Influencing Waiting Lists

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Abstract

In many settings, randomly arriving objects are offered successively to agents according to their fixed order in a waiting list. Though it is costly to wait, an agent may prefer to decline an offered object in order to wait for a better one (e.g. as in perhaps the most significant of such environments, the assignment of donor organs to queued patients). We consider the welfare consequences of influencing such accept/decline decisions. Examples of influence could be: encouraging agents to accept lower quality objects than they otherwise would; disallowing the right to defer offered objects; disallowing agents in certain waiting list positions to accept certain types of objects; etc. We consider a general, abstract definition of “influence” capturing these examples as special cases.

We show that such influence necessarily leads to weakly Pareto-dominated outcomes: influencing behavior in any one position in the list cannot improve the expected continuation payoff to any single position in the waiting list. The same conclusion holds under (CARA) risk-aversion. These results also can be viewed as generalizations of a (reinterpreted) optimization result for the problem of assigning jobs to a set of parallel processors.

We also consider the interests of a party averse to uncertainty in an agent’s waiting time (e.g. a treatment provider whose efficiency

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improves with better prediction). We show that such a party *would* benefit from influencing agents earlier in the waiting list to accept more objects (of any quality), even though this would *not* change the former agent’s average wait. Hence the interests of such a party would not be aligned with agents’ welfare. Our work is relevant to current policy discussions on the excessive time that harvested organs spend being offered to and declined by agents early in the queue, and the subsequent wasteful spoilage of such organs (NY Times (2012)).

1 A Brief Introduction

Many goods are rationed on a first-come-first-served basis through the use of waiting lists. Though waiting lists are ubiquitous, their use is particularly prevalent in the allocation of certain goods on behalf of a society or government: the allocation of transplant organs to patients, public housing to those in need, openings in treatment programs for drug abusers, positions in publicly run nurseries, etc. In such applications we are concerned not only with the *efficiency* of resource allocation, but also with the distribution of welfare in general.1

To illustrate the potential tradeoff between efficiency and distributional concerns consider the allocation of transplant organs, where a consequence of using waiting lists is spoilage. Patients awaiting organ transplants are prioritized to a waiting list based on various patient characteristics.2 When a donor organ arrives, it is sequentially offered to agents according to this priority order, each of whom may decline the current offer. In the case of cadaveric kidneys, for example, this results in waste: some organs that could have benefitted patients later in the waiting list end up spoiling due to the time spent processing offers to patients earlier in the list.3 Phrased differently, the *object-deferral option* exercised by agents earlier in the waiting list—the right to reject an offered organ and wait for one of better quality—results in an efficiency loss due to a form of transaction costs.

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1In contrast, when waiting lists are used in the private sector (e.g. positions in privately owned nursing homes or daycare facilities), the user’s primary objective might simply be revenue maximization.

2Queues for kidneys, livers, and other organs could be based on health, age, objective probability of survival, time of arrival to the queue, etc.

3In the case of cadaveric kidneys, see the article of Sack (2012) in the NY Times.
While there are various potential solutions to such a problem, each solution also impacts the distribution of agents’ welfare in different ways. Imagine a policymaker inquiring about the following (of the many) ways of addressing this problem.

- “Should we encourage patients relatively early in the waiting list to accept organs more frequently?”
- “Should we disallow the deferral option, requiring the first patient in the list to take any acceptable organ?”
- “Should we reserve organs of certain qualities specifically for patients in later list positions?”

While these alternatives could plausibly improve some measure of efficiency in organ allocation, each would also affect the welfare of individual patients in multiple ways. For example, under the first of the above alternatives, patients earlier in the list are somehow influenced to accept organs they otherwise would have declined, which imposes a welfare loss on those agents. A second (positive) welfare effect follows from the increased rate of organ allocation, speeding up the average rate at which patients later in the list progress through it. A third effect, however, comes from the fact that those same later agents may arrive at earlier positions on the waiting list and find their own acceptance decisions being influenced, providing another negative effect. (Similar opposing effects would exist for the policymaker’s other alternatives listed above.) Hence even if such solutions improve efficiency in some form, each of them leaves us with the the distributional question of who wins and who loses.

Our objective in this paper is to consider the welfare-distribution effects of an arbitrary exertion of influence over agents’ deferral decisions when they are offered objects under some waiting list priority ordering. Objects of varying quality arrive randomly over time and are offered sequentially to agents in the list for immediate assignment.\(^4\) Agents are assumed to make (arbitrarily fixed) deferral decisions based on their position in the waiting list. The generality of our class of deferral decisions—defined as waiting list

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\(^4\)Implicitly, the objects are non-storable, such as transplant organs. Furthermore they do not “spoil” as in our motivating example. Such spoilage can be modeled as a constraint as we hope to show in the forthcoming Subsection 6.2.
policies in Subsection 2.1—captures the above policymaker’s hypothetical alternatives as special cases.

It is important to note that we do not initially concern ourselves with how agents arrive at these deferral decisions. They could arise as equilibrium behavior in a model of rational, forward looking agents; indeed the decisions resulting from such behavior end up playing an important role in our results. Alternately these decisions could be the result of behavior that has somehow been influenced as in the examples above. Furthermore there could be many ways of implementing influence over an agent’s deferral decisions, such as simple coercion through legal constraints (i.e. the second and third of the policymaker’s solutions above), motivating physicians to give particular kinds of advice to their queued patients, partially concealing information about the object being allocated, etc. Our results are derived regardless of why agents end up making the decisions they do, hence these specifications do not appear in our model.

A second important point regarding our model has to do with the question of why a policymaker might wish to influence agents’ deferral decisions. When agents are queued and waiting for arriving objects, one can think of there being two major welfare effects that follow when agents have deferral options: the right to defer an object and wait for a better one. The first, which is the one we concern ourselves with here, involve timing and consumption externalities. When agents earlier in the waiting list accept, say, only some set of “better” objects, agents later in the list of wait longer to have access to those types of objects (but also wait less time for “worse” objects). Thus the earlier agents’ decisions impose externalities on the later agents. A second effect could arise through coordination inefficiency, an issue addressed extensively by Leshno (2014). To put it roughly, suppose $A$-type and $B$-type agents are waiting for $A$- and $B$-type objects to arrive. Social efficiency would dictate some degree of correlation in the assignment of object types to agent types. However, in a waiting list environment where agents can defer objects at will, such correlation need not occur naturally.

We wish to address the former externality-type effect in isolation from the latter coordination inefficiency. Therefore, in order to draw our conclusions as cleanly as possible, we eliminate this coordination inefficiency from our model by removing the horizontal differentiation of objects that appears, for instance, in Leshno (2014), assuming instead that agents have identical tastes over the (vertically differentiated) objects.
1.1 Overview of Main Results

By combining our first results (Section 3) we demonstrate that, holding all else constant, the exertion of influence on agents’ acceptance/deferral decisions harms welfare on a position-by-position basis. The simplest statement of this result advocates for a laissez-faire approach. Namely, by exerting no influence on agents’ deferral decisions, we obtain a probability distribution over outcomes that (ex ante and position-by-position) *Pareto-dominates* any other distribution of outcomes that could be achieved through some policy of influencing agents’ decisions. Phrased differently, under a policy of influencing the behavior of agents in any fixed position of the waiting list, the expected payoff to an agent in any other position of the list must (weakly) decrease. This considerably strengthens ideas in related literature (described below) that decentralized, selfish behavior in such settings maximizes total payoffs to agents in some pre-existing queue.

In fact the results imply the following stronger conclusion. Imagine that agents who reach some early position $k$ in the waiting list are somehow influenced to accept objects of marginal quality that, otherwise, only would have been accepted by agents in some later position $\ell > k$. Corollary 1 implies that such a policy change has precisely no effect on the expected payoff to an agent who begins in any position $\ell$ or later. Since the effect for an agent in position $k$ is clearly negative (and the same is quickly shown for positions $k < k' < \ell$), the Pareto-dominance result above follows.

These results have implications for the organ spoilage problem mentioned earlier. One solution to that problem might be to encourage the “agents” to be less selective about their acceptance of “objects,” thereby reducing the chance that an object is offered too many times (and spoils). Our results show that any efficiency gains from such encouragement would be partially offset by welfare losses of this type, to the agents currently in the waiting list.

Next we imagine a party that is averse to uncertainty in any particular agent’s waiting time (e.g. a treatment provider whose efficiency improves with better prediction, or an insurance company, etc.). We show that such a party *would* benefit from a policy in which agents earlier in the waiting list are influenced to accept objects of marginally lower quality than they otherwise would accept; this is true even though such influence *would not* change the original agent’s expected waiting time. At the extreme, a party who wished only to minimize the variance of waiting time of an agent starting in position
A series of related papers starting with Agrawala et al. (1984) are motivated by a seemingly unrelated application. Specifically, they minimize the total processing time for a set of jobs of unknown (exponentially distributed) lengths using a set of processors of varying speeds. Agrawala et al. derive the optimal policy for sequentially assigning jobs to a set of available, parallel processors. As we explain in Section 7, this problem yields a special case of our model in which the value of an object is determined exactly by its arrival rate. Their result can then be reinterpreted in our context as an optimal *utilitarian* policy.\footnote{Coffman et al. (1987) consider the same setup with the object of minimizing the *makespan* (total wait and processing time for the last job to finish). In the context of welfare analysis this would be directly interpreted as a Rawlsian criterion.} Furthermore, they observe that if the jobs were selfish individuals, then this policy would be the equilibrium outcome of a game where the queued jobs can sequentially decide whether to utilize a newly available processor or wait for a better opportunity. Both of these results are
corollaries of our main results. Kumar and Walrand (1985) show that the
results of Agrawala et al. also hold for more general probability distributions,
and with job arrivals.

In a paper much more closely related to ours, Su and Zenios (2004) ex-
plicitly model the assignment of arriving kidneys to a waiting list of strategic
patients using (uninfluenced) first-come-first-served priorities. Extending the
idea mentioned above, they show that (when the set of patients is already
queued and waiting) equilibrium behavior in the induced game maximizes
the expected sum of agent payoffs. On the other hand, this equivalence is
known to break down when agents may continue to arrive to the waiting list.

