa_t \quad \text{for some } \kappa_t \in \partial\upsilon_{t-1}\left(x_t^{-\beta}\frac{c_t}{c_{t-1}}\right).$$

7.2 Pricing in Terms of Market Returns

Pricing restrictions implied by the optimality of the reference consumption plan c can alternatively be formulated in terms of the market returns of a claim on c, which in the representative-agent interpretation correspond to the returns of the market portfolio. To spell out how this can be done, we henceforth assume that the consumption plan c is financed by the allocation policy (ϱ, ψ), which generates the wealth process W. In this section we establish a variant of Theorem 21 in which the role of c_t/c_{t-1} is assumed by $R_t[\psi_t]$.

The key link between consumption and the returns $R_t[\psi_t]$ is provided by the assumption that the consumption policy ϱ is *optimal given* ψ, meaning that there is no allocation policy of the form $(\tilde{\varrho}, \psi)$ that finances a consumption plan $\tilde{c}$ such that $U_0(\tilde{c}) > U_0(c)$. The following characterization is a corollary of Lemma 5 and Theorem 6, using the argument of Remark 7.

Lemma 23 *The consumption policy* ϱ *is optimal given* ψ *if and only if there exists some* $\pi \in \partial U_0(c)$ *such that*

$$\mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}}R_t[\psi_t]\right] = 1. \tag{40}$$

We are now ready to characterize all members of $\partial U_0(c)$ that satisfy (40) in terms of a backward recursion. For notational simplicity, this recursion is stated in terms of the shadow-price-of-wealth process λ, as opposed to the consumption-to-CE ratio process x used in Theorem 21. Given the assumed Condition 17, one can easily go back and forth between x and λ, since $\lambda_t = g_t'(x_t)$ and $x_t = \mathcal{I}_t(\lambda_t)$. Recall that g_t^* denotes the convex dual of g_t defined in (32).

Theorem 24 *Suppose the consumption plan c is financed by the allocation policy (ϱ, ψ) and ϱ is optimal given ψ. Then the corresponding shadow-price-of-wealth process λ is determined by the following backward recursion, which also computes q along the way:*

$$q_{t-1} = \frac{g_{t-1}^{*}(\lambda_{t-1})}{\lambda_{t-1}} = \frac{1}{\upsilon_{t-1}(\lambda_t R_t[\psi_t])}, \quad \lambda_T = 1. \tag{41}$$

Suppose further that $\pi \in \mathcal{L}_{++}$ is such that (40) holds. Then $\pi \in \partial U_0(c)$ if and only if $\pi_0 = \lambda_0$ and

$$\frac{\pi_t}{\pi_{t-1}} = q_{t-1}\kappa_t\lambda_t \quad \textit{for some } \kappa_t \in \partial\upsilon_{t-1}(\lambda_t R_t[\psi_t]). \tag{42}$$

Proof. Suppose that $\pi \in \partial U_0(c)$ satisfies (40). Such a π exists by Lemma 23. By Theorem 6(a) and Lemma 18, we have

$$U_t = \lambda_t W_t \quad \text{and} \quad \varrho_t = h_t(x_t). \tag{43}$$

The first equation in (43) and the budget equation (6) imply

$$U_t = W_{t-1}\left(1 - \varrho_{t-1}\right)\lambda_t R_t[\psi_t]. \tag{44}$$

The first equation of (41) follows from the definition (34) of q_t and expression (32) for the convex dual g_t^* with $x_t = \mathcal{I}_t(\lambda_t)$. Substituting ϱ_t for $h_t(x_t)$ in the definition of q_t and using (44) and the homogeneity of υ_t proves the second equation of (41).

Lemma 6.41 of S09 shows that the function $z \mapsto g_t^*(z)/z$ maps $(0, \infty)$ one-to-one onto itself (as a consequence of Condition 17). Thus for any choice of $\lambda_t \in L(\mathcal{F}_t)_{++}$, there is a unique $\lambda_{t-1} \in L(\mathcal{F}_{t-1})_{++}$ solving (41), and therefore the entire processes λ is uniquely determined by recursion (41) given its terminal value $\lambda_T = 1$.

