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CONDITIONING AND AGGREGATION OF PREFERENCES

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This paper develops a general framework for modeling choice under uncertainty that extends subjective expected utility to include nonseparabilities, state-dependence, and the effect of subjective or ill defined consequences. This is accomplished by not including consequences among the formal primitives. Instead, the effect of consequences is modeled indirectly, through conditional preferences over acts. The main results concern the aggregation of conditional utilities to form an unconditional utility, including the case of additive aggregation. Applications, obtained by further specifying the structure of acts and conditional preferences, include disappointment, regret, and the subjective value of information.

KEYWORDS: Utility theory, nonseparable preferences, nonadditive utility, disappointment, regret, preferences for information.

1. INTRODUCTION

THIS PAPER PRESENTS a general framework for modeling choice under uncertainty that overcomes some of the limitations of Savage's (1954) approach, but also includes the latter as a special case.

Preferences in Savage's formulation are "separable," in the sense that the ranking of two acts given an event does not depend on the consequences of these acts at states outside the event. The relaxation of separability has been discussed in connection to systematic empirical violations of the expected utility hypothesis, notably by Machina (1989). Numerous "non-expected" utility theories (surveyed by Fishburn (1988) and Karni and Schmeidler (1990)) implicitly or explicitly violate the assumption of separability. Savage's theory also assumes that preferences are state-independent, that is, a consequence is valued the same no matter in what state it obtains. Aumann's (1971) letter to Savage clearly demonstrates the limitations of this assumption. Models of state-dependent preferences are surveyed by Karni (1987) and Karni and Schmeidler (1990) (see also Karni (1993a, b)). In defense of Savage's theory, there is the argument that apparent limitations of the theory disappear once consequences are interpreted broadly enough (see, for example, Raiffa (1968) and Hammond (1989)). A problem with this argument, however, is that it necessitates taking as primitive subjective consequences, such as disappointment, regret, frame of mind, and so on. As recognized by Aumann (1971) and others, such subjective consequences are likely to be ill defined or "fuzzy," and in Savage's setting they tend to

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demand of the decision maker to imagine and evaluate impossible acts (such as an act whose consequence in the state of excellent weather is disappointment at the weather).

This paper suggests a general unifying approach that indirectly incorporates consequences in the wide sense, but avoids the above mentioned pitfalls by not actually including consequences among its formal primitives. Just as subjective probability motivates but does not directly enter Savage's axioms, in this paper subjective consequences motivate but are not part of the primitives of the formal theory. This is accomplished by assuming that acts do not necessarily have a state-contingent structure, or if they do, only objective consequences (such as monetary payoffs) are allowed. Conditional preferences over acts are taken as primitive, with the understanding that in expressing these conditional preferences the decision maker takes into consideration subjective consequences not explicitly modeled as part of the acts. The main results of the paper concern the manner in which conditional utilities, to be thought of as subjective valuations of consequences in the wide sense, can be "aggregated" to form unconditional utilities. A special type of aggregation discussed is additive aggregation, which, because conditional utilities can be nonseparable, is more general than state-dependent expected utility. A companion paper (Skiadas (1996a)) shows how subjective probabilities can be defined in the context of additive aggregation, again, without assuming separability or state-independence of preferences.

The general approach will be illustrated with three applications concerning disappointment, regret, and the subjective value of information. Preferences with disappointment have been formulated by Bell (1985), Loomes and Sugden (1986), and Gul (1991), with the common motivation of explaining observed violations of expected utility theory as exhibited, for instance, in Allais' well known examples. As a result, these papers seek minimal parametric representations that explain the data, as opposed to a general decision theoretic definition of disappointment that is provided here. Our formulation generates a version of Dekel's (1986) class of preferences (which contains Gul's representation) that allows for both subjective probabilities and state-dependence. Preferences with regret have been introduced by Bell (1982) and Loomes and Sugden (1982), again to explain observed violations of expected utility theory, and have been further developed in a literature reviewed by Fishburn (1988). Our formulation differs from that literature in that it is cast in terms of transitive preferences, and it involves choice out of general opportunity sets, as opposed to binary choices only. A general advantage of this paper's approach is that it allows the underlying choice objects to have whatever objective structure is more natural for the application at hand. This is illustrated in the example of preferences with regret, as well as in our third application, on the subjective value of information, where it is natural to model choice objects as algebras of events. Finally, the approach of this paper can be extended to a dynamic setting to model preferences with various forms of intertemporal dependencies. Such an extension is carried out in Skiadas (1996b).

The rest of this paper is organized in three sections and two appendices. Section 2 introduces and motivates conditional preferences. Section 3 contains the basic aggregation theorem, while the case of additive aggregation is discussed in Section 4. Appendix A extends the theory to the case of an infinite number of events. Appendix B contains proofs.

2. CONDITIONAL PREFERENCES

In this section we introduce the notion of conditional preferences, we formally distinguish between separable and nonseparable preferences, and we provide some illustrating examples.

This paper concerns atemporal choice situations with uncertainty, in which the decision maker's actions do not influence the way uncertainty resolves. Uncertainty is modeled by a finite *state space* Ω, representing all possible resolutions of uncertainty faced by a decision maker. The assumption that Ω is finite is made only to avoid technicalities; the infinite state space case is treated in Appendix A. Elements and *nonempty* subsets of Ω will be called *states* and *events*, respectively. The decision maker expresses preferences over *acts* that are taken to be elements of the space X. Acts need not have any specific structure; they are merely names of the possible courses of action.

The decision maker perceives certain "consequences" of choosing act x if nature "chooses" event F. These consequences should be interpreted in the broadest sense possible; they should include any subjective state of mind that may ensue, such as disappointing, regret, lack of awareness, and so on. The ex-ante perception of the consequences of act x on event F may differ from the perception of the consequences of act x given the firm knowledge that event F has or will occur. For example, I now perceive that if F happens I will be disappointed. On the other hand, if I know that F will happen with certainty, I will not be disappointed. Moreover, we do not assume that the occurrence of F implies that the decision maker will know that F has happened. For example, consider a situation in which an undetected computer error leads to undesirable consequences, exactly because of the lack of knowledge of the event that a computer error has occurred.

As discussed in the Introduction, the incorporation of consequences in the wide sense among the primitives is problematic, mainly due to the ill defined and subjective nature of such consequences. We will therefore keep the notion of a consequence informal, and use it only to motivate the formal structure introduced below. On the other hand, many consequences are objective and tangible in nature, a common example being monetary payoffs. When we talk of such consequences, we will specify them as objective consequences. Objective consequences may of course coexist with subjective consequences: I just received \$10, but I am disappointed I have not received more.

Although the decision maker may be unable to precisely communicate the nature of consequences, we assume that he or she is always able to assess which one of any two acts will lead to more desirable consequences if any given event occurs. We formalize that by postulating, for every event F, a preference order (complete and transitive relation[2]), $\succeq^F$, on X. The statement "$x \succeq^F y$" has the interpretation that, ex ante, the decision maker regards the consequences of act x on event F no less desirable than the consequences of act y on the same event. We denote by $\succ^F$ and $\sim^F$ the strict preference and indifference relations,[3] respectively, corresponding to $\succeq^F$.

An act, x, is *state contingent* if it is a function of the form $x: \Omega \to M$, for a given set M that we fix throughout the paper. The interpretation of $x(\omega)$, where $\omega \in \Omega$, is that of an objective consequence, for example, a monetary payoff. The set of all state-contingent acts will be denoted M^Ω. Assuming that $X \subseteq M^\Omega$, we define preferences to be *separable* if, for every event F,

(1) $\quad (x = y \text{ on } F) \Rightarrow x \sim^F y, \qquad x, y \in X.$

The interpretation of preference separability is that the objective consequences of x on F adequately represent all the relevant consequences of x given the occurrence of F. Savage's (1954) theory is a theory of separable preferences. Here we are interested in preferences that violate separability, as in the following example.

EXAMPLE 1 (Disappointment): In this example we assume that $X \subseteq M^\Omega$, and that M is the real line, representing monetary payoffs, although the discussion extends readily to more general spaces. We wish to model the preferences of an agent for whom the consequences of an act on an event may include a subjective feeling of disappointment or elation, depending on whether the act's payoffs on the event are better or worse than anticipated. Formally, we weaken (1) to the following condition: for every event F,

(2) $\quad \left(x = y \text{ on } F \text{ and } y \succeq^\Omega x\right) \Rightarrow x \succeq^F y, \qquad x, y \in X.$

This is a statement of (weak) disappointment aversion: Suppose that acts x and y turn out to have the same payoff on F, but x was expected to yield overall consequences no more desirable than y. Then the consequences of x on F are no less desirable than those of y, since the former may include a sense of elation (because the payoff did not turn out to be worse than that of y), while the latter may include a sense of disappointment. A strict version of (2), modeling strict disappointment aversion, states that, for every event F,

(3) $\quad (x = y \text{ on } F \text{ and } y \succ^\Omega x) \Rightarrow x \succ^F y, \qquad x, y \in X.$

Finally, preferences can be further restricted by requiring that more money without disappointment is (weakly) preferred to less money with disappointment. This is formalized by the condition: for every event F,

(4) $\quad \left(x \geq y \text{ on } F \text{ and } y \succeq^\Omega x\right) \Rightarrow x \succeq^F y, \qquad x, y \in X.$

[2] A relation $R \subseteq X^2$ is *complete* if not xRy implies yRx; it is *transitive* if xRy and yRz implies xRz.