Work as early as Naor (1969) has observed that when agents are decid-
ing whether to join a waiting list with costly waiting, decentralized (selfish)
behavior need not be socially optimal. The agent who joins the queue fails
to internalize the waiting cost he imposes on any future agents that might
arrive. A similar effect can occur in our model: when agents are arriving, it
could improve total payoffs if the last agent in the waiting list is assigned an
object he would selfishly consider marginally unacceptable, but that would
have been accepted by an additional agent in the next position in the queue.\footnote{If another agent arrives shortly after this object has been discarded, the social planner regrets not having that object available immediately for assignment, and hence regrets not having assigned it to the agent who was already present.}

Hassin (1985) concisely explains that this problem can be solved by
switching the priority structure to a last-come-first-served protocol, which
gives priority to agents in the reverse order of their arrival. This forces each
arriving agent to internalize the probabilistic arrival of future agents, induc-
ing them to reduce their acceptance thresholds to socially optimal levels. Su
and Zenios (2004) demonstrate this in their kidneys model, estimating the
hypothetical welfare gains from using such a method. They also point out,
as does Hassin, that such LCFS methods are manipulable in various ways,
increase risk to the agents, are inequitable, and are unlikely to be politically
acceptable.

Within the economic literature there has been a recent surge of interest
on this topic and closely related ones, as an offshoot of the now large lit-
erature on static assignment and matching. As we have mentioned above,
Leshno (2014) considers the discrete-time arrival of two types of objects to
queued agents with heterogeneous preferences. With the objective of reduc-
ing the coordination inefficiency we mentioned earlier under the constraint
of incentives, he considers optimizing the buffer policy, i.e. commitment to a method of allocating to agents who have already rejected an offered object that need not respect the original priority order of the waiting list.

Bloch and Cantala (2014) also consider the discrete-time arrival of objects, but without the persistent preferences of Leshno. They analyze equilibria under a mechanism that probabilistically offers arriving objects to agents, where earlier agents in a waiting list have a greater chance of receiving offers. The mechanism designer’s leverage here is to give agents a disincentive ever to arrive at early position in the waiting list by giving them less than a 100% probability of being offered the next arriving object.

With the motivation of a public housing application, Thakral (2014) examines a type of school choice model in which school slots (i.e. apartments) arrive stochastically in discrete time periods, with inherent priorities over the waiting agents. Thakral proposes a multiple wait list procedure, in which an arriving apartment “proposes” to its highest priority agent, which in turn gives that agent the option to either take the apartment, or join a waiting list for any other single apartment type. This idea, incorporated with the “you want my house I get your turn” concept in Abdulkadiroğlu and Sönmez (1999), yields a strategyproof mechanism with a desirable efficiency property that also respects the apartments’ priority orders.

In the kidney exchange model of Ünver (2010), agent and objects arrive in pairs, where the agents have implicit property rights over their initial endowments. With the motivation of organ compatibility, certain trades are feasible (or more desirable than others). The analysis covers both pairwise and multi-way exchanges.

If we generalize the concept of arriving agents and objects to arriving agents and agents, we obtain the related, burgeoning literature on dynamic 2-sided matching. Doval (2014) considers stability in 2-sided matching when agents who arrive in different periods may postpone their arrivals. Akbarpour et al. (2014) analyze the limit behavior of a market in which agents randomly arrive (and depart) a market to be pairwise matched, comparing mechanisms that do and do not assign agents immediately upon arrival.

2 Model

There is a finite set of agents $\mathcal{A}$. Each agent has unit demand over indivisible objects which arrive randomly over time.
There is a set of object types $\mathcal{O} = \{1, 2, \ldots, n\}$.\(^7\) Objects of any particular type arrive according to a Poisson process, independently of the arrivals of other types. Formally, for each $i \in \mathcal{O}$, the time between arrivals of $i$-type objects is exponentially distributed with parameter $\mu_i > 0$ (i.e. with mean $1/\mu_i$), and the arrival times are independent of the arrival of times of object types in $\mathcal{O} \setminus \{i\}$.\(^8\)

An object of type $i \in \mathcal{O}$ provides value of $v_i \in \mathbb{R}$ to an agent (if any) who consumes it; hence agents are homogeneous. To simplify the statement of some results, we assume that values are distinct ($i \neq j$ implies $v_i \neq v_j$); otherwise two nominal types could be combined into a single real type without loss. Furthermore we order the labels so that $v_1 > v_2 > \cdots > v_n$.

Finally, agents have identical, linear waiting costs. If an agent is assigned an $i$-type object after waiting $w$ units of time, then the agent’s total payoff is $v_i - w$. Ex ante, both $v_i$ and $w$ could be uncertain for a queued agent. We initially assume that agents are risk-neutral in total payoff, and consider constant absolute risk aversion in Section 5. From our use of independent Poisson processes, at any point in time past waiting costs can be viewed as sunk, so we often evaluate expected continuation payoffs rather than total payoffs. We anticipate no confusion from this.

### 2.1 Waiting List Policies

One of the examples considered in our introduction was the idea of a policymaker asking “Should we encourage agents relatively early in the waiting list to accept objects more frequently?” In a world of heterogeneous agents, the implementation of such influence would need to be tailored to the specific assumptions about what the planner knows about the agents’ respective types, along with their relative (heterogenous) preferences for various types of objects.\(^9\) Since we are motivated (as in the Introduction) to model homogenous agents, however, it is natural to consider “consistent” influence, in the sense that an agent’s acceptance decision is influenced only as a function of his current position in the queue. This motivates the following definition.

\(^7\)While we assume $\mathcal{O}$ is finite, the results easily extend to cases where $\mathcal{O}$ is countably infinite or a continuum.

\(^8\)Obviously an equivalent specification of the model is that generic object arrival times are exponentially distributed with parameter $\sum \mu_i$, and an independent random process determines that each object’s type is $i$ with probability $\mu_i / \sum_{j \in \mathcal{O}} \mu_j$.

\(^9\)See Leshno (2014).
Definition 1. A waiting list policy is a monotonic function $W$ mapping queue positions into nonempty sets of object types, i.e. $W: \mathbb{N} \rightarrow 2^O$, where $k < \ell$ implies $W(k) \subseteq W(\ell)$ and where $W(1) \neq \emptyset$.

The definition should be interpreted as follows. An agent in position 1 of the waiting list is to be allocated the next object type to arrive that belongs to $W(1)$. If such an object arrives, it is assigned to this agent, who leaves, and every other agent moves up one position in the waiting list. If the next object to arrive is of type $i \notin W(1)$ then this agent does nothing.\footnote{The requirement that $W(1) \neq \emptyset$ guarantees that the agent eventually receives an object. Without this requirement such agents would wait forever, and our results would be restricted to agents in positions $k$ such that $W(k) \neq \emptyset$.}

If the next object to arrive is $i \in W(2) \setminus W(1)$, then it is assigned to the agent in position 2, who departs. All agents in later positions move up one position in the list. Similarly an object belonging to $i \in W(k) \setminus W(k-1)$ is assigned to the agent in position $k$, etc. (Since $W()$ is monotonic, there is at most one such $k$; if an object belongs to no $W(k)$ then it is discarded.)

At this point we reemphasize that the actual act of influencing waiting lists appears no where in this definition. We do not address whether a planner could implement any particular $W$ since our analysis covers all waiting list policies. Indeed implementation could be a separate question for future research. A planner could be constrained by—or possess the tools of—legal force, the choice of mechanism, the ability to withhold or limit information that agents have about the arriving objects, etc. Influence is an abstract, unconstrained ability at this point of our analysis.

One also can interpret $W$ simply as the cumulative function of a kind of anonymous and Markovian social choice function, $f$. Arriving objects are assigned to agents based solely on their current position $k$ in the waiting list, defining $f(i) = k$ if and only if $i \in W(k) \setminus W(k-1)$. Our results are best expressed directly in terms of this cumulative function $W$ rather than the function $f$; hence we dispense with the notation and terminology of such a social choice function.

Examples

The most natural example of a waiting list policy is the one that would obtain as the result of uninfluenced behavior. Specifically, imagine an extensive form game (alluded to in Subsection 3.3) in which rational agents in a waiting
list can foresee the strategic, selfish behavior of the other agents, and make equilibrium acceptance decisions that result.\cite{SuZenios2004} Though we are primarily concerned with analyzing the distribution of welfare, the waiting list policy \( W^* \) that results from such an “uninfluenced game” plays a role in such results, beginning with Theorem 2.

To illustrate another (albeit extreme) example of a waiting list policy, imagine a policymaker removing the agents’ right to defer acceptance decisions, i.e. the agent at the front of the list must accept whatever object arrives next. One such policy is defined by \( W(1) = \mathcal{O} \), and hence \( W(k) = \mathcal{O} \). More generally, a policy satisfying \( W(k) = \mathcal{O}' \subseteq \mathcal{O} \) assigns the object types in \( \mathcal{O}' \) to the agent in position 1 and discards the rest. While such a policy need not sound reasonable, we formally define this subclass of waiting list policies since it plays a role in our results.

\textbf{Definition 2.} A waiting list policy \( W \) is a \textbf{constant standards policy} if, for all \( k, \ell \in \mathbb{N} \), \( W(k) = W(\ell) \).

There are, of course, many other examples of waiting list policies. Imagine a policy reflecting coercive influence, such as a legal constraint that reserves some desirable class of object types \( \mathcal{O}' \subseteq \mathcal{O} \) for agents whose waiting list position \( \ell \) exceeds some value \( k \). This is reflected by any policy where \( \ell < k \) implies \( W(\ell) \cap \mathcal{O}' = \emptyset \). In general, the set \( W(1) \) need not be “better” than any other \( W(k) \) in any sense at all.