Since $\partial\upsilon_{t-1}$ is homogeneous of degree zero, condition (44) implies

$$\partial\upsilon_{t-1}(U_t) = \partial\upsilon_{t-1}(\lambda_t R_t[\psi_t]). \tag{45}$$

Lemma 13 completes the proof. ■

Example 25 (Pricing with SI Kreps-Porteus Utility) *Suppose that $\upsilon_t = u_\gamma^{-1}\mathbb{E}_t u_\gamma$, where u_γ is defined by (23) for some $\gamma \in (0, \infty)$. The corresponding derivative is given in (24) (with $Q = P$). Since the utility is differentiable, $\partial U_0(c) = \{\pi\}$ for some $\pi \in \mathcal{L}_{++}$. Assume that ϱ is optimal given ψ, and therefore π satisfies (40). Theorem 24 implies that $\pi_0 = \lambda_0$ and*

$$\frac{\pi_t}{\pi_{t-1}} = (q_{t-1}\lambda_t)^{1-\gamma}\left(\frac{1}{R_t[\psi_t]}\right)^{\gamma}, \qquad t = 1, \ldots, T, \tag{46}$$

with q and λ given by (41). If $\gamma = 1$, then $\pi_t/\pi_{t-1} = 1/R_t[\psi_t]$.

To better understand the relationship between the last two theorems, as well as for later use, let us introduce the ratio of consumption growth to market returns:

$$Z_t = \frac{c_t}{c_{t-1}} \frac{1}{R_t[\psi_t]} = \left(\frac{1}{\varrho_{t-1}} - 1 \right) \varrho_t, \tag{47}$$

where the last equation follows easily from the budget equation. Note that given $\left(\varrho_{t-1}, \varrho_t\right)$, c_t/c_{t-1} and $R_t[\psi_t]$ carry the same information. If ϱ is optimal given ψ, then $\varrho_t = h_t(x_t) = h_t(\mathcal{I}_t(\lambda_t))$. Therefore, given x or λ, the ratio Z_t is also determined, which explains why either consumption growth or market returns can be used for pricing.

Example 26 (Unit EIS) *As in Example 22, suppose $g_t(x) = x^{1-\beta}$, and therefore $h_t(x) = 1 - \beta$. Assuming ϱ is optimal given ψ, the second equation in* (43) *implies that $\varrho_t = 1 - \beta$. Recursion* (41) *becomes*

$$\lambda_{t-1} = (1-\beta)^{1-\beta} \left(\beta \upsilon_{t-1}\left(\lambda_t R_t[\psi_t]\right)\right)^{\beta}, \qquad \lambda_T = 1.$$

Equation (47) *in this context gives*

$$\frac{c_t}{c_{t-1}} = \beta R_t[\psi_t] \qquad \textit{(assuming unit EIS).}$$

Since $\lambda_t = (1-\beta)\, x_t^{-\beta}$, the above recursion for λ is easily seen to be equivalent to the recursion for x of Example 22.

7.3 The Case of Invertible Proportional-Aggregator Elasticity

As in the last section, we assume that c is financed by the allocation policy (ϱ, ψ) and ϱ is optimal given ψ. As we saw in (43), this means that $\varrho_t = h_t(x_t)$. The key to this section's analysis is the inversion of this equation to write x, and therefore also q and λ, as a function of ϱ. As the unit-EIS case demonstrates, Condition 17 does *not* imply the invertibility[8] of the proportional aggregator elasticity function h_t. For the remainder of this section, we assume that $h_t : (0, \infty) \to (0, 1)$ is invertible, with $\mathcal{J}_t : (0,1) \to (0, \infty)$ denoting the corresponding inverse function: $h_t(\mathcal{J}_t(z)) = z$ for all $z \in (0,1)$. Using the identity $\varrho_t = h_t(x_t)$, we can write

$$x_t = \mathcal{J}_t(\varrho_t), \quad q_t = \left(\frac{1}{\varrho_t} - 1\right) \mathcal{J}_t(\varrho_t) \quad \text{and} \quad \lambda_t = g_t'(\mathcal{J}_t(\varrho_t)). \tag{48}$$

Therefore, if ϱ can be treated as an observable process, so can x, q and λ. Each element π of $\partial U_0(c)$ satisfying (40) is then characterized either in terms of consumption growth or market returns by

$$\pi_0 = \lambda_0, \quad \frac{\pi_t}{\pi_{t-1}} = q_{t-1} \lambda_t \kappa_t \quad \text{for some } \kappa_t \in \partial \upsilon_{t-1}\left(\frac{g_t(x_t)}{x_t} \frac{c_t}{c_{t-1}}\right) = \partial \upsilon_{t-1}\left(\lambda_t R_t[\psi_t]\right). \tag{49}$$

This follows from Theorems 21 and 24, and identities (39) and (45).

The preceding conclusions apply in particular under a constant but non-unit EIS. In this case the ratio π_t/π_{t-1} in (49) can be expressed entirely as a function of the pair $(c_t/c_{t-1}, R_t[\psi_t])$.