[3] Formally, $x \succ^F y \Leftrightarrow$ not $y \succeq^F x$, and $x \sim^F y \Leftrightarrow x \succeq^F y$ and $y \succeq^F x$.

The natural strict versions of (4) are left to the reader. A utility representation of disappointment aversion is discussed in the following section.

The lack of the assumption that acts are state-contingent in our setting allows us to consider whatever objective act structure is natural for the application at hand. This flexibility is illustrated in the following two examples.

EXAMPLE 2 (Regret): In this example we model preferences with regret, a situation in which the ex-post utility of a payoff is influenced by the unrealized ex-post utility of ex-ante available but unselected alternatives. As usual, M is a payoff set, and M^{Ω} is the set of all functions from Ω to M. We take as primitive a set, $\mathscr{C}$, of subsets of M^{Ω}. (For example, with $M = \mathbb{R}$, we could take $\mathscr{C}$ to be the set of all measurable subsets of the Euclidean space M^{Ω}.) The set of acts is then defined to be $X = \{(c,C): c \in C \in \mathscr{C}\}$. Act (c,C) represents the choice of c out of the opportunity set C. Given any event F and opportunity sets $C, D \in \mathscr{C}$, we write $D \geq^F C$ if for any given $c \in C$, there exists a $d \in D$ such that $(d,D) \succeq^F (c,C)$. In other words, $D \geq^F C$ means that, conditionally on F, every choice out of C is (weakly) dominated by some choice out of D. Separability, as defined in (1), remains meaningful in this context if we interpret the term "$(c,C) = (d,D)$ on F" to mean $c = d$ on F. Here we weaken separability to the following condition: for every event F,

$$(5) \qquad (c = d \text{ on } F \text{ and } D \geq^F C) \Rightarrow (c,C) \succeq^F (d,D), \qquad (c,C),(d,D) \in X.$$

This is a statement of (weak) regret aversion: Suppose that state-contingent acts c and d yield identical payoffs on F, but d is chosen out of a set, D, that includes opportunities whose overall consequences on F are at least as attractive as those resulting from acts selected out of the set, C, from which c was chosen. Then the overall consequences of act (c,C) on F entail no more regret than those of (d,D) on F, and are therefore no less desirable. Condition (5) is analogous to condition (2), and the reader will have no difficulty formulating and interpreting the strict versions of (5) that are analogous to (3) and (4). A utility representation of regret aversion is discussed in the following section.

EXAMPLE 3 (Subjective Value of Information): One usually thinks of information as being valuable indirectly, because it presents planning advantages that result in higher utility of state-contingent consumption. But information is also "consumed" and valued directly: for example, it can be entertaining or distressing. Even when an economic agent is in principle only interested in the indirect value of information, it is usually impractical to ex ante consider and evaluate all possible actions consistent with that information. For example, the subscriber to a news report does not contemplate all optimal plans that are contingent on all possible resolutions of uncertainty provided by the report. Instead, the subscriber places a direct subjective value on the report, conditionally on coarse events representing broadly defined scenarios.

With this motivation, we assume that X is a set of algebras. An *algebra* in this context is a set of subsets of Ω that contains Ω, and is closed with respect to disjoint unions and complementation. An act then represents information in the usual sense. In expressing a preference of the form $x \succeq^F y$, the decision maker contemplates the consequences of having information x or information y on the event that F occurs. It is not necessary to assume that $F \in x \cap y$. In fact, the lack of knowledge of some event F on the event that F occurs may be a significant factor in evaluating information (recall the computer example). Separability is also meaningful in the context of this example. Given any $x, y \in X$ and $F \in x \cap y$, we write "$x = y$ *on* F" if $\{G : G \subseteq F\}$ has the same intersection with x as with y. We then define preferences to be *separable* if (1) holds whenever $F \in x \cap y$. In the same vein, the reader will have no difficulty adapting the notions of disappointment and regret of Examples 1 and 2, respectively, to the present context.[4] The example will be continued in Section 4, where an additive utility representation will be discussed.

Let us now summarize our primitives and introduce some convenient terminology and notation. The state space Ω and the set of preference orders $\{\succeq^F : F \subseteq \Omega\}$ are fixed here for the remainder of the main part of the paper. A *partition* is a set of mutually exclusive events whose union is Ω, and should be thought of as a coarser description of the possible resolutions of uncertainty than that provided by Ω. The set of all partitions will be denoted Π.

DEFINITION 1: Let $\mathscr{G}$ be a partition. A *conditional utility given* $\mathscr{G}$ is a function of the form $U^{\mathscr{G}} : \Omega \times X \to \mathbb{R}$ with the following two properties:

(a) For every $x \in X$, $U^{\mathscr{G}}(\cdot, x)$ is $\mathscr{G}$-measurable, meaning that

$$\omega_1, \omega_2 \in G \in \mathscr{G} \Rightarrow U(\omega_1, x) = U(\omega_2, x).$$

(b) For every $G \in \mathscr{G}$ and $\omega \in G$, we have

$$x \succeq^G y \Leftrightarrow U^{\mathscr{G}}(\omega, x) \geq U^{\mathscr{G}}(\omega, y), \qquad x, y \in X.$$

An *unconditional utility* is a conditional utility given the trivial partition, which is just an ordinal utility representation of $\succeq^\Omega$. A *conditional utility family* is a set of the form $\{U^{\mathscr{G}} : \mathscr{G} \in \Pi\}$, where $U^{\mathscr{G}}$ is a conditional utility given $\mathscr{G}$, and $U^{\mathscr{G}}(\omega, \cdot) = U^{\mathscr{H}}(\omega, \cdot)$ whenever $\omega \in F \in \mathscr{G} \cap \mathscr{H}$.

A conditional utility, $U^{\mathscr{G}}$, is just an efficient way of representing a set of ordinal utility representations, one for every $\succeq^G$ as G ranges over $\mathscr{G}$. Given any function of the form $U : \Omega \times X \to \mathbb{R}$, and any $x \in X$, we will write $U(x)$ to denote

[4] In fact, in the finite state space case considered here, we can regard algebras as state contingent acts by taking M to be the set of all events. Not every function from Ω to M would qualify for an act, however, and the view of algebras as state-contingent acts does not extend to the case of an infinite state space (considered later in the paper). Either one of these limitations makes Savage's setting not compatible with this example.

the random variable $U(\cdot, x)$. The set of all random variables will be denoted L, and is (partially) ordered by $\geq$, where $V \geq W$ means $V(\omega) \geq W(\omega)$ for all $\omega \in \Omega$.

3. COHERENCE AND AGGREGATION

Having defined conditional preferences, we now introduce a minimal consistency condition, called "coherence," and we discuss how coherence allows us to aggregate conditional utilities to form unconditional utilities.

Let $\mathscr{F}$ be any fixed partition representing an agent's perception of possible resolutions of uncertainty. A weak form of "coherence" is the following condition:

$$(6) \qquad \left(x \succeq^F y \text{ for all } F \in \mathscr{F}\right) \Rightarrow x \succeq^{\Omega} y, \qquad x, y \in X.$$

The interpretation is that if x yields no less desirable consequences than y under every possible scenario, then x is overall no less desirable than y. In utility terms, we have the following proposition.

PROPOSITION 1: *Suppose that u is an unconditional utility, and U is a conditional utility given $\mathscr{F}$. Condition* (6) *is then equivalent to the existence of an increasing*[5] *mapping, $A: L \to \mathbb{R}$, such that*

$$(7) \qquad u(x) = A[U(x)], \qquad x \in X.$$

PROOF: Without loss, we assume that u is bounded (otherwise, we can make the argument with $\tan^{-1} \circ u$ in place of u). Under (6), A is well defined by (7), and increasing, on the domain $\{U(x): x \in X\}$. A monotone extension of A to the whole of L is obtained as follows. Let $D = \{V \in L: V \geq U(x) \text{ for some } x \in X\}$, and define $A[V] = \sup\{u(x): V \geq U(x),\ x \in X\}$ for all $V \in D$, and $A[V] = \inf\{A[W]: W \geq V, W \in D\}$ for all $V \notin D$. The converse is immediate. *Q.E.D.*

The mapping A in (7) is only restricted by monotonicity. For example, it can be an expectation operator, or it can be a Choquet integral, as in Gilboa (1987) or Schmeidler (1989).