3 Expected payoffs

3.1 Payoff equivalence

Our first main result, \textbf{Theorem 1}, calculates the expected (continuation) payoff to an agent in any \( k \)th position of the waiting list, for an arbitrary waiting list policy \( W \), showing it to be a function only of the set \( W(k) \). Among other things this means that we can analyze the set of achievable expected payoffs to position \( k \) even while restricting attention to the class of constant standards policies, which are simple to write down.

An intuition for this and other results in this section can be seen in the following 2-position example. Focusing on position \( k = 2 \), consider a

\footnote{Such a game is the central part of Su and Zenios (2004); see also the literature following Agrawal et al. (1984) described earlier.}
policy \( W \) such that \( W(1) \subsetneq W(2) \). Suppose we offer the agent in position \( k = 2 \) the choice of (i) using policy \( W \) or (ii) using a policy \( W' \) satisfying \( W'(1) = W'(2) = W(2) \). Under the latter policy, the agent in position 2 must wait until reaching position 1 to receive an object, but still could receive any object type in \( W(2) \).

A first observation (generalized later in Theorem 4) is that \( W \) and \( W' \) induce the same probability distribution over the object type \( i \in W(2) \) ultimately consumed by the agent in position \( k = 2 \). Under policy \( W \), he consumes an object from \( W(1) \) if and only if the next object arrival that belongs to \( W(2) \) also belongs to \( W(1) \subset W(2) \); otherwise he consumes an object from \( W(2) \setminus W(1) \). Under policy \( W' \), he consumes an object from \( W(1) \) if and only if the second object arrival that belongs to \( W(2) \) also belongs to \( W(1) \subset W(2) \); otherwise he consumes an object from \( W(2) \setminus W(1) \). These are equally likely events. This “object equivalence” holds in general, as formalized in Theorem 4.

Therefore the only potential difference between \( W \) and \( W' \) involves the waiting time. Under \( W' \) the agent incurs 2 “rounds” of waiting for an arrival from \( W(2) \), which has an expected time of \( 2/\sum_{j \in W(2)} \mu_j \). Under \( W \) he first waits for an object \( i \in W(2) \) to appear, taking on average \( 1/\sum_{j \in W(2)} \mu_j \). If \( i \in W(2) \setminus W(1) \) it is assigned to him with no additional wait; if \( i \in W(1) \) he advances to position 1 and additionally waits for the next arrival from \( W(1) \). The latter event has probability \( \sum_{j \in W(1)} \mu_j / \sum_{j \in W(2)} \mu_j \), and expected additional wait time of \( 1/\sum_{j \in W(1)} \mu_j \). Therefore his expected wait under \( W \) is

\[
\frac{1}{\sum_{j \in W(2)} \mu_j} + \frac{\sum_{j \in W(1)} \mu_j}{\sum_{j \in W(2)} \mu_j \sum_{j \in W(1)} \mu_j} \frac{1}{\sum_{j \in W(2)} \mu_j} = \frac{2}{\sum_{j \in W(2)} \mu_j}
\]

which is the same as under \( W' \). That is, under \( W \) there is some probability of drawing two rounds of slightly longer wait times, plus some probability of waiting only one round of shorter wait time. Because the probabilities perfectly align with the wait time proportions, terms cancel out under the exponential distribution to yield the same expected wait time as two rounds of “intermediate” wait time under \( W' \).

**Theorem 1** (Expected-payoff equivalence). Fix a policy \( W \). For any position \( k \in \mathbb{N} \), the expected continuation payoff to \( k \) under \( W \), \( \Pi(k; W) \), satisfies

\[
\Pi(k; W) = \frac{\sum_{j \in W(k)} \mu_j v_j - k}{\sum_{j \in W(k)} \mu_j}
\]  \hspace{1cm} (1)

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Consequently, position $k$’s expected payoff is a function only of $W(k)$.

**Proof.** The proof is by induction. For $k = 1$, the agent consumes the first arrival from $W(1)$, so the expected object value minus the expected waiting time is

$$\Pi(1; W) = \frac{\sum_{j \in W(1)} \mu_j v_j}{\sum_{j \in W(1)} \mu_j} - \frac{1}{\sum_{j \in W(1)} \mu_j} = \frac{\sum_{j \in W(1)} \mu_j v_j - 1}{\sum_{j \in W(1)} \mu_j}$$

agreeing with Equation 1.

Fix $k \in \mathbb{N}$ and suppose that Equation 1 holds for $k - 1$. The next object-type to arrive that belongs to $W(k)$ either belongs to $W(k - 1)$ or to $W(k) \setminus W(k - 1)$. In the former case the agent in position $k$ moves to position $k - 1$ and continues with an additional expected continuation payoff $\Pi(k - 1; W)$. In the latter case the agent is assigned the object, receiving payoff $v_j$. Accounting for these two possibilities, along with the expected waiting time for the arrival from $W(k)$, we have the following.

$$\Pi(k; W) = \frac{\sum_{j \in W(k-1)} \mu_j \cdot \Pi(k - 1; W)}{\sum_{j \in W(k)} \mu_j} + \frac{\sum_{j \in W(k) \setminus W(k-1)} \mu_j v_j}{\sum_{j \in W(k)} \mu_j} - \frac{1}{\sum_{j \in W(k)} \mu_j}$$

$$\quad= \frac{\sum_{j \in W(k-1)} \mu_j \cdot \left(\frac{\sum_{j \in W(k-1)} \mu_j v_j - (k - 1)}{\sum_{j \in W(k-1)} \mu_j}\right) + \sum_{j \in W(k) \setminus W(k-1)} \mu_j v_j - 1}{\sum_{j \in W(k)} \mu_j}$$

$$\quad= \frac{\sum_{j \in W(k-1)} \mu_j v_j - (k - 1) + \sum_{j \in W(k) \setminus W(k-1)} \mu_j v_j - 1}{\sum_{j \in W(k)} \mu_j}$$

$$\quad= \frac{\sum_{j \in W(k)} \mu_j v_j - k}{\sum_{j \in W(k)} \mu_j}$$

proving the result. \qed

This result implies that any feasible payoff for (fixed) position $k$ can be achieved under some common standards rule.

**Corollary 1.** For any policy $W$, the expected continuation payoff to position $k$ under $W$ is the same as the expected continuation payoff under the common standards policy $W'$ defined by $W'(\ell) \equiv W(k)$ for all $\ell \in \mathbb{N}$.

This illustrates the perfect offset of two opposing effects that come about when agents in earlier positions use different “standards” in accepting objects.
than do agents in later positions. For instance, if an agent in position $k$ could “force” agents ahead of him to have “lower standards” in choosing which objects to accept, this would (i) force the agent to wait longer even if he eventually consumes a lower quality object, but (ii) potentially reduce the time it takes to reach earlier queue positions (by increasing the speed at which earlier agents depart). It turns out that these effects precisely offset.

### 3.2 Pareto dominance for risk neutral agents

We determine the policies that maximize the expected continuation payoff for an agent in a fixed position $k$. Using Corollary 1, we prove the result by restricting attention to constant-standards policies. Not surprisingly, the “best” such policy from position $k$’s perspective is defined by a “threshold set” consisting of the best $i^*_k$ objects, where the value of object $i^*_k$ must be no smaller than the expected continuation payoff itself. Recall that objects $O$ are indexed in decreasing order of $v_i$’s.

**Lemma 1** ($k$’s favorite policy). For any $k \in \mathbb{N}$, the expected payoff $\Pi(k; \cdot)$ is maximized by any policy $W^*$ that satisfies $W^*(k) \equiv \{1, 2, \ldots, i^*_k\}$, where

$$i^*_k \equiv \max \left\{ i \in O : v_i \geq \frac{\sum_{j=1}^{i-1} \mu_j v_j - k}{\sum_{j=1}^{i-1} \mu_j} \right\} \quad (2)$$

Furthermore, $k < \ell$ implies $i^*_k \leq i^*_\ell$.

**Proof.** Fix $k$, and for any subset of types $C \subseteq O$, consider the constant standards policy $W$ defined by $W(\ell) \equiv C \neq \emptyset$. Rather than writing $\Pi(k; W)$, let $\pi(C)$ denote the expected payoff to position $k$ under such a policy, since we consider varying $C$.

From Theorem 1,

$$\pi(C) = \frac{\sum_{j \in C} \mu_j v_j - k}{\sum_{j \in C} \mu_j} \quad (3)$$

and for any $i \in O \setminus C$, adding $i$ to $C$ yields a payoff of

$$\pi(C \cup \{i\}) = \frac{\sum_{j \in C} \mu_j v_j - k + \mu_i v_i}{\sum_{j \in C} \mu_i + \mu_i}$$

which (weakly) improves on $\pi(C)$ if and only if $v_i \geq (\sum_{j \in C} \mu_j v_j - k) / \sum_{j \in C} \mu_j$. Since object types are in decreasing order of the $v_i$’s, any $W^*$ defined via
Equation 2 maximizes $\Pi(k; \cdot)$.\(^{12}\)

Finally, observe that the right-hand side of the inequality within Equation 2 is decreasing in $k$. Therefore the type index $i_k^*$ is indeed increasing in the position index $k$. \(\square\)

The monotonicity of $i_k^*$ with respect to $k$ is intuitive: agents in later positions—with greater expected wait time—would prefer to set “a lower bar” when determining a set of “acceptable” objects $W^*(k)$. However this monotonicity also leads us to a more significant fact: it is feasible to simultaneously maximize the expected payoff to every position. Imagine if we were to successively ask each agent in each successive position $k$ to list the policies $W$ that maximize $\Pi(k; W)$. There would exist a unique\(^{13}\) policy that appears in all of their lists.

We say that policy $W$ (position-wise) Pareto dominates rule $W'$ if, for all $k \in \mathbb{N}$, $\Pi(k; W) \geq \Pi(k; W')$. Though Pareto dominance is an incomplete relation over policies, there is one policy that Pareto dominates all others.

