A Pedagogical Example of Non-concavifiable Preferences

James Schummer*
Department of Economics
University of Rochester
Rochester, NY 14627

March 27, 1998

Abstract

Mostly as a pedagogical exercise, I (non-rigorously) describe an example of strictly convex, monotonic, smooth, continuous preferences on R^2_+ that are not represented by any concave utility function. The construction for higher dimensions is straightforward. The existence of such preferences is a sort of folk-knowledge (Kannai, 1977; MasColell, 1985), but I have not yet seen a clearly explained example. The construction is based on a similar one by William Thomson (1990) for non-concavifiable "single-peaked" preferences over points on a line.

I also comment on why "spiral staircase" preferences are non-concavifiable.

*Comments from John H. Boyd, III and Matt Mitchell are appreciated. This note builds on previous work by William Thomson.

Non-concavifiable Preferences

A bundle is a point $(x_1, x_2) \in \mathsf{R}^2_+$. A preference relation is a weak order on the set of bundles, R^2_+. We now construct a preference relation R that is strictly convex, monotonic, smooth, continuous, and not represented by a concave function.

Consider the "budget line" $B \equiv \{x \in \mathsf{R}^2_+ : x_1 + x_2 = 4\}$. Let $a = (1,3)$, $b = (2,2)$, and $c = (3,1)$. For all $x \in B$, define $\lambda(x)$ to satisfy

$$
\begin{aligned}
0 \le x_1 < 1 &\implies x = \lambda(x)(0,4) + (1-\lambda(x))a \\
1 \le x_1 \le 2 &\implies x = \lambda(x)b + (1-\lambda(x))a \\
2 < x_1 \le 3 &\implies x = \lambda(x)b + (1-\lambda(x))c \\
3 < x_1 &\implies x = \lambda(x)(4,0) + (1-\lambda(x))c
\end{aligned}
$$

Let the strictly convex preference relation R be such that for all $x, x' \in B$ such that $0 < x_1 < 2 < x'_1 < 4$,

$$
xIx' \iff \lambda(x) = \lambda(x')^2 \text{ and } \begin{bmatrix} x_1 \le 1, x'_1 \ge 3, \text{ or} \\ 1 < x_1 < 2 < x'_1 < 3 \end{bmatrix}
$$

This defines a single-peaked preference relation on B, with its peak at b. It is not representable by a concave function on B (see Thomson, 19??). The idea is that if the "left half" of u were made concave, the resulting function would have a slope of 0 at c, which would violate concavity on the "right half."

All that remains is completing the construction of R for all of R^2_+. That this can be done in a way consistent with strict convexity, monotonicity, smoothness, and continuity should be clear by observing Figure 2.

Why Spiral-Staircase Preferences are Non-concavifiable

Examples of weakly convex, monotonic, continuous preferences that are not represented by a concave utility function appear in Kannai (1977) and Mas-

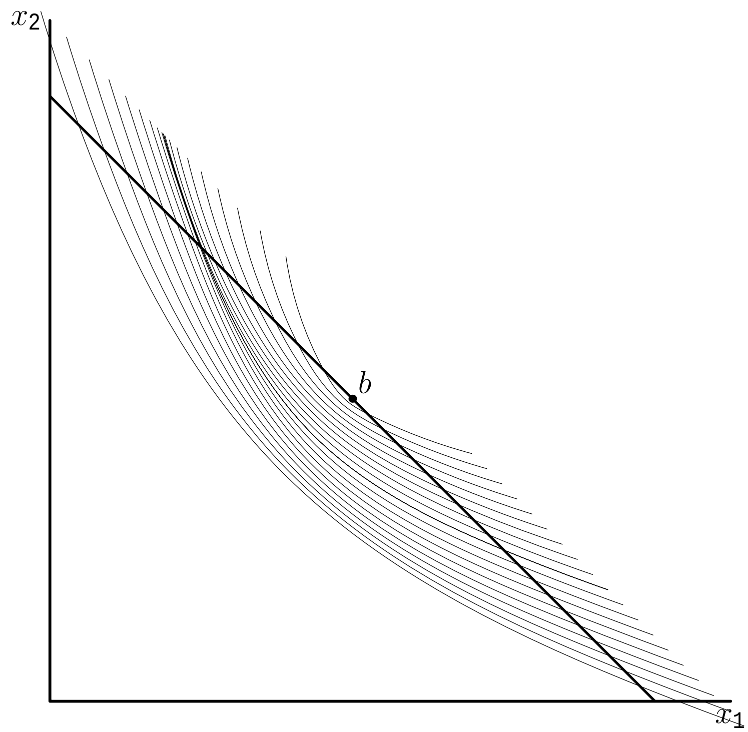

Figure 2: Some indifference curves of non-concavifiable preferences.

Colell (1985). The indifference curves are linear, but not parallel. For example, we can construct such preferences on the convex hull of the points $(0,0)$, $(2,0)$, $(1,1)$, and $(0,1)$. Let C denote that convex hull. Let R be a monotonic preference relation on C such that: for all $x, x' \in C$, $x \; I \; x'$ if and only if $\lambda x + (1-\lambda)x' = (0,2)$ for some $\lambda \in \mathsf{R}$. Suppose by contradiction that such preferences are represented by a concave function u. We will show that the slope of u at various points in C is infinite, leading to a contradiction.

First we will calculate the slope of u at $(0,1)$ in the direction of $(1,0)$. Let s denote that slope. Without loss of generality, let $u(0,0) = 0$ and $u(1,1) = 1$. Then by concavity, $s \geq 1$.

However, concavity also implies $u(1/2,1/2) \geq 1/2 \cdot u(0,0) + 1/2 \cdot u(1,1)$, therefore $u(1/3,1) = u(1/2,1/2) \geq 1/2$. Thus by concavity, $s \geq (1/2)/(1/3) =$

3/2.

By a similar argument concerning $u(1/4, 1/4)$, concavity implies $s \geq (3/2)^2$. Continuing *ad infinitum*, $s \geq (3/2)^k$ for all k, *i.e.* s must be infinite. However the same type of argument can be used at any point in the *interior* of C, calculating the slope of u in the direction that is normal to the indifference curve passing through that point. Hence u can not exist.

References

[1] Jordan, J.S. (1982) A Property of the Demand Correspondence of a Concave Utility Function, *Journal of Mathematical Economics* **9**, 41–50.

[2] Kannai, Y. (1977) Concavifiability and Constructions of Concave Utility Functions, *Journal of Mathematical Economics* **4**, 1–56.

[3] Mas-Colell, A. (1985) *The Theory of General Economic Equilibrium: A Differential Approach*, Cambridge University Press, Cambridge.

[4] Thomson, W. (1990) On the existence of concave representations of single-peaked preferences, University of Rochester mimeo.