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Aggregation, Determinacy, and Informational Efficiency for a Class of Economies with Asymmetric Information

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We identify and analyze a class of economies with asymmetric information that we call quasi-complete. For quasi-complete economies we determine equilibrium trades, show that the set of fully informative equilibria is a singleton, and give necessary and sufficient conditions for the existence of partially informative equilibria. Besides unifying some familiar settings, such as those of Grossman (1976) and Milgrom and Stokey (1982), the following new results are proved: (a) The same restrictions that deliver Gorman aggregation under symmetric information are sufficient for Gorman aggregation under asymmetric information, even under partially informative prices; (b) the traditional assumptions of quadratic utilities and endowment spanning that result in the CAPM under symmetric information deliver a conditional CAPM under asymmetric information with prices that need not be fully informative; (c) the linear equilibrium in Grossman's (1976) model is the only equilibrium (linear or not), while minor changes in the normality assumptions result in indeterminacy and partially informative equilibria; and (d) if there is no aggregate endowment risk, asymmetrically informed agents with common priors sell the risky part of their endowment in every equilibrium. *Journal of Economic Literature* Classification Numbers: D82; G14; C62. © 1998 Academic Press

1. INTRODUCTION

In this paper we identify and analyze a class of economies with asymmetric information that we call quasi-complete. Quasi-complete economies have many of the properties commonly associated with complete markets, but unlike the latter they may support equilibria that do not perfectly aggregate agents' private information. They encompass some basic models

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of asset pricing theory that are well understood in the symmetric information case, but not under asymmetric information. For quasi-complete economies, we will characterize equilibrium trades, we will prove that the set of fully informative[1] equilibria is a singleton, and we will give necessary and sufficient conditions for the existence of partially informative equilibria.

Besides providing a unified framework for some familiar results, such as the no-trade theorem of Milgrom and Stokey [26], the theory of quasi-complete economies delivers several new results. A principal application is the theory of demand aggregation in the sense of Gorman [14], but under asymmetric information and prices that may not perfectly aggregate private information. Special cases include a new version of the CAPM of Sharpe [34] and Lintner [23] that incorporates asymmetric information without distributional assumptions, and the model of Grossman [15, 16] with exponential utilities and normally distributed payoffs and signals. Resolving a long-standing question, we will show that Grossman's economy admits a unique equilibrium (linear or not), while minor changes in the normality assumptions result in partially informative equilibria and indeterminacy, without the assumption of noise traders. Analogous results will also be shown under lognormality, and agents with constant relative risk aversion. In another instance of a quasi-complete economy, we argue that if there is no aggregate endowment risk in the economy, risk averse agents will always sell the risky part of their endowment, no matter what private information they receive.

Our general setting is that of possibly incomplete Arrow–Debreu markets with a single good, and asymmetrically informed agents with rational expectations. Within this setting, quasi-complete economies are quite special and are essentially defined by two properties. The first property is that all equilibrium allocations are interim efficient given private information and the information revealed by prices. The second property is the existence of Arrow–Debreu prices, or "equivalent martingale measures" (EMM). Interim efficiency will allow us to compute equilibrium trades given any equilibrium prices, while EMM existence will allow us to characterize all equilibrium prices as equilibria in a fictitious risk-neutral economy, independently of equilibrium trades. While interim efficiency has been used in the context of no-trade theorems (see, for example, the exposition of Fudenberg and Tirole [13]), its use in computing non-zero equilibrium trades appears to be new. Moreover, our technique

[1] A fully informative equilibrium is one in which agents act as if they possessed all other agents' pooled information, even if this information is not in fact available to them in equilibrium. In some widely cited papers, such as Milgrom and Stokey [26], or much of the Finance literature, "fully revealing" is used instead of "fully informative." Following Radner [31], we use the term "fully revealing" to describe an equilibrium in which agents can invert prices to determine the state of the world.

differs significantly from Grossman's [15] standard method of computing an equilibrium assuming full revelation and then arguing that equilibrium prices do in fact act as a sufficient statistic of aggregate information. For example, Madrigal and Smith [24] recently used Grossman's approach to show the existence of a fully informative equilibrium for a type of quasi-complete economy admitting Gorman aggregation. Grossman's method, however, only works in the fully informative case, while in this paper we characterize all equilibria, fully informative or not.

The equilibrium definition we use throughout the paper is essentially the standard rational expectations equilibrium (REE) notion: Agents choose optimal trades, taking prices as given and using all of the information provided by their private signals and prices. Unlike much of the REE literature, however, we do not make the additional assumption that prices are a deterministic function of the pooled signals. Instead, we allow for prices to depend upon extrinsic "noise," *not* observed by any of the agents. Without an explicit model of the price formation process, it seems unnatural to rule out the possibility of such noise. Of course, the expanded definition of REE prices makes this paper's results stronger, with the only exception being the necessity of our conditions for the uniqueness of the fully informative equilibrium. While our equilibrium notion results in a more elegant formulation in this regard, we also provide alternative results on necessity that require prices to be deterministic functions of the pooled signals, as in the traditional REE literature.[2]

The REE literature for general abstract economies has focused mainly on the question of existence, and is surveyed by Jordan and Radner [21], and Allen and Jordan [3]. Our approach is quite different than the arguments surveyed by Allen, Jordan, and Radner, and neither set of results dominates the other. For example, while the abstract REE literature relies on genericity arguments involving the relative dimensionality of price and signal spaces, the latter play no role in our arguments. On the other hand, our technique applies to a very special type of economy. Indeed, our main application assumes that agents have linear risk tolerance (LRT) with the same coefficient of marginal risk tolerance, a condition familiar from the theory of Gorman aggregation. Essentially, we show that quasi-complete economies have many of the properties of complete markets, even though they can support partially informative equilibria.

Another distinction that needs to be emphasized is that available general existence results consistent with our REE definition concentrate on fully revealing prices, that is, prices that can be inverted to reveal the realization of all private signals. A fully informative equilibrium is one that remains an

[2] Since the first draft of this paper circulated, Dutta and Morris [12] have independently considered a version of this paper's more general definition of REE prices.

equilibrium if all information is made public. That is not the same as saying that in equilibrium agents do in fact know what other agents know. So while fully revealing prices are fully informative, the converse need not be true (also see footnote one). Abstract existence proofs of partially revealing REE have invariably modified the underlying REE definition in significant ways. Special parametric models of partially informative REE, similar to Grossman's [15] model but with noise traders are surveyed by Admati [1]. These are models for which a partially informative linear REE is known to exist, but equilibrium determinacy is still unresolved. Robust examples of partially revealing REE without noise traders have been constructed by Ausubel [5], and Polemarchakis and Siconolfi [29]. Finally, numerical techniques for computing equilibria under asymmetric information, including those of quasi-complete economies, are discussed by Bernardo and Judd [7].

The remainder of this paper is organized in four sections. In Section 2 we introduce the primitives of the model, and we define the REE equilibrium notion. In Section 3 we define quasi-complete economies, and we develop their basic properties. Section 4 is devoted to economies with "linear risk tolerance," a special type of a quasi-complete economy admitting Gorman aggregation, with the CAPM being a special case. Finally, Section 5 discusses the closely related questions of equilibrium uniqueness, and existence of partially informative equilibria.

2. RATIONAL EXPECTATIONS EQUILIBRIUM

We consider a one-period competitive economy with n asymmetrically informed agents, receiving random endowments at time one in terms of a single consumption good. At time zero there are $m+1$ securities available for trade, with specified time-one payoffs in terms of the single good. Each agent chooses an optimal trade conditionally on private information, as well as information that can be inferred from the equilibrium security prices. This section provides the formal details of such a model, and defines the relevant rational expectations equilibrium notion.

2.1. *Uncertainty*

Uncertainty is represented by the probability space $(\Omega, \mathscr{F}, P)$, and the σ-algebra $\mathscr{R} \subseteq \mathscr{F}$ that should be thought of as all economy-relevant information. That is, $\mathscr{R}$ describes all uncertainty regarding security payoffs, beliefs, preferences, endowments, and private information sets. For now, the reader may choose to simply assume that $\mathscr{R} = \mathscr{F}$. We will discuss the role of irrelevant information below and later in the paper. The probability P specifies null events, and beyond that need not represent any beliefs. For example, if $\mathscr{F}$ is finite, P can be arbitrary up to null events. For an infinite $\mathscr{F}$, P will be used to express "infinite sums" as an expectation operator, denoted E. The term *a.s.* (almost surely) should be interpreted to mean "with P-probability one."

State-contingent consumption, asset payoffs, and prices, will be restricted to lie in a linear space, L, of random variables.[3] For a finite state space, which is sufficient for most of the economic intuition of the paper, one can safely assume that L consists of all random variables. For an infinite state space, required to discuss Grossman's [15] and other examples, we will need to place further technical integrability restrictions on the elements of L. The space L is ordered as usual ($X \geqslant Y$ if and only if $X \geqslant Y$ a.s.), and is assumed to contain the constant random variables, and to have the property that $|X| \geqslant |Y|$ and $X \in L$ implies $Y \in L$.

