Rationing through Classification*

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Abstract

Basing resource allocation on the recipients' actions can improve allocation decisions but distort action choices. A mechanism designer optimizes this tradeoff by adjusting how intensely actions affect outcomes. We show that market power—in the form of centralized action choices—affects this design problem in a novel way by considering settings where strategic agents choose actions on behalf of multiple recipients. In sufficiently competitive settings the mechanism designer optimizes against agents' equilibrium incentive to distort action choices of a single "marginal" recipient type, as in related models with no market power. However with sufficiently few agents (strong market power), a novel, second class of equilibria can arise where agents inefficiently distort action choices for multiple recipient types. Computations show that such equilibria can even be welfare optimal. Our work demonstrates how sufficiently strong market power can overturn qualitative insights obtained in standard, competitive mechanism design settings.

Keywords: strategic classification, mechanism design, market power.

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1 Introduction

By observing recipients' actions beforehand, a planner may be able to allocate resources more effectively. For example, transplant organs can be targeted toward the most vulnerable patients by identifying those who obtain the most intensive medical treatments. Poverty alleviation programs provide transfers to poor households based on regular school attendance or medical checkups, while school choice priorities are impacted by the choice of residential location.[1] Promotions in the private sector and the military can be affected by decisions to participate in costly projects or high-risk missions.

From a mechanism design perspective, letting actions impact allocation decisions creates a welfare tradeoff between improving allocation outcomes and distorting action choices. These distortions lower welfare both by making actions choices less efficient and by weakening the planner's ability to infer who takes which actions. Recent work on strategic classification (e.g., Braverman and Garg (2020), Perez-Richet and Skreta (2022, 2023)) considers this tradeoff in settings where individual recipients selfishly choose their own actions. However in some settings this tradeoff is further complicated by the presence of *market power*: the actions of multiple recipients are chosen by a single strategic agent who acts on their behalf. A prominent class of such settings is transplant organ allocation, where transplant centers choose medical treatments on behalf of their multiple patients, with each choice impacting allocation rates for all patients.

Our objective is to examine the mechanism design interplay between this form of market power and the welfare tradeoff described above. We ask not only to what degree observed actions should impact the planner's allocation decisions, but also how this question is impacted by the degree of competition: the extent to which recipients' action choices are centralized.

The simplest model that allows us to do this involves two types of recipients who obtain Low or High value from an object and, respectively, obtain negative or positive benefit from taking an observable "Treatment" action (e.g. intensive medical intervention in the organ allocation example). A planner who allocates by fully prioritizing recipients taking the Treatment action may distort the action choices of Low-type recipients. A planner who ignores action choices altogether—eliminating priorities—

[1] See Martinelli and Parker (2003) and Park and Hahm (2023).

removes these distortions but fails to utilize welfare-improving information. More generally a planner can partially prioritize, by rationing a fraction of objects to recipients taking the Treatment action, allocating the rest to those who do not. Our questions above correspond to how the choice of this ration and the degree of market power jointly affect equilibrium behavior and welfare.

A baseline "perfect competition" case of our model—where individual recipients choose their own actions—is related to the strategic classification literature mentioned above in which market power is set aside. In this baseline case (Section 3) equilibrium welfare is maximized when the planner maximizes the ration of objects allocated to Treatment recipients subject to a no-distortions constraint: Low type recipients must not have the incentive to choose the Treatment action. Any higher ration induces Low types to "game the system," which can be shown to necessarily lower welfare.

To address the impact of market power, our general "imperfect competition" model (Section 4) introduces n strategic agents who each choose actions on behalf of a subset of recipients. These agents can represent, for example, transplant centers in the organ allocation example above that choose medical treatments on behalf of multiple patients.

The centralization of decision making evokes the following intuition in our setting. First, each agent partially internalizes the congestion effect (crowding out High types) from "wrongly" choosing the Treatment action for any of its Low type recipients. Since higher market power (lower n) increases this internalization, the planner can further increase the ration of objects targeted toward recipients who take the Treatment action while still avoiding any distorted action choices.[2] Our results show that this intuition is *partially* true in that, *assuming it remains welfare-optimal to eliminate distortions* (as in the baseline perfect competition case), an increase in market power improves both resource allocation and welfare (Proposition 3).

Surprisingly, however, there are cases where it may not be welfare-optimal to eliminate distortions, overturning the qualitative result of the baseline case. To begin with, the presence of market power can lead to a novel form of what we call "Inversion" equilibria in which agents' action choices are doubly distorted: each agent chooses the Treatment action for some of its *Low* types and simultaneously *fails* to choose this

[2] Similarly, an increase in market power decreases the intensity of such distortions, holding the ration fixed. Parker et al. (2018) provide empirical evidence of this type of effect in the context of heart allocation: patient over-treatment is more prevalent in regions with more (competing) transplant centers.

action for some of its *High* types. Though convexities in the payoff functions complicate the analysis of precisely when such equilibria arise, computations reveal that they do when there is a low relative cost for Low types to take the Treatment action, i.e. when it would be difficult for the planner to screen in the first place. In addition, counterintuitively, such equilibria can be welfare-optimal among *all* equilibria.

For some intuition underlying these "Inversion" equilibria, fix a planner's rationing decision and imagine that agents anticipate a relatively low availability of objects for recipients who take the Treatment action. This could lead the agents (i) to increase some or all of their High type recipients' object allocation rate by *not* choosing Treatment for them, and (ii) to avoid further reducing those High types' allocation rate by choosing the Treatment action for some or all of its Low types. If this behavior leads to a disproportionate overall fraction of recipients taking the Treatment action—and thus a relatively low availability of objects for those recipients—then this is plausible equilibrium behavior. This intuition is strongest when it is not too costly for Low types to take the Treatment action; our computations reveal such equilibria in precisely these cases.

Our results highlight three points relevant for the design of mechanisms that prioritize recipients on the basis of their action. First, in some cases, *partial* prioritization via rationing allows the planner to finely adjust the tradeoff between improving allocation decisions and distorting action choices. Not surprisingly this includes situations where market power is low, as in the strategic classification literature that assumes no market power. Second, in these cases an increase in market power reduces distortions, which alters the optimal mechanism in a way that further enhances welfare. Finally, there are other cases where strong market power leads to unexpected equilibrium behavior. Our contribution is not only to point out that market power can lead to this new form of "Inversion" equilibrium in contrast to the baseline case, but that in some cases such equilibria can even be welfare-optimal.

1.1 Related Literature

Though we consider a specific problem of object allocation via classification, our work addresses the broader question of how market power impacts mechanism design. This interplay is one that has not yet been widely considered in the literature, as pointed out by Agarwal and Budish (2021). They also note the exception of the auction literature, which demonstrates how market power can hinder the designer's objectives. Notably we draw the opposite conclusion in our setting.[3]

The work closest to our specific setting is that on strategic classification (Hardt et al., 2016). This literature focuses on the case where individual recipients strategically choose their own actions, analogous to our baseline "perfect competition" model (Section 3) that ignores market power. We distinguish ourselves from this work by allowing for "imperfect competition" (Section 4), centralizing multiple decisions under one strategic agent.[4] Our contribution is to show the extent to which results in the baseline model extend to the general one and how this is impacted by the *level* of competition.

Generally speaking, models in this literature have the following characteristics. A planner wishes to correctly classify an agent's type as being above or below some threshold (high or low). All agents desire a high classification and can misrepresent their privately known type at some cost. In a continuous-type version of our baseline model (Section 3), Braverman and Garg (2020) maximize equilibrium classification accuracy net of agents' manipulation costs. Under some assumptions they show that optimal classifiers (i) typically require randomization, and (ii) induce no manipulation. Our setup necessarily induces randomness by the nature of our budgeted rationing problem but its quantification is determined endogenously by equilibrium behavior. Nevertheless, our Theorem 2 is analogous to their result.

Perez-Richet and Skreta (2022) allow the planner to commit to a probabilistic testing function that maps (misrepresented) types into randomized signals. The planner uses realized observations to make optimal classification decisions. Under an increasing-returns assumption on misrepresentation costs, accuracy-maximizing mechanisms "raise the bar" by offering the greatest chance of high classification only to observed types above some artificially high threshold. The agents achieving this threshold in equilibrium are precisely those whose true type is above the planner's desired threshold. Other types engage in no misrepresentation, being compensated with enough probability of high classification to offset the benefit of doing so. Perez-Richet and Skreta (2023) impose this no-misrepresentation condition as a constraint

[3]Competition can be shown to harm welfare in various settings outside of mechanism design, such as multi-sided platforms (Tan and Zhou, 2021) and competitive search (Lester et al., 2019; Mekonnen and Pakzad-Hurson, 2024).

[4]An additional difference is that we impose a "classification budget" representing a fixed supply of resources to allocate.

under which they find optimal allocation mechanisms.

In a machine learning context Hardt et al. (2016) provide efficient, near-optimal algorithms for classification accuracy against strategic agents both when the classification objective is known and when it first must be learned by the algorithm through existing data. In a related model, Milli et al. (2019) analyze the tradeoff between accuracy and the resulting manipulation costs imposed on "high type" agents.

Other work examines variations on mechanism-, scoring-, or ratings-design under costly misrepresentation. Frankel and Kartik (2021) consider agents who vary both in type as above and in misrepresentation costs. This dual heterogeneity leads the planner to under-weight observed information to improve accuracy in equilibrium. When types are multidimensional Ball (forthcoming) shows that the planner benefits by under-weighting some dimensions and over-weighting others. Lee and Suen (2023) consider allocating university seats based on exam scores obtained naturally (high types) or through wasteful tutoring (low types), showing that increasing resources (seats) can increase distortions (wasteful tutoring). Finally, Akbarpour et al. (2024) provide a more distinct setting in which rationing is optimal even when the use of transfers is available.

In a dynamic setting, Munoz-Rodriguez (2024) studies a model that is somewhat like an overlapping generations version of our no-market-power model but where action choices are costly for only one type. The optimal dynamic mechanism improves outcomes by granting option value to low types who forgo early assignment.

