An Ascending Vickrey Auction for Selling Bases of a Matroid[*]

Sushil Bikhchandani[†]   Sven de Vries[‡]   James Schummer[§]

Rakesh V. Vohra[¶]

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Abstract

Consider selling bundles of indivisible goods to buyers with concave utilities that are additively separable in money and goods. We propose an ascending auction for the case when the seller is constrained to sell bundles whose elements form a basis of a matroid. It extends easily to polymatroids. Applications include scheduling (Demange, Gale, and Sotomayor, 1986), allocation of homogeneous goods (Ausubel, 2004), and spatially distributed markets (Babaioff, Nisan, and Pavlov, 2004). Our ascending auction induces buyers to bid truthfully, and returns the economically efficient basis. Unlike other ascending auctions for this environment, ours runs in pseudo-polynomial or polynomial time. Furthermore we prove the impossibility of an ascending auction for nonmatroidal independence *set*-systems.

Keywords Matroid, Vickrey, multi-item, combinatorial, auction, polymatroid.

1 Introduction

Consider selling bundles of indivisible objects to buyers with utility which is additively separable in money and individual items. In particular, we analyze such situations where the seller is constrained to sell a subset of objects that forms a basis with respect to some underlying matroid structure (described below) on the objects. While this may seem like an abstract class of problems in the mechanism design literature, it turns out to contain various interesting special cases. Examples (see Section 5) include: scheduling matroids (Demange et al., 1986), the allocation of homogeneous objects (Ausubel, 2004), pairwise kidney exchange (Roth, Sönmez, and Ünver, 2005), spatially distributed markets (Babaioff et al., 2004), bandwidth markets (Tse and Hanly, 1998), and multiclass queuing systems (Shanthikumar and Yao, 1992).

The primary objective is to allocate the objects efficiently; i.e. to allocate the objects which form a maximum-weight basis, where the weights are the agents' valuations for the objects. However it is a particular list of additional *design constraints* that motivates us. In this paper we present an ascending auction that both meets these constraints and allocates objects to form a maximum-weight basis.

One design constraint involves incentives. If the agents' valuations were known to the seller, it would be easy to determine the economically efficient outcome by using the greedy algorithm. When valuations are *private information*, however, the greedy algorithm does not apply. This constrains us to design a method that gives agents an incentive to reveal accurate information.

There is a well-known *sealed-bid* auction which can achieve both the incentives and efficiency objectives: the VCG mechanism (Vickrey, 1961, Clarke, 1971, Groves, 1973). It operates by asking agents for their valuations of all objects, computing an efficient allocation, and charging Vickrey prices (described below) to bidders. Nevertheless, there are reasons for eschewing sealed-bid auctions in favor of ascending auctions. Informally, an ascending auction has the auctioneer announcing prices and bidders reporting their demands at the announced prices. If the reported demands can be feasibly satisfied, the auction ends. Otherwise, the prices are adjusted upwards. An agent who reduces demand is not allowed to increase it later.

Ascending auctions differ according to the kinds of prices that are used (e.g. linear, non-anonymous, or non-linear) and the rules used to adjust prices. Ausubel (2004) and Cramton (1998) list some of the advantages

of ascending auctions, including that
- they are more transparent,
- bidders reveal less private information,
- communication and computation costs for bidders may be lower, and
- they may be better even with mild interdependencies among bidders'
valuations.

This motivates our second design constraint: to search for an *ascending* auction in our environment.

This constraint, too, has been addressed in the literature. Previous work—described in Section 1.1—has developed ascending auctions for environments more general than ours. However, that generality comes at the price of complexity. The auctions discussed in Section 1.1 rely on a number of prices that is exponential in the number of objects. Our third concern, therefore, is to design an auction with low complexity. Unlike other combinatorial environments, we show that the matroid environment admits an ascending auction that runs in polynomial time. Furthermore, we prove the impossibility of an ascending auction for nonmatroidal independence *set*-systems.

We meet our objectives in this paper by describing an ascending auction for the matroid environment that implements the outcome of a sealed-bid VCG auction. Specifically,

1. Bidding truthfully is an ex-post equilibrium of the auction. It should be noted that this result relies neither on proxy bidding schemes that restrict the bids that bidders make nor on consistency checks on bids.

2. The truthful equilibrium results in the efficient outcome.

3. The complexity of the auction scales polynomially in the number of elements and units.

The sealed-bid VCG auction mentioned above gives bidders the incentive to report their valuations truthfully because it charges Vickrey prices. To describe this auction for general environments, let A be a set of abstract alternatives. Each agent $j \in N$ has a valuation $v^j(a)$ for each alternative $a \in A$. *Economic efficiency* requires choosing an alternative a^* that yields total value $V(N) := \max_{a \in A} \sum_{j \in N} v^j(a)$.

The VCG auction finds an allocation that is efficient with respect to the *reported* valuations. Each bidder is induced to truthfully report his valuations, however, because he is charged the net effect his presence has on

the other bidders, in the following sense. If bidder $j \in N$ were absent, economic efficiency would require finding an alternative that provides value $V(N \setminus j) := \max_{a \in A} \sum_{k \in N \setminus j} v^k(a)$. Therefore, the net effect that j's presence has on the other bidders is

$$V(N \setminus j) - \sum_{k \in N \setminus j} v^k(a^*)$$

which defines bidder j's *Vickrey payment*. Bidder j's net *Vickrey payoff* is therefore

$$v^j(a^*) - \left[V(N \setminus j) - \sum_{k \in N \setminus j} v^k(a^*) \right] = V(N) - V(N \setminus j).$$

That is, his net payoff equals his net *contribution* to attainable social surplus, which is why this amount is also called bidder j's *marginal product* (see Makowski and Ostroy (1987) for the connection between marginal product, perfect competition, and the Vickrey mechanism). The payments in a sealed-bid VCG auction can be found by solving $n + 1$ optimization problems: one to find $V(N)$ (and a^*) and n more to find each $V(N \setminus j)$.

1.1 Related Literature

There is a substantial literature on multi-object, ascending-price auctions. We describe some of this previous work.

Models in which bidders have gross substitute preferences over heterogeneous objects have received particular attention. Kelso and Crawford (1982) and Gul and Stacchetti (2000) present ascending auctions in this setting. While these auctions return the efficient outcome when bidders bid truthfully, truthful bidding is not incentive compatible. The dynamic auction of Ausubel (2006) returns the efficient outcome in equilibrium and is incentive compatible. It works by running several dummy auctions (each of which is an instance of the one in Gul and Stacchetti 2000) and then uses the prices from each of these runs to determine the Vickrey payments. In contrast, our paper achieves the same outcome without the necessity of running dummy auctions. These dummy auctions not only increase the duration, but also substantially increase the potential for collusive behavior (see de Vries, Schummer, and Vohra, 2007).

Since preferences in our model also satisfy the gross substitutes condition, the "Agents are Substitutes" condition (Bikhchandani and Ostroy, 2002)

holds. The ascending auctions of Parkes and Ungar (2000a), Ausubel and Milgrom (2002), de Vries et al. (2007), and Mishra and Parkes (2007) rely on this condition. These auctions attain (with the appropriate consistency checks) an efficient outcome in equilibrium but they rely on an exponential (in the number of goods) number of prices. This would make the complexity of running such auctions exponential in the number of goods.

Our auction in contrast produces the same outcome without relying on a proxy bidding scheme (e.g. Parkes and Ungar (2000b), Ausubel and Milgrom (2002)) and with a complexity that is *polynomial* in the number of elements and units $|E| + \rho(E)$.

1.2 Motivating Example: Selling a Tree from a Graph

Primarily to provide intuition for our analysis, we describe an instance of a simple matroid environment: selling a spanning tree of a graph. While this example may appear abstract, a procurement version of it has applications to constructing communication networks. Other applications are discussed in Section 5.

Let $G = (V, E)$ be a complete graph with vertex set V and edge set E. For simplicity, suppose each agent $j \in N$ is interested only in a single edge $e \in E$, so that we can identify edges with agents. Let v_e be agent e's value from obtaining edge e. Our goal is to derive an ascending Vickrey auction to sell off edges that form a maximum weight spanning tree.[1] Since G is complete, no single agent is initially in a position to prevent a spanning tree from forming.

While there is no competition for edges between agents, there is competition between different bidders to cover the same minimal cut (and minimal cuts are cocircuits of the underlying graphical matroid). To see this let us compute the marginal product of an agent e that is part of a maximum weight spanning tree T. To determine agent e's marginal product we identify the reduction in weight of the spanning tree when we remove agent e and replace her with a (next best) edge. If $f \notin T$ is the largest weight edge such that

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[1]Though we speak in terms of *selling* edges, one interpretation for this problem involves a procurement setting, where the auctioneer wants to *purchase* the right to use an edge and the bidder incurs some cost $(-v_e)$ when it is used (e.g. constructing a complete communications network at minimal total social cost). For consistency with the rest of the paper, we avoid procurement terminology and speak of selling.

$T \cup f$ contains a cycle through e, then the maximum weight spanning tree that excludes e is $(T \setminus \{e\}) \cup f$. Thus agent e's marginal product is $v_e - v_f$.

Not all algorithms for finding a maximum weight spanning tree lend themselves to an ascending auction interpretation or generate Vickrey prices. The "greedy out" algorithm does: Starting with the complete set of edges, delete edges in order of increasing weight, whilst keeping the graph connected. An edge is spared deletion when all smaller weight edges that could cover the same cut have been deleted.

This algorithm can be interpreted as an auction which begins with a price $p = 0$ on each edge. Throughout the auction, this price is increased. At each point in time, each agent announces whether he is willing to purchase his edge at the current price.