[8] In fact, h_t need not even be monotone; the proportional aggregator $g_t(x) = \sqrt{x} \exp\left(e^{-1} - e^{-x}\right)$ satisfies condition 17 but defines the elasticity function $h_t(x) = (1/2) + x e^{-x}$, which is hump-shaped.

Theorem 27 (Pricing with Constant Non-Unit EIS) *Suppose the consumption plan c is financed by the allocation policy (ϱ, ψ), and ϱ is optimal given ψ. Assume the constant-EIS proportional aggregator of Example 19 with $\delta \neq 1$, and let Z_t be the period-t ratio of consumption growth to market return, defined in* (47). *For every $\pi \in \mathcal{L}_{++}$ satisfying* (40), *$\pi \in \partial U_0(c)$ if and only if*

$$\pi_0 = \lambda_0 \quad \text{and} \quad \frac{\pi_t}{\pi_{t-1}} = \beta^{1/(1-\delta)} Z_t^{-\delta/(1-\delta)} \kappa_t \quad \text{for some } \kappa_t \in \partial \upsilon_{t-1}\left(Z_t^{-\delta/(1-\delta)} R_t[\psi_t]\right). \tag{50}$$

Proof. Inverting $\varrho_t = h_t(x_t)$ results in

$$\mathcal{J}_t(\varrho) = \left(\frac{\beta}{1-\beta}\frac{\varrho}{1-\varrho}\right)^{1/(1-\delta)}. \tag{51}$$

Expressions (48) for q_t and λ_t become

$$q_t = \left(\frac{\beta}{1-\beta}\right)^{1/(1-\delta)} \left(\frac{\varrho_t}{1-\varrho_t}\right)^{\delta/(1-\delta)} \quad \text{and} \quad \lambda_t = (1-\beta)^{1/(1-\delta)} \left(\frac{1}{\varrho_t}\right)^{\delta/(1-\delta)}.$$

These equation together with (47) imply that

$$q_{t-1}\lambda_t = \beta^{1/(1-\delta)} Z_t^{-\delta/(1-\delta)} \quad \text{and} \quad \partial \upsilon_{t-1}(\lambda_t R_t[\psi_t]) = \partial \upsilon_{t-1}\left(Z_t^{-\delta/(1-\delta)} R_t[\psi_t]\right). \tag{52}$$

Therefore, condition (49) becomes (50). ■

Remark 28 *To the conclusion of the preceding theorem we can also add the identity*

$$\upsilon_{t-1}\left(Z_t^{-\delta/(1-\delta)} R_t[\psi_t]\right) = \beta^{-1/(1-\delta)}. \tag{53}$$

This follows from the first equation in (52) *and recursion* (41).

Expression (50) generalizes a well-known pricing formula of Epstein and Zin (1991) and Weil (1989), which is extended below to a version that adds robustness in the sense of Example 15.

Example 29 (Robust Extension of Epstein-Zin-Weil Pricing) *In addition to the assumptions of Theorem 27, suppose that the conditional CE takes the SI ambiguity-averse form of Example 15, assuming the validity of* (28) *for $p > 1$. The value* (25) *is therefore achieved as a minimum. Let $\mathcal{Q}^*$ be the set of every $Q \in \mathcal{Q}$ that achieves the minimum in* (25). *Because of identity* (53), *the superdifferential expression* (26) *implies*

$$\partial \upsilon_{t-1}\left(Z_t^{-\delta/(1-\delta)} R_t[\psi_t]\right) = \left\{\left(\beta^{1/(1-\delta)} Z_t^{-\delta/(1-\delta)} R_t[\psi_t]\right)^{-1} \frac{\xi_t^Q}{\xi_{t-1}^Q} : Q \in \mathcal{Q}^*\right\}.$$

Applying (27) *with $U_t = Z_t^{-\delta/(1-\delta)} R_t[\psi_t]$, expression* (50) *becomes*

$$\frac{\pi_t}{\pi_{t-1}} = \frac{1}{R_t[\psi_t]}\left(1 + \frac{p-1}{p}\chi \log\left(k_{t-1}\left(\frac{c_t}{c_{t-1}}\right)^{\delta/(1-\delta)}\left(\frac{1}{R_t[\psi_t]}\right)^{1/(1-\delta)}\right)\right)^{1/(p-1)}, \tag{54}$$

where k_{t-1} is the unique element of $L(\mathcal{F}_{t-1})_{++}$ that makes condition (40) valid. Letting p converge to one in (54) with $k_{t-1} = 1/\beta$, the expression for π_t/π_{t-1} reduces to