We formally define a *prior* to be any probability, Q, that is equivalent[4] to P, and whose density with respect to P, dQ/dP, is square-integrable ($E[(dQ/dP)^2] < \infty$). The equivalence restriction means that we will only consider situations in which all agents agree on what events have probability zero, while the square-integrability restriction is purely technical. $\mathscr{P}$ denotes the set of all priors, and E^Q denotes the expectation operator with respect to $Q \in \mathscr{P}$.

2.2. *The Economy*

The economy is defined by the agent characteristics, and the security payoff structure. We now introduce these primitives in turn.

The agents of the economy are indexed by elements of the set $I = \{1, ..., n\}$. Agent $i \in I$ is characterized by the primitives $(P_i, u_i, \mathscr{S}_i, e_i)$, where $P_i \in \mathscr{P}$ is a prior representing the agent's beliefs, $u_i: \Omega \times C_i \to \mathbb{R}$ is a (possibly state-dependent) von Neumann–Morgenstern utility function, $\mathscr{S}_i$ is a σ-algebra representing the agent's private information, and e_i is a C_i-valued element of L representing the agent's time-one endowment. The set $C_i \subseteq \mathbb{R}$ is an open interval representing admissible consumption levels for agent i. (Typically, $C_i = \mathbb{R}_{++}$ or $C_i = \mathbb{R}$.) The expectation operator corresponding to P_i will be denoted E_i. We let $\mathscr{S} = \bigvee_{i \in I} \mathscr{S}_i$ denote the pooled agent information. (Formally, $\mathscr{S}$ is the smallest σ-algebra containing each of the $\mathscr{S}_i$.) The σ-algebra $\mathscr{S}_0 = \bigcap_{i \in I} \mathscr{S}_i$ represents information that is common knowledge as a result of agents observing their private information alone. The aggregate endowment is denoted $e = \sum_{i \in I} e_i$.

[3] A *random variable* is any $\mathscr{F}$-measurable, real valued function on Ω.

[4] Two probabilities P and Q, defined on $(\Omega, \mathscr{F})$, are *equivalent* if $P(F) = 0 \Leftrightarrow Q(F) = 0$ for all $F \in \mathscr{F}$. We write dQ/dP for the Radon–Nikodym derivative (or density) of Q with respect to P.

The following conditions will be assumed to hold for all $i \in I$, throughout the paper:

(a) dP_i/dP and e_i are $\mathscr{R}$-measurable, u_i is $\mathscr{R} \otimes \mathscr{B}(C_i)$-measurable, where $\mathscr{B}(C_i)$ denotes the Borel subsets of C_i, and $\mathscr{S}_i \subseteq \mathscr{R}$.

(b) Either $u_i(\omega, \cdot)$ is strictly increasing, differentiable, and strictly concave for all $\omega \in \Omega$, or $u_i(\omega, \cdot)$ is the identity function for all $\omega \in \Omega$.

(c) $E[u_i(e_i)^2] < \infty$, and $E[(u_i'(e_i)c)^2] < \infty$, for every $c \in L$.

Condition (a) is consistent with the interpretation of $\mathscr{R}$ as relevant information. Condition (b) implies that every agent is either strictly risk averse at all wealth levels, or risk neutral at all wealth levels. Condition (c) is purely technical. The notation should be interpreted as follows: $u_i'(\omega, x)$ denotes the derivative of $u_i(\omega, \cdot)$ at $x \in C_i$, $u_i(c)$, where $c \in L$, denotes the random variable $u_i(\cdot, c(\cdot))$, and analogously for $u_i'(c)$. A consequence of assumption (c) is that, for any $Q \in \mathscr{P}$, and any C_i-valued $c \in L$, $E^Q[|u_i(c)|] \equiv E[(dQ/dP)\,|u_i(c)|] < \infty$. (This follows from the gradient inequality and the Cauchy–Schwarz inequality.)

Time-one payoffs of the m risky securities are represented by the random vector $V = (V_1, ..., V_m) \in L^m$, naturally assumed to be $\mathscr{R}$-measurable. There is also a riskless bond that pays one unit of consumption at time one. Without loss of generality, time-zero security prices will be expressed in terms of the riskless bond, whose price is normalized to one. The asset payoffs complete the specification of the *economy* $\mathscr{E} = ((P_i, u_i, \mathscr{S}_i, e_i)_{i \in I}, V)$.

The set $\mathscr{R}$ can be taken to be the smallest σ-algebra with respect to which all these primitives are measurable, as detailed above. What is important for our purposes, however, is that, from the point of view of any agent, information that is stochastically independent of $\mathscr{R}$ is also stochastically independent of the primitives of the economy. Formally, we have

LEMMA 1. *Suppose that $\mathscr{G}$ is any sub-σ-algebra of $\mathscr{R}$ (for example, $\mathscr{S}$ or the trivial σ-algebra), and Q is any probability equivalent to P such that dQ/dP is $\mathscr{R}$-measurable (for example, $Q = P_i$). Then a random variable is conditionally P-independent of $\mathscr{R}$ given $\mathscr{G}$ if and only if it is conditionally Q-independent of $\mathscr{R}$ given $\mathscr{G}$.*

Proof. Given any random variable X, the assumption that dP/dQ is $\mathscr{R}$-measurable implies that, for any Borel set B,

$$P[X \in B \mid \mathscr{R}] = \frac{E^Q[(dP/dQ)\, 1_{\{X \in B\}} \mid \mathscr{R}]}{E^Q[dP/dQ \mid \mathscr{R}]}$$

$$= Q[X \in B \mid \mathscr{R}], \quad \text{a.s.}$$

If X is conditionally P-independent of $\mathscr{R}$ given $\mathscr{G}$, $P[X \in B \mid \mathscr{R}] = P[X \in B \mid \mathscr{G}]$ a.s. (recall that $\mathscr{G} \subseteq \mathscr{R}$), implying that $Q[X \in B \mid \mathscr{R}]$ is $\mathscr{G}$-measurable, and therefore $Q[X \in B \mid \mathscr{R}] = Q[X \in B \mid \mathscr{G}]$ a.s. This proves that X is conditionally Q-independent of $\mathscr{R}$ given $\mathscr{G}$. Reversing the roles of P and Q shows the converse. ∎

In the remainder of this section we define an equilibrium notion for the type of economy just formulated.

2.3. *Admissible Prices*

A *price vector*, p, is any element of L^m. The interpretation of p_j, $j \in \{1, ..., m\}$, is that of a price for risky security j. A price vector is *admissible* if it is conditionally independent of $\mathscr{R}$ given $\mathscr{S}$, and *pure* if it is $\mathscr{S}$-measurable. Independence here is taken with respect to the underlying probability P, which need not represent the beliefs of any agent. Lemma 1 shows, however, that the definition of admissibility would remain the same if we replaced P by any of the P_i. What we call here "pure prices" is a standard restriction in the literature of rational expectations equilibria, implying that prices cannot reveal more information than what is collectively known to the agents (see, for example, Kreps [22] and Radner [31]).

In this paper we will require equilibrium prices to be admissible, but not necessarily pure. Of course if $\mathscr{R} = \mathscr{F}$, that is, if all information is "relevant," a price is admissible if and only if it is pure, and our setting reduces to the standard one. Our only results that require $\mathscr{F}$ to be larger than $\mathscr{R}$ concern the necessity of our conditions for uniqueness discussed in Section 5 (and further in DeMarzo and Skiadas [10]). For these results, we also give alternative versions that apply with pure prices only.

Admissible but non-pure prices may reflect noise that is not part of the pooled signals. One can think of the "Walrasian auctioneer" as being allowed to set prices using independent randomizing devices whose outcomes are not observed by the agents. Without an explicit price formation mechanism, it seems unreasonable to rule out such randomization of prices. Notice, however, that noise in prices appears endogenously as part of the equilibrium. This is to be contrasted to exogenously specified noise in prices as in Anderson and Sonnenschein [4] and Allen [2]. Also, our equilibrium notion differs from the usual sunspot equilibrium notion (see Shell [35]). If prices are thought of as depending on sunspots, these sunspots are not observable by the agents.[5]

[5] Since the first version of this paper circulated, the paper by Dutta and Morris [12] appeared that has as its central theme a version of the randomized REE prices used here. Their paper can be profitably consulted for further related discussion and examples.

2.4. *Trade and Equilibrium*

A *trade* is any bounded[6] $\mathbb{R}^m$-valued random vector, and represents the number of shares traded in each of the risky securities. The number of riskless bonds traded is determined by budget feasibility. Given price vector p, $\theta \cdot p = \sum_{j=1}^{m} \theta_j p_j$ is the value of the risky part of the trade, while $-\theta \cdot p$ is the value of the riskless position. The overall time-one consumption for agent i with trade θ is therefore $e_i + \theta \cdot (V - p)$. Alternatively, one can think of the m risky securities as being forward (or futures) contracts, with p denoting the corresponding forward prices. Under this interpretation, there is no need for the riskless security.