As the price increases, agents drop out of the auction when the price exceeds their value v_e for the edge, reducing the connectivity of the graph. At some point an agent becomes *critical*: removing the agent disconnects the graph. The edge of the critical agent is a *bridge* of the subgraph of remaining edges. At this point the auctioneer immediately sells the edge to the critical agent at the current price. This edge is to be part of the final maximum weight spanning tree and does not drop out. The auction continues with other agents dropping out or becoming critical and ends when the last critical agent is awarded an edge and a tree is formed.

A critical agent acquires his edge at the price where another bidder dropped out of the auction. That bidder's edge is the best alternative to the critical edge. The Vickrey auction would charge just the price of this best alternative.

It is clear that this should generalize from trees to matroids. However, care is necessary regarding details of the implementation:

- Can an arbitrary cut in the graph be picked and its price increased until it contains only one element?

- What procedure should be followed if multiple agents become critical at the same time?

- Suppose an agent is interested in multiple elements, say b and b', that are part of a maximum weight basis. Suppose also that an element f of interest to a second agent is a second-best replacement for both b and b'. Can the value the second agent assigns to f be used to set the Vickrey price for b *and* b'?

These issues are addressed in the next section.

2   Selling Bases of a Matroid

We begin with a review of basic matroid concepts. Subsequently we describe the model and discuss simplifying assumptions that are without loss of generality. In Subsection 2.3, we recall a nonstandard algorithm for finding the maximum weight basis of a matroid due to Dawson (1980). This algorithm selects a sequence of cocircuit and element pairs that certifies feasibility and optimality simultaneously. We specialize this algorithm to select *VCG-sequences* that also certify optimality for all marginal allocation problems obtained after removing one buyer. The marginal allocation problems are essential to implementation of Vickrey payments and hence to efficiency and desirable incentive properties of the auction.

After establishing that these sequences yield Vickrey prices, we present Auction 3 which is an ascending-price implementation of the sealed-bid VCG auction. In Subsection 2.9 we investigate its runtime and prove in Subsection 2.10 that truthful bidding is an ex post equilibrium in this auction.

2.1   Matroid Basics

We use the standard notions of matroid theory; see Oxley (1992). A *matroid* $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ of a finite ground set E and a set $\mathcal{I}$ of subsets of E satisfying the axioms: (I1) $\emptyset \in \mathcal{I}$, (I2) if $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$, and (I3) if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Subsets of E that belong to $\mathcal{I}$ are called *independent*; all other sets are called *dependent*. Minimal dependent sets of a matroid $\mathcal{M}$ are called *circuits*; the set of all circuits of $\mathcal{M}$ is denoted $\mathcal{C}(\mathcal{M})$. Circuits consisting of a single element are called *loops*. A set $\mathcal{C}$ is the set of circuits of a matroid if and only if it satisfies these three properties: (C1) $\emptyset \notin \mathcal{C}$. (C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$. (C3) If $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$, $f \in C_1 \setminus C_2$, then there exists a $C_3 \in \mathcal{C}$ such that $f \in C_3 \subseteq (C_1 \cup C_2) - e$ *(strong circuit elimination)*.

A maximal independent set is called a *basis* of the matroid; the set of all bases of $\mathcal{M}$ is denoted $\mathcal{B}(\mathcal{M})$. All $B \in \mathcal{B}(\mathcal{M})$ have the same cardinality. The *rank* $r(S)$ of $S \subseteq E$ is the size of a largest independent set contained

in S. Matroids are characterized amongst all independence systems by the property that for any additive weighting $w\colon E \mapsto \mathbb{R}$, the greedy algorithm finds an optimal (maximum-weight) basis.

Let $\mathcal{M}$ be a matroid, $E(\mathcal{M})$ its ground set, and $\mathcal{B}^{\star}(\mathcal{M}) := \{E(\mathcal{M}) - B \colon B \in \mathcal{B}(\mathcal{M})\}$. Then $\mathcal{B}^{\star}(\mathcal{M})$ is the set of bases of a matroid on $E(\mathcal{M})$, called the *dual matroid of* $\mathcal{M}$, denoted $\mathcal{M}^{\star}$.[2] Independent sets and circuits of $\mathcal{M}^{\star}$ are called *coindependent sets* and *cocircuits of* $\mathcal{M}$ respectively. A well-known fact we use later is that any cocircuit of a matroid intersects each of its bases, and each element of a basis belongs to at least one cocircuit.

For $\mathcal{M} = (E, \mathcal{I})$ and $X \subseteq E$ the *deletion of X from $\mathcal{M}$* is the matroid defined by $\mathcal{M} \setminus X := (E \setminus X, \{I \subseteq E \setminus X \colon I \in \mathcal{I}\})$. The *contraction of X in $\mathcal{M}$* is defined by $\mathcal{M}/X := (\mathcal{M}^{\star} \setminus X)^{\star}$. A matroid $\mathcal{M}'$ that is derived by contractions and deletions from a matroid $\mathcal{M}$ is called a *minor of* $\mathcal{M}$. Some properties of minors follow.

Proposition 1 (Oxley, 1992, Prop. 3.1.11). *The circuits of $\mathcal{M}/T$ consist of the minimal non-empty members of $\{C - T : C \in \mathcal{C}(\mathcal{M})\}$.*

Proposition 2 (Oxley, 1992, Cor. 3.1.25). *$\mathcal{M} \setminus e = \mathcal{M}/e$ if and only if e is a loop or coloop of $\mathcal{M}$.*

Proposition 3 (Oxley, 1992, 3.1.14). *For every subset T of a matroid $\mathcal{M}$:*

$$\mathcal{C}(\mathcal{M} \setminus T) = \{C \subseteq E \setminus T \colon C \in \mathcal{C}(\mathcal{M})\}.$$

Lemma 4. *Let $C^{\star}$ be a cocircuit of the matroid $\mathcal{M}$ and $e \in E(\mathcal{M})$:*
(i) If $e \notin C^{\star}$ then $C^{\star}$ is a union of cocircuits of $\mathcal{M} \setminus e$.
(ii) If $e \in C^{\star}$ and $\{e\}$ is a not a coloop of $\mathcal{M}$, then $C^{\star} - e$ is a cocircuit of $\mathcal{M} \setminus e$.

Lemma 4 is the dual of Exercise 2 in Section 3.1 of Oxley (1992). It is proven in the online e-Companion to this article and implies the following.

Corollary 5. *Let $C^{\star}$ be a cocircuit of the matroid $\mathcal{M}$. If $T \subseteq E(\mathcal{M})$ contains no coloops then $C^{\star} - T$ is a union of cocircuits of $\mathcal{M} \setminus T$.*

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[2]See Oxley (1992) for this and other results.

2.2 The Economic Model

We consider a finite matroid $\mathcal{M} = (E, \mathcal{I})$ and set of agents N. The ground set E of $\mathcal{M}$ is partitioned into sets $(E_j)_{j \in N}$. Interpret E_j as the set of elements that agent $j \in N$ may feasibly purchase from the auctioneer. Let $o(e)$ denote the *prospective buyer* (owner) of element e, i.e. the unique index such that $e \in E_{o(e)}$. Since the E_j's *partition* E, $o(e)$ denotes the *index* j with $e \in E_j$.

If agent j acquires some element $e \in E_j$ it would provide him a non-negative value $v_e \in \mathbb{Q}_+$. Acquiring a set $S \subseteq E_j$ provides value $v(S) := \sum_{e \in S} v_e$. The auctioneer is constrained to sell combinations of elements that form a basis of the matroid; for an extension see Section 4. Social surplus is maximized by identifying an optimal (maximum weight) basis of $\mathcal{M}$ with respect to the values v_e.

Until Subsection 2.6 we assume that all values are distinct.

We assume as a generalization of the bridge-condition of the motivating example, the following *no-monopoly condition*, which states that it is possible to form a basis without allocating any elements to any given bidder j: $r(\mathcal{M}) = r(\mathcal{M} \setminus E_j)$ for all $j \in N$. This is equivalent to requiring that no cocircuit $C^\star$ of $\mathcal{M}$ belongs to any bidder j: $C^\star \not\subseteq E_j$ for all $j \in N$. If this condition were violated for some j, then every basis of $\mathcal{M}$ would contain elements of E_j, and $V(N \setminus j)$ would be undefined $(-\infty)$. Bidder j's Vickrey payment would also be undefined $(-\infty)$.

The assumption that agents are interested in distinct items is without loss of generality. In any environment where two agents are interested in the same item e, one can add an element e' parallel to e, and impose that the first agent can only acquire e, and the second only e'. Setting $\mathcal{I}' = \mathcal{I} \cup \{I \cup \{e'\} : I \cup \{e\} \in \mathcal{I}\}$; $\mathcal{I}'$ fulfills the matroid axioms, while any independent set contains at most one of $\{e, e'\}$.

Our auction also applies to settings where the seller is permitted to sell any independent set (when agents can have negative values). To handle this case we can introduce a dummy bidder and, for every element of $\mathcal{M}$, a parallel element that the dummy bidder values at 0.

2.3 Computing the Optimal Basis

For any cocircuit $C^\star$ of $\mathcal{M}$ let $b_{C^\star} := \arg\max\{v_e : e \in C^\star\}$ be its highest-valued element. Algorithm 1—due to Dawson (1980)—finds an optimal basis for a matroid, i.e. one that maximizes $\sum v_e$. As values are distinct, $\mathcal{M}$ and

its minors have unique optimal bases.