**Theorem 2** (Pareto-dominance). For all $k \in \mathbb{N}$ let $W^*(k) \equiv \{1, 2, \ldots, i_k^*\}$ where $i_k^*$ is defined by Equation 2. Then $W^*$ is a feasible policy, and it Pareto dominates every other policy.

**Proof.** Since $i_k^*$ is increasing in $k$, $k < \ell$ implies $W(k) \subseteq W(\ell)$, so $W^*$ is feasible. Pareto dominance is immediate from Lemma 1. \(\square\)

As a corollary of the theorem, $W^*$ maximizes the sum of agents’ expected payoffs, which is analogous to a result of Su and Zenios (2004)\(^{14}\) and to results discussed in Section 7.

### 3.3 Decentralized choice

Imagine a dynamic game in which, upon the arrival of any object, queued agents are sequentially asked whether to leave with the current object or to

---

\(^{12}\)(Ties are irrelevant.) In the nongeneric case that $v_{i_k^*} = \pi(W^*(k))$, it is easy to see that $W'(k) \equiv W^*(k) \setminus \{i_k^*\}$ also maximizes $k$’s payoff. This impacts neither the Lemma nor any other results of the paper.

\(^{13}\)Unique up to the indifference described in footnote 12.

\(^{14}\)Su and Zenios further consider the possibility of agent arrivals, in which case $W^*$ is no longer socially optimal, as observed by Naor (1969).
continue waiting. Equilibrium behavior in this game is uniquely\textsuperscript{15} described by the Pareto-dominant rule $W^*$ defined in Theorem 2.\textsuperscript{16}

**Theorem 3.** The decentralized game yields unique subgame perfect Nash equilibrium (SPE) continuation payoffs. Specifically, under any SPE and following any history of object arrivals, an agent in position $k$ has the expected continuation payoff $\Pi(k, W^*)$, where $W^*$ is the Pareto-dominant rule described in Theorem 2.

**Proof.** Proof to be written, but this essentially follows constructively as in the proof of Lemma 1.

Theorem 3 yields a (loose) fairness argument to deferral options under FCFS allocation. In the decentralized game, an agent could complain that the deferral option of agents in front of him in the queue impose an unfair burden on him, in that he only gets exposure to “poor” objects over short time horizons. Imagine offering this agent the chance to remove the deferral option from all players in the game (including his future self, were he to advance to an earlier position in the queue), and instead requiring all players to accept objects only from some agreed upon set $T \subseteq O$. Would the player want to switch to this new game? What “constant standards” $W'(\cdot) = T$ would that player impose on everyone? Theorem 2 and Theorem 3 show that the agent cannot improve upon the decentralized outcome under any policy that consistently imposes choice behavior (as a function of queue position) consistently across all agents.

4 Object and waiting time distributions

For any policy $W$, Theorem 1 tells us that the expected continuation payoff to an agent starting from position $k$ is a function only of $W(k)$. However this does not tell us anything about the probability distribution over such payoffs, nor specifically about the distributions of waiting times and object consumption. In the application of organ waiting lists, for example, the distribution of waiting time could be of interest both to doctors treating the patients

\textsuperscript{15}Unique up to indifference; see footnote 12.

\textsuperscript{16}Related equilibrium derivations have been obtained under additional assumptions, e.g. by restricting attention ex ante to strategies that are symmetric and Markovian. Such assumptions turn out to be unnecessary.
and to insurance companies paying for patient maintenance while waiting. Specifically, though outside the scope of our model, a doctor’s treatment of a waiting patient could vary depending on whether there is substantial probability of a long wait time for a transplant, versus whether the arrival time of a donor organ could be estimated with little variance.  

Similarly, by altering the variance of patients’ waiting times, the choice of policy \( W \) could affect risk assessment for an insurance company responsible for covering those patients’ waiting costs (e.g. dialysis costs for kidney patients).

We begin with an “object-equivalence” analog of Theorem 1: for an agent in position \( k \), the probability of eventually consuming a particular object type depends only on \( W(k) \). The proof is straightforward.

**Theorem 4** (Object equivalence). For any policy \( W \), the probability that an agent in position \( k \) ultimately consumes an object of type \( i \in W(k) \) is \( \mu_i / \sum_{j \in W(k)} \mu_j \).

**Proof.** This is obviously true for \( k = 1 \). Using induction, fix \( k \) and suppose the statement is true for \( k - 1 \). Nothing happens for the agent in position \( k \) until the arrival of some \( i \in W(k) \). Conditional on the arrival of such an object, the probability it is of type \( i \) is \( \mu_i / \sum_{j \in W(k)} \mu_j \). If \( i \in W(k) \setminus W(k-1) \) then the agent consumes that object proving the claim.

Otherwise, \( i \in W(k-1) \), and the agent moves into position \( k-1 \); that is, the total probability of moving into position \( k - 1 \) is \( \sum_{j \in W(k-1)} \mu_j / \sum_{j \in W(k)} \mu_j \). By the induction assumption, the probability of eventually consuming \( i \in W(k-1) \) given that the agent is already in position \( k - 1 \) is \( \mu_i / \sum_{j \in W(k-1)} \mu_j \). Hence the probability of consuming \( i \) starting from position \( k \) is

\[
\frac{\sum_{j \in W(k-1)} \mu_j}{\sum_{j \in W(k)} \mu_j} \cdot \frac{\mu_i}{\sum_{j \in W(k-1)} \mu_j} = \frac{\mu_i}{\sum_{j \in W(k)} \mu_j}
\]

yielding the result.

Thus, ex ante, \( k \)'s object consumption cannot be (probabilistically) affected by the specification of \( W(\ell) \subseteq W(k) \) for \( \ell \neq k \). Meanwhile Corollary 1 tells us that \( k \)'s expected payoff also depends only on \( W(k) \). Therefore \( k \)'s expected wait time must depend only on \( W(k) \).

\[17\] We say that this is beyond the scope of our model because in such a world, an agent’s waiting cost per unit time is endogenously affected by the distribution of waiting time, rather than an exogenous constant.
**Corollary 2.** For any policy $W$ and starting from any position $k \in \mathbb{N}$, an agent has an expected waiting time of $k / \sum_{j \in W(k)} \mu_j$.

**Proof.** Follows from Theorem 1 and Theorem 4. □

One might naturally wonder whether a stronger result holds, namely, whether the probability distribution of wait time from position $k$ depends only on $W(k)$. This turns out not to be the case; $W \neq W'$ with $W(k) = W'(k)$ could indeed yield different distributions of waiting time for $k$. Following our earlier motivation above for the case of organ waiting lists, this means that certain parties (e.g. insurance companies) could be affected by the policy choice even in a world where the agents (patients) are risk-neutral and care only about the expectation given in Corollary 2.

In general, it can be difficult to describe the distribution of waiting times for arbitrary policies $W$. An exception is the simple case of common-standards policies, which yield waiting times that have an Erlang distribution.

**Lemma 2 (Waiting times for common-standards).** Consider a common-standards policy $W$, i.e. where $W(k) \equiv \hat{O} \subseteq O$ for all $k \in \mathbb{N}$. The distribution of waiting time for an agent in position $k$ is described by an Erlang distribution, with mean $k / (\sum_{\hat{O}} \mu_i)$ and variance $k / (\sum_{\hat{O}} \mu_i)^2$.

**Proof.** The wait time for an arrival of a single object from $\hat{O}$ is exponentially distributed with parameter $\sum_{\hat{O}} \mu_i$. An agent in position $k$ receives such an object precisely after $k$ i.i.d. such arrivals, hence the mean and variance calculations follow directly. Furthermore the sum of $k$ i.i.d. exponentially distributed variables follows an Erlang distribution. □

Despite the difficulty of describing waiting time distributions in general, it is possible to describe the variance of waiting times for any policy. Under certain assumptions, such information could be sufficient in the above example of risk assessment for an insurer whose costs vary with the amount of time patients wait for transplants (e.g. dialysis costs for kidney patients).

To understand the idea behind the proofs, it is simplest to consider the case of two object types $O = \{1, 2\}$, and a policy $W$ where $W(1) = \{1\}$ and $W(2) = \{1, 2\}$.

How long will agent 2 wait to receive an object? Denote this (random variable) wait time as $\hat{w}_2$, and observe that it consists of two parts. First
we wait \( w' \) units of time for the first object; note that \( w' \sim \exp(\mu_1 + \mu_2) \).
If that first object is of type 2, then agent 2 departs with it (and “waits” an additional \( w'' = 0 \) units of time). If it is of type 1, then the agent must wait for a second object of type \( 1 \in W(1) \), which requires additional wait of \( w'' \sim \exp(\mu_1) \). So \( w'' \) is either distributed \( \exp(\mu_1) \) with probability \( p_1 \equiv \mu_1/(\mu_1 + \mu_2) \), or is identically zero with the remaining probability. Furthermore, \( \hat{w}_2 = w' + w'' \).

The variance of \( w'' \), which can be computed in a few different ways\(^{18} \) is

\[
\frac{2}{\mu_1(\mu_1 + \mu_2)} - \frac{1}{(\mu_1 + \mu_2)^2}
\]

Since \( w' \) is exponentially distributed and independent of \( w'' \) we have

\[
\text{Var}(w' + w'') = \frac{1}{(\mu_1 + \mu_2)^2} + \left( \frac{2}{\mu_1(\mu_1 + \mu_2)} - \frac{1}{(\mu_1 + \mu_2)^2} \right)
\]

\[
= \frac{2}{\mu_1(\mu_1 + \mu_2)}
\]

which exceeds the corresponding variance in Lemma 2 of \( 2/(\mu_1 + \mu_2)^2 \) under a policy where \( W'(2) = W'(1) = \emptyset \). Thus \( W \) and \( W' \) yield different waiting time variances even though both rules provide the same expected payoff and the same (probabilistic) object assignment to an agent in position 2.