Given price-vector p, agent i observes the information generated by p, as well as the private information $\mathcal{S}_i$. Accordingly, a trade, θ_i, is *feasible for agent i given price vector p* if it is $\mathcal{S}_i \vee \sigma(p)$-measurable,[7] and $e_i + \theta_i \cdot (V - p) \in C_i$ a.s. We let Θ_i^p denote the set of all trades that are feasible for agent i given p. A trade $\theta_i \in \Theta_i^p$ is *optimal for agent i given price vector p* if for every $\theta \in \Theta_i^p$,

$$E_i[u_i(e_i + \theta_i \cdot (V - p)) \mid \mathcal{S}_i, p] \geqslant E_i[u_i(e_i + \theta \cdot (V - p)) \mid \mathcal{S}_i, p] \quad \text{a.s.}$$

The usual first-order conditions for optimality of θ_i given prices p are

$$E_i[u_i'(e_i + \theta_i \cdot (V - p))(V - p) \mid \mathcal{S}_i, p] = 0 \quad \text{a.s.} \qquad \text{(FOC)}$$

By the gradient inequality, it follows that (FOC) is always sufficient for optimality, and for a finite state space, (FOC) is necessary as well. Technical regularity conditions can be formulated to ensure the necessity of (FOC) with an infinite number of states. A general set of such conditions, suitable for all our applications at once, would create too distracting a set of technicalities at this point. Instead, we make the first order conditions part of the equilibrium definition below. (In some cases, for example, in our extension of Grossman's [15] model, the equivalence of optimality and the first order conditions will be automatic.)

A *trade profile* is any n-tuple of trades. The short-hand notation (θ_i) will represent the trade profile $(\theta_1, ..., \theta_n)$. For any price vector p, the trade profile (θ_i) is *market-feasible given p* if $\theta_i \in \Theta_i^p$ for all i, and $\sum_{i \in I} \theta_i = 0$ a.s., the latter being the usual market clearing condition.

The equilibrium notion considered in this paper is defined as follows:

Definition 1. A rational expectations equilibrium (REE) of the economy $\mathcal{E}$ is a collection $((\theta_i), p)$, where p is an admissible price vector,

[6] A random vector, X, is *bounded* if $|X_i| \leqslant K$ a.s. for some $K \in \mathbb{R}$ and all i.

[7] Given any random vector X, the notation $\sigma(X)$ denotes the smallest σ-algebra with respect to which X is measurable.

and (θ_i) is a trade profile that is market feasible given p and satisfies the first order conditions (FOC). A REE of $\mathscr{E}$ is fully informative if it is also a REE of the economy $((P_i, u_i, \mathscr{S}, e_i)_{i\in I}, V)$, obtained from $\mathscr{E}$ by substituting the agents' pooled information for every agent's private information.

For simplicity, we will use the term "REE" without specifying the underlying economy, except when we are referring to an economy other than $\mathscr{E}$. A (*fully informative*) *REE price vector* is any price vector that is part of some (fully informative) REE. The term *partially informative* means not fully informative. In the REE literature, the qualification "fully revealing" is commonly applied to prices that fully reveal the agents' pooled information, $\mathscr{S}$. The notion of fully informative prices is weaker, since it only requires that prices act as a sufficient statistic of the agents' pooled information, even though agents may not in fact possess information $\mathscr{S}$ in equilibrium.

A main concern of this paper is the determinacy of REE. Given any REE price vector, the optimal consumption of a (strictly) risk averse agent is uniquely determined in equilibrium. The corresponding trade, financing this consumption, is uniquely determined only if there are no redundant assets. Risk neutral agents on the other hand expect zero profits in equilibrium, and their trades are only restricted by market feasibility. With this motivation, we call two REE, $((\theta_i), p)$ and $((\hat{\theta}_i), \hat{p})$, *equivalent* if $p = \hat{p}$ a.s. and $e_i + \theta_i \cdot (V - p) = e_i + \hat{\theta}_i \cdot (V - \hat{p})$ a.s. for every risk averse agent i. Clearly, we can only hope to prove uniqueness of a REE up to equivalence.

3. QUASI-COMPLETE ECONOMIES

The unifying property of economies we consider in this paper is that of "quasi-completeness," formally defined as follows:

DEFINITION 2. The economy $\mathscr{E}$ is *quasi-complete* if there exists a probability, Q, with the property: For any admissible price vector p, there exist random variables, λ_i^p, $i \in I$, and market feasible (under p) trade profile (θ_i^p), such that

$$\lambda_i^p u_i'(e_i + \theta_i^p \cdot (V - p)) = \frac{dQ}{dP_i} > 0 \quad \text{a.s.,} \qquad i \in I,$$

and, for each i, λ_i^p is $\mathscr{S}_i \vee \sigma(p)$-measurable, and $E[(\lambda_i^p \, dP_i/dP)^2] < \infty$.

If the λ_i^p were assumed to be deterministic, the above equations can be recognized as the familiar first-order conditions for ex ante Pareto optimality of the allocation $(e_i + \theta_i^p \cdot (V - p))$. As elaborated on below, the weaker measurability restrictions on the λ_i^p correspond to interim efficiency given the information revealed by prices.[8] That is, quasi-completeness requires that, for any admissible price vector p, there exists a market feasible trade profile (given p) that results in an interim efficient allocation given private information and the information revealed by p. (The notion of interim efficiency in more general settings is discussed by Holmström and Myerson [19], Hahn and Yannelis [17], and the references therein.) In addition, quasi-completeness requires that the probability Q appearing in the first-order conditions for interim efficiency can be chosen independently of the price vector p. A trivial case is when the initial allocation is interim efficient relative to the agents' private information (in which case we can take $\theta_i^p = 0$).

In this section we show that the probability Q of Definition 2 can be used to characterize REE prices, and that, given an REE price vector p, the trade profile (θ_i^p) of Definition 2 together with p form an REE. In the case of an interim efficient initial allocation, we recover a standard no-trade result, but we will also consider an example of a quasi-complete economy in which agents sell their initial endowment of risky assets. Our principal application, involving economies with linear risk tolerance, is the topic of Section 4.

3.1. *Equilibrium Trades and Interim Efficiency*

For quasi-complete economies, equilibrium trades can be characterized in terms of prices through the following result:

PROPOSITION 1 (Trade Theorem). *Suppose that $\mathscr{E}$ is quasi-complete, $((\theta_i), p)$ is a REE, and the trade profile (θ_i^p) is as in Definition 2. Then $((\theta_i^p), p)$ is also a REE, and therefore $((\theta_i), p)$ is equivalent to $((\theta_i^p), p)$.*

Proof. We call a trade profile, (θ_i), *interim efficient given price vector p*, if it is market-feasible given p, and there exists no trade profile (θ_i') that is market-feasible given p and satisfies, for all $i \in I$,

$$E_i[u_i(e_i + \theta_i' \cdot (V - p)) \mid \mathscr{S}_i, p] \geqslant E_i[u_i(e_i + \theta_i \cdot (V - p)) \mid \mathscr{S}_i, p] \quad \text{a.s.},$$

[8] Ex ante efficiency can be recovered in an alternative interpretation. One can think of the weights λ_i^p of Definition 2 as defining new priors Q_i^p, where $dQ_i^p/dP_i \propto \lambda_i^p$, under which the allocation $(e_i + \theta_i^p \cdot (V - p))$ is ex ante efficient. Replacing the original priors, P_i, with the priors Q_i^p results in the same equilibrium posterior beliefs given the REE price vector p. Therefore, every equilibrium of a quasi-complete economy is also an equilibrium of a quasi-complete economy (with different priors) in which the corresponding equilibrium allocation is ex ante efficient.

with at least one of the inequalities being strict with positive probability. We use this concept through the following observation:

TRADE LEMMA. *Suppose that p is a REE price vector. Then* $((\theta_i), p)$ *is a REE if and only if* (θ_i) *is interim efficient given p.*

Quasi-completeness and the gradient inequality implies that (θ_i^p) maximizes the central planner objective

$$E\left[\sum_{i=1}^{n} \frac{dP_i}{dP} \lambda_i^p u_i(e_i + \theta_i \cdot (V - p))\right]$$
$$= \sum_{i=1}^{n} E_i(\lambda_i^p E_i[u_i(e_i + \theta_i \cdot (V - p)) \mid \mathcal{S}_i, p]),$$

as (θ_i) ranges over the set of market-feasible trade profiles given p. Therefore, (θ_i^p) is interim efficient given p. Finally, the proposition follows from the trade lemma, and utility concavity for the risk averse. ∎

The logic of the trade lemma in this proof is familiar from the no-trade theorem of Milgrom and Stokey [26]. Milgrom and Stokey assumed ex ante efficiency, but, as Holmström and Myerson [19] and others have remarked, their proof only relies on interim efficiency. What is novel here is that we will use this argument to compute equilibria involving non-zero trades.