While the imperfect competition aspect of our main model (Section 4) is novel from the perspective of the above literature, it also leads to a generalization of congestion games pioneered by Wardrop (1952). Increasing one's allocation probability through the "route" of misrepresentation necessarily decreases someone else's. Fixing the planner's choice of ration in our baseline model of Section 3, equilibrium existence for example would follow immediately from that literature (Konishi, 2004), though of course we go beyond this by evaluating welfare as we vary the ration.

Allowing a finite set of agents each to control a mass of recipients, our general model becomes a type of atomic congestion game (ACG) for any fixed choice of ration. An existing literature derives existence and uniqueness results for such games as long as they are sufficiently structured, e.g. if all traffic is of a single type and the network is sufficient simple (Bhaskar et al., 2015; Harks and Timmermans, 2018). In our "two-traffic-type" model, however, payoffs violate the typical concavity assumptions that lead to these kind of results; in fact they locally violate concavity everywhere. Despite this technical challenge we provide an existence result under fairly weak additional assumptions on our primitives (Theorem 4).

In a one-type model, Wan (2012) shows that in ACG's with two nodes, total equilibrium welfare increases when a fixed amount of traffic is split amongst fewer atomic agents. Here there is no resource to be allocated; agents are merely trying to minimize transportation costs on a fixed network. Nevertheless, our Proposition 3 draws an analogous conclusion in our two-type model, the difference being that our planner adjusts the optimal rationing of resources with respect to the number of agents. The intuition discussed earlier applies in both cases: with fewer agents, a greater share of congestion costs are internalized by each agent.

2 Model

2.1 Primitives

There is a continuum of *recipients* having two possible types: a mass $r_\ell > 0$ of low type recipients and a mass $r_h > 0$ of high type recipients. Each recipient takes one of two actions, N ("Non-treatment") or T ("Treatment"). A recipient's action is strategically chosen either by the recipient (in Section 3) or an (atomic) *agent* who chooses actions for a subset of recipients (formalized in Section 4). A mass $\phi < r_\ell + r_h$ of objects is assigned to recipients as described in Subsection 2.2.

A recipient's welfare depends on their type, action, and whether they receive an object. Any recipient receiving an object obtains welfare L^*, normalized to be independent of type and action. Otherwise a recipient of type $i \in \{\ell, h\}$ who takes action $a \in \{N, T\}$ obtains welfare L_i^a. We assume

$$L_h^N < L_h^T < L_\ell^T < L_\ell^N < L^* \tag{1}$$

Thus high types obtain relatively greater benefit from an object and (in the absence of distortions) would be the ones taking the "targeted" Treatment action T.[5]

An economy is summarized by primitives $(r_h, r_\ell, \phi, L_h^N, L_h^T, L_\ell^T, L_\ell^N, L^*)$. In Section 4 we add the additional primitive $n > 0$, representing the number of agents

[5]In the context of organ allocation there is typically a desire to prioritize high-risk patients, who are also the ones who benefit from intensive interim medical treatment.

choosing actions on behalf of their respective recipients.

2.2 Rationing through Classification

The planner observes recipients' actions but not their types. Therefore the planner is restricted to assigning some fraction k of the objects to recipients who took action T and assigning the remaining mass $(1-k)\phi$ of objects to recipients whose action is N. We define the process of *Rationing through Classification* as one where (i) the planner publicly commits to a ration $k \in [0,1]$, (ii) each recipient takes an action N or T (chosen individually in Section 3 or by their agent in Section 4), and (iii) $k\phi$ objects are (uniformly randomly) assigned to recipients taking action T, and the rest assigned to those taking N. The ration k represents the degree to which the planner uses classification information as a basis for allocation. Our main question is how the choice of k impacts the structure of equilibria and equilibrium welfare.

Naively, a planner might expect to maximize utilitarian welfare by choosing k maximally, thinking this would maximize the allocation rate to high types. Of course this ignores the possibility that this induces low types also to take action T, achieving neither a first-best assignment nor first-best action choices. At another extreme a planner could attempt to induce efficient action choices by choosing a "proportional" value of k. Namely, let

$$\bar{k} \equiv r_h/(r_\ell + r_h)$$

denote the percentage of recipients who are of high type. Even when ration $\bar{k}$ induces efficient action choices,[6] object assignment is far from optimal since objects are assigned to all recipients with equal probability. Our work examines not only *how* adjustments to k can fine-tune this trade-off, but *whether* fine-tuning occurs at all.[7]

Our restriction to RTC is technically a restriction on feasible mechanisms; e.g. the planner could commit to choosing ration k *ex post*, as a function of all recipients' realized action choices. We consider this restriction insignificant for two reasons. First, RTC is without loss of generality in our baseline "perfect competition" model because infinitesimal agents are "price takers."[8] Second, any general mechanism that is *not* RTC requires the planner to observe a *profile* of realized actions before making alloca-

[6] It may not; see Section 4.

[7] It need not; see Section 4.

[8] Any equilibrium under a general mechanism that results (ex post) in ration k is an equilibrium under RTC when the planner (ex ante) commits to constant k.

tion decisions. While this is feasible within our model (made static for tractability), it is less practical in a dynamic setting.

3 Baseline Case: Perfect Competition

3.1 Equilibrium structure

Fixing a ration k, each recipient selfishly chooses action N or T. A **strategy profile** $p = (p_\ell, p_h)$ denotes the fractions of low- and high-type recipients that choose action T. In the equilibrium analysis we can restrict attention to *non-wasteful* profiles, i.e. where $(1-p_\ell)r_\ell + (1-p_h)r_h \geq (1-k)\phi$ and $p_\ell r_\ell + p_h r_h \geq k\phi$. Such a profile induces the following **allocation probabilities** for recipients who have chosen N or T:[9]

$$\pi^N(p) = \frac{(1-k)\phi}{(1-p_\ell)r_\ell + (1-p_h)r_h} \qquad \pi^T(p) = \frac{k\phi}{p_\ell r_\ell + p_h r_h} \tag{2}$$

When a profile p is clear from the context we may simply write π^N and π^T.

A recipient's payoff is their expected welfare using the values in (1). A profile p is an **equilibrium** if it satisfies the following incentive compatibility conditions.

$$\begin{aligned}
\pi^N L^* + (1-\pi^N)L_\ell^N &< \pi^T L^* + (1-\pi^T)L_\ell^T &&\implies && p_\ell = 1\\
\pi^N L^* + (1-\pi^N)L_\ell^N &> \pi^T L^* + (1-\pi^T)L_\ell^T &&\implies && p_\ell = 0\\
\pi^N L^* + (1-\pi^N)L_h^N &< \pi^T L^* + (1-\pi^T)L_h^T &&\implies && p_h = 1\\
\pi^N L^* + (1-\pi^N)L_h^N &> \pi^T L^* + (1-\pi^T)L_h^T &&\implies && p_h = 0
\end{aligned}$$

Observe that if some type has an incentive not to choose its "natural action" (N for low types, T for high types), then it receives a strictly higher allocation probability at its non-natural action. Since this cannot hold for both types simultaneously, we have the following. (Formal proofs are in the appendix.)

Lemma 1. *If (p_ℓ, p_h) is an equilibrium then at least one type chooses its natural action with certainty, i.e. $p_\ell = 0$ or $p_h = 1$ (or both).*

Also intuitive is that an increase in k should induce more recipients to choose T, and that high type recipients are induced more easily than low types. (Interestingly

[9]Define $\pi^N(1,1) = 0 = \pi^T(0,0)$; these particular values are not significant in the analysis.

this intuition fails to hold in Section 4.) With Lemma 1 this leads to the following description of equilibria.

Proposition 1 (Unique equilibrium)**.** *For any $k \in [0,1]$ there exists a unique equilibrium $p^*(k)$. Furthermore $p^*()$ is weakly increasing in k and satisfies*

$$\begin{aligned} k < k' &\Longrightarrow p^*_\ell(k) = 0,\ p^*_h(k) < 1 &\qquad \textit{(biased toward N)} \\ k' \leq k \leq k^* &\Longrightarrow p^*_\ell(k) = 0,\ p^*_h(k) = 1 &\qquad \textit{(separating)} \\ k > k^* &\Longrightarrow p^*_\ell(k) > 0,\ p^*_h(k) = 1 &\qquad \textit{(biased toward T)} \end{aligned}$$

where

$$k' = \max\left\{0, \frac{r_h}{\phi}\frac{\phi(L^* - L^N_h) + r_l(L^N_h - L^T_h)}{r_h(L^* - L^N_h) + r_l(L^* - L^T_h)}\right\} < \bar{k} \equiv \frac{r_h}{r_\ell + r_h} \tag{3}$$

$$k^* = \min\left\{1, \frac{r_h}{\phi}\frac{\phi(L^* - L^N_l) + r_l(L^N_l - L^T_l)}{r_h(L^* - L^N_l) + r_l(L^* - L^T_l)}\right\} > \bar{k} \equiv \frac{r_h}{r_\ell + r_h} \tag{4}$$

In fact $p^*()$ is constant only on $[k', k^*]$. For some primitives it is possible that $k' = 0$ or $k^* = 1$. In particular the proof of Proposition 1 implies

$$k' > 0 \quad \Leftrightarrow \quad \frac{L^T_h - L^N_h}{L^* - L^N_h} < \frac{\phi}{r_l} \tag{5}$$

$$k^* < 1 \quad \Leftrightarrow \quad \frac{L^N_l - L^T_l}{L^* - L^T_l} < \frac{\phi}{r_h} \tag{6}$$

Intuition driving (6) is that low types are more easily induced to choose T (via an increase in k) when (i) object supply is increased, (ii) there are less competing high types, (iii) the cost of choosing T is decreased, or (iv) the benefit of receiving an object conditional on choosing T is higher. Analogous intuition drives (5).

3.2 Equilibrium welfare

For any $k \in [0,1]$, denote the **equilibrium fraction of objects allocated to high types** as

$$f(k) = (1-k)\frac{(1 - p^*_h(k))r_h}{(1 - p^*_h(k))r_h + (1 - p^*_\ell(k))r_\ell} + k\frac{p^*_h(k)r_h}{p^*_h(k)r_h + p^*_\ell(k)r_\ell}$$

where $p_\ell^*(k), p_h^*(k)$ is the unique equilibrium for k. An increase in k affects welfare both by increasing the fraction of objects allocated to recipients choosing T, and by (weakly) increasing the percentage of recipients choosing T. While the total effect can be positive or negative, an obvious case is when $k \in [k', k^*]$. Since $p^*()$ is constant in this range, an increase in k simply increases $f()$, increasing total welfare.