Algorithm 1 Optimal basis for a matroid (Dawson, 1980)

Require: Matroid $\mathcal{M}$ on ground set E with distinct values.
1: $i \leftarrow 0$
2: **while** $\mathcal{M}$ has a cocircuit that is disjoint from $\{b_1, \ldots, b_i\}$ **do**
3:     $i \leftarrow i + 1$
4:     Let $C_i^\star$ be such a cocircuit
5:     Let $b_i = b_{C_i^\star}$
6: **end while**
7: $r \leftarrow i$

Proposition 6 (Dawson, 1980, Thm. 1). *For any matroid $\mathcal{M}$ with distinct values, Algorithm 1 determines its optimal basis B and its rank $r = r(\mathcal{M})$.*

The way in which cocircuits $C_i^\star$ are chosen in Line 4 of Algorithm 1 is arbitrary (when more than one cocircuit is disjoint from $\{b_1, \ldots, b_i\}$) according to Dawson's Theorem. Intuitively, the algorithm works despite this because any cocircuit of a matroid intersects each of its bases and each element of a basis belongs to at least one cocircuit. From this it can be shown that if a cocircuit is disjoint from a subset of the optimal basis, then the highest-valued element in the cocircuit is part of the optimal basis.

We show that cocircuits $C_i^\star$ must be chosen in a *particular* order so that we obtain an algorithm that can be interpreted as an ascending auction with Vickrey prices. As we formalize next, this order is monotonic with respect to the highest-valued element of $C_i^\star \setminus E_{o(b_{C_i^\star})}$, that is, the highest-valued element of $C_i^\star$ *ignoring* elements which can be purchased by the owner of $b_{C_i^\star}$.

2.4 Sequences and Certificates

For any cocircuit $C^\star$ of $\mathcal{M}$, we define $b_{C^\star}$ to be the highest-valued element in $C^\star$. Excluding the elements which can be owned by agent $o(b_{C^\star})$, denote the highest-valued remaining element as $f_{C^\star} := \arg\max\{v_e \colon e \in C^\star \setminus E_{o(b_{C^\star})}\}$. This is the best element in $C^\star \setminus E_{o(b_{C^\star})}$; it is "second-best" in $C^\star$ in the sense

of being the best element of $C^\star$ that is associated with a bidder distinct from $o(b_{C^\star})$. The element $f_{C^\star}$ is well-defined because the no-monopoly condition ensures $C^\star \setminus E_{o(b_{C^\star})} \neq \emptyset$.

Notice that in the tree case in Subsection 1.2, $f_C^\star$ is just the element with value equal to the current price whose removal makes another agent critical. Similarly, $o(b_{C^\star})$ is the agent that becomes critical. Given that agent $o(b_{C^\star})$ might be interested in multiple elements, it may not be, that removal of $b_{C^\star}$ makes forming a maximum weight basis impossible (since the rank drops) but that only removal of all his elements (from $C^\star$) decreases the rank; nevertheless, of his elements in $C^\star$ he want $b_{C^\star}$ the most.

A sequence of cocircuit-element pairs $((C_1^\star, b_1), (C_2^\star, b_2), \ldots, (C_j^\star, b_j))$ is called *suitable* if it can be generated during the execution of Algorithm 1. That is, such a sequence is suitable for $\mathcal{M}$ if for all i

(S1) $C_i^\star \in \mathcal{C}^\star(\mathcal{M})$ and $C_i^\star \cap \{b_1, \ldots, b_{i-1}\} = \emptyset$, and

(S2) $b_i = \arg\max_{e \in C_i^\star} v_e$.

(S1) ensures that $C_i^\star$ satisfies the condition in Line 2 of Algorithm 1. It is easy to see that a list of elements $\{b_1, \ldots, b_{r(\mathcal{M})}\}$ is an optimal basis if and only if there is a sequence of $r(\mathcal{M})$ cocircuits with which it is suitable.

Although there is considerable flexibility in choosing cocircuits in a suitable sequence, our choice is constrained by the requirement that the sequence should also yield Vickrey prices for elements as they are allocated to buyers. It will be shown that the values of second-best elements of selected cocircuits, $v_{f_{C_i^\star}}$, $i = 1, 2, \ldots, r$ determine Vickrey prices. The ascending auction will satisfy: (i) selected cocircuits, together with their best elements, form a suitable sequence, (ii) the value of second-best elements in the selected cocircuits is non-decreasing, and (iii) the increase in $v_{f_{C_i^\star}}$ along the sequence is minimal. That is, in any suitable sequence $((C_1^\star, b_1), (C_2^\star, b_2), \ldots, (C_j^\star, b_j))$ selected by the auction, $v_{f_{C_i^\star}} \geq v_{f_{C_{i-1}^\star}}$ and for each i there does not exist $C^\star$ which satisfies (S1) and $v_{f_{C_i^\star}} > v_{f_{C^\star}} \geq v_{f_{C_{i-1}^\star}}$. In addition, if at any stage $v_{f_{C_i^\star}} > v_{f_{C_{i-1}^\star}}$ then it will turn out that elements $e \in E$ such that $v_{f_{C_i^\star}} > v_e \geq v_{f_{C_{i-1}^\star}}$ can be removed from consideration; these elements will never be either best or second-best elements of cocircuits in the remaining portion of the suitable sequence. We collect such elements in sets D_i in the definition below; hence D_i collects the elements withdrawn between the award of b_{i-1} and b_i.

A sequence

$$((C_1^\star, b_1, D_1), (C_2^\star, b_2, D_2), \ldots, (C_r^\star, b_r, D_r), (D_{r+1}))$$

$(r = r(\mathcal{M}))$ is a *VCG-sequence* for $\mathcal{M}$ if for all i,

(V1) $C_i^\star \in \arg\min_{\{C^\star \in \mathcal{C}^\star(\mathcal{M}): v_{f_{C^\star}} \geq v_{f_{C_{i-1}^\star}} \text{ and } C^\star \cap \{b_1, \ldots, b_{i-1}\} = \emptyset\}} v_{f_{C^\star}}$

(For ease of notation, let $v_{f_{C_0^\star}} = -\infty$)

(V2) $b_i = \arg\max_{e \in C_i^\star} v_e$, and

(V3) $D_i = \{e \in E \setminus \{b_1, \ldots, b_{i-1}\}: v_{f_{C_{i-1}^\star}} \leq v_e < v_{f_{C_i^\star}}\}$, $1 \leq i \leq r$ and

$D_{r+1} = E \setminus \left(\{b_1, \ldots, b_r\} \dot{\bigcup}_{j=1}^{r} D_j \right)$

In addition to ensuring that the value of second-best elements of cocircuits is minimally non-decreasing along the sequence, (V1) implies (S1). Therefore, if we omit the component D_i of a VCG-sequence then $((C_1^\star, b_1), (C_2^\star, b_2), \ldots, (C_r^\star, b_r))$ forms a suitable sequence. The resulting set of elements $B := \{b_1, \ldots, b_r\}$ forms the optimal basis, while the D's partition the complement of B. Let $p_i = v_{f_{C_i^\star}}$, the value of the second-best element in $C_i^\star$.

Call a cocircuit $C^\star$ of $\mathcal{M}$ *feasible at* $p \geq 0$ if $v_{f_{C^\star}} = p$. We say that $C^\star$ is *feasible at* $p \geq 0$ *via* e if $C^\star$ is feasible at p and $e = f_{C^\star}$.

A VCG-sequence can be identified by an algorithm whose i^{th} iteration is described below.[3]

> 1: Let $\{e_1, e_2, \ldots\} = \{e \in E: v_e \geq v_{f_{C_{i-1}^\star}}\} \setminus \{b_1, \ldots, b_{i-1}\}$ be in increasing order of v_e. Let $k \leftarrow 1$ and $e \leftarrow e_k$
> 2: **while** every $C^\star \in \mathcal{C}^\star(\mathcal{M})$ with $f_{C^\star} = e$ intersects $\{b_1, \ldots, b_{i-1}\}$ **do**
> 3: $\quad D_i \leftarrow D_i \cup \{e\}$; $k \leftarrow k+1$; $e \leftarrow e_k$
> 4: **end while**
> 5: Choose $C_i^\star \in \mathcal{C}^\star(\mathcal{M})$ so that $e = f_{C_i^\star}$ and $C_i^\star \cap \{b_1, \ldots, b_{i-1}\} = \emptyset$.
> 6: $b_i \leftarrow b_{C_i^\star}$

It would be convenient to discard elements of the matroid as we go along; for that we introduce the next definition. A sequence $((C_1^\star, b_1, D_1), (C_2^\star, b_2, D_2),$

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[3] We follow the convention that the **while** condition in step 2 below is satisfied if there exists no $C^\star \in \mathcal{C}^\star(\mathcal{M})$ with $f_C^\star = e$.

$\ldots, (C_r^\star, b_r, D_r), (D_{r+1}))$ is called a *condensed VCG-sequence* if $D_{r+1} = E \setminus$ $\left(\{b_1, \ldots, b_r\} \dot{\cup} \bigcup_{j=1}^r D_j \right)$ for $r = r(\mathcal{M})$ and (with $\mathcal{M}_1 := \mathcal{M}$ and $\mathcal{M}_{i+1} := \mathcal{M}_i / b_i \setminus D_i$) we have for all $1 \leq i \leq r$:

(CV1) $f_i = \arg\max_{\{f \in E(\mathcal{M}_i) : \mathcal{M}_i \setminus \{e : v_e < v_f\} \text{ fulfills the no monopoly condition}\}} v_f$ and
$\qquad D_i = \{e \in E(\mathcal{M}_i) : v_e < v_{f_i}\}$,

(CV2) $C_i^\star$ is a cocircuit of $\mathcal{M}_i \setminus D_i$ with $f_{C_i^\star} = f_i$, and

(CV3) $b_i = \arg\max_{e \in C_i^\star} v_e$.

Clearly, a condensed VCG-sequence can be identified by an algorithm whose i^{th} iteration (starting with $D_i = \emptyset$) is described in Procedure 2.

Procedure 2 Determining one step of a condensed VCG sequence.