EXAMPLE 1 (Continued): In the context of Example 1, we assume that (6) holds, and we let u, U, and A be as in Proposition 1. We then have the following representation of disappointment aversion:

PROPOSITION 2: *Condition* (2) *holds for every $F \in \mathscr{F}$ if and only if there exists a function $f: \Omega \times M \times \mathbb{R} \to \mathbb{R}$, nonincreasing in its last argument, such that*

$$(8) \qquad U(\omega, x) = f(\omega, x(\omega), u(x)), \qquad (\omega, x) \in \Omega \times X.$$

[5] The function $A: L \to \mathbb{R}$ is *increasing* if $V \geq W$ implies $A[V] \geq A[W]$.

Given such an f, u uniquely satisfies

$$u(x) = A[f(x, u(x))], \qquad x \in X. \tag{9}$$

PROOF: The proof of the first claim is analogous to that of Proposition 1. Combining (7) and (8), we obtain (9). To show that $u(x)$ uniquely solves (9), suppose that for a given x the equation $v = A[f(x,v)]$ is satisfied for $v = v_1$ and $v = v_2$, where $v_1 < v_2$. Then the monotonicity properties of A and f imply that $v_1 = A[f(x,v_1)] \geq A[f(x,v_2)] = v_2$, a contradiction. *Q.E.D.*

One obtains separability (condition (1)) or strict disappointment aversion (condition (3)) if f is constant or strictly monotone, respectively, in its utility argument. Condition (4) corresponds to monotonicity of f in its payoff argument.

EXAMPLE 2 (Continued): In the context of Example 2, we assume that (6) holds, and we let u, U, and A be an in Proposition 1. Since U is purely ordinal, we can and do assume without loss in generality that U is bounded. The following is a utility characterization of regret aversion:

PROPOSITION 3: *Condition* (5) *holds for every $F \in \mathscr{F}$ if and only if there exists a function $f: \Omega \times M \times \mathbb{R} \to \mathbb{R}$ that is nonincreasing in its last argument, and has the following property*: *For every $C \in \mathscr{C}$, the random variable $\hat{C}$ is uniquely defined by*

$$\hat{C}(\omega) = \sup_{c \in C} f(\omega, c(\omega), \hat{C}(\omega)), \qquad \omega \in \Omega, \tag{10}$$

and U is given by

$$U(\omega,(c,C)) = f(\omega, c(\omega), \hat{C}(\omega)), \qquad (\omega,(c,C)) \in \Omega \times X. \tag{11}$$

PROOF: For each $C \in \mathscr{C}$, we define $\hat{C}(\omega) = \sup_{c \in C} U(\omega,(c,C))$, $\omega \in \Omega$. Since U is assumed bounded, $\hat{C}$ is a finite-valued random variable. Moreover, for all $F \in \mathscr{F}$, $C \geq^F D$ if and only if $\hat{C} \geq \hat{D}$ on F. Suppose now that (5) holds. Arguing analogously to the proof of Proposition 1, it follows that there is a function $f: \Omega \times M \times \mathbb{R} \to \mathbb{R}$, nonincreasing in its last argument, such that (11) holds. Combining this with the definition of $\hat{C}$ gives (10). The uniqueness statement follows from the monotonicity of f in its utility argument (just as in Proposition 2). The converse is immediate. *Q.E.D.*

In the above representation, the function f and the aggregator A completely specify the unconditional utility u through (10), (11), and (7). The random variable $\hat{C}$ has the interpretation of the highest utility level that can be achieved by choosing from C with hindsight, that is, knowing the true event in $\mathscr{F}$.

A more general type of aggregation than that of Proposition 1 involves the familiar decision theoretic construct of a tree, and the procedure of dynamic programming. The decision maker formulates a number of scenarios, then each scenario is further refined to sub-scenarios, and so on up to the "leaves" of the resulting "tree." In evaluating an act, the decision maker can "prune" the tree, replacing a "branch" with a single number that evaluates the act's consequences at each one of the discarded branch's leaves. More pruning can then reduce the size of the tree further, until the tree is reduced to its root, resulting in an overall evaluation of the act. It should be emphasized that no temporal aspect of the decision process is being modeled here. The tree is just a way of organizing a set of scenarios as perceived ex ante. If conditional utilities are to represent the evaluation of consequences in the wide sense, we would like to be able to choose them in a way that makes the above procedure valid.

To make this type of aggregation possible, we strengthen condition (6):

ASSUMPTION 1 (Coherence): *For any disjoint events F and G, and any $x, y \in X$,*

$$x \succeq^F y \text{ and } x \succeq^G y \quad \textit{implies} \quad x \succeq^{F \cup G} y, \qquad \textit{and}$$

$$x \succ^F y \text{ and } x \succ^G y \quad \textit{implies} \quad x \succ^{F \cup G} y.$$

Coherence is interpreted as follows: If the broadly defined consequences of x are no worse than those of y under either one of two sub-scenarios of a given scenario, then the decision maker concludes that the consequences of x are no worse than those of y under the given scenario. This is essentially what Savage (1954) informally described as the "sure-thing principle." In stating and formalizing the sure-thing principle, however, Savage relied on the additional assumption of state-contingent acts and separability. In the following section we will see that by dropping the requirement of separability, while preserving coherence, we can accommodate preferences that cannot be represented by an expected utility, but for which conditional utilities can still be aggregated additively.

Given coherence, one can apply the logic of Proposition 1 to construct an "aggregator," analogous to A of equation (7), at every node of a tree. Such an aggregator computes the conditional utility at a node in terms of the conditional utilities at the immediate successor nodes. In fact, we will show that conditional utilities can be chosen in such a way so that the *same* aggregator at a node can be used to aggregate the conditional utilities not only at the immediate successor nodes, but also at any set of successor nodes that are mutually exclusive and exhaustive. To formalize this, we define the set $S(\mathcal{G})$, where $\mathcal{G}$ is a partition, to include any mapping f with the following properties: (a) the domain of f is a subset of L; (b) the image space of f consists of $\mathcal{G}$-measurable random variables (that is, random variables that are constant over any given event of $\mathcal{G}$); and (c) for any random variables V and W in the domain of f, and any G in $\mathcal{G}$, $V = W$ on G implies that $f(V) = f(W)$ on G. Aggregation is then defined as follows:

DEFINITION 2: A conditional utility family, $\{U^{\mathcal{G}} : \mathcal{G} \in \Pi\}$, is said to *admit aggregation* if there exists a set of the form $\{A[\cdot|\mathcal{G}] : \mathcal{G} \in \Pi\}$, such that, for all $\mathcal{G}$

and $\mathscr{H}$ in Π, $A[\cdot|\mathscr{G}]$ is a member of $S(\mathscr{G})$, and

(12) $\mathscr{G} \subseteq \mathscr{H}$ implies $U^{\mathscr{G}}(x) = A[U^{\mathscr{H}}(x)|\mathscr{G}]$ for all $x \in X$.

To relate the above formalism to our earlier discussion, we notice that the leaves of a tree correspond to a partition, and a conditional utility of an act given that partition represents the valuations of the consequences of the act at those leaves. Condition (12) therefore expresses the fact that, while pruning the tree down to $\mathscr{G}$, the same aggregation rule, $A[\cdot|\mathscr{G}]$, is used for any original form of the tree (corresponding to $\mathscr{H}$). Moreover, the requirement that $A[\cdot|\mathscr{G}] \in S(\mathscr{G})$ guarantees that when a branch is eliminated the valuation that replaces it depends only on the conditional valuations on that branch (even if preferences are not separable relative to some objective state-contingent structure of acts).

Given coherence, aggregation is possible if we only assume essentially what we normally assume in the first place in order to obtain some utility representation. This is demonstrated in the following result, which is a corollary of Theorem A2.

THEOREM 1: *Given coherence (Assumption* 1), *there exists a conditional utility family that admits aggregation if either one of the following two conditions holds*: (*a*) *X is countable* (*or finite*); (*b*) *X is a connected, separable topological space* (*say* $\mathbb{R}^n$), *and every* $\succeq^F$ *is continuous.*[6]

An obvious candidate for an aggregator mapping is an expectation operator, in which case we have *additive aggregation.* Other aggregators are possible; for example, the decision maker could consider a worst case scenario at every node of a tree. An independent characterization of aggregator families is given in Skiadas (1995). A constructive characterization of all functional forms that aggregators can assume is, however, an open problem. Sufficient conditions for additive aggregation are given in the following section.