For $k \in [k^*, 1]$, it turns out that an increase in k disproportionately increases the number of low types choosing T to the extent that $f()$ decreases. Analogously, for $k \in [0, k']$ a *decrease* in k disproportionately increases the number of high types choosing N, increasing $f()$.

Theorem 1. *The equilibrium fraction $f()$ of objects allocated to high types is*

- *decreasing in k for $k \in [0, k']$;*
- *increasing in k for $k \in [k', k^*]$;*
- *decreasing in k for $k \in [k^*, 1]$.*

Furthermore $f()$ is maximized at k^*.

For an intuition, imagine primitives are such that when $k = 1$, (i) *all* recipients choose T in equilibrium, but (ii) any low type recipient is indifferent between the lottery they face—receiving an object (L^*) or or not (L_ℓ^T)—and deviating to choose N (a payoff of L_ℓ^N). Note that every recipient faces the same probability $\pi^T = \phi/(r_\ell + r_h)$ of receiving an object.

Next consider a small decrease in k and a "proportional" change in the strategy profile such that (i) a mass ϵ of low type recipients instead choose N and (ii) a mass $\pi^T\epsilon$ of objects is rationed amongst those low type recipients choosing N. Note that once again every recipient (at N or T) has probability π^T of receiving an object. However this means that low types choosing T are strictly worse off than low types choosing N (since their welfare is lower conditional on not receiving an object). In order to restore equilibrium indifference, a *greater* than proportional number of low types must choose N. In other words, $p_\ell^*(k)$ must be disproportionately sensitive to changes in $k \in [k^*, 1]$. An analogous argument applies to $p_h^*(k)$ for $k \in [0, k']$.

Theorem 1 has immediate welfare implications. For $k \in [k', k^*]$ welfare increases in k since actions remain fixed while $f()$ increases. For $k \in [k^*, 1]$, an increase in k reduces $f()$ and increases $p_\ell^*()$, necessarily decreasing welfare. Finally for $k \in [0, k']$, an increase in k decreases $f()$ but also improves welfare by reducing $p_h^*()$. Either effect can dominate, breaking symmetry with the case $k \in [k^*, 1]$. Nevertheless, one can separately prove that equilibrium welfare at any $k \in [0, k']$ is inferior to that obtained at k^*.

Theorem 2. *Utilitarian welfare (total recipient equilibrium payoffs) is*

- *increasing in* k *for* $k \in [k', k^*]$*;*
- *decreasing in* k *for* $k \in [k^*, 1]$*;*
- *maximized at* k^* *among all* $k \in [0, 1]$*.*

We next incorporate market power into this environment by letting agents each decide actions on behalf of their own share of recipients. Notably Lemma 1 does not extend to that setting. Even when equilibria do resemble those of Proposition 1, the arguments proving Theorem 2 also no longer apply (because $f()$ loses a monotonicity property). We instead extend Theorem 2 in the form of Theorem 5.

4 Imperfect Competition

4.1 Atomic agents

We capture the idea of market power by specifying a number n of atomic agents who choose actions on behalf of their own recipients. Formally, there are n agents, each choosing actions on behalf of a mass r_ℓ/n of low-type recipients and a mass r_h/n of high-types. A **strategy** for agent i is a pair $p_i = (p_{i\ell}, p_{ih}) \in [0, 1]^2$ specifying the percentages of its low-type and high-type recipients taking action T. A **strategy profile** is denoted $p = (p_1, p_2, \ldots, p_n)$. We let p_{-i} denote the list of strategies for all agents other than i.

An agent's **payoff** is the total expected welfare of its recipients as in (1). We continue to interpret the parameters in (1) as individual recipient welfare, in which case payoffs are that of a utilitarian agent that puts weight only on its own recipients. However, since individual recipients play no strategic role in this section, one could go well beyond a utilitarian interpretation. For example, the parameters in (1) could represent the profit an agent receives based on its recipients' types, actions, and assignment outcome; or they could represent some combination of such profits and welfare, or payoffs more generally as long as they are additive across recipients. Such interpretations might lead to different planner objectives than those we consider here.

Generalizing concepts from earlier, a profile p is **non-wasteful** (for k) when there are no more objects than recipients at N or at T, i.e.

$$(1 - \sum p_{i\ell}/n)\, r_\ell + (1 - \sum p_{ih}/n)\, r_h \geq (1-k)\phi \text{ and } (\sum p_{i\ell}/n)\, r_\ell + (\sum p_{ih}/n)\, r_h \geq k\phi$$

Analogous to (2), a non-wasteful profile p induces **allocation probabilities**

$$\pi^N = \frac{(1-k)\phi}{(1 - \sum p_{i\ell}/n)\, r_\ell + (1 - \sum p_{ih}/n)\, r_h} \qquad \pi^T = \frac{k\phi}{(\sum p_{i\ell}/n)\, r_\ell + (\sum p_{ih}/n)\, r_h}$$

The **payoff** to agent i at profile p is

$$\begin{aligned} u_i(p) = \frac{1}{n} \Big[&(1-p_{i\ell}) r_\ell (\pi^N L^* + (1-\pi^N) L_\ell^N) + (1-p_{ih}) r_h (\pi^N L^* + (1-\pi^N) L_h^N) \\ &+ p_{i\ell} r_\ell (\pi^T L^* + (1-\pi^T) L_\ell^T) + p_{ih} r_h (\pi^T L^* + (1-\pi^N) L_h^T) \Big] \end{aligned} \tag{7}$$

In standard fashion, p_i is a **best response** to p_{-i} if $p_i \in \arg\max u_i(\cdot, p_{-i})$, and p is a (pure Nash) **equilibrium** if, for each i, p_i is a best response to p_{-i}.

4.2 Equilibrium structure and intuition

The perfect competition setting yields intuitive equilibria: low types take action N and high types take action T, with the possible exception that recipients of *one* type instead take the opposite action. The imperfect competition setting admits the possibility of equilibria with the "inverse" structure: agents assign low types to action T and high types to action N, again with a possible exception for only one type.

Theorem 3. *Fix k and suppose p is an equilibrium. There exists an equilibrium p^* that is payoff-equivalent to p, is symmetric, and satisfies one of the following.*

- *(Non-inversion) For every agent i, $p^*_{i\ell} = 0$ or $p^*_{ih} = 1$.*
- *(Inversion) For every agent i, $p^*_{i\ell} = 1$ or $p^*_{ih} = 0$.*

"Interior" equilibria are ruled out since, at any strategy profile, all agents face the same linear incentive to "swap" the opposite actions of opposite-type recipients. At any interior profile, all agents would prefer executing the same such swaps until reaching the same kind of a corner solution (a Non-inversion or Inversion strategy).

For an intuition behind symmetry, note that the set of Non-inversion strategies is a monotonic, one-dimensional set: a decision of how many recipients to send to

Treatment, prioritizing high-type recipients over low types. If agent i sends fewer recipients to Treatment than agent j, then i has a greater marginal incentive than j to send *additional* recipients to Treatment since doing so crowds out fewer of i's own recipients. Since both agents should face the same marginal incentive in equilibrium, they must choose symmetric Non-inversion strategies; the same argument applies to Inversion.

Inversion equilibria exhibit a surprising "double distortion" in that each agent simultaneously chooses T for at least some of its low type recipients and chooses N for at least some of its high type recipients. For an intuition as to how such equilibria arise, consider agent i's best response when i's competitors choose T for a "large" percentage of their recipients. First, excess congestion at T could conceivably lead i to choose N for (at least some of) its *high* types in order to give those (high marginal value) recipients better odds of an object. Given this, it is conceivable that i prefers to choose T for its *low* types (and remaining high types) to avoid congesting its high types at N. This possibility is most conceivable when low types' welfare is relatively insensitive both to object assignment and to action choice. If i thus chooses T for a "large" percentage of recipients, the profile may be an equilibrium.

Computational analysis demonstrates that one or both forms of equilibria might exist (Subsection 4.3). One factor contributing to this—and complicating equilibrium analysis in general—is that payoff functions (7) exhibit convexities[10] contrasting typical assumptions made in the literature on atomic congestion games. While our computational results suggest a general existence result, our analytical result below specifically shows that Non-inversion equilibria exist under mild additional assumptions. The first is that any recipient's benefit of receiving an object exceeds the welfare difference between any two non-receiving recipients.

Assumption 1 (Objects are sufficiently valuable)**.** $L^* - L^N_\ell > L^N_\ell - L^N_h$.

Under this assumption, and for "reasonable" rations k (i.e. no less than the proportional value $\bar{k} = r_h/(r_\ell + r_h)$), exactly one Non-inversion equilibrium exists as long as, relative to object scarcity, high types benefit significantly from Treatment (8), and Treatment makes high and low type recipients similar (9).[11]

Theorem 4. *Fix $n \geq 3$ and suppose that Assumption 1 holds. If $k \geq \bar{k}$ and*

[10] In fact they locally violate concavity everywhere; see the online appendix.

[11] Even weaker assumptions are used in the proof, but require concepts from Subsection 4.4.

$$\frac{L_h^T - L_h^N}{L_\ell^N - L_h^N} \geq \frac{\phi}{(r_\ell + r_h)n - \phi(n-1)} \tag{8}$$

and

$$\frac{L_\ell^T - L_h^T}{(L_h^T - L_h^N) + (L_\ell^N - L_\ell^T)} < 1 - \frac{\phi}{(r_\ell + r_h)\frac{n-1}{n}} \tag{9}$$

then there exists a unique Non-inversion equilibrium.

The proof cannot be applied to the full range of the model's primitives, and does not apply to Inversion equilibria due to complications arising from convexities in payoff functions. We therefore turn to computational analysis to investigate the prevalence of either type of equilibrium.