In the proof of Theorem 5, the two “objects” used above become classes of objects, \( W(1) \) and \( W(2) \setminus W(1) \), or more generally \( W(k-1) \) and \( W(k) \setminus W(k-1) \). In addition \( w' \) represents the wait for the first object from \( W(k) \) and \( w'' \) represents the entire continuation waiting time (either zero or a continued wait from position \( k-1 \)). This gives us a recursive expression for \( \hat{w}_k \) which can be solved explicitly.

**Theorem 5** (Wait time variance). For any policy \( W \), the waiting time from position \( k \), \( \hat{w}_k \), has a variance of

\[
\text{Var}(\hat{w}_k) = \frac{1}{\sum_{j \in W(k)} \mu_j} \left( \frac{\sum_{\ell=1}^{k} 2\ell}{\sum_{j \in W(\ell)} \mu_j} - \frac{k^2}{\sum_{j \in W(k)} \mu_j} \right). \quad (4)
\]

\(^{18}\)Being a weighted density of exponentials, \( w'' \) follows a hyper-exponential distribution (see Appendix) which has a known expression for variance. It can also be computed using Theorem 5 or its proof.
Proof. The wait time \( \hat{w}_k \) is the sum of two independent random variables: the initial wait \( w' \) until the arrival of the next object \( i \in W(k) \), and the remaining wait \( w'' \), which either has the same distribution as \( \hat{w}_{k-1} \) (if \( i \in W(k-1) \)) or is degenerately \( w'' = 0 \) (if \( i \in W(k) \setminus W(k-1) \)).

For all \( \ell \in \mathbb{N} \) define \( M_\ell \equiv \sum_{j \in W(\ell)} \mu_j \). Since \( w' \) is exponentially distributed,

\[
\text{Var}(w') = 1/M_k^2.
\]

To consider the variance of \( w'' \), we first recall the following easily proven fact. Let a random variable \( Y \) equal the value of some r.v. \( X \) with probability \( p \) and be degenerately \( Y = 0 \) with probability \( 1 - p \). Then

\[
\text{Var}(Y) = p\text{Var}(X) + (p - p^2)E(X)^2.
\]

Here,

\[
\text{Var}(w'') = \frac{M_{k-1}}{M_k} \text{Var}(\hat{w}_{k-1}) + \left( \frac{M_{k-1}}{M_k} - \left( \frac{M_{k-1}}{M_k} \right)^2 \right) E(\hat{w}_{k-1})^2.
\]

By Corollary 2, \( E(\hat{w}_{k-1}) = (k-1)/M_{k-1} \). Therefore

\[
\text{Var}(\hat{w}_k) = \text{Var}(w') + \text{Var}(w'') = \frac{1}{M_k^2} + \frac{M_{k-1}}{M_k} \text{Var}(\hat{w}_{k-1}) + \left( \frac{M_{k-1}}{M_k} - \left( \frac{M_{k-1}}{M_k} \right)^2 \right) \frac{(k-1)^2}{M_{k-1}^2}
\]

which we can solve recursively.

For any rule, \( \hat{w}_1 \) is exponentially distributed with variance of \( 1/M_1^2 \) which coincides with Equation 4. We show that if Equation 4 holds for some arbitrary \( k - 1 \) then it holds for \( k \). Substituting into Equation 5,

\[
\text{Var}(\hat{w}_k) = \frac{M_{k-1}}{M_k} \text{Var}(\hat{w}_{k-1}) + \frac{(k-1)^2}{M_{k-1}M_k} - \frac{k^2 - 2k}{M_k^2}
\]

\[
= \frac{1}{M_k} \left( \sum_{\ell=1}^{k-2} \frac{2\ell}{M_\ell} + \frac{2(k-1) - (k-1)^2}{M_{k-1}} \right) + \frac{(k-1)^2 - k^2 - 2k}{M_{k-1}M_k}
\]

\[
= \frac{1}{M_k} \left( \sum_{\ell=1}^{k-2} \frac{2\ell}{M_\ell} + \frac{2k - k^2}{M_k} \right) = \frac{1}{M_k} \left( \sum_{\ell=1}^{k-1} \frac{2\ell}{M_\ell} + \frac{-k^2}{M_k} \right)
\]

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proving the result. \(\square\)

Thus for any \(k\) and any \(\ell < k\), increasing \(M_\ell \equiv \sum_{j \in W(\ell)} \mu_j\)—expanding the set \(W(\ell)\)—decreases the variance of \(\hat{w}_k\).\(^{19}\) That is, the variance of \(k\)'s waiting time weakly decreases monotonically with respect to increases (in terms of set inclusion) in \((W(\ell))_{\ell < k}\). At the extreme, the variance is minimized (subject to fixing \(W(k)\)) by using a constant-standards policy.

**Corollary 3** (Waiting time variance decreases in \(W\)). Fix a position \(k \in \mathbb{N}\) and policy \(W\). Let \(W'\) be a policy satisfying \(W'(k) = W(k)\) and for all \(\ell < k\) \(W'(\ell) \supseteq W(k)\). Then the variance of \(k\)'s waiting time under \(W'\) is at least as great as it is under \(W\) \(\text{Var}(\hat{w}_k^W) \geq \text{Var}(\hat{w}_k^{W'})\). Therefore among all policies \(W'\) satisfying \(W'(k) = W(k)\), the constant standards policy \(A'(\ell) \equiv W(k)\) minimizes the variance of \(k\)'s waiting time.

An interpretation of this result is that, holding all else constant, a party that benefits from lower waiting time variance for, say, agents later in the queue, has the incentive to influence waiting lists so that agents earlier in the queue are allocated more objects than they otherwise would accept. If we consider starting from the decentralized, Pareto-dominant rule \(W^*\) described in Theorem 2 (where, in effect, agents simply optimize their individual behavior), influencing or coercing earlier agents to accept more objects would typically lower their expected payoffs, and would have no effect on the expected payoff of agents in later positions. Thus the cost of lower the waiting-time variance to position \(k\) is an unambiguously negative impact on expected payoffs. There is a tension between the expected-payoff benefits from \(W^*\) and a “smoother” rule that lowers the variance of wait times for later agents.

Expanding \(W(k)\) could increase or decrease the variance of \(k\)'s waiting time, depending on the parameters \(\mu_j\). Consider \(O = \{1, 2\}\), normalizing \(\mu_1 = 1\), and start with constant standards policy \(W(\ell) = \{1\}\) for all \(\ell \in \mathbb{N}\). The variance of waiting time from position \(k\) is simply \(k\). If we add the second object to \(W(k)\), giving \(W'(k) = \{1, 2\}\), there are two, possibly opposing, intuitive effects. First, adding the object tends to lower the variance of \(\hat{w}_k\) because we increase the rate at which (assignable) objects arrive. Second, however, the added object could arrive “quickly” relative to overall average waiting time, having a positive effect on variance. In this example it turns out

\(^{19}\)Of course in isolation, \(W(\ell)\) can only be expanded up to the set \(W(\ell + 1) \supseteq W(\ell)\) before an increase in \(W(\ell + 1)\) is required to maintain the definition of a policy.
that the latter effect indeed dominates whenever $\mu_2$ is sufficiently small (and $k \geq 4$).\footnote{It can be shown that if $k \leq 3$, then an expansion of $M_k$ necessarily lowers the variance of $\hat{w}_k$.} As $\mu_2$ is increased, the first affect becomes increasingly dominant.

4.1 Payoff Variance

Despite Theorem 4 (object equivalence) and Corollary 3 (waiting time monotonicity in $W$), these results do not determine the variance to the total (continuation) payoff to an agent in position $k$. The latter result addresses only one of two opposing effects on payoff variance.

Effect 1: Corollary 3 tells us that, fixing $W(k)$, the variance of $k$’s waiting time decreases as we expand $M_\ell (\ell < k)$.

Effect 2: By expanding some $M_\ell (\ell < k)$, we change the correlation between the value of assigned objects and the waiting cost incurred to receive that object.

Effect 1 exists, and can be derived, independently of the values $v_i$ of the object types. To illustrate effect 2, however, consider a policy that assigns “better” objects to agents in earlier positions and “worse” objects to those in later positions. Such a rule positively correlates (for agents starting later in the queue) the value of the assigned object with realized waiting time (i.e. a negative correlation between value and cost), lowering payoff variance.

It is simple to see that Effect 1 can dominate Effect 2 via the (uninteresting) case in which all objects have (approximately) the same value. Under such parameters the agents are essentially indifferent about which object they receive, so increasing the arrival rate is the only thing that can lower variance.\footnote{We omit formalizing a resulting implication of Corollary 3: for any $k$, if object values are sufficiently similar, then a constant standards policy minimizes the variance of $k$’s payoff.}

Similarly, Effect 2 can dominate Effect 1.

Example 1 (Effect 2 dominates Effect 1.). Consider a 2-type example $\mathcal{O} = \{1, 2\}$, where $v_1 = 3$, $\mu_1 = 1$, $v_2 = 1$, and $\mu_2 = 2$. It can be shown (e.g. using Theorem 6) that when $W(1) = \{1\}$ and $W(2) = \{1, 2\}$, an agent in position $k = 2$ has a (continuation) payoff variance of $2/3$. Similarly a policy with
$W'(1) = W'(2) = \emptyset$ yields a variance of $10/9$.\(^{22}\) (The policy $W''(1) = \{2\}$, $W''(2) = \{1, 2\}$, which allocates the better object to position 2, correlates longer waits with worse objects, and not surprisingly gives an even higher payoff variance of $5/3$.)

The careful reader might note another occurrence in the previous example. Adding object type 2 to $W(1)$ not only increased the variance of position 2’s payoff, but also decreased the expected payoff to position 1. (Rather than accept the object with value $v_2 = 1$, an agent in position 1 would prefer to wait an expected one unit of additional time to obtain $v_1 = 3$.) This combination of effects turns out not be a coincidence, as we now show.

The variance of position $k$’s continuation payoff has three components: the variance associated with the uncertain value of the object ultimately to be consumed; a term related to the variance of the waiting time from position $k$; and a term that compares the expected payoffs to position $k$ and $\ell = 1, \ldots, k - 1$.