A natural question is whether coherence places any significant restrictions on preferences. Almost tautologically, coherence is valid as long as one is willing to interpret conditional preferences as being derived from consequences defined broadly enough. In this sense, the framework of this paper provides great flexibility, but that does not mean that there are no instances in which a good modeling approach is inconsistent with aggregation in the sense of Definition 2. As an example, consider "ambiguity" or "Knightian uncertainty," as discussed by Knight (1921), and illustrated by Ellsberg's (1961) well known examples. In principle, one can postulate that ambiguity is a subjective consequence, and is therefore reflected in the value of conditional utilities (as explained in Skiadas (1995)). Some of the more structured and satisfying existing models of ambiguity aversion, however, assign utilities only to objective consequences, and have ambiguity reflected in the form of aggregation. (For surveys see Camerer and

[6] A preference order, $\succeq$, is *continuous* if all sets of the form $\{x : x \succeq y\}$ and $\{x : y \succeq x\}$ are closed.

Weber (1992), and Karni and Schmeidler (1990).) The latter approach is inconsistent with coherence or aggregation in the sense of Definitions 1 and 2. The weaker form of coherence expressed by (6) still applies, however, as do Propositions 1, 2, and 3. For example, A in (7) and (9) could be a Choquet integral.

4. ADDITIVE AGGREGATION

In this final section we consider some sufficient conditions for additive aggregation, relying on Debreu's (1960) theorem. The representation we will obtain is as follows:

DEFINITION 3: The pair (U, P) is an *additive representation* if U is a function of the form $U: \Omega \times X \to \mathbb{R}$, P is a probability, and for every event F,

$$(13) \qquad x \succeq^F y \Leftrightarrow \int_F U(x)\, dP \geq \int_F U(y)\, dP, \qquad x, y \in X.$$

The representation (U, P) is *unique* if any other additive representation of the form $(\tilde{U}, P)$ satisfies $P[\tilde{U} = \alpha U + \beta] = 1$ for some $\alpha \in (0, \infty)$ and $\beta \in L$. (U, P) is *continuous* if $\int_F U dP$ is continuous for every event F.

Clearly, an additive representation induces a corresponding conditional utility family that admits additive aggregation, that is, the aggregators in (12) are conditional expectations. That an additive representation need not exist follows from well known counterexamples dating back to Scott and Suppes (1958), formulated in the context of state-contingent acts and separable preferences.

Given state-contingent acts, additive aggregation reduces to a possibly state-dependent expected utility representation if and only if preferences are separable. Because state-dependence is allowed, however, a unique additive representation does not uniquely determine the underlying probability: P can be chosen arbitrarily (up to events of probability zero it defines) by simply rescaling U state by state. In Skiadas (1996a) the present setting is extended to give conditions under which there is a unique probability measure consistent with additive aggregation, and that measure has the interpretation of subjective probability.

To motivate our assumptions, it is helpful to consider the basic idea of our approach. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be the partition that generates the algebra of all events. We wish to provide sufficient conditions, under which we can consistently define a preference order $\succeq$ on X^n by letting $(x_1, \ldots, x_n) \succeq (y_1, \ldots, y_n)$ $\Leftrightarrow x \succeq^{\Omega} y$, where $x \sim^{F_i} x_i$ and $y \sim^{F_i} y_i$ for all i. Moreover, we wish to be able to apply Debreu's additive representation theorem on $\succeq$. To this end we will employ a connectedness-continuity assumption, a strict version of coherence, and a "solvability" assumption. We begin by formally stating the latter:

ASSUMPTION 2: *The following conditions hold*:

(a) (*Solvability*) *Given any mutually exclusive events* $F_1, \ldots, F_n$, *and any acts* $x_1, \ldots, x_n$, *there exists an act* x *such that* $x \sim^{F_i} x_i$ *for all* i.

(b) X *is a compact topological space*.

With separable preferences over state-contingent acts, solvability is automatically satisfied, because one can construct x to yield the same payoffs as x_i on F_i, for every i. In its general form, solvability can be thought of as the availability of acts that exactly compensate for not following certain courses of action under corresponding scenarios. Act x could, for example, be a state-contingent monetary payoff that, on the occurrence of event F_i, would exactly compensate for not having followed act x_i, for every i. Of course, such an act may be an artifact, not naturally occurring in the initial formulation of the problem. It is not hard to see that under solvability and coherence the preference order $\succeq$ on X^n introduced above is well defined. Part (b) of Assumption 2 has as its main purpose to make $\succeq$ continuous. Alternatively, we can relax compactness and strengthen solvability to reflect the intuition that sufficiently small perturbations of the acts x_i correspond to small perturbations of the "compensating" act x:

ASSUMPTION 3: (*Continuous Solvability*) *Given any mutually exclusive events* $F_1, \dots, F_n$, *and any acts* $x_1, \dots, x_n$, *there exists an act* x *with the following property: for every neighborhood* N *of* x, *there exist neighborhoods* $N_1, \dots, N_n$ *of* $x_1, \dots, x_n$, *respectively, such that* $x_i' \in N_i$ *for all* i *implies that there exists* $x' \in N$ *such that* $x' \sim^{F_i} x_i'$ *for all* i.

We are now ready to state an additive aggregation theorem. Following Debreu, we call an event F *essential* if $x \succ^F y$ for some $x, y \in X$.

THEOREM 2: *Suppose that the following conditions hold*:
(a) (*Continuity*) X *is a connected topological space, and each* $\succeq^F$ *is continuous.*
(b) (*Strict Coherence*) *For any disjoint events* F *and* G, *and any acts* x *and* y,

$$x \succeq^F y \text{ and } x \succeq^G y \text{ implies } x \succeq^{F \cup G} y, \qquad \text{and}$$

$$x \succ^F y \text{ and } x \succeq^G y \text{ implies } x \succ^{F \cup G} y.$$

(c) (*Essential Events*) *There are at least three mutually exclusive essential events.*
(d) (*Solvability +*) *Either Assumption* 2 *or Assumption* 3 *holds.*
Then there exists a continuous and unique additive representation.

Theorem 2 can be applied in the context of any of the examples introduced so far, with Theorem 1 in Skiadas (1996a) providing the necessary extension for the subjective probability interpretation. Of course an appropriate topology on X has to be defined in each case. Additive aggregation in Example 1, where the standard Euclidean topology can be used, leads to a version of Dekel's (1986) utility representation that incorporates subjective probability and possible state-dependence (see Skiadas (1996a) and Appendix A). We conclude with brief discussions of the application of Theorem 2 to our other two examples.

EXAMPLE 2 (Continued): Specializing Example 2, we now describe one of many possible topologies on X that allow the application of Theorem 2. For simplicity, we make several restrictive assumptions, briefly indicating generalizations later on. We assume that $M = \mathbb{R}^k$ for some integer k, an element of M representing a payoff in k commodities. We take $B \subseteq M^{\Omega}$ to be the set of all state-contingent acts that are valued in $[0,1]^k$, and we view B as a "box" in a standard Euclidean space. The set of measurable subsets of B (the Borel subsets) is denoted $\mathscr{B}$. We regard $\mathscr{B}$ as a (semi)metric space, where the "distance" between two sets is taken to be the $(k \times |\Omega|)$-dimensional volume of their symmetric difference. This (semi)metric may at first appear inadequate as a measure of "closeness" of opportunity sets, since attaching any single choice to an opportunity set does not "move" the set by any distance. This pathology disappears, however, if we assume that every opportunity set allows free disposal. Formally, we let $\mathscr{C}$ be the set that is composed of every $C \in \mathscr{B}$ with the *free-disposal property*: for every $c \in C$ and $c' \in B$, $c' \leq c$ implies $c' \in C$ (where $c' \leq c$ means $c'_i(\omega) \leq c_i(\omega)$ for all $\omega \in \Omega$ and $i \in \{1,\ldots,k\}$). The topology on $X = \{(c,C): c \in C \in \mathscr{C}\}$ is defined as the relative product topology.

The above setting is easily generalized to include opportunity sets that do not allow free disposal, under the implicit assumption that, for every commodity, more is preferred to less. Suppose that $(c,C) \in B \times \mathscr{B}$ satisfies $c \in C$, but C does necessarily have the free-disposal property. Let $C^{\downarrow} = \{c \in B : c \leq c'$ for some $c' \in C\}$. Assuming monotone preferences, it is reasonable to impose the restriction: $(c,C) \sim^F (c, C^{\downarrow})$ for every event F. The interpretation is that regret is experienced only relative to the maximal elements of the closure of an opportunity set, and the closures of C and $C^{\downarrow}$ have the same maximal elements. Under this assumption we can identify, in utility terms, (c,C) with $(c,C^{\downarrow})$, reducing the setup to the case of free disposal. Finally, it is worth noting that the definition of the topology on $\mathscr{B}$ requires little of the special structure assumed above. Given any nonatomic measure space, $(B,\mathscr{B},\mu)$, where μ is a finite measure, the semimetric defined on $\mathscr{B}$ by $d(C_1,C_2) = \mu(C_1 \triangle C_2)$ renders $\mathscr{B}$ a connected Polish space (see Halmos (1974, Section 40)).[7] Of course, there are other interesting metrics on sets of opportunity sets, a well known one being the Hausdorff metric (see, for example, Aliprantis and Border (1994, Section 3.16)).