4.3 Computational analysis: equilibria and welfare

Across a wide range of economies and of rations (k) we search for all (approximate) equilibria, classify each as Non-inverting or Inverting, and evaluate its welfare. We find that Inversion equilibria can exist—even exclusively—arising when objects are not too scarce and $L_\ell^T - L_h^T$ is relatively large. Not surprisingly these conditions run counter to inequalities (8) and (9). Furthermore such equilibria can be welfare optimal when $L_\ell^N - L_\ell^T$ is small, which also counters (9).

We next describe the primitives we consider, providing technical details and additional computations in the online appendix. Normalizing $r_\ell + r_h \equiv 1$, we consider a full (discretized) range of values for $0 < \phi < 1$. Since our analytical results (Theorem 3, Theorem 4) require $n \geq 3$, we consider the case of maximal market power by setting $n = 3$; larger n are considered in the online appendix (see Observation 5 below).

Normalizing recipient welfare values $L_h^N \equiv 0$ and $L_\ell^N \equiv 0.5$, we set L^* equal to the lower bound given by Assumption 1, i.e. $L^* = 2L_\ell^N - L_h^N \equiv 1$. Larger L^* are considered in the online appendix (see Observation 4 below). With these three welfare values fixed, we consider the full (discretized) range of values for L_h^T and L_ℓ^T satisfying Equation 1.

For all such instances we compute approximate equilibria, classifying each as an Inversion or Non-inversion profile. We present these results using the following nota-

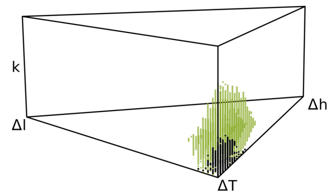

Figure 1: Equilibrium structure when $r_\ell = 0.3, \phi = 0.5$. Black: unique Inversion equilibrium. Unshaded: unique Non-inversion equilibrium. Green: one of each equilibrium type.

tion.

$$\begin{aligned} \Delta_h &\equiv L_h^T - L_h^N \geq 0 \qquad & \Delta_T &\equiv L_\ell^T - L_h^T \geq 0 \\ \Delta_\ell &\equiv L_\ell^N - L_\ell^T \geq 0 \qquad & \Delta_* &\equiv L^* - L_\ell^N \geq 0 \end{aligned} \tag{10}$$

Our normalizations of L_h^N, L_ℓ^N, and L^* imply $\Delta_h + \Delta_T + \Delta_\ell = 0.5$ (and $\Delta_* = 0.5$). Thus the set of feasible choices of L_h^T and L_ℓ^T can be visualized as a 2-dimensional simplex representing feasible choices of the triplet $(\Delta_\ell, \Delta_T, \Delta_h)$, as in Figure 1. The vertical dimension of the prism represents the ration $k \in [0, 1]$.

As one example of our computations, the prism in Figure 1 shows which (approximate) equilibrium types exist when $r_\ell = 0.3$ and $\phi = 0.5$. More generally Figure 2 displays prisms for various r_ℓ and ϕ. Consistent with Theorem 4, a unique Non-inversion equilibrium exists for sufficiently large k and sufficiently small Δ_T. However they exist beyond the set of primitives assumed in that theorem (but with uniqueness implied by Proposition 2). Even when one does not exist, a unique Inversion equilibrium does; thus we find (approximate) equilibrium existence across the full range of parameters despite the poorly behaved payoff functions that complicate more general analytical results. We now summarize our computational findings.

Observations. For the parameters discussed above we find:

1. For each instance considered, there exists a unique Non-inversion equilibrium or a unique Inversion equilibrium (or both).
2. Inversion equilibria are more prevalent as Δ_T becomes large. Since a large Δ_T ($\Delta_\ell, \Delta_h \approx 0$) implies a low cost to taking the "wrong" action for either type. An agent may be induced to do so if this increases the rate at which its high types receive objects. If in addition $k = 0.5$, a Non-inversion strategy profile is almost payoff equivalent to its mirror (Inversion) profile obtained by reversing all recipients' actions; Figure 2 exhibits equilibrium multiplicity in these cases.
3. Inversion equilibria are more prevalent when objects are more plentiful. When ϕ

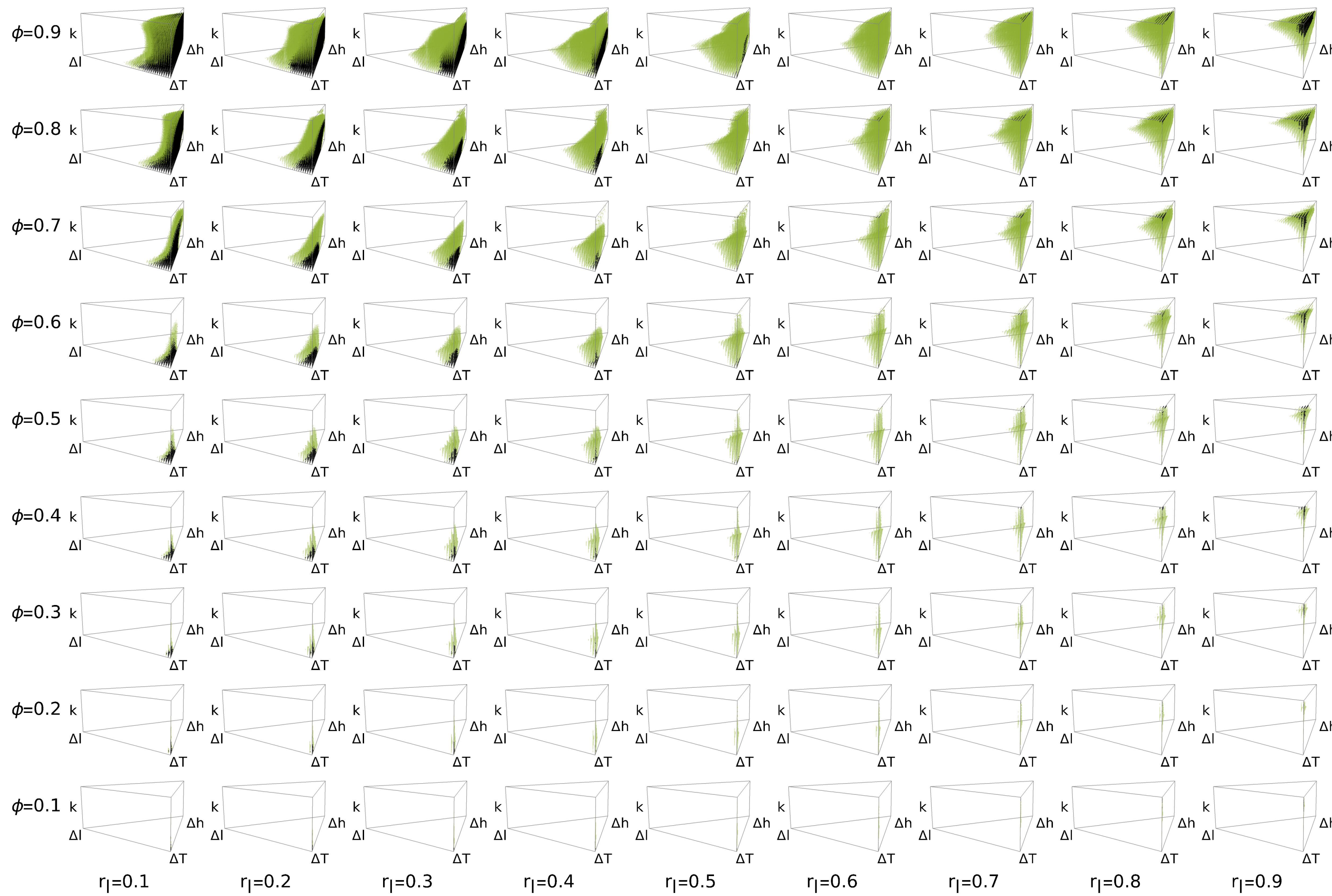

Figure 2: Equilibrium forms in various economies. Black: one Inversion equilibrium only. Unshaded: one Non-inversion equilibrium only. Green: one of each.

is large, the intuition for Inversion equilibria (Subsection 4.2) becomes stronger.

4. Inversion equilibria are less prevalent as L^* increases (see online appendix). Intuitively, as the distinction between types and actions disappears we converge to a one-type congestion model with known existence and uniqueness results.
5. Inversion equilibria are less prevalent as n increases (see online appendix), consistent with their non-existence under Perfect Competition (Proposition 1).

Observations 3 and 4 suggest that Inversion equilibria arise in cases where the planner's allocation problem has lower stakes, i.e. objects are more plentiful or provide lower value. In remaining cases (where Δ_T is large) it makes little sense to Ration through Classification in the first place, since action choices have little screening power: N and T approximate cheap-talk messages.

The remaining question is on welfare: What ration k (and equilibrium form) maximizes welfare? Though intuition suggests that welfare maximizing equilibria should be Non-inverting, it is logically possible for Inversion equilibria to be optimal. To see why, suppose $\Delta_\ell \approx 0$ (low types have a low cost for Treatment), and start from some arbitrary Non-inversion profile at which the allocation probabilities satisfy $\pi^N < \pi^T$. If agent i increases $p_{i\ell}$, its payoff changes primarily in two ways: (i) it gains since its low-type recipients receive objects more frequently and (ii) it loses because it crowds out its own high-type recipients, who receive objects less frequently. If we had started from an Inversion profile where $\pi^N > \pi^T$, the same conclusion would follow from a decrease in $p_{i\ell}$, *but with effect (ii) becoming stronger*. The asymmetry in effect (ii) is because crowding out high types from receiving objects is *more costly* when those high types are taking the "wrong" action, N. Interestingly, this introduces the possibility for Inversion profiles to provide a *stronger* incentive for an agent *not* to crowd out its own (and others') high types from receiving objects, because doing so is more costly than if actions were reversed.

Can this improvement in the high types's share of objects outweigh the "wrong" choices being made over actions under Inversion? Our computations (Figure 3) illustrate that it can in some cases where Δ_ℓ is small. However if objects are scarce or if low types non-negligibly distinguish between actions N and T, Non-Inversion equilibria are welfare-optimal; we consider them next.