Quasi-completeness has much stronger implications than were used in Proposition 1. Given any admissible price vector p, if $\mathcal{E}$ is quasi-complete, then the allocation $(e_i + \theta_i^p \cdot (V - p))$ is interim efficient given p, in the sense that there exists no feasible allocation (c_i) such that $E_i[u_i(c_i) \mid \mathcal{S}_i, p] \geqslant E_i[u_i(e_i + \theta_i^p \cdot (V - p)) \mid \mathcal{S}_i, p]$ a.s., with at least one of the inequalities being strict with positive probability.[9] Allocative efficiency is a stronger condition than trade profile efficiency if markets are incomplete. Conversely, assuming a finite state space in order to avoid technicalities, if $(e_i + \theta_i^p \cdot (V - p))$ is interim efficient given p, then the first order conditions of Definition 2 must hold, but in general the measure Q will depend on the choice of p. Quasi-completeness further demands that the measure Q is the same for any choice of an admissible price vector p, clearly a strong assumption, which we will however show to be satisfied in a number of

[9] A *feasible allocation* is any $(c_i) \in L^n$, such that c_i is C_i-valued and $\sum_i c_i \leqslant e$. As in Proposition 1, the above claim follows by showing that $(e_i + \theta_i^p \cdot (V - p))$ solves the central planner objective $\sum_i E_i[\lambda_i^p u_i(c_i)]$ as (c_i) ranges over the set of feasible allocations. In a finite state space, all interim efficient allocations can be characterized as a solution to a central planner problem of this form (see Holmström and Myerson [19]).

interesting economies. The pricing implications of this assumption are discussed later in this section.

3.2. *Examples*

Below we present two simple instances of a quasi-complete economy. The first one is the context of a familiar no-trade theorem, while the second one involves non-zero equilibrium trades, and is apparently new.

EXAMPLE 1 (No-Trade Theorem). Assuming for simplicity a finite state space, the discussion of the last subsection implies that if the initial allocation, $(e_1, ..., e_n)$, is interim efficient given the agents' initial private information, then $\mathscr{E}$ is quasi-complete. In this context, Proposition 1 reduces to the well-known no-trade result of Milgrom and Stokey [26] (stated in terms of interim efficiency by Holmström and Myerson [19]). More generally, given an equilibrium allocation of a quasi-complete economy, Proposition 1 shows that if agents receive new asymmetric information and markets reopen, there will be no further trade by the risk averse.

Next we present an example in which the Trade Theorem (Proposition 1) is used to compute non-zero equilibrium trades.

EXAMPLE 2 (A Trade Result). We consider a common-prior economy in which individual endowments are traded, and either there is no aggregate endowment risk, or there is a risk neutral agent. In such an economy, we show that all risk averse agents sell the risky part of their endowment in equilibrium, even if prices are partially informative:

PROPOSITION 2. *Suppose that, for all $i \in I$,*

(a) *dP_i/dP is $\mathscr{S}_i$-measurable, and u_i is $\mathscr{S}_i \otimes \mathscr{B}(C_i)$-measurable;*

(b) *$e_i = a_i + b_i \cdot V$, where (a_i, b_i) is a $\mathscr{S}_i$-measurable bounded random vector;*

(c) *either $b \equiv \sum_{i \in I} b_i = 0$, or agent one is risk neutral and b is $\mathscr{S}_1$-measurable;*

(d) *marginal utilities and prior densities are bounded away from zero.*[10]

Then the economy $\mathscr{E}$ is quasi-complete, and every REE *is equivalent to one in which all the risk averse agents sell their endowment of risky assets.*

Proof. Given any price vector p, let the trade profile (θ_i^p) be defined by $\theta_i^p = -b_i$ for all $i > 1$, and $\sum_{i \in I} \theta_i^p = 0$. Market clearing implies that if $b = 0$, then $\theta_1^p = -b_1$. On the other hand, if $b \neq 0$, agent one is risk neutral, and

[10] That is, $\inf\{u_i'(\omega, x), dP_i/dP(\omega) \colon \omega \in \Omega, x \in C_i\} > 0$ for all i.

simply collects the aggregate risky endowment. The quasi-completeness condition of Definition 2 is satisfied with $\lambda_i^p = u_i'(a_i + b_i \cdot p)^{-1}(dP/dP_i)$ and $Q = P$, and the result follows from Proposition 1. ∎

Part (a) of the proposition's assumptions is satisfied if agents have a common prior and state-independent utilities. The measurability restriction on P_i implies that in any equilibrium the posterior beliefs of agent i would be the same whether his prior were P_i or P. Effectively, we are therefore assuming common priors. Part (b) states that endowments are tradeable, but endowed positions are only privately known. Part (c) states that either there is no aggregate endowment risk, or that there is a risk neutral agent with knowledge of the aggregate endowment of shares of the risky securities. Finally, part (d) is a purely technical strengthening of the assumptions of strict utility monotonicity and prior equivalence.

The above examples are extreme in that they either involve no trade, or full trade, no matter how risk averse agents are. In Section 4, we analyze a class of quasi-complete economies in which equilibrium trades depend on the agents' coefficients of risk aversion.

3.3. *Equivalent Martingale Measure*

So far we have explained that given a REE price vector p, the corresponding equilibrium trades in a quasi-complete economy can be determined by computing the interim efficient trades given p. In this subsection we use the fact that Q in the definition of quasi-completeness is independent of the choice of p to characterize REE prices in a way that is independent of trades, through the notion of an equivalent martingale measure (EMM).

Under symmetric information, the lack of arbitrage opportunities is known to imply the possibility of "risk neutral pricing." That is, we can regard equilibrium prices as being established in a fictitious economy in which all agents are risk neutral and have an artificial common prior Q. In the terminology of Harrison and Kreps [18], Q is an equivalent martingale measure (EMM). Alternatively, we can think of the density dQ/dP as an Arrow–Debreu state price density (provided of course that the riskless bond is the numeraire, as we assume throughout this paper).

Applying the same idea in our setting of asymmetric information, an "EMM" Q should satisfy the first order conditions of optimality for risk neutral agents with prior Q, namely, $p = E^Q[V \mid \mathcal{S}_i, p]$ for every i. As the example of Duffie and Kan [11] shows, this set of equations need not have a solution in Q for every arbitrage-free equilibrium price vector p. Lack of arbitrage alone is not sufficient to guarantee the existence of an EMM under asymmetric information. For quasi-complete economies, however, an EMM always exists in equilibrium, even in the following stricter sense:

DEFINITION 3. An *Equivalent Martingale Measure* (EMM) is any probability, Q, satisfying the following conditions:

(a) For every admissible price vector p, p is a REE price vector if and only if $p = E^Q[V \mid \mathscr{S}_i, p]$ for all $i \in I$.

(b) For every REE price vector p, p is fully informative if and only if $p = E^Q[V \mid \mathscr{S}, p]$.

(c) Q is equivalent to P, dQ/dP is $\mathscr{R}$-measurable, and $E^Q[|V_j|] < \infty$ for all j.

Notice that if Q is an EMM and p is a REE price vector, it is also true that $p = E^Q[V \mid \mathscr{S}_0, p]$ and $p = E^Q[V \mid p]$. (Recall that $\mathscr{S}_0 = \bigcap_i \mathscr{S}_i$.) This follows by applying the operators $E^Q[\cdot \mid \mathscr{S}_0, p]$ and $E^Q[\cdot \mid p]$ on any of the equations in part (a) of the EMM definition.

EMM existence is an easy consequence of quasi-completeness:

PROPOSITION 3 (EMM Theorem). *Every quasi-complete economy has an* EMM. *In particular, the probability Q of Definition* 2 *is an* EMM.

Proof. Suppose that $\mathscr{E}$ is quasi-complete, p is an admissible price vector, and Q and (θ_i^p) are as in Definition 2. By Proposition 1, if p is a REE price vector, then $((\theta_i^p), p)$ is a REE. The corresponding first order conditions for agent optimality (FOC) and quasi-completeness imply that $p = E^Q[V \mid \mathscr{S}_i, p]$ for all i. Conversely, the latter equations and quasi-completeness imply (FOC), and therefore that $((\theta_i^p), p)$ is a REE. This confirms part (a) of Definition 3. The same arguments apply after substituting the pooled information, $\mathscr{S}$, for each agent's private information, confirming part (b) of Definition 3.

The first two properties of Q in part (c) of Definition 3 are immediate by construction. Finally, we confirm the integrability condition of Definition 3(c). Given any REE price vector p, let $c_i^p = e_i + \theta_i^p \cdot (V - p)$, and $\Omega_i = \{c_i^p \geqslant e_i\}$. Since $\sum_i c_i^p = \sum_i e_i$ a.s., Ω differs from $\bigcup_i \Omega_i$ by at most a null event. By utility concavity and quasi-completeness, we have, for any i, j,

$$E^Q[|V_j| \, 1_{\Omega_i}] = E_i[\lambda_i^p u_i'(c_i^p) \, |V_j| \, 1_{\Omega_i}] \leqslant E[(dP_i/dP) \, \lambda_i^p u_i'(e_i) \, |V_j| \, 1_{\Omega_i}] < \infty,$$

where the last inequality follows from assumption (c) of Section 2.2, and the Cauchy–Schwarz inequality. Adding up over i, we conclude that $E^Q[|V_j|] < \infty$. ∎

An immediate implication of this proposition is that to characterize REE prices of quasi-complete economies, it suffices to do so for the risk-neutral case, under a common prior. The latter task is undertaken in DeMarzo and Skiadas [10], where we show that, under risk neutrality, the set of fully informative REE prices is a singleton, and we give necessary and sufficient conditions for the existence of partially informative REE. Based on that analysis, below we discuss the existence of fully informative REE of quasi-complete economies, while the possible existence of partially informative REE is the topic of Section 5.