EXAMPLE 3 (Continued): For simplicity, we have assumed up to this point a finite state space, which makes Theorem 2 inapplicable in the context of Example 3 (since X is discrete). Appendix A shows, however, that all our results extend to an infinite state space. We now assume that $(\Omega,\mathscr{F},P)$ is an infinite probability space, and we define X to be a subset of Φ, the space of all

[7] To prove connectedness, it suffices to show that each $C \in \mathscr{B}$ is pathwise connected to the empty set. Using a standard construction on nonatomic spaces, one can construct an increasing function $p: Q \cap [0,\mu(C)] \to \mathscr{B}$, where $\mathscr{B}$ is ordered by inclusion and Q is the set of rationals, such that $\mu(p(r)) = r$ for all $r \in Q$. Since p is uniformly continuous and $\mathscr{B}$ is a complete metric space, p has a continuous extension that defines a path connecting the empty set to C.

complete[8] sub-σ-algebras of $\mathscr{F}$. Besides the fact that Ω is now infinite, our discussion in Example 3 of Section 2 remains valid word for word. Conditional preferences can be defined either with respect to a finite number of events, belonging to a fixed partition, or with respect to any event in $\mathscr{F}$, as explained in Appendix A. A suitable topology on Φ, under which Φ becomes a Polish space, is that suggested by Cotter (1986) (assuming that $\mathscr{F}$ is countably generated). Under Cotter's topology, "nearby" information sets result in posterior estimates that are close in the L^1-norm. Moreover, the space of all finite partitions (or, more precisely, the corresponding complete σ-algebras) is dense in Φ. One can easily define subsets of Φ that are connected in the Cotter topology. A simple example is obtained by letting X be generated by a multivariate Brownian motion. Then any two points in X are connected by a path indexed by time, since (by the martingale convergence theorem) the passage of time causes posterior estimates to change continuously.

It is also worth noting that, under Cotter's topology, separability (as defined for this example in Section 2) implies continuous solvability (Assumption 3). Moreover, additive aggregation together with separability implies that the value of a partition can be computed as the sum of the value of its elements, in the following sense: Let X^f consist of all elements of X that are generated by finite partitions (and the null events), and let $\pi(x)$ denote the partition generating $x \in X^f$. Under additive aggregation and separability, one can easily show that there exists a function $V: \mathscr{F} \to \mathbb{R}$ such that the unconditional utility, u, representing $\succeq^{\Omega}$ satisfies

$$u(x) = \sum \{V(F) : F \in \pi(x)\}, \qquad x \in X^f.$$

If u is continuous in the Cotter topology, a part of the conclusions of both Theorem 2 and Theorem A3, then this expression completely determines u on X, since X^f is dense in X. Such an additive representation of u on X^f was also discussed by Gilboa and Lehrer (1991). Their result is cardinal in character, however, taking u as a primitive, and assuming an additivity-type condition on u directly. Finally, the extension in Skiadas (1996a) can be applied here to write $V(F)$ as a product of a utility of F and a subjective probability of F.

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APPENDIX A: THE CASE OF INFINITELY MANY EVENTS

This appendix outlines an extension of the theory of this paper to the case of an infinite number of events.

[8] In this context, a σ-algebra is *complete* if it contains all the P-null events.

We take as primitive a probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is a σ-algebra of subsets of Ω, called *events*. The (countably additive) probability P on $\mathscr{F}$ matters only through the *null events* (events of probability zero) it defines. As before, the space of *acts*, X, is a set with no specific structure, unless otherwise specified. A *preference order* is a complete and transitive relation on X.

DEFINITION A1: A *coherent preference family*, or *CPF* for short, is a set of preference orders of the form $\{\succeq^F : F \in \mathscr{F}\}$, satisfying the following conditions for all non-null $F, G \in \mathscr{F}$:

(a) For any countable set, $\{F_1, F_2, \dots\}$, of mutually exclusive events whose union is F, and any $x, y \in X$,

$$x \succeq^{F_n} y \text{ for all } n \text{ implies } x \succeq^F y, \text{ and } x \succ^{F_n} y \text{ for all } n \text{ implies } x \succ^F y.$$

(b) If $F \triangle G$[9] is null, then $\succeq^F = \succeq^G$.

A fixed CPF, $\{\succeq^F : F \in \mathscr{F}\}$, will be taken as primitive throughout, unless otherwise indicated. The relations $\succ^F$ and $\sim^F$ corresponding to $\succeq^F$ are defined as usual.

Suppose that $\mathscr{G}$ is a partition representing a finite number of scenarios. Given any acts x and y, the state space decomposes into two events, denoted $\{x \succeq^{\mathscr{G}} y\}$ and $\{y \succ^{\mathscr{G}} x\}$, that are uniquely characterized by the requirements that they are both unions of elements of $\mathscr{G}$, and that they satisfy:

$$\text{(14)} \qquad \text{for all } G \in \mathscr{G}, \quad \left(G \subseteq \left\{x \succeq^{\mathscr{G}} y\right\} \Rightarrow x \succeq^G y\right) \text{ and } \left(G \subseteq \left\{y \succ^{\mathscr{G}} x\right\} \Rightarrow y \succ^G x\right).$$

The events of the form $\{x \succeq^{\mathscr{G}} y\}$ can be represented as a function $\succeq^{\mathscr{G}} : X^2 \to \mathscr{G}$ that contains the same information as the sub-CPF $\{\succeq^G : G \in \mathscr{G}\}$, and can therefore be sensibly called the conditional preference given $\mathscr{G}$. Completely analogously, we now define the conditional preference with respect to a σ-algebra representing a possibly infinite number of scenarios.

DEFINITION A2: Let $\mathscr{G}$ be any σ-algebra of events. A *conditional preference given* $\mathscr{G}$ is any mapping $\succeq^{\mathscr{G}} : X^2 \to \mathscr{G}$ such that all events of the form $\{x \succeq^{\mathscr{G}} y\} = \succeq^{\mathscr{G}}(x, y)$ and $\{y \succ^{\mathscr{G}} x\} = \Omega \setminus \{x \succeq^{\mathscr{G}} y\}$ satisfy (14).

THEOREM A1: *For every σ-algebra of events, $\mathscr{G}$, a conditional preference, $\succeq^{\mathscr{G}}$, given $\mathscr{G}$, exists and is unique in the sense that if $\succeq_1^{\mathscr{G}}$ is another conditional preference given $\mathscr{G}$, then $\{x \succeq^{\mathscr{G}} y\} \triangle \{x \succeq_1^{\mathscr{G}} y\}$ is null for all $x, y \in X$.*

An *information class* is any set of complete[10] σ-algebras of events, and should be thought of as a set of ways in which the decision maker can describe the possible scenarios. An *algebra-CPF* is any set of conditional preferences of the form $\{\succeq^{\mathscr{G}} : \mathscr{G} \in \Phi\}$, where Φ is an information class. An algebra-CPF completely determines the underlying CPF if the corresponding information class contains all σ-algebras generated by finite partitions (and the null events). An independent characterization of algebra-CPFs, that does not utilize an underlying CPF, is given in Skiadas (1995).

We now provide a natural extension of Definition 1. The following notational conventions will be used throughout the remainder of the paper. For every algebra $\mathscr{G}$ and event F, $\mathscr{G} \cap F = \{G \cap F : G \in \mathscr{G}\}$. If V and W are random variables, we say that $V = W$ *on* (event) F if $F \setminus \{V = W\}$ is null.

[9] We define $F \triangle G = (F \setminus G) \cup (G \setminus F)$, the symmetric difference of F and G.

[10] We have defined a σ-algebra of events to be *complete* if it contains all null events.

DEFINITION A3: A *utility representation* of an algebra-CPF, $\{\succeq^{\mathscr{G}} : \mathscr{G} \in \Phi\}$, is a family, $\{U^{\mathscr{G}} : \mathscr{G} \in \Phi\}$, of functions such that, for all $x, y \in X$, and $\mathscr{G}, \mathscr{H} \in \Phi$,

(a) $U^{\mathscr{G}} : \Omega \times X \to \mathbb{R}$, and $U^{\mathscr{G}}(x) = U^{\mathscr{G}}(\cdot, x)$ is $\mathscr{G}$-measurable;

(b) $\{x \succeq^{\mathscr{G}} y\} \triangle \{U^{\mathscr{G}}(x) \geq U^{\mathscr{G}}(y)\}$ is a null event;

(c) $F \in \mathscr{G} \cap \mathscr{H}$ and $\mathscr{G} \cap F = \mathscr{H} \cap F$ implies $U^{\mathscr{G}}(x) = U^{\mathscr{H}}(x)$ on F.