**Theorem 6** (Payoff variance). The variance of the payoff to position $k$ under a policy $W$ is

$$\text{Var}(v_i | i \in W(k)) + \frac{k^2}{M_k^2} - \frac{2}{M_k} \sum_{\ell=1}^{k-1} (\Pi(\ell; W) - \Pi(k; W))$$

(6)

where $M_k = \sum_{j \in W(k)} \mu_j$ and $\text{Var}(v_i | i \in W(k))$ is the variance of the value of an object drawn from $W(k)$.

**Proof.** An agent’s (continuation) payoff from position $k$ is $v - t$ where $v$ is the value of the object ultimately consumed from $W(k)$ and $t$ is the realized waiting time. The variance of $v - t$ is $\text{Var}(v) + \text{Var}(t) - 2\text{Cov}(v, t)$. Following Theorem 4, the expected value and variance of $v$ can be computed easily. The expected value and variance of $t = w_k$ are provided in Corollary 2 and Theorem 5. Therefore what remains is to find $E(vt)$ in order to derive $\text{Cov}(v, t)$.

Via the law of total expectation we take a weighted average of $E(vt | J)$’s, where $J \leq k$ is the position in which the agent starting in position $k$ ultimately is assigned an object (from $W(k)$). Denoting $M_\ell \equiv \sum_{j \in W(\ell)} \mu_j$,\(^{22}\) when $v_1 = 2$ it appears that the two rules give $k = 2$ the same payoff variance, and when $v_1 = 1$ the previous corollary tell us that $A'$ must give a weakly lower variance, though it turns out to be strictly lower.

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Theorem 4 states the probability distribution of $J$.

\[ \forall j \in \{1, 2, \ldots, k\}, \quad P(J = j) = (M_j - M_{j-1})/M_k \]

Observe that for a fixed $J$, $v$ and $t$ are (conditionally) independent; that is, $E(vt|J) = E(v|J)E(t|J)$, greatly simplifying the problem. Those conditional expectations are the following.

\[
E(v|J) = \frac{\sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i}{\sum_{i \in W(J) \setminus W(J-1)} \mu_i} = \frac{\sum_{i \in W(J) \setminus W(J-1)} v_i \mu_i}{M_j - M_{j-1}} \\
E(t|J) = \frac{1}{M_k} + \frac{1}{M_{k-1}} + \cdots + \frac{1}{M_J} 
\]

So

\[
E(vt) = \sum_{j=1}^{k} E(v|J = j)E(t|J = j) \cdot P(J = j) \\
= \sum_{j=1}^{k} \left( \frac{\sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i}{M_j - M_{j-1}} \right) \left( \frac{1}{M_k} + \frac{1}{M_{k-1}} + \cdots + \frac{1}{M_j} \right) \cdot \frac{M_j - M_{j-1}}{M_k} \\
= \sum_{j=1}^{k} \left( \frac{\sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i}{M_k} \right) \left( \frac{1}{M_k} + \frac{1}{M_{k-1}} + \cdots + \frac{1}{M_j} \right) 
\]

From previous results we have

\[
E(v)E(t) = \left( \frac{\sum_{i \in W(k)} v_i \mu_i}{M_k} \right) \left( \frac{k}{M_k} \right) = \sum_{j=1}^{k} \left( \frac{\sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i}{M_k} \right) \left( \frac{k}{M_k} \right) 
\]

Subtracting Equation 8 from Equation 7,

\[
\text{Cov}(v, t) = E(vt) - E(v)E(t) \\
= \sum_{j=1}^{k} \left( \frac{\sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i}{M_k} \right) \left( \frac{1}{M_j} + \frac{1}{M_{j+1}} + \cdots + \frac{1}{M_k} - \frac{k}{M_k} \right) \\
= \frac{1}{M_k} \sum_{j=1}^{k} \left( \frac{\sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i}{M_k} \right) \left( \frac{k}{M_k} - \sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i \right) \\
= \frac{1}{M_k} \left[ \left( \frac{k}{M_k} \sum_{i \in W(j) \setminus W(j-1)} v_i \mu_i \right) - k \cdot \frac{\sum_{i \in W(k)} v_i \mu_i}{M_k} \right] \\
= \frac{1}{M_k} \sum_{i=1}^{k} \left( \frac{\sum_{i \in W(\ell) \setminus W(\ell-1)} v_i \mu_i}{M_\ell} - \frac{\sum_{i \in W(k)} v_i \mu_i}{M_k} \right) 
\]

\[ (9) \]
Using Equation 4 and Equation 9,

\[
\text{Var}(v - t) = \text{Var}(v) + \text{Var}(t) - 2\text{Cov}(v, t)
\]

\[
= \text{Var}(v) + \frac{1}{M_k} \left( \left( \sum_{\ell=1}^{k} \frac{2\ell}{M_t} - \frac{k^2}{M_k} \right) - \frac{2}{M_k} \sum_{\ell=1}^{k} \left( \frac{\sum_{i\in W(\ell)} v_i \mu_i}{M_t} - \frac{\sum_{i\in W(k)} v_i \mu_i}{M_k} \right) \right)
\]

\[
= \text{Var}(v) + \frac{2}{M_k} \left( \sum_{\ell=1}^{k} \left( \frac{\ell - \sum_{i\in W(\ell)} v_i \mu_i}{M_t} + \frac{\sum_{i\in W(k)} v_i \mu_i}{M_k} \right) - \left( \frac{k^2/2}{M_k} \right) \right)
\]

\[
= \text{Var}(v) + \frac{2}{M_k} \left( \sum_{\ell=1}^{k} \left( \frac{\ell - \sum_{i\in W(\ell)} v_i \mu_i}{M_t} + \frac{\sum_{i\in W(k)} v_i \mu_i - k}{M_k} \right) + \left( \frac{k^2/2}{M_k} \right) \right)
\]

\[
= \text{Var}(v) + \frac{k^2}{M_k^2} - \frac{2}{M_k} \sum_{\ell=1}^{k-1} (\Pi(\ell; W) - \Pi(k; W))
\]

completing the proof. \( \square \)

Fixing \( W(k) \) (and hence \( M_k \)), all terms in (6) become constants with the exception of the terms

\[
- \sum_{\ell=1}^{k-1} \Pi(\ell; W)
\]

It is now straightforward to see that, for a fixed \( W(k) \), the variance of position \( k \)'s payoff is minimized by maximizing the sum of expected payoffs to the positions that proceed \( k \)! With Theorem 2, this means that, subject to maximizing \( k \)'s expected payoff, variance is indeed minimized by using the policy \( W^* \) which was obtained from the decentralized queueing game described in Subsection 3.3.

**Corollary 4.** Conditional on maximizing \( k \)'s expected payoff, the policy \( W^* \) minimizes the variance of position \( k \)'s payoff. More generally, requiring \( W(k) \) to be fixed, a policy that minimizes the variance of \( k \)'s payoff is any policy \( W' \) satisfying \( W'(\ell) = W^*(\ell) \cap W(k) \) for all \( \ell < k \).

A warning is in order. This result does not imply that risk-averse agents would implement policy \( W^* \) in an equilibrium of the decentralized game (as would risk-neutral agents à la Theorem 3). For instance even an agent in position 1 might want to accept an object \( i \notin W^*(1) \) whose value is slightly below \( \Pi(1, W^*) \) in order to avoid the variance of further waiting. Therefore we explicitly examine welfare under a class of risk-averse preferences in Section 5.

25
Relatedly, the covariance Equation 9 is interesting: fixing \( M_k (W(k)) \), the covariance between object value and waiting time is the sum of the “gaps” in expected object value in \( W(k) \) and each of the \( k \) sets \( W(\ell) \) (\( \ell \leq k \)). This is consistent with the intuition that, to maximize this covariance (and hence reduce payoff variance), one wants to correlate good objects with long waiting times, which necessarily correlates worse objects with short waiting times.

Going further, if we fix \( W(k) \), maximizing the covariance of object value and waiting time involves maximizing the sum of the expected values of objects from \( W(1), W(2), \ldots, W(k-1) \). This (at least in a model of continuous object types) would be accomplished by assigning only the (infinitesimally small subset of) the highest-value objects to those \( W(\ell) \)'s, and the rest of the objects to \( W(k) \). Hence covariance maximization is in a sense directly opposed to minimizing wait time variance, which is done by assigning all object types to \( W(1) \). Overall, payoff variance minimization strikes a balance between these two observations, and assigns to each position precisely the objects that would maximize that position’s expected payoff.

5 Risk-averse agents

Suppose agents have constant absolute risk aversion, i.e. utility of the form \( u(v,t) = -e^{-\alpha(v-t)} \) for risk parameter \( \alpha > 0 \). Assume that \( \alpha < \mu_1 \), to guarantee bounded expected utility from waiting for the best object.\(^{23}\)

It is worth noting some attributes of CARA expected utility, which can be proven fairly easily. First, if \( t \) is exponentially distributed with parameter \( \mu \), then

\[
E(u(-t)) = \int_0^\infty -e^{-\alpha(-t)} df(t) = -\frac{\mu}{\mu - \alpha} \tag{10}
\]

Second, if \( x_1, \ldots, x_k \) are independent random variables, then

\[
E(u(\sum x_i)) = \int \int -e^{-\alpha(\sum x_i)} df(x_1) \cdots df(x_k) = -\prod \int e^{-\alpha x_i} df(x_i) = -\prod -E(u(x_i)) \tag{11}
\]

Using equations (10) and (11), we can derive the expected utility to an agent sitting in position \( k \) under any rule \( W \), which we denote \( U^W_k \). For

\(^{23}\)The weaker assumption \( \alpha < \sum \mu_i \) would suffice, but would complicate the statement of some results
instance, when \( k = 1 \), the agent must wait to receive the next object from \( W(1) \). The waiting time for this is exponentially distributed with parameter \( M_1 \). Thus the expected utility is

\[
U^W_1 \equiv E(u(v - t)) = -(-E(u(v))) \cdot (-E(u(-t)))
\]

\[
= E(u(v)) \cdot \frac{M_1}{M_1 - \alpha}
\]

\[
= \frac{M_1}{M_1 - \alpha} \sum_{i \in W(1)} \frac{\mu_i}{M_1} (-e^{-\alpha v_i})
\]

An agent in position 2 ultimately receives an object from \( W(2) \). However, the agent’s ultimate waiting time depends on whether that object belongs to \( W(1) \) or to \( W(2) \setminus W(1) \). In the former case, the agent’s waiting time is \( t_2 + t_1 \), where \( t_\ell \) represents the (exponentially distributed) waiting time for an arrival of an object from \( W(\ell) \). Nevertheless, for all \( k \), we can decompose the expected utility into a distribution over expected utilities that can be simplified using equations (10) and (11).