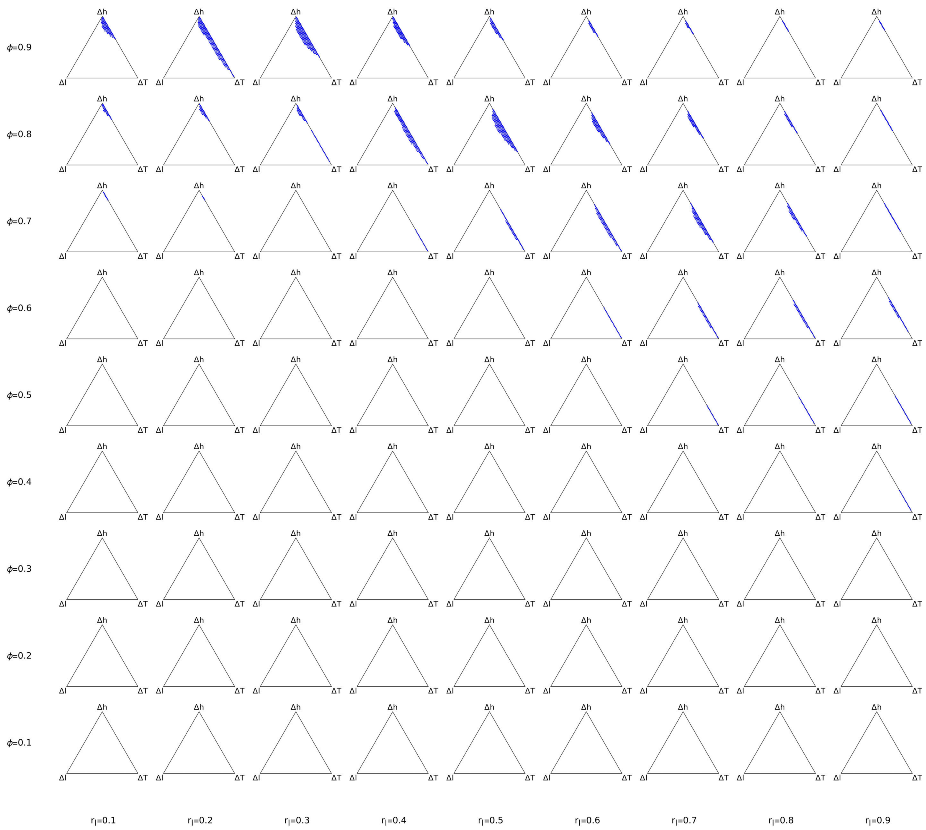

Figure 3: Across all values of k, the welfare-maximizing equilibrium is an Inversion equilibrium for economies in the blue regions.

4.4 Non-inversion equilibrium

We next show how the ideas of Section 3 hold in the presence of market power whenever Non-inversion equilibria prevail, the challenge being that such equilibria may not exist (Figure 2) or may be suboptimal (Figure 3). To overcome this technical issue, we first show general existence and uniqueness of "NI-candidates:" symmetric, Non-inversion strategy profiles satisfying certain *local* IC constraints. Since any NI equilibrium must be an NI-candidate, this proves at most one NI equilibrium can exist (as in our computations). We then prove our main results: the conclusions of Theorem 1 and Theorem 2 hold for any NI-candidate, and thus for any NI equilibrium.

Definition 1 (NI-candidate)**.** Fixing k, a symmetric profile p^* is an NI-candidate for k when any one of the following holds.

- (1-NI) $p^*_{i\ell} \equiv 0$ and $\frac{\partial u_i}{\partial p_{ih}}(p^*) \equiv 0$.
- (1-NI corner) $p^*_{i\ell} \equiv 0$, $p^*_{ih} r_h \equiv k\phi$, and $\frac{\partial u_i}{\partial p_{ih}}(p^*) \leq 0$.
- (2-NI) $p^*_{i\ell} \equiv 0$, $p^*_{ih} \equiv 1$, $\frac{\partial u_i}{\partial p_{ih}}(p^*) \geq 0$, and $\frac{\partial u_i}{\partial p_{i\ell}}(p^*) \leq 0$.
- (3-NI) $p^*_{ih} \equiv 1$ and $\frac{\partial u_i}{\partial p_{i\ell}}(p^*) \equiv 0$.
- (3-NI corner) $p^*_{ih} \equiv 1$, $(1 - p^*_{i\ell}) r_\ell \equiv (1-k)\phi$, and $\frac{\partial u_i}{\partial p_{i\ell}}(p^*) \geq 0$.

The following result extends Proposition 1 to NI-candidates.

Proposition 2 (Unique NI-candidate)**.** *Fix $n \geq 3$ and suppose Assumption 1 holds. For any $k \in [0,1]$ there exists a unique NI-candidate $p^*(k)$. Furthermore $p^*()$ is weakly increasing in k, and*

$$
\begin{aligned}
k < k'_n &\implies \forall i,\ p^*_{i\ell}(k) = 0 \text{ and } p^*_{ih}(k) < 1 && \text{(Region NI-1)} \\
k'_n \leq k \leq k^*_n &\implies \forall i,\ p^*_{i\ell}(k) = 0 \text{ and } p^*_{ih}(k) = 1 && \text{(Region NI-2)} \\
k > k^*_n &\implies \forall i,\ p^*_{i\ell}(k) > 0 \text{ and } p^*_{ih}(k) = 1 && \text{(Region NI-3)}
\end{aligned}
$$

where

$$
k'_n = \max\left\{0, \frac{-(L^T_h - L^N_h) + \frac{\phi}{r_\ell}\left(\frac{n-1}{n}L^* + \frac{1}{n}L^N_\ell - L^N_h\right)}{\frac{\phi}{r_\ell}\left(\frac{n-1}{n}L^* + \frac{1}{n}L^N_\ell - L^N_h\right) + \frac{\phi}{r_h}\frac{n-1}{n}(L^* - L^T_h)}\right\} \tag{11}
$$

$$
k^*_n = \min\left\{1, \frac{(L^N_\ell - L^T_\ell) + \frac{\phi}{r_\ell}\frac{n-1}{n}(L^* - L^N_\ell)}{\alpha}\right\} > k' \tag{12}
$$

$$
\alpha = \frac{\phi}{r_\ell}\frac{n-1}{n}(L^* - L^N_\ell) + \frac{\phi}{r_h}\left[\frac{n-1}{n}L^* + \frac{1}{n}L^T_h - L^T_\ell\right] > 0
$$

It can be checked that k'_n and k^*_n converge to (3) and (4) as $n \to \infty$. The result is obtained by proving a limited form of concavity among Non-Inversion profiles (Lemma 5). Notably, a version of that lemma cannot hold for Inversion profiles, so we do not have corresponding results for such equilibria.

Extending the main conclusions of Theorem 1 and Theorem 2 to NI-candidates, both the fraction of objects allocated to high types and utilitarian welfare are single-peaked on $k \in [k'_n, 1]$ and are maximized at $k = k^*_n$.[12]

Theorem 5. *Fix $n \geq 3$ and suppose that Assumption 1 holds. Among all NI-candidates for $k \in [0, 1]$, both the equilibrium fraction of objects allocated to high types and agents' equilibrium payoffs are*

- *increasing in k for $k \in [k'_n, k^*_n]$,*
- *decreasing in k for $k \in [k^*_n, 1]$, and*
- *maximized at k^*_n (defined in Equation 12).*

While this result shows, for example, that an increase in $k \in [k^*_n, 1]$ leads to a disproportionate increase in low types to T, note that the intuition provided after Theorem 1 does not apply here. Namely, as $k \in [k^*_n, 1]$ increases, low types shift to T at a slower rate since an agent internalizes the increasing congestion effect on its own high types. Plausibly, an increase in k might then increase the equilibrium fraction of objects allocated to high types. Nevertheless, Theorem 5 rules this out.[13]

Proposition 3 (Competition lowers welfare)**.** *The ration k^*_n is decreasing in n. Therefore maximal welfare across all NI-candidates is decreasing in n.*

The simple proof of this (which we omit) is apparent in the following comparison between the perfect and imperfect competition scenarios. In the perfect competition case we can rewrite k^* as defined in (4) using the welfare differences defined in (10).

$$k^* = \frac{\Delta_\ell + \frac{\phi}{r_\ell}\Delta_*}{\frac{\phi}{r_\ell}\Delta_* + \frac{\phi}{r_h}(\Delta_* + \Delta_\ell)}$$

[12] One inconsequential contrast to Theorem 1 is a possible non-monotonicity on $k \in [0, k'_n]$.

[13] The expressions in the proof are extensive but yield the following intuition. At an NI-candidate, an agent considering whether to choose T for (infinitesimally) additional low types to T sees zero marginal gain when summing the three payoff effects: (i) increased benefit for those low types, (ii) increased congestion cost for its high types at T, and (iii) decreased congestion cost for its low types remaining at N. If an increase in k leads agents to increase the mass of low types at T *less* than proportionately at the new NI-candidate, benefits (i) and (iii) would be higher and cost (ii) would be lower than before. This contradicts the new NI-candidate requirement to satisfy the zero-marginal-gain condition.

We also can rewrite k_n^* (see proof of Proposition 4).

$$k_n^* = \frac{\Delta_\ell + \frac{\phi}{r_\ell}\frac{n-1}{n}\Delta_*}{\frac{\phi}{r_h}\frac{n-1}{n}\Delta_\ell + \left(\frac{\phi}{r_\ell} + \frac{\phi}{r_h}\right)\frac{n-1}{n}\Delta_* - \frac{\phi}{r_h}\frac{1}{n}\Delta_T}$$

Thus imperfect competition shrinks the numerator of k^* by less than a factor of $(1/n)$, and the denominator by more than that factor; hence $k_n^* > k^*$. However this expression also highlights the role of Δ_T: the agent internalizes the congestion its low types impose on its own high types at Treatment via Δ_T, the relative welfare difference between the types.

4.5 All-or-nothing prioritization

Though fractional rationing ($k < 1$) is typically optimal, in practice this approach may be infeasible for a variety of reasons (institutional or political constraints, complexity, etc.). Regardless of the reason, we compare the following two extremes of our approach more commonly seen in practice.