1: Let $\{e_1, e_2, \ldots, \} = E(\mathcal{M})$ be in increasing order of
$\quad$ v. Let $k \leftarrow 1$ and $e \leftarrow e_k$
2: **while** $\mathcal{M} \setminus (D_i \cup \{e\})$ fulfills the no-monopoly condition **do**
3: $\quad$ $D_i \leftarrow D_i \cup \{e\}$; $k \leftarrow k + 1$; $e \leftarrow e_k$
4: **end while**
5: Choose $C_i^\star \in \mathcal{C}^\star(\mathcal{M} \setminus D_i)$ with $e = f_{C_i^\star}$.
6: $b_i \leftarrow b_{C_i^\star}$ and $p_i \leftarrow v_e$
7: $\mathcal{M} = \mathcal{M}_{i+1} \leftarrow \mathcal{M}/b_i \setminus D_i$

Notice that if $\mathcal{M} \setminus D_i$ fulfills the no-monopoly condition, then the **while** condition is equivalent to that every $C^\star \in \mathcal{C}^\star(\mathcal{M})$ with $f_C^\star = e$ intersects $\{b_1, \ldots, b_{i-1}\}$.

Our goal is to prove that for every condensed VCG sequence, it is possible to find a VCG sequence that has the same D_i, b_i, and second best element $f_{C_i^\star}$, and vice versa.

Theorem 7. *For every VCG-sequence* $((C_1^\star, b_1, D_1), (C_2^\star, b_2, D_2), \ldots,$ $(C_r^\star, b_r, D_r), (D_{r+1}))$ *there exists a condensed VCG-sequence* $((C_1''^\star, b_1', D_1'),$ $(C_2''^\star, b_2', D_2'), \ldots, (C_r''^\star, b_r', D_r'), (D_{r+1}'))$ *with* $(b_k', D_k', f_{C_k''^\star}) = (b_k, D_k, f_{C_k^\star})$ *for* $1 \leq k \leq r+1$. *Also for a condensed VCG-sequence there exists a corresponding VCG-sequence.*

The fairly routine proof is given for the reader's convenience in the e-Companion A.2.

2.5 Vickrey Prices

We show that an element $f_{C^\star}$ can be "second-best" for at most one cocircuit per bidder. This property is important for determining Vickrey payments. If it did not hold then one bidder might be awarded two elements b, b', when another bidder, $o(f_{C^\star})$, drops his interest in $f_{C^\star}$; the removal of b, b' would require that $f_{C^\star}$ be inserted *twice* into the basis, which is not possible.

Lemma 8. *If a VCG-sequence contains two distinct cocircuits $C_i^\star, C_j^\star$ with $o(b_i) = o(b_j)$ then $f_{C_i^\star} \neq f_{C_j^\star}$.*

Proof. Recall from Procedure 2, Step 6 that $p_i = v_{f_{C_i^\star}}$. If $p_i \neq p_j$ then $f_{C_i^\star} \neq f_{C_j^\star}$.

Assume $p_i = p_j$. Suppose that $f_{C_i^\star} = f_{C_j^\star}$ and $o(b_i) = o(b_j)$, and without loss of generality that $v_{b_i} > v_{b_j}$. By strong circuit elimination, there exists a cocircuit $C^\star \subseteq (C_i^\star \cup C_j^\star) - f_{C_i^\star}$ that contains b_i. Since b_i is the highest valued element of $C_i^\star \cup C_j^\star$, it follows that $b_{C^\star} = b_i$.

Furthermore, since $f_{C_i^\star}$ is the highest valued element of $(C_i^\star \cup C_j^\star) \setminus E_{o(b_i)}$, it follows that $v_{f_{C^\star}} < v_{f_{C_i^\star}}$ (because $f_{C_i^\star} \notin C^\star$, and values are distinct). Therefore $C^\star$ must have been selected in an earlier step $k < \min\{i, j\}$ of the sequence, and $b_k = b_i$. This contradicts $C_i^\star \cap \{b_1, \ldots, b_{i-1}\} = \emptyset$. $\square$

The next lemma shows that no element is best for one cocircuit and second-best for another cocircuit in the same VCG-sequence.

Lemma 9. *For a VCG-sequence and all i, j holds $b_i \neq f_{C_j^\star}$.*

Proof. Since $C_j^\star$ is disjoint from $\{b_1, \ldots, b_{j-1}\}$, $i < j$ implies $f_{C_j^\star} \neq b_i$. Clearly $b_i \neq f_{C_i^\star}$ (covering the case $i = j$). Finally, $i > j$ implies $v_{b_i} > v_{f_{C_i^\star}} \geq v_{f_{C_j^\star}}$, proving $b_i \neq f_{C_j^\star}$. $\square$

To determine Vickrey payments for the optimal basis $B = \{b_1, \ldots, b_r\}$ derived from a VCG-sequence, we need information about the optimal bases for each marginal matroid $\mathcal{M}^{-j} := \mathcal{M} \setminus E_j$. To prove that the p_i's are Vickrey payments for the b_i's we show that the optimal basis of each $\mathcal{M}^{-j}$ is

$$B^{-j} := (B \setminus E_j) \cup \{f_{C_i^\star} : b_i \in B \cap E_j\}. \tag{1}$$

We argue that B^{-1} is an optimal basis of $\mathcal{M}^{-1}$, the proof of optimality of the other B^{-j}'s is analogous. Given the VCG-sequence $((C_1^\star, b_1, D_1),$

$(C_2^\star, b_2, D_2), \ldots, (C_r^\star, b_r, D_r), (D_{r+1}))$ consider the sequence $B^{-1} = (b_1', \ldots, b_r')$:

$$b_i' = \begin{cases} b_i & \text{if } o(b_i) \neq 1 \\ f_{C_i^\star} & \text{if } o(b_i) = 1. \end{cases}$$

Lemmas 8 and 9 imply that $b_1', \ldots, b_r'$ are distinct. To prove that B^{-1} is optimal for $\mathcal{M}^{-1}$ we need to construct a sequence of cocircuits that makes B^{-1} suitable for $\mathcal{M}^{-1}$.

As $\mathcal{M}^{-1}$ is a deletion minor of $\mathcal{M}$ and $\mathcal{M}$ has no coloops, it follows from Corollary 5 that for every cocircuit $C^\star$ of $\mathcal{M}$, the set $C^\star \setminus E_1$ is the union of cocircuits of $\mathcal{M}^{-1}$. For $1 \leq i \leq r$ let $C_i'^\star \subseteq C_i^\star$ be the cocircuit of $\mathcal{M}^{-1}$ that contains b_i'. The sequence $((C_1'^\star, b_1'), (C_2'^\star, b_2'), \ldots, (C_r'^\star, b_r'))$ has the property that $b_i' \in C_i'^\star$ and $b_i' = \arg\max_{e \in C_i'^\star} v_e$ by construction. As all b_i' are distinct, the $C_i'^\star$ are also distinct. However, this sequence need not be suitable, as there could be indices $i < j$ with $b_i' = f_{C_i^\star} \in C_j'^\star$. The next result establishes that this difficulty can be overcome.

Lemma 10. *Suppose a sequence $\mathcal{K} := ((C_1'^\star, b_1'), \ldots, (C_r'^\star, b_r'))$ of $\mathcal{M}^{-1}$ is such that*

(1) all b_i' are different,

(2) $b_i' = \arg\max_{e \in C_i'^\star} v_e$ for $1 \leq i \leq r$,

(3) and the sequence $((C_1'^\star, b_1'), \ldots, (C_j'^\star, b_j'))$ is suitable for some $1 \leq j < r$.

Then the cocircuit $C_{j+1}'^\star$ can be modified so that conditions (1)–(3) hold for $j + 1$ and $\mathcal{M}^{-1}$.

Proof. If $((C_1'^\star, b_1'), \ldots, (C_{j+1}'^\star, b_{j+1}'))$ is suitable we are done; otherwise consider the smallest $i < j + 1$ with $b_i' \in C_{j+1}'^\star$. Using strong circuit elimination we can choose a cocircuit $C^\star$ of $\mathcal{M}^{-1}$ in $(C_i'^\star \cup C_{j+1}'^\star) - b_i'$ that contains b_{j+1}'. We replace $C_{j+1}'^\star$ by $C^\star$. By assumption, $b_k' \notin (C_i'^\star \cup C_{j+1}'^\star) - b_i'$ for every $k < i$. Therefore, $\{b_1', \ldots, b_i'\} \cap C^\star = \emptyset$, and we can replace $C_{j+1}'^\star$ by $C^\star$. Let

$$\mathcal{K}' = ((C_1'^\star, b_1'), \ldots, (C_j'^\star, b_j'), (C^\star, b_{j+1}'), (C_{j+2}'^\star, b_{j+2}'), \ldots, (C_r'^\star, b_r')).$$

Now $\mathcal{K}'$ fulfills (1)–(3) for j and $\{b_1', \ldots, b_i'\} \cap C_{j+1}'^\star = \emptyset$. Either $\mathcal{K}'$ fulfills (3) for $j + 1$, or there exists another index $i' > i$ with $b_{i'}' \in C_{j+1}'^\star$, in which case we repeat the procedure until $\mathcal{K}'$ fulfills (1)–(3) for $j + 1$. □

The hypotheses of Lemma 10 are satisfied by a VCG-sequence and $j = 1$. Repeated application of the lemma proves there are cocircuits $C_1''^\star, C_2''^\star, \ldots,$ $C_r''^\star$ that make the sequence $((C_1''^\star, b_1'), (C_2''^\star, b_2'), \ldots, (C_r''^\star, b_r'))$ suitable in $\mathcal{M}^{-1}$; hence B^{-1} is optimal for $\mathcal{M}^{-1}$ and B^{-i} is optimal for $\mathcal{M}^{-i}$.