The notion of aggregation in this setting is essentially the same as in the finite case. Given σ-algebra $\mathscr{G}$, we let $S(\mathscr{G})$ be the set of all functions, f, mapping random variables to $\mathscr{G}$-measurable random variables, with the property that $V = W$ on $G \in \mathscr{G}$ implies $f(V) = f(W)$ on G. The definition of aggregation is then exactly as in Definition 2, with Π representing an information class, instead of a set of partitions. Theorem 1 is a corollary of a more general aggregation result that we now state. Given an algebra-CPF $\mathscr{A} = \{\succeq^{\mathscr{G}} : \mathscr{G} \in \Phi\}$, the set $Z \subseteq X$ is *$\mathscr{A}$-dense* if it satisfies the following condition: For every $x, y \in X$ and $\mathscr{G} \in \Phi$, if $G \in \mathscr{G} \cap \{x \succ^{\mathscr{G}} y\}$ is non-null, then there exists a $z \in Z$ such that $\{x \succeq^{\mathscr{G}} z \succeq^{\mathscr{G}} y\} \cap G$ is also non-null.

THEOREM A2: *(a) Any algebra-CPF, $\mathscr{A}$, for which there exists a countable $\mathscr{A}$-dense subset of X has a utility representation that admits aggregation. (b) If a countable algebra-CPF, $\mathscr{A}$, has a utility representation, then there exists a countable $\mathscr{A}$-dense subset of X.*

Theorem 1 follows from Theorem A2 by taking the countable $\mathscr{A}$-dense subset to be X in case (a), and any countable topologically dense subset of X in case (b).

Finally, we extend Theorem 2 to include additive aggregation with an infinite number of events. For this purpose, we generalize our setting a little. Taken as primitive are $(\Omega, \mathscr{F})$, where $\mathscr{F}$ is a σ-algebra of subsets of Ω, and a set of preference orders $\{\succeq^F : F \in \mathscr{F}\}$ (not assumed to be a CPF, yet). The probability P is no longer among the primitives; instead, we will derive it as part of the representation result below. In this context, L represents the set of $\mathscr{F}$-measurable functions from Ω to $\mathbb{R}$. The definition of an *additive representation* (U, P) is just as in Definition 3, with the additional technical requirements that (a) $U(x) \in L$ for every $x \in X$; (b) P is countably additive; and (c) $\int_\Omega |U(x)|\, dP < \infty$ for every $x \in X$. The definition of uniqueness and continuity of an additive representation is the same as in Definition 3.

THEOREM A3: *Suppose that, in addition to conditions (a) through (c) of Theorem 2, the following are satisfied:*

(d) *Assumption 2.*

(e) *Suppose that $\{x_n\}$ is a sequence of acts converging to some act x, and that $\{E_n\}$ is an increasing ($E_{n+1} \supseteq E_n$ for all n) sequence of events such that $\bigcup_{n=1}^{\infty} E_n = \Omega$. If there is $y \in X$ such that $x_n \sim^{E_n} y$ for all n, then $x \sim^{\Omega} y$.*

Then there exists a continuous and unique additive representation.

Notice that the additional assumption (e) is necessary. A version of Theorem A3 that does not require compactness of X appears in Skiadas (1995). Closely related results are also given by Vind and Grodal (1990), and Skiadas (1996a).

EXAMPLE 1 (Continued): Continuing our discussion of disappointment aversion, we now provide a representation with additive aggregation and infinitely many events. We take as primitive a set, $\{\succeq^F : F \in \mathscr{F}\}$, of preference orders on X, and we assume the following:

(a) $(\Omega, \mathscr{F}, P)$ is a nonatomic probability space, the role of P being only the definition of null events.

(b) For simplicity, $M = [0, 1]$. The space of acts, X, consists of all M-valued random variables, and is endowed with the relative weak topology of $L^1(\Omega, \mathscr{F}, P)$ (see Dunford and Schwartz (1988, IV.8.7)).

(c) The assumptions of Theorem A3 hold. Compactness and connectedness of X are automatic by its definition (see Dunford and Schwartz (1988, IV.8.9 and V.6.1)).

(d) (Extreme Acts) Let $\underline{x}(\omega)=0$ and $\bar{x}(\omega)=1$ for all $\omega \in \Omega$. Then $\bar{x} \succ^F \underline{x}$ and $\bar{x} \succeq^F x \succeq^F \underline{x}$ for all $x \in X$ and non-null $F \in \mathcal{F}$.

(e) (Monotonicity and Disappointment Aversion) Condition (4) holds for all $F \in \mathcal{F}$.

By Theorem A3, there exists a continuous and unique additive representation. Fixing such a representation (U, P), we define the unconditional utility $u = \int_\Omega U dP$, with $I = [u(\underline{x}), u(\bar{x})]$ denoting its image (a closed interval since u is continuous and X is compact).

THEOREM A4: *Given (a)–(e) above, there exists a function $f: \Omega \times M \times I \to \mathbb{R}$, nondecreasing in its second argument and nonincreasing in its third argument, such that $P[U(x) = f(x, u(x))] = 1$. Given any such function f, u uniquely solves*

$$u(x) = E[f(x, u(x))], \qquad x \in X. \tag{15}$$

Moreover, f can be chosen to be strictly decreasing in its last argument if and only if (3) is satisfied, and nondependent on its last argument if and only if (1) holds.

For an example satisfying all of the above assumptions, suppose that $f: \Omega \times M \times [0,1] \to [0,1]$ is continuous in its last two arguments, strictly increasing in its second argument, and nonincreasing in its last argument. Suppose further that $f(\omega, 0, v) = 1 - f(\omega, 1, v) = 0$ for all $(\omega, v) \in \Omega \times [0,1]$. We claim that a utility u is well defined by (15). Uniqueness follows just as in Proposition 2. To show existence, we observe that $E[f(x,v)] - v$ is a continuous function of v that is nonnegative for $v=0$ and nonpositive for $v=1$, and therefore vanishes somewhere on $[0,1]$. Given u, we define the entire CPF through the conditional utilities: $U^{\mathcal{G}}(x) = E[f(x, u(x)) | \mathcal{G}]$. It is easy to check then that all the assumptions of Theorem A4 hold, including solvability.

APPENDIX B: PROOFS

This appendix outlines proofs of the paper's theorems. Additional details can be found in Skiadas (1995). The following lemma will be of repeated use.

LEMMA B1: *Given probability space $(\Omega, \mathcal{F}, P)$, suppose that all non-null events are designated one of light or dark in a way so that (a) countable disjoint unions of light (respectively, dark) events are light (respectively, dark), and (b) any two non-null events whose symmetric difference is null are either both light or both dark. Let an event be white (respectively, black) if all its non-null subevents are light (respectively, dark). Then $\Omega = B \cup W$, where B is a black event, W is white, and $B \cap W = \varnothing$. This decomposition is unique up to P-null events.*

PROOF: This is a proof outline, similar to Halmos' (1974, Theorem 29A) proof of the Hahn decomposition theorem, which this lemma generalizes. Suppose F is non-null and not white. Let D_1 be a dark subevent of F. If D_1 is not black, let n_1 be the smallest integer for which there exists a light subevent $L_1 \subseteq D_1$ such that $P(L_1) \geq 1/n_1$. Then $D_2 = D_1 \setminus L_1$ is dark. Repeating, we either arrive at a non-null black event D_n, or produce a disjoint sequence, $\{L_k\}$, of light events and a divergent sequence, $\{n_k\}$, of integers. In the latter case an argument by contradiction shows that $D_1 \setminus \bigcup_{k=1}^\infty L_k$ is non-null and black. Therefore, every non-null event that is not white (black) has a non-null black (white) subevent. Let $\{B_n\}$ be a sequence of black events such that $P(B_n) \to \sup\{P(B): B \text{ is black}\}$. An argument by contradiction shows that $\Omega \setminus \bigcup_{n=1}^\infty B_n$ is white, and the theorem follows. *Q.E.D.*

PROOF OF THEOREM 2: We let $\{F_1, \ldots, F_n\}$ be the partition that generates the algebra of all events, and we write $\succeq_i = \succeq^{F_i}$ and $\sim_i = \sim^{F_i}$. Without loss in generality, we assume that each F_i is essential. By solvability, there exists a map $\xi: X^n \to X$ such that $\xi(x_1, \ldots, x_n) \sim_i x_i$ for all i. Fixing such a ξ, we define the preference order $\succeq$ on X^n, by letting $\mathbf{x} \succeq \mathbf{y} \Leftrightarrow \xi(\mathbf{x}) \succeq^\Omega \xi(\mathbf{y})$. We now show that $\succeq$ is continuous in the product topology of X^n. This is immediate under Assumption 3.