(FP) *Full prioritization* of one classification of recipients over the other.

(NP) *No Prioritization* of either classification over the other.

In our model FP corresponds to setting $k = 1$ (as long as $\phi \leq r_h$). NP uniformly rations objects across all recipients regardless of action, i.e. each recipient receives an object with probability $\phi/(r_\ell + r_h)$. Under perfect competition, NP is the equilibrium result of setting $k = \bar{k} = r_h/(r_\ell + r_h)$ (see Proposition 1). Under imperfect competition, the same is typically true (e.g. when Equation 8 holds).

It follows intuitively (assuming Non-inversion equilibria) that FP achieves close to the optimal level of welfare when k_n^* is larger (closer to 1) and NP is closer to optimal when k_n^* is lower (closer to $\bar{k}$). Therefore we reframe the comparison between FP and NP as a question of whether k_n^* is large or small. The following comparative statics use the welfare differences defined in (10).[14]

[14] An alternative approach is to directly compute welfare at (i) the NI-candidate profile when $k = 1$ and (ii) the separation profile $(p_{i\ell}, p_{ih}) \equiv (0, 1)$ when $k = \bar{k}$. It tediously involves two roots when solving for $p_{i\ell}$ in case (i), and offers little additional insight to Proposition 4.

Proposition 4 (Conditions justifying FP over NP)**.** *Consider varying primitives* $L_h^N, L_h^T, L_\ell^T, L_\ell^N, L^*$ *in a way that varies only one of the differences* $\Delta_\ell, \Delta_T, \Delta_h, \Delta_*$*, defined in Equation 10, keeping the rest constant.*

- k_n^* *is increasing in* Δ_ℓ*.*
- k_n^* *is increasing in* Δ_T*.*
- k_n^* *is constant in* Δ_h*.*
- k_n^* *is decreasing in* Δ_**.*

Furthermore k_n^* *decreases in* ϕ*.*

Natural intuition suggests why FP should be better than NP under these conditions. Large Δ_ℓ makes it costly for a low type receiving Treatment to fail to receive an object. Large Δ_T increases the value of objects to high types, increasing the agent's internalized cost or congesting them with low types. At k_n^*, an agent's marginal decision (at Non-inversion profiles) does not involve high types, so Δ_h is irrelevant. Increasing Δ_* is analogous to reducing differences between types and actions; if the agent's objective becomes object-share maximization it is harder to maintain type separation. Finally when ϕ decreases there is less to be gained by choosing Treatment for low types.

5 Conclusion

When biasing resource allocation toward recipients who take a particular action, the planner potentially improves the way resources are allocated but distorts decisions whether to take that action. Intuitively, these distortions can be reversed by reducing the ration of resources dedicated to recipients who take the action. This is indeed the case in a perfectly competitive, "no market power" model where recipients decide actions for themselves; by fine-tuning the ration, the planner can continuously fine-tune this tradeoff and maximize equilibrium welfare (Theorem 2).

We show that the presence of market power—the centralization of action choices among a few strategic agents—has multiple effects in this setting. An intuitive effect is that such agents internalize the congestion effect one recipient imposes by crowding out others who take the same action. It follows that, when equilibria have the same structure as in the perfectly competitive model above, an increase in market power leads to a new optimal ration that improves welfare (Proposition 3).

A second, less intuitive effect is that sufficiently strong market power can lead to a second form of ("Inversion") equilibria, where each agent makes inefficient action choices for multiple types of recipients in a way that disproportionately improves resource allocation rates for its high-value recipients. Even more surprising is that such highly distorted equilibria can, in some cases, maximize equilibrium welfare. Taken together, these results show that the presence of market power can have unanticipated effects on equilibrium behavior and therefore on the design of mechanisms. A corollary of this is that optimal mechanisms (and the behavior they induce) can vary across different markets when those markets embody different competitive structures. Designing and then applying a *single* mechanism across such disparate markets can lead to varying levels of inefficient behavior (e.g., Parker et al., 2018) and, consequently, misallocation of resources.

References

Nikhil Agarwal and Eric Budish. Market design. In *Handbook of Industrial Organization*, pages 1–79. Elsevier, 2021.

Mohammad Akbarpour, Piotr Dworczak, and Scott Duke Kominers. Redistributive allocation mechanisms. *Journal of Political Economy*, 132(6):1831–1875, 2024.

Ian Ball. Scoring strategic agents. *American Economic Journal: Microeconomics*, forthcoming.

Umang Bhaskar, Lisa Fleischer, Darrell Hoy, and Chien-Chung Huang. On the uniqueness of equilibrium in atomic splittable routing games. *Mathematics of Operations Research*, 40(3):634–654, 2015.

Mark Braverman and Sumegha Garg. The Role of Randomness and Noise in Strategic Classification. In *1st Symposium on Foundations of Responsible Computing (FORC 2020)*, volume 156 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 9:1–9:20. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020.

Alex Frankel and Navin Kartik. Improving Information from Manipulable Data. *Journal of the European Economic Association*, 20(1):79–115, 2021.

Moritz Hardt, Nimrod Megiddo, Christos Papadimitriou, and Mary Wootters. Strategic classification. In *Proceedings of the 2016 ACM conference on innovations in theoretical computer science*, pages 111–122, 2016.

Tobias Harks and Veerle Timmermans. Uniqueness of equilibria in atomic splittable polymatroid congestion games. *Journal of Combinatorial Optimization*, 36:812–830, 2018.

Hideo Konishi. Uniqueness of user equilibrium in transportation networks with heterogeneous commuters. *Transportation Science*, 38(3):315–330, 2004.

Frances Xu Lee and Wing Suen. Gaming a selective admissions system. *International Economic Review*, 64(1):413–443, 2023.

Benjamin Lester, Ali Shourideh, Venky Venkateswaran, and Ariel Zetlin-Jones. Screening and adverse selection in frictional markets. *Journal of Political Economy*, 127(1):338–377, 2019.

César Martinelli and Susan W Parker. Should transfers to poor families be conditional on school attendance? a household bargaining perspective. *International Economic Review*, 44(2):523–544, 2003.

Teddy Mekonnen and Bobak Pakzad-Hurson. Competition, persuasion, and search. *arXiv preprint arXiv:2411.11183*, 2024.

Smitha Milli, John Miller, Anca D Dragan, and Moritz Hardt. The social cost of strategic classification. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pages 230–239, 2019.

Edwin Munoz-Rodriguez. Ethical organ allocation under dynamic asymmetric information, 2024.

Minseon Park and Dong Woo Hahm. Location choice, commuting, and school choice. *Commuting, and School Choice (January 18, 2023)*, 2023.

William F. Parker, Allen S. Anderson, Donald Hedeker, Elbert S. Huang, Edward R. Garrity, Jr, Mark Siegler, and Matthew M. Churpek. Geographic variation in the treatment of u.s. adult heart transplant candidates. *Journal of the American College of Cardioloy*, 71(16):1715–1725, 2018.

Eduardo Perez-Richet and Vasiliki Skreta. Test design under falsification. *Econometrica*, 90(3):1109–1142, 2022.

Eduardo Perez-Richet and Vasiliki Skreta. Fraud-proof non-market allocation mechanisms. 2023.

Guofu Tan and Junjie Zhou. The effects of competition and entry in multi-sided markets. *The Review of Economic Studies*, 88(2):1002–1030, 2021.

Cheng Wan. Coalitions in nonatomic network congestion games. *Mathematics of Operations Research*, 37(4):654–669, 2012.

John Glen Wardrop. Some theoretical aspects of road traffic research. *Proceedings of the Institution of Civil Engineers*, 1(3):325–362, 1952.

6 Proofs Appendix

6.1 Perfect Competition

Proof of Lemma 1. If $p_\ell > 0$ then low types weakly prefer choosing Treatment:

$$\pi^N L^* + (1-\pi^N)L_\ell^N \leq \pi^T L^* + (1-\pi^T)L_\ell^T$$

Since $L^* > L_\ell^N > L_\ell^T$ (and $\min\{\pi^N, \pi^T\} < 1$) this would imply $\pi^N < \pi^T$. Similarly $p_h < 1$ would imply $\pi^N > \pi^T$. Hence $p_\ell = 0$ or $p_h = 1$. □

Lemma 2. *For any k there is a unique equilibrium $p_\ell(k), p_h(k)$. Furthermore $p_\ell()$ and $p_h()$ are weakly increasing in k.*

Proof of Lemma 2. For any k, equilibrium existence follows from standard arguments and is omitted. To prove uniqueness and monotonicity, fix $k, \tilde{k}$ with $k \leq \tilde{k}$ and let (p_ℓ, p_h) and $(\tilde{p}_\ell, \tilde{p}_h)$ be arbitrary equilibria for k and $\tilde{k}$ respectively, with allocation probabilities $\pi^N, \pi^T, \tilde{\pi}^N, \tilde{\pi}^T$. We show monotonicity $(p_\ell, p_h) \leqq (\tilde{p}_\ell, \tilde{p}_h)$ which also implies uniqueness $(k = \tilde{k})$.

Claim: either $(p_\ell, p_h) \leqq (\tilde{p}_\ell, \tilde{p}_h)$ or $(p_\ell, p_h) \geqq (\tilde{p}_\ell, \tilde{p}_h)$. If $p_\ell = \tilde{p}_\ell$ (or $p_h = \tilde{p}_h$) the claim follows immediately. If $p_\ell < \tilde{p}_\ell$ then Lemma 1 implies $\tilde{p}_h = 1 \geq p_h$. Similarly $p_\ell > \tilde{p}_\ell$ implies $p_h = 1 \geq \tilde{p}_h$, proving the claim.