$$\frac{\pi_t}{\pi_{t-1}} = \left(\beta\left(\frac{c_t}{c_{t-1}}\right)^{-\delta}\right)^{\phi}\left(\frac{1}{R_t[\psi_t]}\right)^{1-\phi}, \quad \text{where} \quad \phi = \frac{1-\gamma}{1-\delta}, \quad (\delta \neq 1), \tag{55}$$

where $\gamma = 1+\chi$. This is the familiar pricing formula of Epstein and Zin (1991) and Weil (1989), as it should be given our earlier discussion of the case $p = 1$ in Example 15. In fact, it is straightforward to confirm that if U_0 takes the Epstein-Zin-Weil form of Example 20 with prior $Q = P$, then Theorem 27 reduces to the claim that for any $\pi \in \mathcal{L}_{++}$ satisfying (40) and $\pi_0 = \lambda_0$, $\pi \in \partial U_0(c)$ if and only if π satisfies (55).

The analysis in Skiadas (2008b) suggests that pricing using (54) is consistent with Epstein-Zin-Weil pricing in the limiting case of a Brownian filtration, but it has quantitatively different implications in the presence of Poisson jumps if $p \neq 1$. In a calibration, the pair of parameters (p, χ) provides the flexibility to price Brownian and Poissonian risks differently.

8 Optimal Consumption and Portfolio Choice

As before, we consider an agent with a utility function $U_0 : \mathcal{L}_{++} \to \mathbb{R}$ satisfying Condition 17, who takes as given the market introduced in Section 3. We use the analysis of Section 7.2 to establish a recursive method for computing every optimal allocation policy.

Theorem 30 *Suppose Condition 17 is satisfied. An allocation policy (ϱ, ψ) is optimal if and only if it can be computed by the following recursive procedure:*

1. *(Initialization) Set $\lambda_T = \varrho_T = 1$ and $t = T$.*

2. *(Recursion) With λ_t already computed, select $\psi_t \in L(\mathcal{F}_{t-1})_{++}$ so that $\psi_t \in A_t$ and*

$$\upsilon_{t-1}(\lambda_t R_t[\psi_t]) = \max_{\alpha_t \in A_t} \upsilon_{t-1}(\lambda_t R_t[\alpha_t]). \tag{56}$$

Also compute $\lambda_{t-1} \in L(\mathcal{F}_{t-1})_{++}$ as the unique solution to

$$\frac{\lambda_{t-1}}{g^*_{t-1}(\lambda_{t-1})} = \upsilon_{t-1}(\lambda_t R_t[\psi_t]), \tag{57}$$

and set $\varrho_{t-1} = h_{t-1}(\mathcal{I}_{t-1}(\lambda_{t-1}))$.

3. *While $t > 1$, decrease t by one and repeat Step 2.*

The process λ defined in this procedure is the shadow-price-of-wealth process corresponding to the optimal consumption plan c that is financed by (ϱ, ψ). Finally, if W is the wealth process generated by (ϱ, ψ), the utility process of $c = \varrho W$ is given as $U = \lambda W$.

Example 31 (Unit Relative Risk Aversion) *Suppose υ is an SI Expected-Utility CE with unit relative risk aversion, that is, $\upsilon_t = \exp \mathbb{E}_t \log$. Then equation (56) implies that an optimal allocation can be determined myopically, independently of the value of λ_t :*

$$\mathbb{E} \log \left(R_t[\psi_t] \right) = \max_{\alpha_t \in A_t} \mathbb{E} \log \left(R_t \left(\alpha \right) \right).$$

The optimality of the myopic portfolio allocation is not, however, generally valid if one adds robustness by adopting the SI conditional CE of Theorem 5 in Skiadas (2013) with $\gamma = 1$.

The preceding procedure for computing optimal consumption-portfolio strategies under trading constraints takes the form of a direct backward recursion on the information tree, in contrast to the duality approach of Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992). Small-risk conditional-CE approximations of the type discussed in Skiadas (2008a,b) can be used to relate the discrete backward recursion of Theorem 30 to continuous-time solutions in terms of BSDEs, as in Schroder and Skiadas (2003, 2008). The basic idea behind the relationship between a backward recursion and a BSDE is explained in Skiadas (2008a,b), where one can also find some references to a large mathematics literature that develops numerical BSDE solution methods, as well as the close relationship between BSDEs and associated PDEs. The discrete consumption/portfolio theory offers a simplified entry point to the continuous-time methodology, analogous to the simplification of the Black-Scholes theory achieved by the binomial model of option pricing.