Suppose now that Assumption 2 holds, and let $\mathbf{y}=(y_1,\dots,y_n)\in X^n$. We will show that $\{\mathbf{x}\in X^n:\mathbf{x}\succeq\mathbf{y}\}$ is closed in the product topology. (That $\{\mathbf{x}\in X^n:\mathbf{y}\succeq\mathbf{x}\}$ is closed is shown the same way.) Suppose that $\{x_i^\gamma\}$ is a net of acts converging to x_i, for all $i\in\{1,\dots,n\}$, and $\mathbf{x}^\gamma=(x_1^\gamma,\dots,x_n^\gamma)\succeq\mathbf{y}$ for all γ. Since X is assumed compact, the net $\{\xi(\mathbf{x}^\gamma)\}$ has a subnet $\{\xi(x^{\gamma(\alpha)})\}$ that converges to some $x\in X$. Since X is assume Hausdorff, the subnet $\{x_i^{\gamma(\alpha)}\}$ converges to x_i for every i. By continuity of each of the preferences $\succeq_i$, we then have that $x\sim_i x_i$ for all i. Continuity of $\succeq^\Omega$ implies that $x\succeq^\Omega y$. Therefore, $(x_1,\dots,x_n)\succeq\mathbf{y}$. This proves continuity of $\succeq$.

By coherence, it now follows easily that $\succeq$ satisfies the assumptions of Theorem 6.13 in Krantz et al. (1971) (whose proof was completed by Walker (1988)). Therefore, there exist functions $u_i:X\to\mathbb{R}$ such that $\Sigma_{i=1}^n u_i$ is an ordinal utility representation of $\succeq$. Given any strictly positive probability P, we define $U(x,\omega)=u_i(x)/P(F_i)$ if $\omega\in F_i$. We claim that (U,P) is an additive representation. To see that, fix any event F, and let $I=\{i:F_i\subseteq F\}$. For any $x,\ z\in X$, we denote by $x_F z$ the element of X^n whose ith coordinate is x if $i\in I$, and z otherwise. For any $x,y\in X$, we can pick an arbitrary $z\in X$, and use coherence and the definition of $\succeq$ to write: $x\succeq^F y\Leftrightarrow\xi(x_F z)\succeq^\Omega\xi(y_F z)\Leftrightarrow\Sigma_{i\in I}u_i(x)\geq\Sigma_{i\in I}u_i(y)$. This proves (13). Uniqueness and continuity of (U,P) follow from the corresponding properties of the u_i. *Q.E.D.*

Proof of Theorem A1: Fix any $x,y\in X$ and σ-algebra $\mathscr{G}$, and let a non-null event be *dark* if $x\succeq^G y$, and *light* otherwise. Applying Lemma B1, let $\{x\succeq^{\mathscr{G}}y\}$ be the black event of the black-and-white decomposition of Ω. *Q.E.D.*

Proof of Theorem A2: (a) Suppose that $Z=\{z_1,z_2,\dots\}$ is an $\mathscr{A}$-order dense subset of X. Given any $\mathscr{G}\in\Phi$ and $x\in X$, let $d_n^{\mathscr{G}}(x)=1_{\{x\succeq^{\mathscr{G}}z_n\}}+1_{\{x\succ^{\mathscr{G}}z_n\}}$ (where $1_F=1$ on F and $1_F=0$ on $\Omega\setminus F$). In the case that Z has a last element z_N, let $d_n^{\mathscr{G}}(x)=0$ for $n>N$. Let

$$U^{\mathscr{G}}(x)=\sum_{n=1}^{\infty}\frac{d_n^{\mathscr{G}}(x)}{3^{2n-1}},\qquad x\in X,\ \mathscr{G}\in\Phi.$$

Notice that there is a one-to-one correspondence between values of $U^{\mathscr{G}}(x)$ and of the sequence $\{d_n^{\mathscr{G}}(x)\}$. We will show that $\{U^{\mathscr{G}}:\mathscr{G}\in\Phi\}$ is a utility representation of $\mathscr{A}$ that admits aggregation. Conditions (a) and (c) of Definition A3 follow easily. Because $d^{\mathscr{G}}(x)\geq d^{\mathscr{G}}(y)$ on[11] $\{x\succeq^{\mathscr{G}}y\}$ for all n, $U^{\mathscr{G}}(x)\geq U^{\mathscr{G}}(y)$ on $\{x\succeq^{\mathscr{G}}y\}$. Conversely, suppose that $G=\{U^{\mathscr{G}}(x)\geq U^{\mathscr{G}}(y)\}\cap\{y\succ^{\mathscr{G}}x\}$ is non-null. Since Z is $\mathscr{A}$-dense, $G'=\{y\succeq^{\mathscr{G}}z_m\succeq^{\mathscr{G}}x\}\cap G$ is also non-null for some m. It follows that $d_m^{\mathscr{G}}(y)>d_m^{\mathscr{G}}(x)$ on G', while $d_n^{\mathscr{G}}(y)\geq d_n^{\mathscr{G}}(x)$ on G' for all n. Therefore, $U^{\mathscr{G}}(y)>U^{\mathscr{G}}(x)$ on $G'\subseteq G$, a contradiction. This proves condition (b) of Definition A3. To show that (12) consistently defines an aggregator mapping, it suffices to prove that, for all $x,y\in X$ and $\mathscr{G},\mathscr{H},\mathscr{I}\in\Phi$,

$$(16)\qquad (G\in\mathscr{G}\subseteq\mathscr{H}\cap\mathscr{I}\text{ and }U^{\mathscr{H}}(x)=U^{\mathscr{I}}(y)\text{ on }G)\Rightarrow(U^{\mathscr{G}}(x)=U^{\mathscr{G}}(y)\text{ on }G).$$

We assume that $\mathscr{G}=\mathscr{H}\cap\mathscr{I}$; (16) follows from this special case by coherence. Suppose that $U^{\mathscr{H}}(x)=U^{\mathscr{I}}(y)$ on $G\in\mathscr{G}$. Then $d_n^{\mathscr{H}}(x)=d_n^{\mathscr{I}}(y)$ on G for every n. It follows that, for each n, there exist disjoint (and possibly null) events in $\mathscr{G}$, G_0^n, G_1^n, and G_2^n, whose union is G, and such that $d_n^{\mathscr{H}}(x)=d_n^{\mathscr{I}}(y)=i$ on G_i^n. We claim that $G'=\{x\succ^{\mathscr{G}}y\}\cap G$ is null. To see that, suppose that G' is not null. Since Z is $\mathscr{A}$-dense, $G''=\{x\succeq^{\mathscr{G}}z_n\succeq^{\mathscr{G}}y\}\cap G'$ is non-null for some n. But it is not hard to check that this leads to a contradiction on each of the events $G''\cap G_i^n$, $i\in\{0,1,2\}$. Similarly, $\{y\succ^{\mathscr{G}}x\}\cap G$ is null. Using (b) of Definition A3, we conclude that $U^{\mathscr{G}}(x)=U^{\mathscr{G}}(y)$ on G. This completes the proof of Theorem A2(a).

(b) We assume that $\Phi=\{\mathscr{F}\}$; part (b) follows easily from this case. Suppose that U is a state-contingent utility representing $\succeq=\succeq^{\mathscr{F}}$. Let $\mathscr{K}$ denote the set of bounded open intervals with

[11] If V and W are random variables, F is an event, and R is a relation on $\mathbb{R}$ (such as $=$, $\geq$, or $>$), we write "*VRW on F*" if $F\setminus\{VRW\}$ is null.

rational end points. Using Lemma B1, one can show[12] that, for each $I \in \mathscr{I}$, there exists a sequence $\{z_n^I\} \subseteq X$, and a countable partition $\{F_0^I, F_1^I, \dots\}$ of Ω such that $U(z_n^I) \in I$ on F_n^I for all n, and $\{U(x) \in I\} \cap F_0^I$ is null for all x. Let $Z_1 = \bigcup_{I \in \mathscr{I}} \{z_n^I\}$. We now define what will be the remaining elements of an $\mathscr{A}$-order dense set. Let D consist of all the triples $(x, y; F)$, where $(x, y) \in X^2$, and F is an event such that $\{x \succ y\} \cap F$ is non-null but $\{x \succ z \succ y\} \cap F$ is null for all $z \in X$. We claim that there exists a sequence $C = \{(x_n, y_n)\} \subseteq X^2$ such that, for each $(x, y; F) \in D$, $\{x_n \sim x \succ y \sim y_n\} \cap F$ is non-null for some n. Letting $Z_2 = \{z : (z, x) \in C$ for some $x \in X\}$, it then follows without much difficulty that $Z_1 \cup Z_2$ is a countable $\mathscr{A}$-order dense set. Finally, we show the existence of C, assuming $D \neq \varnothing$ (otherwise the result is trivial). Given $(x, y) \in X^2$, we can apply Lemma B1, with a non-null event F being dark if $(x, y; F) \in D$, to obtain the black-and-white decomposition: $\Omega = B(x, y) \cup W(x, y)$. Consider now the finite measure space $(\Omega \times \mathbb{R}, \mathscr{F} \times \mathscr{B}, \mu)$, where $\mathscr{B}$ is the Borel σ-algebra on $\mathbb{R}$, and $\mu = P \times l$, where l is any finite measure on $\mathbb{R}$ equivalent to Lebesgue measure. For each $x, y \in X$ we define the set $S(x, y) = \{(\omega, r) : \omega \in B(x, y),\ U(x, \omega) > r > U(y, \omega)\}$. The sets $S(x, y)$ are disjoint up to μ-null sets, and therefore at most a countable number of them can be of positive μ measure, say $S(x_1, y_1), S(x_2, y_2), \dots$. Letting $C = \{(x_1, y_1), (x_2, y_2), \dots\}$, the desired properties follow. *Q.E.D.*