Throughout this section we rely on the following notation, which we used intermittently in Section 4. With respect to a given policy \( W \), we write

\[
\forall k \in \mathbb{N} \quad M_k \equiv \sum_{j \in W(k)} \mu_j
\]  

(12)

Since a given \( W \) is clear from context, we omit the dependence of the \( M_k \)’s on \( W \) from the notation.

We derive the expected utility of a policy \( W \) to position \( k \) in the following result.

**Theorem 7.** Fix a rule \( W \) and suppose agents have CARA utility functions with parameter \( \alpha \). The expected utility of an agent in any position \( k \in \mathbb{N} \) is

\[
U^W_k = \sum_{i \in W(k)} \frac{\mu_i}{M_k} (-e^{-\alpha v_i}) \cdot \prod_{\ell = \min W^{-1}(i)}^k \frac{M_{\ell}}{M_{\ell} - \alpha}
\]  

(13)

where the \( M_\ell \)'s satisfy Equation 12.

**Proof.** Fix \( W \), \( \alpha \), and a position \( k \). By Theorem 4 an agent in position \( k \) ultimately consumes object \( i \in W(k) \) with probability \( \mu_i/M_k \). Conditional
on consuming \( i \in W(k) \), the agent’s waiting time is \( t_k + t_{k-1} + \cdots + t_\ell \) where \( i \in W(\ell) \setminus W(\ell - 1) \), and where \( t_j \) is exponentially distributed with parameter \( M_j \). This is because, in order to consume such an \( i \), the agent must first advance to position \( \ell \) in the queue and then receive an object, requiring waits for objects from \( W(k), W(k - 1), \ldots, W(\ell) \).

Denoting \( t \) as the total (unconditional) waiting time and \( v \) as the value of the received object, we have

\[
U^W_k \equiv E(u(v-t)) = \sum_{\ell=1}^{k} \sum_{i \in W(\ell) \setminus W(\ell-1) \setminus W(\ell-2) \setminus \cdots \setminus W(1)} \frac{\mu_i}{M_k} E(u(v_i - \tau_i))
\]

where the second and third lines follow from Equation 11, and the last from Equation 10. For each \( i \), the \( M_j/(M_j - \alpha) \) term appears for each position \( j \leq k \) satisfying \( i \in W(j) \), so the last line yields Equation 13.

To see the effects of modifying a policy, fix \( \ell \leq k \), and consider removing some object \( i \in W(\ell) \setminus W(\ell - 1) \) from \( W(\ell) \) (i.e. assigning the object to position \( \ell + 1 \) rather than \( \ell \)). This increases the value of \( \min W^{-1}(i) \) from \( \ell \) to \( \ell + 1 \) in Equation 13, yielding two effects. First, this subjects that object to one less “penalty term” \( M_{\ell}/(M_{\ell} - \alpha) > 1 \), which increases \( U^W_k \). This is interpreted as follows: the risk averse agent in a later position \( k \) “gains in the short run” when an earlier position must now defer object \( i \) to a later position. Second, however, such a change decreases \( M_\ell \), which in turn increases the \( M_\ell/(M_\ell - \alpha) \) penalty term which applies to all (remaining) objects in \( W(\ell) \setminus \{i\} \). That is, the agent will have to wait longer (increasing risk) in those situations in which he moves from position \( \ell + 1 \) to \( \ell \) without receiving this object \( i \).

To further investigate this tradeoff, we first provide this easily interpreted recursive expression for \( U^W_k \). It can be derived through Equation 13.
Corollary 5. For $k \geq 1$,

$$U_{k+1}^W = \frac{M_{k+1}}{M_{k+1} - \alpha} \left( \frac{M_k}{M_{k+1}} U_k^W + \sum_{i \in W(k+1) \setminus W(k)} \frac{\mu_i}{M_{k+1}} (-e^{-\alpha v_i}) \right)$$

A direct proof of this follows from the observation that an agent in position $k+1$ must (i) endure the waiting time for an object from $W(k+1)$, and then (ii) either experience the additional (continuation) payoff of being in position $k$, or immediately receive an object from $W(k+1) \setminus W(k)$. Equation (10) provides the first term.

Proof. By (14), we need to show that $U_k^W$ is improved by adding $i$ to $W(k)$ whenever $u(v_i) = -e^{-\alpha v_i} > U_k^W$ (This follows intuitively, but can also be derived from (13).)

On the other hand, consider an object type $i \in W(k) \setminus W(k-1)$ for $k \geq 2$. Would $U_k^W$ be improved by moving object $i$ to $W(k-1)$? It turns out that it would if and only if $U_{k-1}^W$ would also be improved.

Lemma 3. Fix policy $W$, position $k \geq 2$, and (if one exists) object type $j \in W(k) \setminus W(k-1)$. Let $W'(k-1) = W(k-1) \cup \{j\}$, and $W'(\ell) = W(\ell)$ for all $\ell \neq k-1$, i.e. $W'$ is obtained from $W$ by allocating $j$ to $k-1$ instead of to $k$. Then $U_k^W \geq U_k^{W'}$ if and only if $U_{k-1}^W \geq U_{k-1}^{W'}$. That is, $U_k$ and $U_{k-1}$ “agree” on whether $j$ should be allocated to position $k-1$ or $k$.

Proof. By (14), we need to show that $U_k^W \geq U_k^{W'}$ if and only if $U_{k-1}^W \geq u(v_i) = -e^{-\alpha v_i}$.

Observe that $M_k^{W'} = M_k^W$ and that $W(k) \setminus W(k-1) = \{j\} \cup (W'(k) \setminus W'(k-1))$. This cancels some terms in the expression of Corollary 5, so that

$$U_k^W \geq U_k^{W'} \iff \frac{M_{k-1}^W}{M_k^W} U_{k-1}^W + \frac{\mu_j}{M_k^W} u(v_j) \geq \frac{M_{k-1}^W}{M_k^{W'}} U_{k-1}^{W'}$$

Since $M_{k-1}^{W'} = M_{k-1}^W + \mu_j$ the latter inequality becomes

$$\frac{M_{k-1}^W}{M_{k-1}^W + \mu_j} U_{k-1}^W + \frac{\mu_j}{M_{k-1}^W + \mu_j} u(v_j) \geq \frac{M_{k-1}^W}{M_{k-1}^{W'}} U_{k-1}^{W'}$$

Hence, for $k \geq 2$, if $U_k^W \geq U_k^{W'}$ and $U_{k-1}^W \geq U_{k-1}^{W'}$ then $U_k^W \geq U_k^{W'}$. This is the direct proof of Lemma 3.
Next we express $U_{k-1}^{W'}$ in terms of $U_{k-1}^{W}$. The following equation can be derived (tediously) from Equation 13; however it can be understood as follows. After adding $j$ to $W(k - 1)$, with probability $\frac{M_{k-1}^W + \mu_j}{M_{k-1}^W + \mu_j - \alpha}$ the agent receives the payoff he would have received under $W$, and with the remaining probability he receives $u(v_j)$. In both cases the term $\frac{M_{k-1}^W + \mu_j}{M_{k-1}^W + \mu_j - \alpha}$ represents the waiting cost utility as in (10). However in the former case, $U_{k-1}^{W}$ is corrected for the fact that the waiting cost utility $\frac{M_{k-1}^W + \mu_j}{M_{k-1}^W + \mu_j - \alpha}$ no longer applies. In summary, we have

\[
U_{k-1}^{W'} = \frac{M_{k-1}^W}{M_{k-1}^W + \mu_j} U_{k-1}^{W} \left[ \frac{M_{k-1}^W - \alpha}{M_{k-1}^W + \mu_j - \alpha} \right] \\
+ \frac{\mu_j}{M_{k-1}^W + \mu_j} u(v_j) \frac{M_{k-1}^W + \mu_j}{M_{k-1}^W + \mu_j - \alpha} \\
= \frac{M_{k-1}^W - \alpha}{M_{k-1}^W + \mu_j - \alpha} U_{k-1}^{W} + \frac{\mu_j}{M_{k-1}^W + \mu_j - \alpha} u(v_j)
\]

Now (15) becomes

\[
\frac{M_{k-1}^W}{M_{k-1}^W + \mu_j} U_{k-1}^{W} + \frac{\mu_j}{M_{k-1}^W + \mu_j} u(v_j) \geq \frac{M_{k-1}^W - \alpha}{M_{k-1}^W + \mu_j - \alpha} U_{k-1}^{W} + \frac{\mu_j}{M_{k-1}^W + \mu_j - \alpha} u(v_j)
\]

which is true precisely when $U_{k-1}^{W} \geq u(v_j)$. □

**Lemma 3** leads to an analog of Theorem 2 under CARA utility, namely that one policy, which we denote $W^\alpha$, Pareto-dominates every other such rule. It is constructed by recursively finding sets $W(k)$ that maximize the expected utility of positions $k = 1, 2, 3, \ldots$.

**Theorem 8.** Suppose agents’ preferences are described by a CARA utility function with (common) parameter $\alpha$. Consider the (generically unique) policy $W^\alpha$ defined by sequentially maximizing $U_k^{W}$ (using equation (13)) for $k = 1, 2, 3, \ldots$. Then $W^\alpha$ Pareto-dominates every other policy.