A Appendix: Remaining Proofs

A.1 Proof of Lemma 1

We already showed $(2 \implies 1)$, we now show $(1 \implies 2)$. Suppose the first condition is satisfied and consider the convex sets

$$A = \{(\alpha, x) : \alpha > \nu(z), \ x \in X\}, \quad B = \{(\beta, y) : \beta \leq \nu(z + y), \ z + y \in (0, \infty)^n\}.$$

If $(\alpha, x) \in A \cap B$, then $\nu(z) < \alpha \leq \nu(z + x)$ for some $x \in X$ satisfying $z + x \in (0, \infty)^n$, which contradicts the assumed condition. Note also that $(\nu(z), 0)$ is in the closure of both sets. Therefore, by the separating hyperplane theorem, there exists some $(r, p) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$[(\alpha, x) \in A \implies r\alpha + (p \mid x) \leq r\nu(z)] \quad \text{and} \quad [(\beta, y) \in B \implies r\beta + (p \mid y) \geq r\nu(z)]. \tag{58}$$

The second part of this condition implies that $r < 0$. Indeed, if $r > 0$ the condition is violated by taking β to minus infinity and if $r = 0$ the condition is violated by taking $y = -\varepsilon p$ for $\varepsilon > 0$ small enough so that $z + y \in (0, \infty)^n$. After rescaling, we can therefore set $r = -1$. Given this normalization, the first part of condition (58) implies that $(p \mid x) \leq 0$ for all $x \in X$, and the second part of (58) implies that $p \in \partial\nu(z)$.

A.2 Proof of Theorem 6

For each period t, we define the function $\delta_t : A_t \to L(\mathcal{F}_{t-1})_{++}$ by

$$1 + \delta_t(\alpha_t) = \mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}} R_t[\alpha_t]\right], \quad \alpha_t \in A_t. \tag{59}$$

For any trading strategy ψ, we write $\delta(\psi)$ for the element of $\mathcal{P}_0$ whose end-of-period-t value is $\delta_t(\psi_t)$.

(a) Suppose W is the wealth process generated by the allocation policy (ϱ, ψ) through the budget equation (6), while V is defined in (8). We are to show that $W = V$ if and only if $\delta_t(\psi_t) = 0$.

Multiplying equations (6) and (3) and applying $\mathbb{E}_{t-1}$ on both sides, we find

$$\mathbb{E}_{t-1}[\pi_t W_t] = \pi_{t-1} W_{t-1} \left(1 - \varrho_{t-1}\right)\left(1 + \delta_t(\psi_t)\right). \tag{60}$$

On the other hand, given that $W_T = c_T$, $W = V$ if and only if W satisfies the recursion

$$\mathbb{E}_{t-1}[\pi_t W_t] = \pi_{t-1} W_{t-1}(1 - \varrho_t).$$

The two displayed equations are equivalent if and only if $\delta_t(\psi_t) = 0$.

(b) The following preliminary result is key.

Lemma 32 *Suppose the consumption plan c is financed by the allocation policy (ϱ, ψ), which generates the wealth process W. Then*

$$\mathbb{E}\left[\sum_{t=0}^{T} \pi_t c_t\right] = \pi_0 w + \mathbb{E}\left[\sum_{t=1}^{T} \pi_{t-1} W_{t-1}\left(1 - \varrho_{t-1}\right)\delta_t(\psi_t)\right].$$

Proof. Let $S_t = W_t - c_t = W_t(1 - \varrho_t)$. Subtracting $\pi_{t-1} S_{t-1}$ on both sides of equation (60), we find

$$\mathbb{E}_{t-1}[\pi_t S_t - \pi_{t-1} S_{t-1} + \pi_t c_t] = \pi_{t-1} S_{t-1} \delta_t(\psi_t).$$

To complete the proof, take unconditional expectations on both sides, apply the law of iterated expectations, add up the resulting equation from $t = 1$ to T, and use the fact that $S_0 = w - c_0$ and $S_T = 0$. ■

Returning to the main proof, suppose first that equations (12) are satisfied and therefore $(\pi \mid c) = \pi_0 w$ by Lemma 32. For any $x \in X(c)$, there exists some admissible allocation policy $(\tilde{\varrho}, \tilde{\psi})$ that finances $c + x$. Applying Lemma 32 to $c + x$ and using the fact that $\delta(\tilde{\psi}) \leq \delta(\psi)$, we infer that $(\pi \mid c + x) \leq \pi_0 w$. Therefore $(\pi \mid x) \leq 0$, proving that π is a SPD at c.