3.4. *Fully Informative REE*

An interesting corollary of our earlier discussion is that in a quasi-complete economy making agents better informed does not enlarge the set of equilibrium price vectors. More precisely, suppose that the economy $\mathscr{E}'$ is the same as $\mathscr{E}$, except that in $\mathscr{E}'$ the private information of agent i is given by $\mathscr{S}'_i$ where $\mathscr{S}_i \subseteq \mathscr{S}'_i \subseteq \mathscr{S}$. We claim that if $\mathscr{E}$ is quasi-complete, then the set of REE prices of $\mathscr{E}'$ is a subset of the set of REE prices of $\mathscr{E}$. This is true because if $\mathscr{E}$ is quasi-complete with EMM Q as given in Definition 2, then $\mathscr{E}'$ is also quasi-complete with the same EMM Q, since the measurability restrictions on the λ_i^p are only weakened if agents are better informed. Hence, if p is a REE price vector of $\mathscr{E}'$, $p = E^Q[V \mid \mathscr{S}'_i, p]$ for all i. Applying the operator $E^Q[\cdot \mid \mathscr{S}_i, p]$ on both sides, we obtain $p = E^Q[V \mid \mathscr{S}_i, p]$ for all i, and therefore p is also a REE price vector of $\mathscr{E}$.

In the extreme case of $\mathscr{S}'_i = \mathscr{S}$, this argument indicates that every quasi-complete economy has a fully informative equilibrium. More directly, we have the following result, based on an argument of DeMarzo and Skiadas [10]:

PROPOSITION 4. *Suppose that Q is an* EMM. *Then $p = E^Q[V \mid \mathscr{S}]$ is a fully informative* REE *price vector, and is almost surely equal to any fully informative* REE *price vector.*

Proof. By the law of iterated expectations, we have

$$E^Q[V \mid \mathscr{S}_i, E^Q[V \mid \mathscr{S}]] = E^Q[E^Q[V \mid \mathscr{S}] \mid \mathscr{S}_i, E^Q[V \mid \mathscr{S}]]$$
$$= E^Q[V \mid \mathscr{S}].$$

Since Q is an EMM, this shows that $E^Q[V \mid \mathscr{S}]$ is a REE price vector. The same argument with $\mathscr{S}$ in place of $\mathscr{S}_i$ shows that $E^Q[V \mid \mathscr{S}]$ is fully informative. Conversely, if p is a fully informative REE price vector, then $p = E^Q[V \mid \mathscr{S}, p]$, and, by Lemma 1, p is conditionally Q-independent of V given $\mathscr{S}$. Therefore, $p = E^Q[V \mid \mathscr{S}]$. ∎

The fully informative REE price vector may or may not be the unique REE price vector. We now give two simple examples, one in which the fully informative REE price vector is unique, and one (borrowed from DeMarzo and Skiadas [10]) in which a non-informative equilibrium exists along with the fully informative one. General necessary and sufficient conditions for the existence of partially informative REE of quasi-complete economies are discussed in Section 5.

EXAMPLE 3 (Unique REE). Suppose that one agent's private information is finer than any other agent's information. That is, $\mathcal{S}_i = \mathcal{S}$ for some i. If p is a REE price vector and Q is an EMM, we have $p = E^Q[V \mid \mathcal{S}_i, p] = E^Q[V \mid \mathcal{S}]$, where the second equality follows from the fact that p is Q-conditionally independent of V given $\mathcal{S} = \mathcal{S}_i$ (see Lemma 1). By Proposition 4, p is therefore the fully informative REE price vector. This proves that, in a quasi-complete economy, if one agent is better informed than all other agents, then the only possible REE price vector is the fully informative one.

EXAMPLE 4 (Partially Informative REE). Suppose that Q is an EMM, $n = 2$, $m = 1$, and that $\mathcal{S}_1$ and $\mathcal{S}_2$ are stochastically independent with respect to Q. The risky security payoff is $V = S_1 S_2$, where, for each $i \in \{1, 2\}$, S_i is a $\mathcal{S}_i$-measurable random variable that has zero mean under Q. Then $E^Q[V \mid \mathcal{S}_i] = 0$ for every i, and therefore $p = 0$ is a non-informative REE price. The fully informative price is of course $p = V$.

4. LINEAR RISK TOLERANCE

Having established the basic properties of quasi-complete economies, in this section we discuss an important parametric class of such economies, that we will refer to as LRT economies, since agents with linear risk tolerance is one of their basic attributes. In the symmetric information case, LRT economies have a long history in economic theory, because they are the class of symmetric information economies in which aggregation in the sense of Gorman [14] is possible. For LRT economies, we will utilize last section's results to derive closed form expressions for equilibrium trades as a function of prices, and for an EMM. Special cases include Grossman's [15] model and extensions, as well as a new version of the CAPM with asymmetric information. A main conclusion will be that Gorman aggregation is possible in LRT economies, even under asymmetric information and partially informative prices. A clear exposition of the corresponding theory for the symmetric information case is given by Magill and Quinzii [25].

Formally, we will consider the following class of economies, where some of the assumptions are a little stronger than needed for expositional simplicity:

DEFINITION 4. $\mathcal{E}$ is an LRT economy if the following conditions are satisfied:

(a) For every $i \in I$, dP_i/dP is $\mathscr{S}_i$-measurable.

(b) There exist constants, $\alpha_i \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, such that, for all $i \in I$,

$$C_i = \{x : \alpha_i + \beta x > 0\}$$

and

$$-\frac{u_i'(\omega, x)}{u_i''(\omega, x)} = \alpha_i + \beta x, \qquad (\omega, x) \in \Omega \times C_i.$$

(c) For every $i \in I$, $e_i = a_i + b_i \cdot V$, where (a_i, b_i) is a $\mathscr{S}_i$-measurable bounded random vector. Moreover, $a \equiv \sum_i a_i$ and $b \equiv \sum_i b_i$ are $\mathscr{S}_0$-measurable.

(d) If $\beta = 0$, $E[\exp(k\,|c|)] < \infty$ for all $k \in \mathscr{R}$ and $c \in L$.

Part (a) was discussed in Example 2. Part (b) is the assumption of *linear risk tolerance* (LRT), or *hyperbolic absolute risk aversion* (HARA). The constant β is the *coefficient of marginal risk tolerance* assumed to be common among agents. Part (c) states that endowments are tradeable, endowed positions are individually known, but the aggregate endowment of each asset is commonly known (recall that $\mathscr{S}_0 = \bigcap_i \mathscr{S}_i$). Part (d) is a purely technical integrability restriction.

The class of utilities with LRT can be expressed explicitly by integrating to obtain, up to a positive affine transformation,

$$u_i(c) = \begin{cases} \dfrac{1}{\beta - 1}(\alpha_i + \beta c)^{(\beta-1)/\beta} & \text{if} \quad \beta \neq 0 \quad \text{and} \quad \beta \neq 1; \\ \log(\alpha_i + c) & \text{if} \quad \beta = 1; \\ -\alpha_i \exp(-c/\alpha_i) & \text{if} \quad \beta = 0 \quad \text{and} \quad \alpha_i > 0. \end{cases}$$

Important special cases include constant absolute risk aversion ($\beta = 0$ and $\alpha_i > 0$), and constant relative risk aversion ($\beta \neq 0$ and $\alpha_i = 0$). Quadratic utility is obtained if $\beta = -1$.

Throughout this section, we assume that $\mathscr{E}$ is LRT, and we freely use the notation of Definition 4.

4.1. *Characterization of Equilibria*

By confirming quasi-completeness, in this section we prove that in every equilibrium of an LRT economy each agent holds a proportion of the aggregate endowment of risky assets equal to the ratio of the agent's equilibrium risk tolerance to the market's aggregate equilibrium risk tolerance. We also show that an EMM density is proportional to the marginal utility of a "representative agent" at the aggregate endowment.

Letting $\alpha = \sum_{i \in I} \alpha_i$, we fix a von Neumann–Morgenstern utility function $u: \{x \in \mathbb{R}: \alpha + \beta x > 0\} \to \mathbb{R}$ such that $-u'(x)/u''(x) = \alpha + \beta x$ for all x. This condition specifies u up to a positive affine transformation, and therefore uniquely specifies a preference relation over consumption plans, that we think of as the preferences of a representative agent. We also recall that $a = \sum_i a_i$ and $b = \sum_i b_i$, where (a_i, b_i) is the endowed portfolio of agent i. Our main conclusion on LRT economies can then be stated as follows:

PROPOSITION 5. *Suppose that $\mathscr{E}$ is an* LRT *economy with representative agent utility u, as specified above. Then $\mathscr{E}$ is quasi-complete, and the following are true*:

(a) *Every* REE *is equivalent to a* REE, $((\theta_i^p), p)$, *with*

$$\theta_i^p = \rho_i b - b_i, \qquad \text{where} \qquad \rho_i = \frac{\alpha_i + \beta(a_i + b_i \cdot p)}{\alpha + \beta(a + b \cdot p)}, \qquad i \in I.$$

(b) *$\mathscr{E}$ has an* EMM, *Q, given by $dQ/dP = u'(e)/E[u'(e)]$.*

Proof. Let (θ_i^p) and Q be as specified in the statement of the proposition. We will confirm the existence of corresponding agent weights, λ_i^p, so that the quasi-completeness condition of Definition 2 is satisfied. This will complete the proof, since Propositions 1 and 3 apply.