Theorem 11. *Given a VCG-sequence. Then $B, B^{-1}, \ldots, B^{-n}$ are the unique optimal bases of $\mathcal{M}, \mathcal{M}^{-1}, \ldots, \mathcal{M}^{-n}$, respectively. The Vickrey payoff of bidder $j \in N$ is $\sum_{i:b_i \in \{b_1, \ldots, b_r\} \cap E_j} (v_{b_i} - v_{f_{C_i^\star}})$, and his Vickrey payment is $\sum_{i:b_i \in \{b_1, \ldots, b_r\} \cap E_j} v_{f_{C_i^\star}} = \sum_{i:b_i \in \{b_1, \ldots, b_r\} \cap E_j} p_i$.*

The theorem shows that if bidder j is awarded element b_i, then his final payment increases by p_i. Therefore, in a dynamic setting he could be charged p_i at the moment he is awarded element b_i.

2.6 Tie Breaking

Ties among valuations presents a slight technical complication which can be resolved without changing the ideas in our results. The reader willing to ignore this technicality may skip this subsection.

When values are not necessarily distinct, we can break ties by choosing sufficiently small positive δ and perturbing v_e by $\epsilon_e := \delta^e$ to $v_e' := v_e + \epsilon_e = v_e + \delta^e$. For sufficiently small δ, the perturbed valuations have the property that $v'(S) \le v'(T)$ implies $v(S) \le v(T)$.

Clearly a basis that is optimal for v' on a minor of $\mathcal{M}$ is also optimal for v on the same minor yielding the following generalization of Theorem 11.

Lemma 12. *Suppose a VCG-sequence for the perturbed valuation v' of a no-monopoly matroid $\mathcal{M}$ with nonnegative valuation v. Then $B, B^{-1}, \ldots, B^{-n}$ are the unique optimal bases with respect to v' and v of $\mathcal{M}, \mathcal{M}^{-1}, \ldots, \mathcal{M}^{-n}$. The Vickrey payment (under v) by bidder $j \in N$ is $\sum_{i:b_i \in \{b_1, \ldots, b_r\} \cap E_j} v_{f_{C_i^\star}}$.*

Lemma 12 allows us to extend the notions of feasible sequence, VCG-sequence, and condensed VCG-sequence, to valuation with ties, by requiring that the sequence conditions are fulfilled for a perturbed valuation. To sidestep the issue of how to determine sufficiently small δ, one could choose an arbitrary strict order $\prec$ on E and use it to break ties with respect to v. Henceforth we will break ties in this way and will demonstrate how this can be achieved *without* relying on the cooperation of the selfish bidders.

2.7 The Ascending Auction

Auction 3 Efficient ascending auction (unit-step)

Require: No-monopoly matroid $\mathcal{M}$ with nonnegative integer valuation

1: $i \leftarrow 1, \quad p \leftarrow 0, \quad r \leftarrow r(\mathcal{M})$
2: Determine an ordering of E (for tie-breaking)
3: **while** $i < r$ **do**
4:     Ask bidders to determine $F = \{f_1, \ldots, f_k\} \leftarrow \{f \in E(\mathcal{M}) : v_f = p\}$ and label the elements of F so that they are increasing according to the tie-breaking.
5:     **for** $\ell \leftarrow 1$ to k **do**
6:         **while** there exists a bidder j and a cocircuit $C^\star$ of $\mathcal{M} \setminus f_\ell$ with $C^\star \subseteq E_j$ **do**
7:             Ask bidder j to determine $\arg\max_{e \in C^\star} v_e$ and let b_i be the most valuable element from this set according to tie-breaking.
8:             Award b_i to j and charge him $p_i \leftarrow p$.
9:             $\mathcal{M} \leftarrow \mathcal{M}/b_i$
10:            $i \leftarrow i + 1$
11:         **end while**
12:         $\mathcal{M} \leftarrow \mathcal{M} \setminus f_\ell$
13:     **end for**
14:     $\mathcal{M}$ and $p \leftarrow p + 1$
15: **end while**
16: $B \leftarrow \{b_1, \ldots, b_r\}$ is the optimal basis

Auction 3 is our main algorithm. Notice that the condition $C^\star \subseteq E_j$ in Line 6 generalizes the condition we had in the motivating example that an agent owns a bridge. Observe in addition that the tie-breaking is carried out *without* the bidders' cooperation: In Lines 4 and 7 the bidders are asked questions regarding their valuations; *afterwards* the auctioneer breaks ties in their answers.

In ascending price auctions, it is standard to ask bidders to reveal their demand at current prices. Lines 4 and 7 are to be interpreted in this spirit. These steps in Auction 3 ask bidders to reveal their demand correspondences

at current prices, and in particular reveal those elements that they are just indifferent between buying or not at current prices. In implementations, this latter information can also be extracted by asking bidders to reveal demand at current prices at the next higher price increment.

It is easy to show that Auction 3 determines a condensed VCG-sequence if bidders behave truthfully, yielding our main result.

Theorem 13. *For every no-monopoly matroid with nonnegative integer valuation, if bidders behave truthfully, then Auction 3 determines an efficient allocation and charges Vickrey prices.*

Proof (Sketch; full proof in e-Companion A.3). Notice that for distinct valuations the auction determines a condensed VCG-sequence.

This would become clear if we add the line

12.5: **if** f_ℓ has not been awarded **then** $D_i \leftarrow D_i \cup \{f_\ell\}$.

By Theorem 7 there is a corresponding VCG-sequence; hence (with Subsection 2.6) the efficient allocation is found. With Theorem 11 it follows that the p_i lead to Vickrey prices. $\square$

The classical English-auction style of increasing p in Line 14 is $p \leftarrow p+1$, which is why we refer to this as the *unit-step* version of our auction. It is clear why this works when valuations are integer. There is another variant, the *long-step* version, that is faster and able to handle arbitrary (noninteger) valuation; it is introduced in Subsection 2.9.

The unit-step auction can be described simply in the following way. Start with the price set at zero where, by assumption, no bidder holds a monopoly. Then (Line 4) ask the bidders whether they have any elements at current price that they are indifferent about (because the price matches their values); such a question is not unusual for an auction. Among the elements that the bidders announce, the auctioneer can break ties in the way he has predetermined to. Hence he can pick 'the' smallest element f among them (and later larger ones). Now Line 6 is a question that requires only knowledge of the matroid and the partition of E into sets of interest E_i; since the auctioneer knows both, checking whether there is a $C^\star$ of $\mathcal{M} \setminus f_\ell$ and a bidder j with $C^\star \subseteq E_j$ is easy for him and requires no further knowledge. Therefore the auctioneer can determine whether the removal of f would give another bidder j a monopoly. If it does, let bidder j determine in Line 7 his best elements from $C^\star$; among these best elements the auctioneer determines

with his tie-breaking plan a unique best element $e \in C^\star$, awards it to j and charges j the current price. (All of this requires no advanced knowledge of bidders valuations). Then contract e and check for any other bidders that might own a cocircuit. Afterwards delete the unnecessary element f from the matroid. If at current price p another element f' was announced (but not yet awarded) then repeat the previous step; otherwise increase p and continue. In the next section, an example illustrates the auction.

2.8 An Example

An example run of the auction on a graphical matroid $\mathcal{M}$ is described in Fig. 1. As shown in Fig. 1(a), there are three bidders $\{a, b, c\}$ and five elements $\{a{:}5, a{:}4, b{:}3, b{:}2, c{:}1\}$; an element encodes the prospective buyer and value. The matroid has rank 3. Since Figs. 1(b)–1(d) show that removal of any single agent does not decrease the rank of the matroid it fulfills the no-monopoly condition.

At price $p = 1$ the situation in Fig. 1(e) results: only bidder c announces that the value of his element $f = c{:}1$ (in gray) is reached and that he is indifferent between paying 1 for it or not getting it. At this moment it has to be checked whether deletion of f would give bidder a a monopoly. As the cut (dashed line) indicates in Fig. 1(f), if f is removed then bidder a owns a monopoly, as removal of $a{:}5, a{:}4$ from $\mathcal{M} - f$ would reduce its rank. Consequentially bidder a is asked for his most valuable element out of $a{:}5, a{:}4$, to which he will answer $a{:}5$. Now the auctioneer awards him $a{:}5$ and charges 1 for it. To remove $a{:}5$ from consideration he contracts $a{:}5$ yielding Fig. 1(g).

Bidder c's indifference has another consequence. As can be seen in Fig. 1(h), removal of bidder b and $c{:}1$ decreases the rank of the matroid, hence b would have a monopoly. He is awarded the better of $b{:}2, b{:}3$ at a price of 1. Again the auctioneer contracts the awarded edge $b{:}3$ yielding Fig. 1(i).

Since no more monopolies occur at $p = 1$, element $c{:}1$ is removed and the price increases to $p = 2$. Bidder b then announces that the value of his element $b{:}2$ is reached; see Fig. 1(j). If $b{:}2$ were removed then bidder a would again have a monopoly; see Fig. 1(k). So bidder a is awarded $a{:}4$ at a price of $p = 2$, element $a{:}4$ is contracted, and $b{:}2$ is deleted; Fig. 1(l) results. As $r(\mathcal{M})$ elements have been awarded, the auction ends.