**Proof.** Obviously no policy can improve the expected utility to position $k = 1$. Fixing $W^\alpha(1)$, we cannot increase $U_2^{W^\alpha}$ by adding any object $j \not\in W^\alpha(2)$ to $W^\alpha(2)$. Fixing $W^\alpha(2)$, Lemma 3 implies that we cannot increase $U_2^{W^\alpha}$ by changing $W^\alpha(1)$ to some $W(1) \subseteq W^\alpha(2)$. Repeating these arguments for $k = 3, 4, \ldots$ proves the result. □
6 Model extensions

6.1 Discounted payoffs

It turns out that with discounted payoffs, an agent in position \( k \) is no longer exactly indifferent among all rules having the same \( W(k) \), i.e. Theorem 1 breaks down. Intuitively this can be seen by comparing, say, a constant standards rule \( W(1) = \cdots = W(k) \), and a rule in which \( W(1) \subset \cdots \subset W(k) \). We know that both rules have equal expected waiting times for the agent in position \( k \) and the same probability distribution over objects (Theorem 4). However the former rule gives the agent any object in \( W(k) \) according to the same probability distribution of waiting time, whereas the latter rule tends to give objects from \( W(1) \) after longer waiting times than objects from \( W(K) \setminus W(1) \). If, for example, \( W(1) \) contains the best objects, the agent in position \( k \) is worse off having to wait longer for (and hence further discount) those valuable objects.

Formally, suppose agents (continuously) discount the future at a nominal interest rate \( r \) per time period. If an agent pays (unit flow) waiting costs for an amount of time that is exponentially distributed with parameter \( \mu \), then the expected NPV of these waiting costs is\(^{24}\)

\[
\frac{1}{r + \mu}.
\]

If an agent is to receive an object worth \( v \) only after waiting some time that is exponentially distributed with parameter \( \mu \), then the expected NPV of this object is

\[
v\mu/(r + \mu).
\]

Therefore, for a fixed waiting policy \( W() \), it is easy to derive the expected NPV (ENPV) of an agent in position \( k = 1 \). The ENPV of an object randomly chosen from \( W(1) \) minus the ENPV of the waiting time for such an object to arrive is

\[
ENPV_1 = \left( \sum_{i \in W(1)} \frac{\mu_i v_i M_1}{M_1 r + M_1} \right) - \frac{1}{r + M_1}
\]

\[
= \frac{(\sum_{i \in W(1)} \mu_i v_i) - 1}{r + M_1}.
\]

\(^{24}\)See appendix for proofs of these fundamentals.
At first glance $ENPV_1$ appears as a simple generalization of $\Pi(1)$, as derived in the proof of Theorem 1. However the general description of $ENPV_k$ turns out not to be as simple. The simplest way to think of $ENPV_2$ is to consider the agent in position 2 acquiring payoffs as follows. First the agent waits for an arrival from $W(2)$. Such an object might be allocated to this agent, but also could be allocated to position 1 (depending on $W(1) \subseteq W(2)$). In the latter case, we can imagine that the agent in position 2 is in fact allocated a pseudo-object with value $ENPV_1$. That is, define $v'_i$ so that $v'_i = ENPV_1$ for $i \in W(1)$ and $v'_i = v_i$ otherwise. Solving as above and then substituting,

$$ENPV_2 = \frac{(\sum_{i \in W(2)} \mu_i v'_i) - 1}{r + M_2}$$

$$= \frac{1}{r + M_2} \left[ M_1 ENPV_1 + (\sum_{i \in W(2) \setminus W(1)} \mu_i v_i - 1) \right]$$

$$= \frac{1}{r + M_2} \left[ \frac{M_1}{r + M_1} (\sum_{i \in W(1)} \mu_i v_i - 1) + (\sum_{i \in W(2) \setminus W(1)} \mu_i v_i - 1) \right]$$

In general,

$$ENPV_k = \frac{1}{r + M_k} \left[ M_{k-1} ENPV_{k-1} + (\sum_{i \in W(k) \setminus W(k-1)} \mu_i v_i - 1) \right]$$

which can be used to prove the following general formula.

**Theorem 9.** Suppose agents continuously discount payoffs at nominal rate $r$ per period. For rule $W$, the expected NPV of payoff to an agent in position $k$ is

$$\Pi(k) = \frac{1}{M_k + r} \sum_{\ell=1}^{k} \left[ \left( \sum_{i \in W(\ell) \setminus W(\ell-1)} \mu_i v_i - 1 \right) \prod_{j=\ell}^{k-1} \frac{M_j}{M_j + r} \right]$$

When there is no discounting ($r = 0$), this reduces to Equation 1. This payoff can be interpreted as follows. The expected discounted payoff to position $k$ can be decomposed into the sum of $k$ position-specific components (positions $\ell = 1, \ldots, k$). Each such component has a “value part” $(\sum_{i \in W(\ell) \setminus W(\ell-1)} \mu_i v_i)$ and a “cost part” $(-1)$ of passing through that position. The final product term represents two things, depending on which
part we look at. For the value part, it is the amount by which the future value must be discounted to the present (the $e^\mu/(r + \mu)$ described above). For the cost part, its denominator represents the discounting of cost flow (the $1/(r + \mu)$ above) while its numerator contributes to the probability of reaching any such position $\ell \leq k$.

### 6.2 Object spoilage

In progress...

### 6.3 Involuntary departures

A simple model of exogenous agent departures (e.g. patient deaths) can be modeled by adding artificial "departure objects" to $O$. Imagine that remaining agents leaves the queue without an object, according to a Poisson process, with each agent having the same departure rate $\lambda$. Artificially add one object $d_k$ with arrival rate $\lambda$ for each position $k$. By restricting attention to policies such that $d_k \in W(\ell) \setminus W(\ell - 1)$ if and only if $\ell = k$, each agent is exposed to the possibility of “death” as long as he is in the queue. Payoffs can be calculated conditional on not having received one of those objects.

### 7 Relation to a parallel-processor problem

The following special case of our model has been analyzed by Agrawala et al. (1984) and related papers. Suppose there are $J$ jobs, each of which must be processed on one of $n$ processors. Each processor $i$ has its own speed $s_i$ and can process one job at a time. Each job $j$ has an unknown length $\ell_j$ which is drawn from an exponential distribution with mean $1/\lambda$ at the instant its processing begins, so the time to process job $J$ on processor $i$ will be $\ell_j/s_i$.

When processing a batch of jobs, a natural optimization question arises: is it better to use all available processors, or should some number $m$ of slowest processors be left idle? How does $m$ depend on the number of remaining jobs to be processed?

When the objective is to minimize the total expected time the $J$ jobs spend in-system (i.e. waiting times plus processing times), a simple threshold-based policy is shown to be optimal by (1984). They also observe that this
socially optimal policy would correspond to equilibrium behavior when the jobs are put in a waiting list and selfishly decide which available server to use, or whether to continue waiting (for a faster server to become available). Kumar and Walrand (1985) extend this equivalence to models with varying assumptions on distributions.

The insight behind the equivalence comes from an intuitive, recursive argument: Imagine that when \( J - 1 \) jobs are queued, socially optimal policies are equivalent to equilibrium behavior. Fixing this behavior, consider the presence of a \( J \)th job, at the end of the queue, that strategically minimizes its own waiting time. Since its behavior cannot have an effect on the preceding jobs, this also minimizes the expected sum of the \( J \) waiting times.

This parallel-processor problem is a special case of our model, where the arrival rates \( \mu_i \) completely determine the values \( v_i \). Specifically, each processor \( i \) finishes any particular job according to an exponential distribution with parameter \( \mu_i \equiv \lambda s_i \). This simultaneously means that (i) an opening at a utilized processor \( i \) arrives with rate \( \mu_i \), and that (ii) the “value” to a job of being assigned to processor \( i \) is \( v_i \equiv -1/\mu_i \), the additional expected waiting time of \( 1/\mu_i \) from being processed by \( i \).

By Theorem 3 we can derive the thresholds of Agrawala et al. (1984) via Equation 2. A job in position \( j \) should accept an offer to use processor \( \tau \) whenever

\[
\frac{-1}{\mu_\tau} \geq \frac{-(\tau - 1) - j}{\sum_{i=1}^{\tau-1} \mu_i}
\]

which Agrawala et al. rewrite as

\[
j \geq \frac{\sum_{i=1}^{\tau-1} \mu_i}{\mu_\tau} - \tau + 1
\]

The result that these thresholds yield socially optimal behavior (minimizing total processing time of \( J \) jobs) is a corollary of Theorem 2.

8 Conclusion

We have considered the welfare implications for that arise from arbitrarily, but consistently, influencing the acceptance or deferral decisions of agents in a waiting list for randomly arriving objects. Our first result shows that the (continuation) welfare to an agent who is in a position \( k \) of the waiting list depends only on the cumulative set of object types that this agent could
potentially receive, and in particular is unaffected by the precise position $\ell \leq k$ in which the agent would be eligible to receive any particular such type. We use this result to show that the exertion of zero influence over the agents yields the unique (position-wise) Pareto-dominant level of influence.\footnote{The result also yields an interesting theoretical curiosity: equilibrium payoff to any position $k$ has a very simple form, equivalent to the payoff in a hypothetical scenario where that agent could dictate the subset of object types available to society, and force every agent at the front of the waiting list to accept any of them.}

Regarding risk, two of our results suggest a gap between the viewpoints of two parties: (i) an agent in the waiting list who receives a payoff via both the object quality and the waiting cost, and (ii) a party whose payoff is determined only through the waiting cost. As above, the former party (i) obtains maximal welfare (under CARA utility) as a result of equilibrium behavior in a decentralized game when all waiting agents have such preferences, again advocating a laissez-faire approach to waiting lists. On the other hand, the latter party (ii) typically obtains minimal risk (in terms of variance) by assigning more object types to the earliest agents in the waiting list, compared to the laissez-faire outcome. Hence such a variance-averse party would be in favor of some form of influence over waiting list behavior.

References