We use the fact that, up to a positive scaling factor (which does not affect our results), marginal utilities are given by

$$u_i'(c) = \begin{cases} (\alpha_i + \beta c)^{-1/\beta}, & \text{if } \beta \neq 0; \\ \exp(-c/\alpha_i), & \text{if } \beta = 0, \end{cases}$$

and analogously for the representative agent utility u. Direct algebra then shows that the quasi-completeness condition is satisfied with

$$\lambda_i^p E[u'(e)] \frac{dP_i}{dP} = \begin{cases} \rho_i^{1/\beta}, & \text{if } \beta \neq 0; \\ \exp((a_i + b_i \cdot p)/\alpha_i - (a + b \cdot p)/\alpha), & \text{if } \beta = 0. \end{cases}$$

There only remains to check integrability restrictions on λ_i^p and $u'(e)$. Suppose first that $\beta \neq 0$. Then $\lambda_i^p \, dP_i/dP$ is clearly square integrable, since $\rho_i \in (0, 1)$. Also, $u'(e)$ is square-integrable, which implies that $Q \in \mathscr{P}$. To see that, recall the assumption that $E[u_i'(e_i)^2] < \infty$ for every i (condition (c) of Section 2.2 with $c = 1$). Defining $\Omega_i = \{\alpha_i + \beta e_i \leqslant \alpha + \beta e\}$ if $\beta > 0$, and $\Omega_i = \{\alpha_i + \beta e_i \geqslant \alpha + \beta e\}$ if $\beta < 0$, we notice that $\Omega = \bigcup_i \Omega_i$. Therefore, $E[(\alpha + \beta e)^{-2/\beta} 1_{\Omega_i}] \leqslant E[(\alpha_i + \beta e_i)^{-2/\beta} 1_{\Omega_i}] < \infty$. Adding up over i, we obtain $E[(\alpha + \beta e)^{-2/\beta}] < \infty$, as claimed. The case of $\beta = 0$ is immediate by assumption (d) of Definition 4, since $p \in L^m$. ∎

The LRT conditions of Definition 4 can be weakened somewhat. In particular, Proposition 5 remains valid if we assume that the coefficients α_i and β are bounded random variables, provided α_i is $\mathscr{S}_i$-measurable, and α and β are $\mathscr{S}_0$-measurable (although the set C_i must now be taken to be state dependent). Finally, if $\beta=0$, Proposition 5 remains valid if the assumption that a is $\mathscr{S}_0$-measurable is removed, except that the expression for the EMM now becomes $dQ/dP \propto \exp(-b \cdot V/\alpha)$.

4.2. *Gorman Aggregation with Asymmetric Information*

An economy admits *Gorman aggregation* if equilibrium prices do not depend on how the aggregate endowment is initially allocated among agents. In his classic paper, Gorman [14] showed, in the context of an exchange economy, that this type of aggregation is possible if and only if asset demands are affine functions of wealth with the same slope, a condition that Pollak [30] fully characterized in terms of allowable utility forms. When Pollak's utilities are interpreted as von Neumann–Morgenstern utilities, they correspond to the LRT utilities of Definition 4, a fact recognized in the theory of fund separation initiated by Tobin [36], and extended by Cass and Stiglitz [9], and others. The theory of Gorman aggregation has been adapted to the Arrow–Debreu setting by Rubinstein [33], who only showed the sufficiency of LRT, and Brennan and Kraus [8] and Milne [27] who pointed out that LRT is a necessary condition for Gorman aggregation. A modern exposition is given by Magill and Quinzii [25], who also give more complete historical and bibliographical background than provided here.[11]

Our contribution to this literature is the generalization of the theory of Gorman aggregation to the case of asymmetric information. Assuming that $\mathscr{E}$ is LRT, Proposition 5 shows that $dQ/dP=u'(e)/E[u'(e)]$ defines an EMM, Q, and therefore prices do not depend on the way that e is initially allocated among agents. This becomes apparent after a change of measure, resulting in the "representative agent" pricing formulas:

$$p=\frac{E[u'(e)\,V\mid \mathscr{S}_i,\,p]}{E[u'(e)\mid \mathscr{S}_i,\,p]}, \qquad i\in I\cup\{0\},$$

and

$$p=\frac{E[u'(e)\,V\mid p]}{E[u'(e)\mid p]}.$$

[11] All of the above references consider economies without special distributional assumptions. The theory of fund separation and aggregation extends to more general preferences, under special distributional assumptions on asset payoffs. Details and further references can be found in Ross [32] and Ingersoll [20].

This type of representative agent pricing is familiar in the symmetric information case, and must clearly hold in a fully informative equilibrium. Here, we have proved the stronger result that, for a LRT economy, the above pricing equations must hold in equilibrium even if prices are partially informative. The fully informative REE price vector is given by $p = E[u'(e)\, V \mid \mathscr{S}] / E[u'(e) \mid \mathscr{S}]$.

Finally, an alternative way of stating our main conclusion is that, in a LRT economy, an admissible price vector p is a REE price vector if and only if, for every $i \in I$, p is a REE price vector of the representative agent economy $((P, u, \mathscr{S}_i, e), V)$. Moreover, if p is a REE price vector, then p is also a REE price vector of the representative agent economies $((P, u, \mathscr{S}_0 \vee \sigma(p), e), V)$ and $((P, u, \sigma(p), e), V)$.

4.3. *The CAPM with Asymmetric Information*

As a final application of Proposition 5, we show that the traditional formulation of the capital asset pricing model (CAPM) of Sharpe [34] and Lintner [23], with quadratic utilities and no distributional assumptions,[12] extends to the case of asymmetric information. Grossman's [15] model is known to generate a fully informative CAPM with asymmetric information, under exponential utilities and normally distributed asset payoffs and signals. Our formulation differs significantly in that it makes no distributional assumptions, and can support partially informative prices, according to the necessary and sufficient conditions of Section 5.

The reader familiar with the usual derivation of the CAPM under quadratic utilities will easily recognize that the pricing equations of the last subsection with u' linear imply a conditional CAPM. For completeness, we provide here a full statement of this observation. We use the notation $\mathrm{Cov}[X, Y \mid \mathscr{I}]$, to denote the conditional covariance of X and Y given $\mathscr{I}$, and analogously $\mathrm{Var}[X \mid \mathscr{I}]$ denotes the conditional variance of X given $\mathscr{I}$.

PROPOSITION 6. *Suppose that* $\mathscr{E}$ *is* LRT *with* $\beta = -1$ (*corresponding to quadratic utilities*), *and that* p *is a* REE *price vector such that, with probability one,* $p_j \neq 0$ *for all* j, *and* $a + b \cdot p \neq 0$. *Let the security excess returns,* $\mathscr{R} = (\mathscr{R}_1, ..., \mathscr{R}_m)$, *and the market excess return,* $\mathscr{R}^e$, *be defined by*

$$\mathscr{R}_j = \frac{V_j}{p_j} - 1, \qquad \mathscr{R}^e = \frac{a + b \cdot V}{a + b \cdot p} - 1.$$

[12] For the symmetric information case, the CAPM with expected utility maximizing agents requires either the type of assumptions made in this section, involving quadratic utilities, or the distributional assumptions discussed by Ross [32], or some mixture of these assumptions as discussed by Berk [6].

Then, for any σ-algebra $\mathcal{I} \in \{\mathcal{S}_1 \vee \sigma(p), ..., \mathcal{S}_n \vee \sigma(p), \mathcal{S}_0 \vee \sigma(p), \sigma(p)\}$, *we have*

$$E[\mathcal{R} \mid \mathcal{I}] = \frac{\mathrm{Cov}[\mathcal{R}, \mathcal{R}^e \mid \mathcal{I}]}{\mathrm{Var}[\mathcal{R}^e \mid \mathcal{I}]} E[\mathcal{R}^e \mid \mathcal{I}].$$

Proof. Proposition 5 applies with $u'(e) = k(\alpha - e)$ for some positive constant k. After some manipulation, the pricing equations of Section 4.2 reduce to

$$p = E[V \mid \mathcal{I}] - \frac{\mathrm{Cov}[e, V \mid \mathcal{I}]}{\alpha - E[e \mid \mathcal{I}]},$$

and therefore

$$a + b \cdot p = E[e \mid \mathcal{I}] - \frac{\mathrm{Var}[e \mid \mathcal{I}]}{\alpha - E[e \mid \mathcal{I}]}.$$

It follows that $E[\mathcal{R} \mid \mathcal{I}] = K\,\mathrm{Cov}[\mathcal{R}, \mathcal{R}^e \mid \mathcal{I}]$ and $E[\mathcal{R}^e \mid \mathcal{I}] = K\,\mathrm{Var}[\mathcal{R}^e \mid \mathcal{I}]$, for some $\mathcal{I}$-measurable and non-vanishing K. Dividing the first equation by the second one, the result follows. ∎

The conclusion of Proposition 6 states that, conditionally on all available information, every agent will find that the CAPM holds in equilibrium, and so will an outside observer with only public information, as well as an observer who only extracts information from prices. The following is an example of a fully informative CAPM equilibrium in which agents disagree on security betas with probability one.