In the sealed-bid VCG auction, the elements $a{:}5, a{:}4, b{:}3$ would be awarded and bidder a would pay 3, bidder b would pay 1, and bidder c pays nothing. These payments match the outcome of our ascending auction carried out
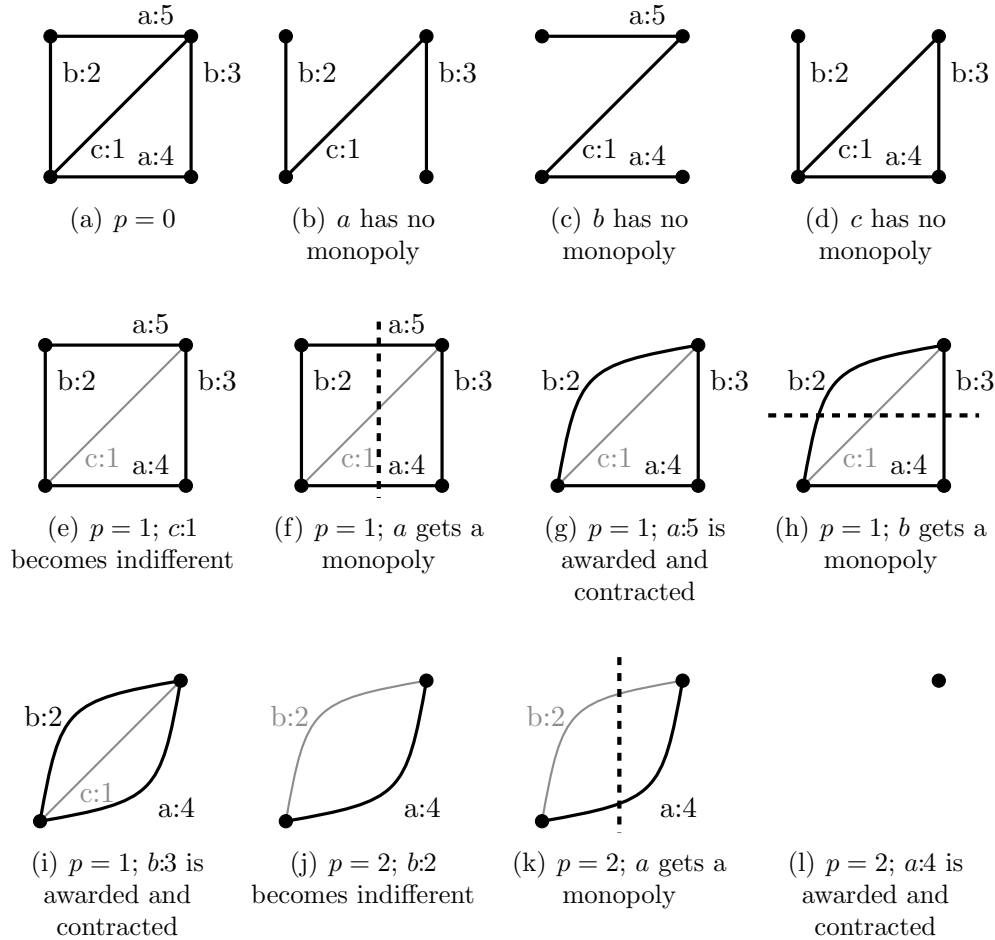
above.

Figure 1: An example illustrating Auction 3 on a graphical matroid with elements $\{a\!:\!5, a\!:\!4, b\!:\!3, b\!:\!2, c\!:\!1\}$ (encoding bidder and value) and bidders a, b, c.

2.9 Polynomial Auction

Not much has been said about the runtime of Auction 3. The problem with the unit-step version is that we have to determine several numbers (the values of the second-best elements) but do so by querying *every* smaller integer. Clearly this yields an auction that is only pseudo-polynomial (in the highest

value). More fundamentally, the classical English ascending auction to sell a single unit of a single good has the same problem. From a computational standpoint, binary search seems to be one way to solve this problem; however, under straight-forward binary search it turns out that truth-telling need not be a weakly dominant strategy. Recently Grigorieva, Herings, Müller, and Vermeulen (2007) proposed the bisection auction for a single good that takes polynomial time and in which truth-telling is again a weakly dominant strategy.

To handle arbitrary values or to be faster, however, the auction can be easily modified to the *long-step* version by changing Line 14 to read

14L: Ask each bidder j to determine $u_j = \min_{e \in E_j \cap \mathcal{M}} v_e > p$; $\mathcal{M}^p \leftarrow \mathcal{M}$ and
$\quad\quad p \leftarrow \min_{j \in N} u_j$.

This step asks each bidder for the value of his least valuable remaining object, then increases the price to the lowest of these reported values.[4] This can be implemented in a variety of ways in practice. For instance, bidders can be asked for "suggested price increases," with the assurance that only the lowest of these suggestions will be implemented in order to speedily move the auction along. If the bidder with the lowest reported price increase suggests too low of a price increase (i.e. under reports his u_j), the auction is delayed as no bidder withdraws at the resulting price.

The long-step version of Auction 3 runs in polynomial time and, since it avoids unit-steps, can accommodate nonnegative *rational* valuations. The outer loop of the long-step auction is executed at most $|E|$ times. It makes only polynomially many calls to an oracle finding a cocircuit of a minor of $\mathcal{M}$ contained in some E_i, and it performs only polynomially many steps.

Theorem 14. *For every no-monopoly matroid with nonnegative valuation, if bidders behave truthfully, then the long-step version of Auction 3 determines an efficient allocation and charges Vickrey prices. The auction goes through a polynomial number of steps (in $|E|$, $|N|$, and the largest encoding length of a value) and makes polynomial use of a cocircuit oracle (for $\mathcal{M}$ and its minors).*

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[4]In the special case of a single object auction, the price simply jumps from zero through all the bidders' values—from lowest to second-highest—and then awards the object to the highest bidder without revealing his bid. If there are more goods, however, then significantly less information is revealed.

Proof. The **while** loop of Lines 3–15 is carried out at most $r(\mathcal{M}) \leq n$ times (because in each iteration at least one element gets contracted as bidders are honest). The inner loops are carried out at most $|E|$ and $|N|$ times respectively. Therefore the polynomial-time result holds. $\square$

Notice, that under our assumption of single 'ownership' we have $N = O(|E|)$ and hence one could say that the auction is polynomial in $|E|$ and the largest encoding length of a value. On the other hand, clearly, the auction is polynomial in the number of rounds it needs, but the latter is only endogenously determined; however, as already mentioned, it can be bounded above by $|E|$.

Theorems 13 and 14 are computational results. That is, it is implicitly assumed that the bidders honestly report their information to the Auction. Of course this assumption is not without loss of generality when bidders are strategic. In the next section, we consider this issue of incentives.

2.10 Incentives

We show that in Auction 3, truthful bidding forms an *ex post equilibrium*. That is, no bidder can benefit by bidding untruthfully provided that all other bidders bid truthfully. This is a stronger equilibrium property than Bayesian Nash equilibrium.

When Vickrey payments are implemented through the use of a *sealed-bid* auction, each bidder maximizes his payoff by bidding truthfully (i.e. truthfully reporting his valuations) regardless of the strategy used by other bidders. In other words, the Vickrey mechanism is *strategy-proof*. Therefore it should not be surprising that an *ascending* auction (i.e. extensive form game) that implements Vickrey payments inherits good incentive properties.

The argument follows the logic of the Revelation Principle. Suppose a bidder behaves in a way which is consistent with some valuation function different from his true one. This causes him to receive elements and make payments that correspond to the Vickrey allocation/payments for the false valuation. By strategy-proofness of the sealed-bid (direct revelation) mechanism, the bidder cannot be better off than if he had behaved truthfully. This yields Lemma 15 below.

Formally, for any possible list of valuations that a bidder could realize, a *strategy* dictates how a bidder should behave after any possible history throughout the auction. While we omit a tedious mathematical definition,

this means that each bidder must decide what information to declare in Lines 4 and 7 of Auction 3, given any possible auction history up to that point (e.g. the order and price at which elements already have been awarded or withdrawn by other bidders). A strategy is a *truthful bidding strategy* if, regardless of the auction history, the bidder truthfully declares which of his elements have value p (at Line 4) and which are in $\arg\max_{e \in C^\star} v_e$ (at Line 7, if the bidder is bidder j).

Lemma 15. *For any profile of valuations, suppose each bidder is using a truthful bidding strategy. Then any bidder receives a payoff at least as great as the payoff he would receive using any other strategy which is truthful with respect to some other valuations.*

The result follows from the strategy-proofness of the VCG sealed-bid auction, as stated above. This argument is not sufficient to prove a dominant-strategy result, however, as bidders could bid in a way which is inconsistent with *any* valuation function. It is conceivable for a bidder to make declarations in Lines 4 and 7 of Auction 3 that are inconsistent with truthful bidding under *any* possible valuations. For example, when a bidder is awarded a cocircuit $\{a, b, c\}$ he might declare that $v_a = v_b > v_c$ and be awarded, say, a. Later if he is awarded the cocircuit $\{b, c, d\}$ but announces $v_c > v_b$, this is inconsistent with *any* valuation. Against strategies that make such inconsistent announcements conditional on the announcements of others, truthful bidding may not be a best response.[5] This is why truthful bidding is not a dominant strategy in dynamic, ascending (i.e. extensive-form) auctions.

Nevertheless, it turns out that it *is* an equilibrium for all bidders to commit to truthful bidding at all times throughout the auction. When all other bidders are behaving truthfully, a bidder can do no better than to bid truthfully. The key to proving this is Lemma 16, which makes nontruthful strategies partially irrelevant.

Lemma 16. *Fix a profile of valuations and a bidder i. Suppose the other bidders are using truthful bidding strategies but bidder i is not. Then there exists a truthful bidding strategy for i (that is truthful with respect to* some *valuation $\tilde{v}$) that yields the same auction outcome as his original strategy.*

Proof. Under the assumptions of the lemma, suppose bidder i is using an arbitrary bidding strategy s. Using s, suppose i is awarded the set of elements

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[5]Similar difficulties appear in the literature on extensive-form implementation. See Moore and Repullo (1988).

$\{e_1, \ldots, e_k\}$, in that order, at prices $p_{e_1}, \ldots, p_{e_k}$, respectively. For every $e \in E_i \setminus \{e_1, \ldots, e_k\}$ that he is not awarded, denote by p_e the price at which he dropped the element from the auction (Line 4).