Conversely, suppose π is an SPD at c. We first argue that $W = V$. Fixing any reference non-terminal spot (F, t), define a new consumption policy $\tilde{\varrho}$ that is equal to ϱ everywhere except for its value at spot (F, t), where it is modified so the proportion saved is scaled by $(1 - s)$, that is, $1 - \tilde{\varrho}(F, t) = (1 - s)(1 - \varrho(F, t))$, where s is any scalar that is consistent with the requirement

$\tilde{\varrho}(F, t-1) \in (0,1)$. The corresponding amount consumed at spot (F, t) is $c(F, t) + sW(F, t)$. After spot (F, t) consumption, the agent can follow the same allocation policy and finance the original consumption plan scaled by $1 - s$. The allocation policy $(\tilde{\varrho}, \psi)$, therefore, finances the consumption plan $c + x$, where $x \in X(c)$ vanishes outside $F \times \{t, \ldots, T\}$, takes the value $sW(F, t)$ at spot (F, t) and is equal to $-sc$ on $F \times \{t+1, \ldots, T\}$. Since $(\pi \mid x) \leq 0$, we have

$$s\mathbb{E}\left[\left(\pi_t W_t - \sum_{s=t}^{T} \pi_s c_s\right) 1_F\right] \leq 0.$$

Since a small enough s can be chosen to be either positive or negative, the last inequality is in fact an equality, which proves that $V = W$. By part (a), we have also shown that $\delta_t(\psi_t) = 0$.

There remains to show that

$$0 = \max_{\alpha_t \in A_t} \delta_t(\alpha_t). \tag{61}$$

Suppose instead that there is some spot $(F, t-1)$ and some portfolio allocation $\alpha^0 \in A(F, t-1)$ such that $\delta_t(\alpha_t^0) > 0$ on the event F. For each $\phi \in [0,1]$, let

$$\alpha^\phi = (1 - \phi)\alpha^0 + \phi\psi(F, t).$$

Since $\delta(\psi) = 0$ and $R_{F,t}$ is concave, it follows that $\delta_t(\alpha_t^\phi) > 0$ on F for every $\phi \in (0,1)$. Moreover, since $R_t[\psi_t]$ is strictly positive and $R_{F,t}$ is continuous, we can select ϕ sufficiently close to one so that $R_{F,t}(\alpha_t^\phi)$ is also strictly positive. Fixing such a choice of ϕ, we define the allocation policy $\tilde{\psi}$ to be the same as ψ except for the value of $\tilde{\psi}_t$ on the event F which we select to be equal to α^ϕ. By construction, $R(\tilde{\psi})$ is strictly positive and $\delta_t(\tilde{\psi}_t) > 0$ on F. Suppose the allocation policy $\left(\varrho, \tilde{\psi}\right)$ finances the consumption plan $c + x$ and generates the wealth process $\tilde{W}$. Note that $\delta(\tilde{\psi})$ vanishes except on $F \times \{t\}$ where it is strictly positive. By Lemma 32,

$$(\pi \mid c + x) = \pi_0 w + E\left[\sum_{t=1}^{T} \pi_{t-1}\tilde{W}_{t-1}\left(1 - \varrho_{t-1}\right)\delta_t(\tilde{\psi}_t)\right] > \pi_0 w = (\pi \mid c),$$

and therefore $(\pi \mid x) > 0$. Since $x \in X(c)$, this contradicts the assumption that π is an SPD at c. This shows (61) and completes the Theorem's proof.

A.3 Proof of Lemma 13

(Sufficiency) Suppose that π satisfies (18). Fixing any y such that $c + y \in \mathcal{L}_{++}$, we will show that

$$U_0(c + y) - U_0(c) \leq (\pi \mid y). \tag{62}$$

Recall the definition of λ and q in (17), and define

$$\Delta_t = \frac{U_t(c + y) - U_t(c)}{\lambda_t}.$$

The utility recursion (14), the gradient inequality for f, and the assumption that $\kappa_t \in \partial\upsilon_t(U_{t+1}(c))$ imply the inequalities

$$\Delta_t \leq y_t + q_t\left[\upsilon_t(U_{t+1}(c + y)) - \upsilon_t(U_{t+1}(c))\right] \leq y_t + q_t\mathbb{E}_t\left[\kappa_{t+1}\lambda_{t+1}\Delta_{t+1}\right].$$

Combined with the assumed condition (18), this proves the recursion

$$\pi_t \Delta_t \leq \pi_t y_t + \mathbb{E}_t \left[\pi_{t+1} \Delta_{t+1} \right].$$

A backward-in-time induction, starting with the terminal condition $\Delta_T = y_T$, shows that

$$U_t(c+y) - U_t(c) = \lambda_t \Delta_t \leq \mathbb{E}_t \left[\sum_{s=t}^{T} \frac{\pi_s}{\pi_t} y_s \right], \quad t = 0, \ldots, T. \tag{63}$$

Since $\pi_0 = \lambda_0$, the preceding inequality for $t = 0$ gives (62).