Claim: $(p_\ell, p_h) \leqq (\tilde{p}_\ell, \tilde{p}_h)$. First suppose instead that $p_\ell > \tilde{p}_\ell$ and hence $p_h \geq \tilde{p}_h$. Since $k \leq \tilde{k}$ this implies $\pi^N > \tilde{\pi}^N$ and $\pi^T < \tilde{\pi}^T$. Since p is an equilibrium for k, low types weakly prefer Treatment in that equilibrium:

$$\pi^T L^* + (1 - \pi^T)L_\ell^T \geq \pi^N L^* + (1 - \pi^N)L_\ell^N$$

This implies a strict such preference at $\tilde{p}$ under $\tilde{k}$:

$$\tilde{\pi}^T L^* + (1 - \tilde{\pi}^T)L_\ell^T > \tilde{\pi}^N L^* + (1 - \tilde{\pi}^N)L_\ell^N$$

This strict preference requires $\tilde{p}_\ell = 1$ in equilibrium, contradicting $p_\ell > \tilde{p}_\ell$. Supposing $p_h > \tilde{p}_h$ leads to a similar contradiction. □

Proof of Proposition 1. By Lemmas 1 and 2 there exist $0 \leq k' \leq k^* \leq 1$ that define the three cases of Proposition 1. When $k = \bar{k} = r_h/(r_\ell + r_h)$, a separating profile $(p_\ell = 0,\, p_h = 1)$ yields $\pi^N = \pi^T$, so $(p_\ell = 0,\, p_h = 1)$ is an equilibrium where each agent has *strict* incentive to choose their natural action. By continuity this would hold for small perturbations of k, thus $k' < \bar{k} < k^*$.

Next, by continuity, k' is the lowest value of k at which the separation profile $(p_\ell = 0,\, p_h = 1)$ induces a high type to choose T, i.e. at which

$$\pi^T L^* + (1 - \pi^T)L_h^T \geq \pi^N L^* + (1 - \pi^N)L_h^N$$

Substituting $\pi^N = (1 - k)\phi/r_\ell$ and $\pi^T = k\phi/r_h$ this becomes

$$k \geq \frac{(L_h^N - L_h^T) + \frac{\phi}{r_\ell}\phi(L^* - L_h^N)}{\frac{\phi}{r_h}(L^* - L_h^T) + \frac{\phi}{r_\ell}(L^* - L_h^N)}$$

yielding k' as in (3). This is positive whenever the numerator is, yielding (5).

Similarly, low types are induced to choose N at the separation profile when

$$\pi^T L^* + (1 - \pi^T)L_\ell^T \leq \pi^N L^* + (1 - \pi^N)L_\ell^N$$

Analogous arguments lead to (4) and (6). □

Proof of Theorem 1. The result is obvious in the range $k \in [k', k^*]$ where $p_l(k) \equiv 0$, $p_h(k) \equiv 1$, and hence $f(k) \equiv k$.

For any $k \in (k^*, 1)$, Proposition 1 implies $p_\ell(k) > 0$ and $p_h(k) = 1$; furthermore $p_\ell(k) < 1$ (otherwise a low type guarantees an object deviating to N). This implies an equilibrium indifference condition for low types. Writing equilibrium allocation probabilities π^N, π^T as functions of k, it is

$$\pi^T(k)L^* + (1 - \pi^T(k))L_\ell^T = \pi^N(k)L^* + (1 - \pi^N(k))L_\ell^N, \text{ or}$$
$$\frac{1 - \pi^N(k)}{1 - \pi^T(k)} = \frac{L^* - L_\ell^T}{L^* - L_\ell^N} > 1$$

where $L_\ell^T < L_\ell^N$ implies the inequality. Therefore $\pi^N(k) < \pi^T(k)$, and $\pi^N(k), \pi^T(k)$ vary in the same direction with a change in $k \in (k^*, 1)$. We show $\pi^T(k)$ (hence f) is decreasing on this range.

Fix $k^* < k < k + \epsilon < 1$ and let $\delta = p_\ell(k + \epsilon) - p_\ell(k) \geq 0$. If instead we have $\frac{k\phi + \epsilon\phi}{p_\ell(k)r_\ell + \delta r_\ell + r_h} = \pi^T(k + \epsilon) \geq \pi^T(k) = \frac{k\phi}{p_\ell(k)r_\ell + r_h}$ then $(\epsilon\phi)/(\delta r_\ell) \geq \pi^T(k) > \pi^N(k)$. This also means $\frac{(1-k)\phi}{(1-p_\ell(k))r_\ell} = \pi^N(k) > \pi^N(k + \epsilon) = \frac{(1-k)\phi - \epsilon\phi}{(1-p_\ell(k))r_\ell - \delta r_\ell}$. (In words, if an increase in k moves "disproportionately" few low types to T to increase π^T, this must decrease $\pi^N < \pi^T$.) This contradicts the fact that π^N, π^T covary; the indifference condition cannot hold at $k + \epsilon$. Therefore (with continuity arguments) π^T decreases in $k \in [k^*, 1]$.

A symmetric argument applies to $k \in [0, k']$ (where $\pi^N > \pi^T$). An increase in k disproportionately increases p_h, increasing π^N, the rate at which *low* types receive objects, hence decreasing $f()$. □

Proof of Theorem 2. On $k \in [k', 1]$ welfare is clearly single-peaked (with peak at k^*) following arguments made in the text. The rest of this proof covers $k \in [0, k']$.

At $k = 0$ we know that (i) all objects go to the agents choosing N, (ii) all low types choose N ($p_\ell = 0$), and (iii) at most all high types choose N ($p_h \leq 1$). Thus the fraction of objects going to high types at $k = 0$ is

$$f(0) = \frac{(1 - p_h(0))r_h}{(1 - p_h(0))r_h + r_\ell} \leq \frac{r_h}{r_h + r_\ell} \equiv \bar{k}$$

i.e. high types receive less than their "proportional share" $\bar{k}$.

At $k = k^*$, agents use a separating profile and thus $f(k^*) = k^* > \bar{k}$ (where the inequality holds from Proposition 1). Thus when comparing k^* to $k = 0$, (i) high types receive more objects and (ii) treatment decisions are more efficient. Welfare is thus higher at k^*.

Finally the same conclusion can be drawn for any $k \in (0, k']$: By Theorem 1 high types receive even fewer objects at such k than at $k = 0$, and thus fewer than at k^*. Furthermore treatment decisions remain less efficient than at k^*. Therefore welfare is higher at k^* than at any $k \in [0, k']$. □

6.2 Imperfect Competition

It is immediate that any equilibrium profile must be non-wasteful, so we restrict attention to non-wasteful profiles henceforth.

6.2.1 Equilibrium structure: Non-inversion/Inversion

We first observe that an agent i's best response must be either a Non-inversion or Inversion *strategy*: a point on the boundary of $[0,1]^2$. Fix a profile p with interior $p_i \in (0,1)^2$, resulting in allocation probabilities π^N, π^T. Consider deviation p'_i obtained from p_i by "swapping" $\epsilon > 0$ mass of low types at T to N with ϵ mass of high types at N to T, i.e.

$$(p'_{i\ell}, p'_{ih}) = (p_{i\ell} - \epsilon n/r_\ell, p'_{ih} + \epsilon n/r_h)$$

Since this deviation does not change the total masses of recipients at N and T it does not change π^N and π^T. Therefore this deviation affects neither the other agents' payoffs nor i's total consumption of objects. The deviation changes i's payoff only in that, among i's recipients who fail to receive an object, some who were assigned to N transform from high types into low types and some who were assigned to T turn from low types into high types. The magnitude of this change in payoff is

$$\epsilon[(1 - \pi^N)(L^N_\ell - L^N_h) + (1 - \pi^T)(L^T_h - L^T_\ell)] \tag{13}$$

While (13) can have any sign, its linearity in ϵ means that payoff functions are ruled surfaces, so a best response is a corner solution (or payoff-equivalent to one).

Lemma 3 (No double-mixing)**.** *Fix k, an agent i, and a profile p at which p_i is a best response to p_{-i}. There exists $p'_i \in [0,1]^2 \setminus (0,1)^2$ such that*

(i) p'_i also is a best response to p_{-i}, and

(ii) for any agent j and any p'_{-i}, $u_j(p_i, p'_{-i}) = u_j(p'_i, p'_{-i})$.

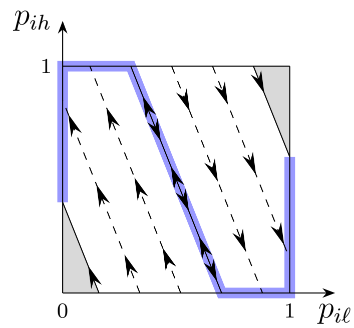

Figure 4: Changing agent i's strategy along a dashed line (northwestly) changes i's payoff according to (13); arrows represent a payoff increase. Best responses lie within the thick blue line. Depending on parameters, there may exist (shaded) regions of wasteful strategies.

Proof of Lemma 3. Fix i and p as in the Lemma. If (13) is positive, i would have the incentive to swap equal masses of low types at T with high types at N if feasible. Since p_i is a best response this must be infeasible: either $p_{i\ell} = 0$ or $p_{ih} = 1$. Similarly if (13) is negative then $p_{i\ell} = 1$ or $p_{ih} = 0$. In either case the result follows immediately by letting $p_i' = p_i$.

Suppose (13) equals zero. If $p_i \notin (0,1)^2$, setting $p_i' = p_i$ proves the result. Otherwise let $(p_{i\ell}', p_{ih}') = (p_{i\ell} - \epsilon n/r_\ell, p_{ih}' + \epsilon n/r_h)$ where, choosing ϵ maximally, $p_i' \notin (0,1)^2$. Since (13) is zero p_i' is also a best response to p_{-i}, proving (i). Furthermore this deviation preserves i's total masses of recipients assigned N and T, implying (ii). $\square$

Fact 1. The following facts about (13) are used below.

(i) Since $|(L_h^T - L_\ell^T)| < (L_\ell^N - L_h^N)$, $\pi^N = \pi^T$ implies that (13) is positive.

(ii) $\pi^N = 1$ ($\pi^T = 1$) implies that (13) is negative (positive).

(iii) When a change in strategy profile increases the total mass of recipients at T, π^N increases, π^T decreases, and thus (13) decreases.

By Lemma 3, a best response is either Non-inverting or Inverting, or it can be replaced with a payoff equivalent such strategy without affecting other agents' payoffs. Figure 4 illustrates these best responses, though we henceforth ignore interior ones. In addition we can prove the following.