We define a valuation $\tilde{v}$ and show that the same outcome would be obtained (for all agents) if bidder i bids truthfully with respect to $\tilde{v}$. For all $e \in E_i$, let

$$\tilde{v}_e = \begin{cases} (1 + p_r) + (k - l) & \text{if } e = e_l \in \{e_1, \ldots, e_k\} \\ p_e & \text{if } e \in E_i \setminus \{e_1, \ldots, e_k\} \end{cases}$$

(where p_r is the highest charged price). Therefore,
- $\tilde{v}_e > \tilde{v}_f$ for all $e \in \{e_1, \ldots, e_k\}$ and $f \in E_i \setminus \{e_1, \ldots, e_k\}$ and
- $\tilde{v}_{e_l} > \tilde{v}_{e_j}$ if $1 \le l < j \le k$.

Let $\tilde{s}$ be the strategy of bidding truthfully with respect to $\tilde{v}$.

The differences between s and $\tilde{s}$ are the following. In Line 4 of the auction when the current price is p, under $\tilde{s}$ bidder i will announce the elements e that he will not win such that $\tilde{v}_e = p$. Under s it could be that he announces additional elements that he is actually awarded in an execution of Line 7 (while the price is still p, but before the element is deleted in Line 12). Suppose that such an element (with smallest index after tie-breaking) is f_m. Since f_m was awarded to him later in Line 7 under s, this happens before the loop in Line 5 gets to $l = m$. Later, at $l = m$, the **while** condition in Line 6 cannot be fulfilled, hence his announcing of f_m has no influence on the course of the computation. So we achieve the same outcome (allocation and prices) for any additionally announced element.

In Line 7, under s, bidder i may announce multiple elements of $C^\star$. However, only one element is awarded to him, and the remaining elements are irrelevant in the computation. Under $\tilde{s}$, he announces that same element only; hence the computations in both cases are identical. $\square$

A profile of strategies is an *ex post equilibrium* if, for any realization of valuations, no bidder has an incentive to change his strategy even if he could fully learn the others' valuations. We have imposed no assumption on the information that bidders possess. At one extreme, bidders may know a lot about each other. At another extreme, bidders may only have probabilistic beliefs about others' valuations; these beliefs may not even be commonly known. The strength of the ex post equilibrium concept is that such assumptions become irrelevant. For any possible realization of valuations, no bidder

regrets having used the strategy he did, all else being fixed. While this is a very strong (hence desirable) concept, we should also point out that it is not quite as strong as the concept of dominant strategies.[6]

Theorem 17. *For any profile of valuations, the profile of truthful bidding strategies is an ex post equilibrium of Auction 3.*

Proof. Suppose all bidders are bidding truthfully. Lemma 16 implies that each bidder i has a best response among the set of strategies which are truthful with respect to *some* (possibly false) valuation $\tilde{v}$. Lemma 15 implies that the truthful one (with respect to *true* valuations) is one best response. Therefore bidder i cannot do better than truthful bidding. □

The long-step version of Auction 3 also possesses strong incentives properties. It is straightforward to show that Lemma 15 also applies to the long-step auction (see the e-Companion). Slight care needs to be taken with Lemma 16, however. For example, consider Line 14. Even if all other bidders are truthful, an untruthful bidder i could announce $u_i = p + 2^{-t}$ in every round t. The auction could then continue indefinitely if the bidders' announcements yield $F = \emptyset$ in Line 4, in each round.

It is simple to fix this by adding a "consistency check" to Line 4 which would require the following: if bidder j announces the minimal utility u_j in Line 14L of the long-step auction, then he must announce at least one element in the next execution of Line 4. Formally, Line 4 could be changed to the following.

4L: Ask bidders to determine $F = \{f_1, \ldots, f_k\} \leftarrow \{f \in E(\mathcal{M}) : v_f = p\}$ and label the elements of F so that they are increasing according to the tie-breaking. Every bidder j with $p = u_j$ must announce at least one element.

This addition guarantees that if all bidders are bidding truthfully, no single bidder can deviate to an arbitrary strategy in order to prolong the auction indefinitely. With this, the arguments behind Lemma 16 and hence Theorem 17 would apply to the long-step auction.

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[6]In fact dominant strategy equilibria are not obtainable in many (reasonable) dynamic-auction settings. It is beyond the scope of this paper to elaborate much on this point, but the essential reason for this is that *no* strategy can be optimal (best response) when another bidder is behaving in a way tailored to "punish" players using that given strategy; hence none can be dominant.

We close this section with a remark on another way one could achieve similar incentives results in practice. When discussing incentives, we have been assuming (implicitly) that information exchanged in the auction is public. Specifically, information elicited from bidders in Lines 4 and 7 of the auction is revealed to everyone. If this communication between the auctioneer and bidders could be made privately, this would reduce the ways in which bidders could react to seemingly inconsistent behavior by other bidders. In game theoretic terms, this would create fewer information sets throughout the auction, making it easier to achieve incentive compatibility. A trade off is that this would also reduce the transparency of the auction.

3   Impossibility

We have exhibited an ascending auction for matroids that returns the Vickrey outcome. The auction is "simple" in the sense that it monotonically adjusts a single parameter—a single price—to determine the entire outcome. Can this approach extend beyond matroids? In this section, we show that it cannot: Outside of the matroid class, such a simple type of auction cannot always find an optimal basis. Of course an auctioneer could use more complicated auctions (with multiple parameters/prices) in order to implement many kinds of outcomes (see Bikhchandani and Ostroy, 2002 and de Vries et al., 2007). But the point of Theorem 18 below is to show that a matroid structure is required to achieve the simplicity exhibited in Auction 3.

To make this result precise, we first describe a more general class of allocation problems. We assume the seller's problem is to allocate a best element $B \in \mathcal{B}'$, where $\mathcal{B}'$ is a set of subsets of the ground set E. Under the nonnegativity assumption on the valuation, we know that only a maximal element of $\mathcal{B}'$ might be chosen, so without loss of generality we can assume that $\mathcal{B}'$ is a clutter (i.e. if $B, C \in \mathcal{B}'$ and $B \subseteq C$ then $B = C$). We will refer to sets from $\mathcal{B}'$ as *bases* although they may not be matroid-bases. Next we characterize the subsets S of E that contain a basis. For this we need the notion of a blocker $\mathrm{bl}(\mathcal{B}')$ that is defined as the set of inclusionwise minimal sets C so that each C intersects all members of $\mathcal{B}'$. Lawler (1966) observed that $\mathrm{bl}(\mathrm{bl}(\mathcal{C}')) = \mathcal{C}'$ if $\mathcal{C}'$ is a clutter; furthermore $\mathrm{bl}(\mathcal{C}')$ is always a clutter. Let $\mathcal{C}' := \mathrm{bl}(\mathcal{B}')$ and refer to its elements as *cocircuits*. For a set $S \subseteq E$ there exists a set $B \in \mathcal{B}'$ with $B \subseteq S$ if and only if for all $C^\star \in \mathcal{C}'$ holds $S \cap C^\star \neq \emptyset$. Say that for $\mathcal{B}'$ the no-monopoly condition holds if for all i the set $E \setminus E_i$

contains a basis; recall, this holds if and only if no cocircuit is contained in a single E_i.

In the following it suffices to focus on the special case in which *all E_i are singletons (and all values are distinct);* clearly any auction beyond matroids should be able to work in this simple environment.

A mechanism that sells the best element from $\mathcal{B}'$ is an *ascending, truthful, 1-parameter mechanism* if

1. there is a single ascending price;
2. bidders reveal which of their items f match the current price;
3. bidders have the incentive to reveal such items truthfully;
4. when a bidder reveals such an item, the seller either deletes that item (y_{i+1}) at the current price or awards it (later) at a price no higher than the current price;
5. the seller must delete that item if the remaining elements (that is $E \setminus \{f, y_1, \ldots, y_i\}$) still contain a basis (otherwise, the mechanism is already unable to always find the optimum basis for matroids).

The seller deletes an element f if and only if the remaining elements $E \setminus \{f, y_1, \ldots, y_i\}$ contain a basis; equivalently this requires that $E \setminus \{f, y_1, \ldots, y_i\}$ intersects every cocircuit (or equivalently that no cocircuit is contained in $\{f, y_1, \ldots, y_i\}$). An ascending auction has to look like Algorithm 4 (though some additional computation for the payments might be necessary). This is simply the well known *worst-out greedy algorithm*.

Theorem 18. *For given $\mathcal{B}'$ an ascending, truthful, 1-parameter mechanism is able to find the optimal basis for all valuations and all buyer partitions for up to $|E|$ agents if and only if $\mathcal{B}'$ fulfills the basis axioms of matroids.*

Proof. (This is a minor variant of a proof of a different result by Korte and Lovász, 1984, Thm. 4.1.) Recall we assume that all bidders are interested only in a single item. Let $\mathcal{M}^\star := \{I \subseteq E : \exists B \in \mathcal{B}' \text{ with } I \cap B = \emptyset\}$. Clearly, a worst-out greedy basis found by Algorithm 4 is of maximum value if and only if the complementary best-in greedy basis has minimum value. The latter can hold for all valuation only if $\mathcal{M}^\star$ is a matroid. But then $\mathcal{B}'$ are the bases of $\mathcal{M}^{\star\star}$ and therefore $\mathcal{B}'$ has to fulfill the basis axioms of matroids. The other direction is obvious. $\square$

Algorithm 4 Optimal allocation from a clutter

Require: Finite no-monopoly clutter $\mathcal{B}'$ with nonnegative, distinct rational valuation, singleton buyership, and blocker $\mathcal{C}' := \mathrm{bl}(\mathcal{B}')$.