Example 5 (CAPM with Asymmetric Betas). We adopt the assumptions of Proposition 6, with two agents ($n = 2$), two risky securities ($m = 2$), and $\alpha = 3$. For example, each agent's utility function can be given by $u_i(\omega, x) = -(1.5 - x)^2/2$, for $x < 1.5$. The aggregate endowment e is assumed to satisfy $E[e] = 1$ and $E[e^2] = 2$. The first agent has no private information ($\mathcal{S}_1$ is trivial), while the second agent observes a signal S (generating $\mathcal{S}_2$), which is assumed to be stochastically independent of the aggregate endowment, e. The security payoffs are $V_1 = e$ and $V_2 = (2e - 1)\,S + 1$. By the discussion of Example 3, the only REE price vector is the fully informative one, that is, $p = (1/2, 1)$. The first agent expects security two to pay off $E[V_2] = E[S] + 1$, while the second agent expects $E[V_2 \mid S] = S + 1$. If $S \neq E[S]$ a.s., then the two agents will disagree on the security's expected return with probability one. Since both agents agree on the market return, they will also disagree on the second security's beta. Prices in this example satisfy $p = E[V \mid \mathcal{S}_i, p] - (1/2)\,\mathrm{Cov}[e, V \mid \mathcal{S}_i, p]$.

The fact that agents agree on the difference of these two quantities, does not imply that they agree on their individual values.

5. UNIQUENESS AND PARTIALLY INFORMATIVE REE

For a quasi-complete economy, we have characterized equilibrium allocations in terms of prices, we have shown the existence of an EMM, and we have proved the existence of a single fully informative equilibrium. All this has been accomplished without any distributional assumptions. Distributional assumptions are important, however, for determining whether partially informative equilibria exist. DeMarzo and Skiadas [10] have developed necessary and sufficient conditions for REE uniqueness under risk neutrality. The existence of an EMM allows us to use these same conditions to characterize equilibrium uniqueness in quasi-complete economies. In this section we review some of the results in DeMarzo and Skiadas [10], as applying to the current context, and we demonstrate their use in some concrete cases. For example, we will show that Grossman's [15] linear equilibrium is the unique equilibrium under his assumptions, but that minor changes in the joint normality condition on payoffs and signals generate partially informative equilibria.

5.1. *Assumptions and the Basic Idea*

Throughout this section, it will be helpful to explicitly model the signals that generate the agents' private information. Formally, we assume that, for each i, $\mathcal{S}_i = \sigma(S_i)$, where S_i is a random variable valued in some measurable space $(A_i, \mathcal{A}_i)$. We let $A = A_1 \times \cdots \times A_n$, $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, and $S = (S_1, ..., S_n)$. In applications, we typically take each A_i to be either finite (with $\mathcal{A}_i$ being the power set of A_i), or the real line (with $\mathcal{A}_i$ being the usual Borel sets). We will see shortly that the key to REE determinacy, given the existence of an EMM Q, lies in properties of the function, $f: A \to \mathbb{R}^m$, giving the fully informative REE prices as a function of the pooled signals: $f(S) = E^Q[V \mid S]$ a.s.

Another standing assumption that we are going to make throughout this section is that there exists a random variable with continuous cumulative distribution function that is independent of $\mathcal{R}$ (with respect to P). This assumption is without loss of generality, since we can always enlarge the underlying probability space in order to satisfy it. On the other hand, it rules out the case $\mathcal{R} = \mathcal{F}$, which we have allowed so far, and there will always be admissible prices that are not pure. For this reason, we will explicitly state results that apply specifically to pure prices.

The basic idea behind the results of this section can be explained briefly as follows. Let us assume, for simplicity, that $V = f(S)$ a.s. for some $f\colon A \to \mathbb{R}^m$, A is finite, and $P[S = s] > 0$ for all $s \in A$. Given any admissible price vector p, and EMM Q, consider the system of (in)equalities:

$$Q[p \neq f(S)] > 0 \qquad \text{and} \qquad p = E^Q[f(S) \mid S_i, p], \qquad i \in I,$$

stating that p is a partially informative REE price vector. Clearly, this system is linear in Q. One can therefore apply a standard duality (or separating hyperplane) argument to prove that no probability, Q, solves the above system, for any given p, if and only if f is "separably oriented" (SO) in a sense to be defined shortly. This result is useful to the extent that if we can show f to be SO, then the fully informative equilibrium is the unique equilibrium (up to equivalence). Conversely, if f is not SO, then for some choice of agent priors, there exists a partially informative pure REE.

More can be claimed if non-pure prices are allowed. Suppose that f is not SO, and therefore there exists some partially informative REE price vector, $\hat{p}$, if the agent priors are $(\hat{P}_i)$, not necessarily the priors, (P_i), we are interested in. Utilizing properly weighted coin tosses (one, for each outcome of S), it is then possible to mix $\hat{p}$ with the fully informative prices, $f(S)$, to obtain a partially informative REE under the original priors (P_i). The idea is that an agent with prior P_i faced with the information revealed by (S_i, p) would form the same posterior beliefs as an agent with prior $\hat{P}_i$ faced with the information revealed by $(S_i, \hat{p})$. This argument shows that f being SO is necessary and sufficient for REE uniqueness in quasi-complete economies, the necessity part relying on A being finite. With an infinite A, REE uniqueness in quasi-complete economies is equivalent to the "approximately separably oriented" (ASO) condition, a slight weakening of the SO condition (defined in DeMarzo and Skiadas [10]), designed to take care of the empty interior problems that arise with duality arguments in infinite-dimensional spaces.

In the remainder of this section, we discuss in more detail the SO condition and its application to quasi-complete economies. We also introduce the related "overlapping diagonals" (OD) condition, which is necessary for f to be SO, and serves as a convenient tool for proving the existence of partially informative REE.

5.2. *The "Separably Oriented" Condition*

We begin with a formal statement of the SO condition, which, in a sense explained above, is a dual condition to REE uniqueness for quasi-complete economies.

DEFINITION 5. The function $f: A \to \mathbb{R}^m$ is *separably oriented* (SO) if there exist product-measurable and bounded functions, $g_i: A_i \times \mathbb{R}^m \to \mathbb{R}^m$, such that

$$f(s) \neq r \quad \Rightarrow \quad (f(s) - r) \cdot \sum_{i \in I} g_i(s_i, r) > 0, \qquad (s, r) \in A \times \mathbb{R}^m.$$

The term "separably oriented" is motivated by the geometric interpretation of the condition: For any given reference point r, the vector $f(s) - r$ forms an acute angle with the vector $\sum_i g_i(s_i, r)$.

PROPOSITION 7. *Suppose that Q is an* EMM, *and $f: A \to \mathbb{R}^m$ is such that $f(S) = E^Q[V \mid S]$ a.s. If f is* SO, *then every* REE *price vector is almost surely equal to $f(S)$. Conversely, if f is not* SO, *A is finite and $P[S = s] > 0$ for all $s \in A$, then the following are true*: (a) *there exists a partially informative* REE, *and* (b) *for some prior profile, $(\hat{P}_i) \in \mathscr{P}^n$, the economy $((\hat{P}_i, u_i, \mathscr{S}_i, e_i)_{i \in I}, V)$ has a partially informative pure* REE.

Proposition 7 is a direct consequence of Lemma 1 and results in DeMarzo and Skiadas [10], based on the ideas outlined in the last subsection. The sufficiency part of the proof is straightforward, and worth reviewing. Suppose that f is SO, Q is an EMM, and p is a REE price vector. By Lemma 1, p is conditionally Q-independent of V given S. Therefore,

$$\begin{aligned} p &= E^Q[V \mid S_i, p] = E^Q[E^Q[V \mid S, p] \mid S_i, p] \\ &= E^Q[f(S) \mid S_i, p], \qquad i \in I. \end{aligned}$$

If f is SO, letting the functions g_i be as in Definition 5 results in the condition $E^Q[(f(S) - p) \cdot g_i(S_i, p)] = 0$ for all i. Adding up over i, it follows that the SO condition can only be valid if $p = f(S)$ a.s. The only REE price vector is therefore the fully informative one.

