Theoretical Foundations of Asset Pricing: Addenda and Errata*

Costis Skiadas

February 7, 2025

Addenda

Proposition 3.2.6.

The proposition's proof has been omitted, since it is straightforward given the hints preceding the proposition's statement. Here it is spelled out.

(1) Suppose $x^i \in \mathcal{D}^i(c^i)$ for all i and let $x \equiv \sum_i x^i$. By Lemma 3.2.4, c^i is Π-optimal and therefore $\Pi(x^i) > 0$ for all i, which implies $\Pi(x) > 0$. Since Π is a present-value function, $x \notin X$.

(2) Suppose $\mathcal{D}(c)$ is convex and (X, c) is an effectively complete market equilibrium. By Proposition 3.1.8(2), there exists a present-value function Π such that c is Π-optimal, and therefore c^i is Π-optimal for every agent i (as shown following Lemma 3.1.7). Let us now confirm the properties defining (Π, c) as an Arrow-Debreu equilibrium. By assumption, c clears the market. Since $c^i - e^i \in X$, $\Pi(c^i - e^i) = 0$ and therefore $\Pi(c^i) = \Pi(e^i)$. Finally, if $x^i \in \mathcal{D}^i(c^i)$ then $\Pi(x^i) > 0$ (by the Π-optimality of c^i) and therefore $\Pi(c^i + x^i) > \Pi(c^i) = \Pi(e^i)$.

(3) In the discussion following Lemma 3.2.4, we saw that for any positive linear functional Π on $\mathcal{L}$, (Π, c) is an Arrow-Debreu equilibrium if and only if (X^Π, c) is an equilibrium, where $X^\Pi \equiv \{x \in \mathcal{L} \mid \Pi(x) = 0\}$. Since X is complete, $X = X^\Pi$, where Π is the unique present-value function for X.

Corollary 3.2.7.

The first part of the corollary is equivalent to

Proposition. If (X, c) is an equilibrium and X is a complete market, then the allocation c is optimal given X.

The proposition follows from the corollary by letting $X = \bar{X}$. Conversely, the proposition implies the first part of Corollary 3.2.7, since optimality of c given $\bar{X}$ implies optimality given any market included in $\bar{X}$.

Here is another proof of the above proposition that does not rely on the use of a present-value function or an Arrow-Debreu equilibrium. Suppose (X, c) is an equilibrium, X is a complete market, and $x^i \in \mathcal{D}^i(c^i)$ for all i.

We are to show that $x \equiv \sum_i x^i \notin X$. We assume instead that $x \in X$ and reach a contradiction. Since X is complete, there exist $\delta^i \in \mathbb{R}$ and $y^i \in X$ such that $x^i = \delta^i 1_{\Omega \times \{0\}} + y^i$. If $\delta^i < 0$ then, by preference monotonicity, $y^i \in \mathcal{D}^i(c^i)$, contradicting the optimality of c^i for $\mathcal{D}^i$ given X. Therefore $\delta^i \geq 0$ for all i, and $x = \delta 1_{\Omega \times \{0\}} + y$, where $\delta \equiv \sum_i \delta^i \geq 0$ and $y \equiv \sum_i y^i \in X$. If $\delta > 0$, then $x - y$ is an arbitrage, contradicting the fact that X is arbitrage-free (by individual optimality and preference monotonicity). Under the assumption $x \in X$, we have shown that $\sum_i \delta^i \geq 0$ and $\delta^i \geq 0$ for all i, which implies $\delta^i = 0$ and therefore $x^i = y^i \in \mathcal{D}^i(c^i) \cap X$, again contradicting individual optimality. Therefore $x \notin X$.

Errata

1. **Section A.4, page 196, line 1.** $x \in (0, \infty)^N$ should be $x \in \mathbb{R}^N$.
2. **Section B.4, page 218, four lines prior to Theorem B.4.1.** "Note that a functional $f : X \to \mathbb{R}$ is linear..." should read: "Note that a linear functional is both concave and convex."