1: $i \leftarrow 0, \quad p \leftarrow \min\{v_e \colon e \in E\}$
2: **while** $p < \infty$ **do**
3:      Let $f \in E$ be the element with $v_f = p$
4:      **if** $E \setminus \{f, y_1, \ldots, y_i\}$ intersects every member of $\mathcal{C}'$ **then**
5:          $i \leftarrow i + 1$
6:          $y_i \leftarrow f$
7:      **end if**
8:      $p \leftarrow \min\{v_e \colon e \in E \text{ with } v_e > p\}$
9: **end while**
10: Award $E \setminus \{y_1, \ldots, y_i\}$.

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4   Selling Bases of a Polymatroid

A matroid's rank function r is integer-valued and satisfies three properties.

(R1) $S \subseteq E \Rightarrow 0 \leq r(S) \leq |S|$.
(R2) $S \subseteq T \subseteq E \Rightarrow r(S) \leq r(T)$.
(R3) r is submodular, i.e. $r(S) + r(T) \geq r(S \cap T) + r(S \cup T)$ for all $S, T \subseteq E$.

Associated with every matroid is a polyhedron

$$\sum_{e \in S} x_e \leq r(S) \quad \forall S \subseteq E$$

$$0 \leq x_e \quad \forall e \in E$$

whose extreme points are 0-1 vectors corresponding to the independent sets of the matroid.

     This class of polyhedrons can be generalized by replacing the rank function on the right-hand side of the first inequality with a non-decreasing, normalized ($\rho(\emptyset) = 0$), submodular function $\rho(\cdot)$. The polyhedrons in the resulting class are called *polymatroids* and play an essential role in combinatorial optimization. The extreme points of a polymatroid are sometimes referred to as the independent vectors of the polymatroid. Our auction for selling a basis of a matroid can be extended to sell a basis of a polymatroid.

As before let E be a set of distinct items and consider a non-decreasing function $\rho\colon 2^E \to \mathbb{N}_0$, normalized, submodular, and integer-valued function $\rho(\cdot)$. The set of feasible allocations is determined by the *polymatroid associated with ρ*: $P_\rho = \{x \in \mathbb{R}^E\colon x \geq \mathbf{0},\, x(S) \leq \rho(S)\, \forall S \subseteq E\}$. It is well known that P_ρ has integral extreme points if, as we assume, ρ is integer-valued. We assume that each agent $j \in N$ has for each $e \in E$ a valuation $v_e^j(x_e)$ that is nondecreasing, concave, and piecewise-linear with integral breakpoints. Agent j's value for $x \in P_\rho \cap \mathbb{N}_0^E$ is $v^j(x) = \sum_{e \in E} v_e^j(x_e)$. The efficient allocation solves

$$\max \sum_{j \in N} v^j(x^j)$$

$$\text{s.t.} \sum_{e \in S} \sum_{j \in N} x_e^j \leq \rho(S) \quad \forall S \subseteq E \qquad \text{(P)}$$

$$x^j \geq \mathbf{0} \qquad\qquad \forall j \in N$$

and we look for an ascending auction that returns the VCG outcome. If the values were known, the algorithms of Fujishige (1980), Groenevelt (1991), Nagano (2007) would apply.

Helgason (1974) describes a pseudo-polynomial reduction from integer polymatroids to matroids; here we follow the description given by Schrijver (2003). Let $X_e = \{e\} \times N \times \{1, \ldots, \rho(\{e\})\}$ for each $e \in E$ and set $X = \bigcup_{e \in E} X_e$. The idea is to replace the element $e \in E$ of the polymatroid, that could occur in quantity up to $\rho(\{e\})$, by the elements of the set X_e (this is the pseudo-polynomial step). A set $I \subseteq X$ is considered independent in the new matroid if, with $x_e := |I \cap X_e|$ for all e, holds $x \in P_\rho$.

We have already duplicated the elements for the different agents, so that agent $j \in N$ is interested only in the elements of type $(e, j, i) \in X$ with $e \in E$ and $1 \leq i \leq \rho(\{e\})$. Consequentially, no two agents are interested in the same element of X. Set the values of the element (e, j, i) by $v_{e,j,1} := v_e^j(1)$ and $v_{e,j,i} := v_e^j(i) - v_e^j(i-1)$ (for $1 < i \leq \rho(\{e\})$). Note, that $v_{e,j,i}$ is nonincreasing in i. If we consider an optimal basis B in the matroid, then because $v_e^j(\cdot)$ is assumed to be nondecreasing and concave, we can assume without loss of generality $\{i : (e, j, i) \in B \cap X_e\}$ is for every (e, j) a set of consecutive integers starting with 1. Consequentially for every optimal basis of the matroid there is an optimal basis of the polymatroid of the same value, and vice versa.

The resulting auction is pseudo-polynomial because of the involved pseudo-polynomial transformation. If bidders are only permitted to announce their

demand for every item e as a consecutive integer set starting at 1 then the profile of truthful bidding turns out to be an ex post equilibrium.

5   Applications

5.1   Matroid Applications

5.1.1   Scheduling Matroids

Our auction applies to scheduling matroids where each agent has a set of unit length jobs with a commonly known release and due date but a privately known value for completion. In the special case in which every agent has only one job to be scheduled, the auction by Demange et al. (1986) applies; so for this scheduling problem, our auction generalizes theirs.

5.1.2   Graphical matroids

When applied to graphical matroids (for an example run of the presented auction on a graphical matroid, see Subsection 2.8), problems like minimal network-planning where agents bid that preferred links become directly connected (instead of just being connected via several hops) can be solved as an ascending auction.

5.1.3   Uniform Matroids

For selling multiple units of a single good to bidders that have decreasing marginal values, an application of our auction to a properly constructed uniform matroid is possible. We omit the details and refer to 5.2.4.

5.1.4   Transversal Matroids for Pairwise Kidney Exchange

Consider the problem of a pairwise kidney exchange. At present the use of monetary transfers is considered unethical. Nevertheless we discuss the possibility because of its interest.

As there are insufficiently many cadaver organs available for the patients in need, other sources are necessary. Traditionally, a patient could be helped only if he finds a suitable (and willing) donor among his closest family or friends. But even that helps only a minority of patients. Even more patients

could be helped by following the approach of Roth et al. (2005) who devise a matching method to maximize the number of transplantations. Their transformation to a matching problem can be taken further to a matching matroid (from Edmonds and Fulkerson (1965) this is a transversal matroid). Allowing patients to bid for priority in being matched yields a situation that can be efficiently solved with our matroid auction.

5.2   Polymatroid Applications

There are a variety of settings where the set of feasible resources to be allocated is a polymatroid. In these cases, the auction developed in this paper will apply.

5.2.1   Spatial Markets

Motivated by Babaioff et al. (2004), consider a capacitated network where each arc has both a cost and a capacity. Agents are identified with disjoint sets of demand nodes. The utility of each agent is the sum of increasing and concave functions of the flow x_i into each of his nodes i. The seller is identified with a source node s. It is well known that an allocation x is feasible if and only if $\sum_{e \in S} x_e \leq \rho(S)$ for all S where $\rho(S)$ is the submodular function giving the value of a minimum s-S-cut (e.g. see Federgruen and Groenevelt, 1986). Therefore, our polymatroid auction applies to such settings. For an application, one can imagine a (streaming) video on demand service, where a single customer represents, say, a group of spatially dispersed family members, each of whom wants to watch various movies.

5.2.2   Bandwidth Markets

In wireless communication settings the resource to be consumed is transmission rate. In a Gaussian multiple access channel the set of achievable rates, called the Cover–Wyner capacity region, forms a polymatroid. See Tse and Hanly (1998). Our polymatroid auction would apply in this setting to allocate transmission rates.

5.2.3   Multiclass Queuing Systems

A wide range of manufacturing and service facilities are modeled as multiclass queuing systems. Job requests arrive from agents and are then scheduled.

The schedule determines the completion times of the various jobs. If x_i is the completion time of the job associated with agent i, then under certain conditions, the set of feasible x_i's forms a polymatroid (see Shanthikumar and Yao, 1992). If agents incur a disutility that is convex in completion time, then our polymatroid auction would apply.

5.2.4 Multiple Units of a Single Good with Decreasing Marginal Value

The problem of allocating up to k units of an identical good (let $E = \{1\}$) amongst n agents with decreasing marginal utilities can be formulated as a polymatroidal optimization problem. Let x_i be the quantity allocated to agent i and $u_i(\cdot)$ the utility of agent i. The problem of finding an efficient allocation is

$$\max \sum_{j \in N} v^j(x^j)$$
$$\text{s.t.} \sum_{e \in \{1\}} \sum_{j \in N} x_e^j \leq k$$
$$x_e^j \geq 0 \qquad \forall j \in N, e \in \{1\}$$

Now if we set $\rho(\{1\}) = k, \rho(\emptyset) = 0$ then it becomes clear, that the feasible region forms a polymatroid and the valuations of the bidders fulfill the conditions. Our auction when applied to this setting yields the auction by Ausubel (2004).

Notice that the case of several goods for which bidders have decreasing valuations separable in goods can be done analogously.

5.2.5 A Seller Subject to Polymatroid Production Constraints

Consider a seller who produces shoe heels and pencil erasers and, due to capacity constraints, can produce only up to $x_h \leq \rho(h)$ heels and up to $x_e \leq \rho(e)$ erasers. If there is limited supply of a common input (e.g. rubber), then the seller could be constrained to produce no more than say $\rho(\{e, h\}) \leq \rho(e) + \rho(h)$ heels and erasers *in total*. Our polymatroid auction can be applied to settings where a seller faces such constraints, when buyers' valuations are both separable in heels and erasers and have decreasing marginal values in quantity.

6   Conclusion

We proposed the first ascending auction for matroids and polymatroids. The auction has truthful bidding as an ex post equilibrium and has various applications. It remains an open question whether a generalization to interdependent valuations is possible similar to the way Perry and Reny (2005) generalize to interdependent valuations the auction of Ausubel (2004).

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