As an illustration, f is trivially SO in the context of Example 3, making uniqueness in that example a corollary of Proposition 7. In the following subsection, we will confirm the SO condition in Grossman's and related examples by exploiting the ordinal additivity of f. We call a real-valued function on A *ordinally additive* if it is of the form $k(\sum_{i \in I} h_i(s_i))$, $s \in A$, for some measurable and bounded functions $h_i: A_i \to \mathbb{R}$ and strictly increasing $k: \mathbb{R} \to \mathbb{R}$. Clearly, if every coordinate of f is ordinally additive, then f is SO, but the converse is false even for a real-valued f.

In the case of an infinite A, the SO condition is not necessary for uniqueness. In DeMarzo and Skiadas [10] we prove that, given EMM existence, REE uniqueness is equivalent to f being "approximately separably oriented" (ASO), a slightly weaker version of the SO condition. If A is finite and $P[S = s] > 0$ for all $s \in A$, then f is SO if and only if it is ASO, but we also give an example of a function f (with infinite domain) that is ASO but not SO.

5.3. *The Grossman and Related Examples*

As a source of interesting applications, we now consider a LRT economy $\mathscr{E}$ (see Definition 4), with the following specifications:

A1 (Exponential Utilities). $u_i(\omega, x) = -\alpha_i \exp(-x/\alpha_i)$, $(\omega, x) \in \Omega \times \mathbb{R}$ $(\beta = 0)$.

A2 (Conditional Normality). The conditional distribution of V given S (under P) is normal with mean $\mu(S)$, for some $\mu: A \to \mathbb{R}^m$, and variance-covariance matrix $\Sigma(S)$, for some $\Sigma: A \to \mathbb{R}^{m \times m}$.

Using the EMM expression of Proposition 5, we can explicitly compute

$$E^Q[V \mid S] = \frac{E[(dQ/dP)\, V \mid S]}{E[dQ/dP \mid S]}.$$

Letting $r = (\sum_i \alpha_i)^{-1}$, a standard formula for the expectation of the exponential of a normal variable gives

$$\begin{aligned} E\left[\frac{dQ}{dP}\,\middle|\, S\right] &= E[\exp(-rb'V) \mid S] \\ &= \exp\left(-rb'\mu(S) + \frac{r^2}{2}\, b'\Sigma(S)\, b\right) \quad \text{a.s.}, \end{aligned}$$

where b is treated here as a column vector with transpose b'. Differentiating both sides with respect to b, and rearranging, we obtain $E^Q[V \mid S] = f(S)$ a.s., where

$$f(s) = \mu(s) - r\Sigma(s)\, b, \qquad s \in A.$$

This simple expression allows us to check whether f has the properties required for REE uniqueness.

Our next proposition resolves the question of uniqueness in Grossman's [15, 16] model. While Grossman, and in greater generality Nielsen [28], have shown uniqueness within the class of linear equilibria, the possibility of nonlinear equilibria has been up to now an open question. Interestingly, the following proof of uniqueness only relies on the easy part of Proposition 7, whose proof was given above.

PROPOSITION 8. *Suppose that* $\mathscr{E}$ *is* LRT *with exponential utilities (as specified in* A1), *and* (V, S) *is jointly normally distributed. Then every* REE *is equivalent to the fully informative* REE.

Proof. Suppose that (V, S) has mean (μ_V, μ_S) and variance-covariance matrix

$$\begin{pmatrix} \Sigma_{VV} & \Sigma_{VS} \\ \Sigma_{SV} & \Sigma_{SS} \end{pmatrix},$$

with the natural interpretation of the components (assumed to be constants). Using standard formulas for normal distributions, we find that A2 is satisfied with

$$\mu(s) = \mu_V + \Sigma_{VS} \Sigma_{SS}^{-1} (s - \mu_S)$$

and

$$\Sigma(s) = \Sigma_{VV} - \Sigma_{VS} \Sigma_{SS}^{-1} \Sigma_{SV}, \qquad s \in A.$$

Hence $\mu(s)$ is additive in s, $\Sigma(s)$ is constant in s, and therefore $f(s) = \mu(s) - r\Sigma(s) b$ is also additive in s. This implies that f is SO, except for the fact that the functions g_i of Definition 5 are not bounded. Given our distributional assumptions, however, the relevant part of the proof of Proposition 7 given earlier still applies, and uniqueness follows. (Alternatively, one can easily confirm that f is ASO in the sense of DeMarzo and Skiadas [10].) ∎

Another interesting special case arises with power or logarithmic utilities and a log-normal conditional distribution of payoffs given the signals. For a brief illustration, consider a LRT economy with the following specializations: (a) $\alpha_i = 0$ for all i, and $\beta > 0$; (b) there is only one risky asset ($m = 1$), and the riskless bond is in zero net supply ($a = 0$); (c) the conditional distribution of $\log(V)$ given S is normal with mean $\mu(S)$ and variance $\sigma^2(S)$. We also normalize the risky asset supply by letting $b = 1$. A calculation analogous to the one for the exponential utility case shows that if Q is an EMM,

$$f(S) = E^Q[V \mid S] = \exp\left(\mu(S) + \left(\frac{1}{2} - \frac{1}{\beta}\right) \sigma^2(S)\right), \qquad \text{a.s.}$$

In particular, if $(\log(V), S)$ is normally distributed, then μ is a linear function of S, while σ does not depend on S. The analogous argument used in

Proposition 8, utilizing the ordinal additivity[13] of f, proves REE uniqueness (up to equivalence) in this case.

The above arguments reveal the role of normality assumptions in delivering (ordinal) additivity of the function f, and hence REE uniqueness. On the other hand, there still remains the question of whether alternative distributions lead to partially informative equilibria or indeterminacy. In our closing subsection, we show that the answer is yes, by utilizing the "overlapping diagonals" (OD) condition.

5.4. *The "Overlapping Diagonals" Condition*

We conclude with a condition on a function $f: A \to \mathbb{R}$ that is necessary for f to be SO, and under regularity assumptions, it is also necessary for f to be ASO. We will then use this condition to construct robust examples of indeterminate partially informative REE of LRT economies.

Given any $x, y \in \mathbb{R}^n$ and $J \subseteq I$, the notation (x_J, y_{-J}) denotes the vector z with $z_i = x_i$ for $i \in J$, and $z_i = y_i$ for $i \notin J$.

DEFINITION 6. The function $f: A \to \mathbb{R}$ has *overlapping diagonals* (OD) if

$$\max\{f(x), f(y)\} \geqslant \min\{f(x_J, y_{-J}), f(y_J, x_{-J})\}, \qquad x, y \in A, \qquad J \subseteq I.$$

The term "overlapping diagonals" is suggestive of the following geometric picture: Imagine the rectangle formed by the four points x, y, (x_J, y_{-J}), and (y_J, x_{-J}), and for each diagonal consider the interval defined by the values of f at the end points of the diagonal. The two intervals overlap for any choice of $x, y \in A$ and $J \subseteq I$, if and only if f has OD. Example 4 is a simple illustration of a function f that does not have OD, making the existence of partially informative REE there a corollary of Proposition 9 below.

In DeMarzo and Skiadas [10] we prove that if $f: A \to \mathbb{R}$ is SO, then it has OD. We also show that if A is a Euclidean space, $f: A \to \mathbb{R}$ is ASO and continuous, and S has a diffuse distribution, then f has OD. (We call a measure, π, on a Euclidean space *diffuse* if every set of π-measure zero has Lebesgue measure zero.) As a corollary, we have the following result on the role of the OD condition in proving the existence of partially informative REE.

PROPOSITION 9. *Suppose that* $m = 1$, Q *is an* EMM, $f: A \to \mathbb{R}$ *is such that* $f(S) = E^Q[V \mid S]$ *a.s., and that either one of the following regularity conditions are valid*:

[13] Strictly speaking, f is not ordinally additive, because of unbounded components in the additive representation. As in Proposition 8, however, the same proof goes through under the present distributional assumptions.

(a) *A is finite and* $P[S=s]>0$ *for all* $s\in S$.

(b) $A_i=\mathbb{R}$ *for all i, f is continuous, and the distribution of S is diffuse.*

If f does not have OD, *then there exist partially informative* REE.

Remark. As with Proposition 7, the partially informative REE in the conclusion of Proposition 9 may not be pure. Conclusion (b) of Proposition 7, however, is also valid under the assumptions of Proposition 9.

For a concrete instance of this result, consider a LRT economy satisfying A1 (exponential utilities) and A2 (conditional normality), as well as $m=1$, $n=2$, $\mu(s)=s_1+s_2$, and $\Sigma(s)=(s_1+s_2)^2$. If $2rb>1$ (r and b being constants), $f(s)=\mu(s)-r\Sigma(s)b$ violates the OD condition for $x=(1, 1)$ and $y=(-1, -1)$. This proves the existence of partially informative REE. Moreover, this violation of the OD condition is robust to small perturbations of the parameters, and therefore so is the existence of a partially informative REE.

Analogous examples of the existence of partially informative REE can be easily constructed in the context of power utilities and payoffs that are log-normally conditionally distributed given the agent signals, as outlined in the last subsection, or in the context of the CAPM.

DeMarzo and Skiadas [10] provide further results on the OD condition, showing, for example, that in certain two-agent economies the OD condition is sufficient for REE uniqueness without even the assumption of quasi-completeness.

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