(Necessity) Suppose that $\pi \in \partial U_0(c)$. Then $\pi_1 \in \partial F(c_1)$, where

$$F(z) = f(0, c_0, \upsilon_0(f(1, z, \upsilon_1(U_2(c))))), \quad z \in L(\mathcal{F}_1).$$

To prove this claim, consider any $\delta \in L(\mathcal{F}_1)$ and define $y \in \mathcal{L}$ by letting $y_1 = \delta$ and $y_t = 0$ for $t \neq 0$. Using the utility recursion and the definitions, we have

$$F(c_1 + \delta) - F(c_1) = U_0(c+y) - U_0(c) \leq (\pi \mid y) = \mathbb{E}[\pi_1 \delta].$$

The idea now is to apply some sort of chain rule to compute $\partial F(c_1)$. For this purpose, we use

Lemma 33 *Suppose that ν is a concave CE on $(0,\infty)^n$ and the function $\phi : (0,\infty) \to \mathbb{R}$ is (strictly) increasing and differentiable. Then for every $z \in (0,\infty)^n$,*

$$\partial(\phi \circ \nu)(z) = \left\{ \phi'(\nu(z)) \kappa : \kappa \in \partial \nu(z) \right\} \quad \textit{and} \quad \partial(\nu \circ \phi)(z) = \left\{ \kappa \phi'(z) : \kappa \in \partial \nu(\phi(z)) \right\},$$

where $\nu \circ \phi$ denotes the function $(z_1, \ldots, z_n) \mapsto \nu(\phi(z_1), \ldots, \phi(z_n))$.

Proof. This follows from Proposition 4.2.5 of Bertsekas (2003) and Exercise 10.7 of Rockafellar and Wets (1998). The reader interested in a simple direct proof may proceed as follows. Let F be $\phi \circ \nu$ or $\nu \circ \phi$. The nonobvious part is showing that every element of $\partial F(z)$ is of the claim form. To this end, the following fact can be used (see Theorem 23.2 of Rockafellar (1970)):

$$p \in \partial F(z) \quad \Longleftrightarrow \quad F'(z; y) \leq (p \mid y) \text{ for all } y \in \mathbb{R}^n,$$

where $F'(z;y) = \lim_{\alpha \downarrow 0} (F(z + \alpha y) - F(z))/\alpha$. The last directional derivative can be computed in each case by a chain-rule type argument. In the case $F = \nu \circ \phi$ this part of the argument is complicated by the fact that the direction $(\phi(z + \alpha y) - \phi(z))/\alpha$ varies with α, although it does converge to $\phi'(z;y)$. This difficulty is easily overcome using the local Lipschitz continuity of concave functions (see Theorem 10.4 of Rockafellar (1970)). ■

Lemma 33 can be applied to first compute the superdifferential of $z \mapsto \upsilon_0(f(1, z, \upsilon_1(U_2(c))))$ at c_1, and then once again to compute $\partial F(c_1)$. Given the definition of λ and q in (17), the result is that $\pi_1 = q_0 \lambda_0 \kappa_1 \lambda_1$, for some $\kappa_1 \in \partial \upsilon_0 (U_1 (c))$. Since $U_0 (c) = f (0, c_0, \upsilon_0 (U_1))$ is differentiable in c_0, it is immediate that $\pi_0 = \lambda_0$. Therefore,

$$\frac{\pi_1}{\pi_0} = q_0 \kappa_1 \lambda_1, \quad \text{for some } \kappa_1 \in \partial \upsilon_0 (U_1 (c)).$$

We have shown condition (18) for the first period only. If the conditional superdifferential condition (63) is assumed, the same argument applies with the role of time zero assumed by any given nonterminal spot, resulting in the full condition (18). We complete the proof by showing that $\pi \in \partial U_0 (c)$ implies (63) for all y such that $c + y \in \mathcal{L}_{++}$. By Lemma 5, the assumption that $\pi \in \partial U_0 (c)$ implies that $U_0 (c) = \max \{U_0 (c + x) : (\pi \mid x) = 0,\ c + x \in \mathcal{L}_{++}\}$. In other words, c is optimal for an agent who at time zero faces a complete market with state-price density π (see Chapter 5 of S09 for details). By the dynamic consistency property of recursive utility (discussed in Chapter 6 of S09), at each time-t spot, the consumption plan c must still be optimal given the complete market defined by π; that is,

$$U_t (c) = \max \left\{ U_t (c + x) : \mathbb{E}_t \left[\sum_{s=t}^{T} \frac{\pi_s}{\pi_t} x_s \right] = 0, \quad c + x \in \mathcal{L}_{++} \right\}.$$

Indeed, if the last condition were violated, one would easily be able to construct an x such that $U_0 (c + x) > U_0 (c)$ and $(\pi \mid x) = 0$. Applying the necessity part of Lemma 5 to each time-t spot confirms (63) and completes the lemma's proof.