PROOF OF THEOREM A3: We will show that Theorem A3 is a consequence of Theorem 2 in Skiadas (1996a). Given any event, G, let $\mathscr{G}$ be the algebra generated by a finite partition that contains G and at least three essential events. Let N be the union of all nonessential events in $\mathscr{G}$. Applying Theorem 2, we can obtain an additive representation $(U^{\mathscr{G}}, P^{\mathscr{G}})$ for the restricted preference family $\{\succeq^G : G \in \mathscr{G}\}$, so that $U^{\mathscr{G}}$ is a conditional utility given $\mathscr{G}$ and $P^{\mathscr{G}}(N) = 0$. Since X is compact, and, for every $\omega \notin N$, $U^{\mathscr{G}}(\omega, \cdot)$ is continuous, the range of the latter is a closed interval. By a change of measure and appropriate choice of an affine transformation, we can assume that this interval is $[0, 1]$. Moreover, this condition uniquely determines the choice of $(U^{\mathscr{G}}, P^{\mathscr{G}})$ up to nonessential events. We now define $U(x, G) = \int_G U^{\mathscr{G}}(x)\, dP^{\mathscr{G}}$. It is easy to see that $U(x, G)$ does not depend on the particular choice of $\mathscr{G}$ used above. For any disjoint events F, G, and any act x, let $F \sim^x G \Leftrightarrow U(x, F) = U(y, G)$. Theorem 2 of Skiadas (1996a) now applies, and the result follows. Alternatively, we could only use, from Skiadas (1996a), Step 5 of the proof of Theorem 1 and the proof of Theorem 2. *Q.E.D.*

PROOF OF THEOREM A4: By an appropriate change of measure and choice of an affine transformation, we can and do assume that $U(\bar{x}) = 1 - U(\underline{x}) = 1$ a.s. This is without loss in generality. Let X^0 consist of all *simple* acts, that is, acts of the form $x = \sum_{n=1}^N m_n 1_{F_n}$, where $m_n \in M$ for all n, and $\{F_1, \dots, F_N\}$ is a partition of Ω. A standard exercise shows that X^0 is dense in X. Our plan is to show the result on X^0, and then use a continuity argument to extend it to the whole of X.

LEMMA B2: *Given any $z \in X$ (respectively, X^0) and $v \in (0, 1)$, there exists a countable partition of Ω, $\{F_n^{z,v}\}$, and a sequence $\{x_n^{z,v}\} \subseteq X$ (respectively, X^0), such that $u(x_n^{z,v}) = v$ and $x_n^{z,v} = z$ on $F_n^{z,v}$, for every n.*

PROOF: Given (z, v), we use Lemma B1 with a non-null event F being light if the statement of the lemma is true with F in place of Ω, and dark otherwise. This gives a black-and-white decomposition: $\Omega = B \cup W$. We claim that B is null, which proves the lemma. Suppose that B is non-null. We will show an $x \in X$ (respectively, X^0) such that $u(x) = v$ and $\{x = z\} \cap B$ is non-null, contradicting the fact that B is black. By the nonatomicity assumption, there is a decreasing sequence $\{F_n\}$ of non-null subevents of B with a null intersection. Given such a sequence, let $\bar{x}_n = z 1_{F_n} + 1_{\Omega \setminus F_n}$ and $\underline{x}_n = z 1_{F_n}$. Since $\{\bar{x}_n\}$ ($\{\underline{x}_n\}$) converges to $\bar{x}$ ($\underline{x}$), and u is continuous with $u(\bar{x}) = 1$ ($u(\underline{x}) = 0$), it follows that for some n, $u(\bar{x}_n) > v > u(\underline{x}_n)$. Fixing this n, and for every

[12] Let a non-null event F be light if and only if the claim is true with F in place of Ω, and with F_0^I null. Then let F_0^I be the black part of Ω.

$\alpha \in (0,1)$, let $x^\alpha = \alpha \bar{x}_n + (1-\alpha)\underline{x}_n$. Then $u(x^\alpha)$ varies continuously with α, and achieves values both above and below v. Therefore, for some α, $x = x^\alpha$ has the required properties. Notice that if z is in X^0, so is x^α. *Q.E.D.*

We define $f: \Omega \times M \times I \to \mathbb{R}$ as follows. First, we let $f(\omega, m, 1) = 1$ and $f(\omega, m, 0) = 0$ for all $(\omega, m) \in \Omega \times M$. For any given $(m, v) \in M \times (0,1)$, we choose events $F_n^{m,v}$ and acts $x_n^{m,v}$, $n = 1, 2, \ldots$, as in Lemma B2, where we have abused notation in denoting by m the act with constant payoff $m \in M$. We then define $f(\omega, m, v) = U(\omega, x_n^{m,v})$ for all $\omega \in F_n^{m,v}$. With f so defined, one can easily check, using (4), that $U(x) = f(x, u(x))$ for all $x \in X^0$, and that f has all the stated monotonicity properties.

We now show that $U(x) = f(x, u(x))$ for all $x \in X$. For fixed $x \in X$, let $\Omega^* = \{U(x) < f(x, u(x))\}$ and $\Omega_* = \{U(x) > f(x, u(x))\}$. We are to show that Ω^* and Ω_* are both null. Suppose that Ω^* is non-null. As in the proof of Lemma B2, the nonatomicity of P implies the existence of a non-null event $F \subseteq \Omega^*$ such that $u(1_F) \le u(x)$. We now fix such an event F, and show that $\int_F U(x)\,dP \ge \int_F f(x, u(x))\,dP$, a contradiction. Let $\{x_n\}$ be a decreasing sequence in X^0 converging to x almost surely (one exists because x is bounded and measurable).

LEMMA B3: *There exists a sequence $\{\tilde{x}_n\}$ in X^0, converging almost surely to an act $\tilde{x}$, such that $u(\tilde{x}_n) = u(\tilde{x}) = u(x)$ and $\tilde{x}_n = x_n$ on F, for all n.*

PROOF: For every $\alpha \in [0,1]$, let $x_n(\alpha) = x_n 1_F + \alpha 1_{\Omega \setminus F}$. By (4), u is nondecreasing. Therefore, $u(x_n(1)) \ge u(x) \ge u(1_F) \ge u(x_n(0))$, where the second inequality is by the choice of F. Since $u(x_n(\alpha))$ is continuous in α, the set $A_n = \{\alpha : u(x_n(\alpha)) = u(x)\}$ is nonempty and closed, for every n. Let $\alpha_n = \max A_n$ and $\tilde{x}_n = x_n(\alpha_n)$. Since $\{x_n\}$ is decreasing, $x_{n+1}(\alpha_n) \le \tilde{x}_n$, and therefore $u(x_{n+1}(\alpha_n)) \le u(\tilde{x}_n) = u(x)$, implying that $\alpha_{n+1} \in [\alpha_n, 1]$. This shows that $\{\alpha_n\}$ is nondecreasing and therefore has a limit $\tilde{\alpha}$. Letting $\tilde{x} = x 1_F + \tilde{\alpha} 1_{\Omega \setminus F}$, the result follows. *Q.E.D.*

Lemma B3 and the monotonicity of f in the payoff argument imply that

$$\int_F f(x, u(x))\,dP \le \int_F f(x_n, u(x))\,dP = \int_F f(\tilde{x}_n, u(\tilde{x}_n))\,dP = \int_F U(\tilde{x}_n)\,dP.$$

(The last equality holds because $\tilde{x}_n \in X^0$.) Letting $n \to \infty$, we obtain

$$\int_F f(x, u(x))\,dP \le \int_F U(\tilde{x})\,dP = \int_F U(x)\,dP,$$

where we have used the continuity of $\int_F U dP$, and (4). This shows that Ω^* is null. A symmetric argument shows that Ω_* is also null. The remaining claims of Theorem A4 are immediate. *Q.E.D.*

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