A.4 Proof of Lemma 18

Recall that U_0 can be viewed as a CE on $(0, \infty)^n$, where n is the total number of spots. Applying the second part of (4), it follows that $U_0 (c) = (\pi \mid c) = \lambda_0 V_0$, where the last equality follows from Lemma 13 and the definition of V. The same argument can be applied on the information subtree rooted at any given nonterminal spot (F, t), concluding that $U (c) (F, t) = \lambda (F, t) V (F, t)$. Finally, for the terminal spots we use the identities $U_T = c_T = V_T$ and $\lambda_T = 1$. This proves that $U = \lambda V$. Using the latter along with the definitions of λ, x and h, and the utility recursion (37), we have

$$\frac{c_t}{V_t} = \frac{c_t g_t' (x_t)}{U_t} = \frac{c_t g_t' (x_t)}{\upsilon_t (U_{t+1}) g_t (x_t)} = h_t (x_t).$$

A.5 Proof of Theorem 30

A first key observation is that condition (56) is equivalent to

$$\mathbb{E}_{t-1} [\kappa_t \lambda_t R_t[\psi_t]] = \max_{\alpha_t \in A_t} \mathbb{E}_{t-1} [\kappa_t \lambda_t R_t[\alpha_t]], \quad \text{for some } \kappa_t \in \partial \upsilon_{t-1} (\lambda_t R_t[\psi_t]). \tag{64}$$

This claim follows[9] from Lemma 1 and will be used without further explanation. Throughout this proof, W denotes the wealth process generated by (ϱ, ψ).

[9] To see why, fix any nonterminal spot $(F, t-1)$ with immediate successor spots $(F_1, t), \ldots, (F_n, t)$. Let $\bar{\upsilon}_{F,t-1} : (0, \infty)^n \to (0, \infty)$ be the CE defined by (13), and for each $\alpha \in A (F, t-1)$ let $\bar{R}_{F,t} (\alpha)$ be the vector in $\mathbb{R}^n$ whose ith component is the realization of $R_{F,t} (\alpha)$ at spot (F_i, t). Lemma 1 applies with $\nu = \bar{\upsilon}_{F,t-1}$, $X = \left\{ x \in \mathbb{R}^n : x \leq \bar{R}_{F,t} (\alpha) - \bar{R}_{F,t} (\psi (F, t)),\ \alpha \in A (F, t-1) \right\}$, and $(x \mid y) = \sum_{i=1}^n x_i y_i P [F_i \mid F]$.

Suppose that c is an optimal consumption plan. Let λ is the corresponding shadow-price-of-wealth process, and let (ϱ, ψ) be any allocation policy financing c. Since ϱ is optimal given ψ, Theorem 24 applies, establishing the validity of recursion (57). (Recall that λ_{t-1} is determined by (57) given λ_t by virtue of Proposition 6.41 of S09.) Also, equations (43) in the proof of Theorem 24 establish the validity of Step 3 of the procedure and that λW is the utility process of c. Applying Lemma 5, select any $\pi \in \partial U_0(c)$ that is also an SPD at c. By Theorem 6, condition (10) is satisfied. The first equation of (10) and the fact that $\pi \in \partial U_0(c)$ gives the representation (42) of Theorem 24, which together with the second equation of (10) gives condition (64). This proves (56) and completes the necessity proof.

Conversely, suppose that the allocation policy (ϱ, ψ) and process λ are constructed by the stated recursion. Let also q, κ and π be defined by (41) and (42) of Theorem 24. By construction,

$$1 = q_{t-1}\upsilon_{t-1}\left(\lambda_t R_t[\psi_t]\right) = q_{t-1}\mathbb{E}_{t-1}\left[\kappa_t \lambda_t R_t[\psi_t]\right] = \mathbb{E}_{t-1}\left[\frac{\pi_t}{\pi_{t-1}} R_t[\psi_t]\right],$$

where the second equation follows by (4). By Lemma 23, we have shown that ϱ is optimal given ψ, and therefore the conclusions of Theorem 24 apply, including the fact that $U = \lambda W$ pointed out in its proof. We have proved that $\pi \in \partial U_0(c)$. Condition (56) implies (64), which in turn implies the rest of the SPD condition (10). By Lemma 5, optimality of c is proved.

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