Lemma 4. *Fix k. If profile p is an equilibrium, there exists a payoff-equivalent equilibrium p^* where either*

- *(Non-inversion) for every agent i, $p^*_{i\ell} = 0$ or $p^*_{ih} = 1$; or*
- *(Inversion) for every agent i, $p^*_{i\ell} = 1$ or $p^*_{ih} = 0$.*

Proof. Fixing such p, if (13) is positive (or negative) all agents are using Non-inversion (or Inversion) strategies (Figure 4). If (13) is zero we can construct a payoff equivalent, Non-inverting profile p' (as in the proof of Lemma 3) at which each individual agent sends the same mass of recipients to T at both profiles. Therefore for all i, all best responses to p_{-i} remain best responses to p'_{-i}, and thus p' also is an equilibrium. □

6.2.2 Equilibrium structure: symmetry

To make the rest of our proofs more concise, we express strategies and the planner's rationing decision in terms of masses rather than percentages. To represent a planner's choice of k we denote the masses of objects rationed to N and T as

$$\phi_N = (1-k)\phi \qquad \phi_T = k\phi$$

Similarly for a given strategy profile p and agent i we write

$$\begin{aligned} A_i &= (1-p_{i\ell})r_\ell/n & D_i &= p_{i\ell}r_\ell/n \\ B_i &= (1-p_{ih})r_h/n & E_i &= p_{ih}r_h/n \\ C_i &= \sum_{j\neq i}[(1-p_{j\ell})r_\ell/n + (1-p_{jh})r_h/n] & F_i &= \sum_{j\neq i}[p_{j\ell}r_\ell/n + p_{jh}r_h/n] \end{aligned}$$

Here A_i, B_i, and C_i are i's low types, high types, and competitors that take action N; D_i, E_i, and F_i correspond to T. We can write (7) (i's payoff u_i) as

$$\begin{aligned} &A_iL^N_\ell + \frac{A_i}{A_i+B_i+C_i}\phi_N(L^*-L^N_\ell) + B_iL^N_h + \frac{B_i}{A_i+B_i+C_i}\phi_N(L^*-L^N_h) \\ &+ D_iL^T_\ell + \frac{D_i}{D_i+E_i+F_i}\phi_T(L^*-L^T_\ell) + E_iL^T_h + \frac{E_i}{D_i+E_i+F_i}\phi_T(L^*-L^T_h) \end{aligned} \tag{14}$$

keeping in mind that $A_i = r_\ell/n - D_i$ and $B_i = r_h/n - E_i$.

While an agent's payoff is not generally concave in p_i, it is concave with respect to p_{ih} and, in some special cases, with respect to $p_{i\ell}$. The proof of the lemma also contains partial derivatives of payoffs utilized in later proofs.

Lemma 5 (Limited concavity.)**.** *Fix k, a non-wasteful profile p, and an agent i.*

(i) $u_i(p)$ *is concave in* p_{ih}.

(ii) If $p_{ih} = 1$ *then* $u_i(p)$ *is either decreasing or concave in (non-wasteful)* $p_{i\ell} \in [0, 1]$.

(iii) If Assumption 1 holds, $n \geq 2$*, and* $p_{jh} = p_{kh}$ *for all* j, k*,*[15] *then* $u_i(p)$ *is concave in* $p_{i\ell}$.

Proof. To prove (i) we show (14) is concave in E_i. Omitting subscript i, its derivative with respect to E (noting $B = r_h/n - E$) is

$$\begin{aligned}
\frac{\partial u_i}{\partial E} &= \frac{A}{(A+B+C)^2}\phi_N(L^* - L_\ell^N) - L_h^N - \frac{A+C}{(A+B+C)^2}\phi_N(L^* - L_h^N) \\
&\quad - \frac{D}{(D+E+F)^2}\phi_T(L^* - L_\ell^T) + L_h^T + \frac{D+F}{(D+E+F)^2}\phi_T(L^* - L_h^T) \\
&= (L_h^T - L_h^N) + \frac{\phi_N}{A+B+C}\left(\frac{A}{A+B+C}(L_h^N - L_\ell^N) - \frac{C}{A+B+C}(L^* - L_h^N)\right) \\
&\quad + \frac{\phi_T}{D+E+F}\left(\frac{D}{D+E+F}(L_\ell^T - L_h^T) + \frac{F}{D+E+F}(L^* - L_h^T)\right) \\
&= (L_h^T - L_h^N) - \phi_N\frac{A(L_\ell^N - L_h^N) + C(L^* - L_h^N)}{(A + r_h/n - E + C)^2} + \phi_T\frac{D(L_\ell^T - L_h^T) + F(L^* - L_h^T)}{(D+E+F)^2}
\end{aligned} \tag{15}$$

Since all bracketed terms are positive, (15) is decreasing in E. Therefore u_i is concave in E (i.e. in p_{ih}).

To show (ii) and (iii), the derivative of (14) with respect to D is

$$\begin{aligned}
\frac{\partial u_i}{\partial D} &= (L_\ell^T - L_\ell^N) + \frac{\phi_N}{A+B+C}\left(\frac{B}{A+B+C}(L_\ell^N - L_h^N) - \frac{C}{A+B+C}(L^* - L_\ell^N)\right) \\
&\quad + \frac{\phi_T}{D+E+F}\left(\frac{E}{D+E+F}(L_h^T - L_\ell^T) + \frac{F}{D+E+F}(L^* - L_\ell^T)\right) \\
&= \underbrace{(L_\ell^T - L_\ell^N)}_{\text{treatment effect}} + \underbrace{\phi_N\frac{B(L_\ell^N - L_h^N) - C(L^* - L_\ell^N)}{(r_\ell/n - D + B + C)^2}}_{N\text{-reallocation effect}} + \underbrace{\phi_T\frac{-E(L_\ell^T - L_h^T) + F(L^* - L_\ell^T)}{(D+E+F)^2}}_{T\text{-reallocation effect}}
\end{aligned} \tag{16}$$

While the treatment effect is negative, the overall sign of (16) depends on the signs of two "reallocation effects." Denote

$$X = B(L_\ell^N - L_h^N) - C(L^* - L_\ell^N) \qquad X' = -E(L_\ell^T - L_h^T) + F(L^* - L_\ell^T)$$

[15] The assumption that both $C_i \geq r_h/n$ and $p_{ih} = 0$ also is sufficient.

When $X > 0$ ($X < 0$) the "N-reallocation effect" is convex and increasing in D (concave, decreasing in D); when $X' > 0$ ($X' < 0$) the "T-reallocation effect" is convex and decreasing in D (concave, increasing in D).

Assumption 1 implies $X < X'$ (see online appendix); thus there are three cases.

- $X < X' \leq 0$: it is immediate that (16) is negative, so u_i is decreasing in D (i.e. in $p_{i\ell}$).
- $X \leq 0 < X'$: both reallocation effects are decreasing in D so (16) is decreasing in D; hence u_i is concave in D (in $p_{i\ell}$).
- $0 < X < X'$: both treatment effects are positive and convex in D, but change in opposite directions with respect to D. Therefore (16)'s sign and its direction of change w.r.t. D are indeterminate.

To prove statement (ii) of the lemma observe that if $p_{ih} = 1$ (i.e. $B = 0$) then $X < 0$ yielding the first and second cases above.

To prove (iii) observe that if $p_{ih} = p_{jh}$ for all j then $B \leq (n-1)C$ and $E \leq (n-1)F$. If $n \geq 2$ then Assumption 1 implies $X < 0 < X'$ yielding the second case above. □

The next lemma implies the intuitive idea that an agent who is sending more recipients to Treatment than another derives lower marginal benefit from sending additional recipients to Treatment due to crowding out more of its own recipients.

Lemma 6. *For any k, any agents i and j, and any non-wasteful profile p,*

$$\begin{aligned}\frac{\partial u_i}{\partial D_i} - \frac{\partial u_j}{\partial D_j} &= \frac{\partial u_i}{\partial E_i} - \frac{\partial u_j}{\partial E_j}\\ &= (D_j - D_i)\left[\frac{\phi_N(L^* - L_\ell^N)}{[r_\ell + r_h - (D_i + E_i + F_i)]^2} + \frac{\phi_T(L^* - L_\ell^T)}{(D_i + E_i + F_i)^2}\right]\\ &\quad + (E_j - E_i)\left[\frac{\phi_N(L^* - L_h^N)}{[r_\ell + r_h - (D_i + E_i + F_i)]^2} + \frac{\phi_T(L^* - L_h^T)}{(D_i + E_i + F_i)^2}\right] \qquad (17)\end{aligned}$$

Proof. Rewriting (15) with $A_i = r_\ell/n - D_i$ and $C_i = (n-1)(r_\ell/n + r_h/n) - F_i$,

$$\begin{aligned}\frac{\partial u_i}{\partial E_i} = (L_h^T - L_h^N) - \phi_N \frac{(r_\ell/n - D_i)(L_\ell^N - L_h^N) + ((n-1)(r_\ell/n + r_h/n) - F_i)(L^* - L_h^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2}\\ + \phi_T \frac{D_i(L_\ell^T - L_h^T) + F_i(L^* - L_h^T)}{(D_i + E_i + F_i)^2} \qquad (18)\end{aligned}$$

An analogous expression holds for j. Since $D_i + E_i + F_i = D_j + E_j + F_j$ (the total mass of recipients receiving Treatment is fixed), the two denominators in (18) are the same as those in the analogous expression for j. Hence

$$\frac{\partial u_i}{\partial E_i} - \frac{\partial u_j}{\partial E_j} = \phi_N \frac{(D_i - D_j)(L_\ell^N - L_h^N) + (F_i - F_j)(L^* - L_h^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} + \phi_T \frac{(D_i - D_j)(L_\ell^T - L_h^T) + (F_i - F_j)(L^* - L_h^T)}{(D_i + E_i + F_i)^2}$$

Since $F_i - F_j = -(D_i - D_j) + (E_j - E_i)$,

$$\frac{\partial u_i}{\partial E_i} - \frac{\partial u_j}{\partial E_j} = \phi_N \frac{(D_i - D_j)(L_\ell^N - L_h^N - L^* + L_h^N) + (E_j - E_i)(L^* - L_h^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} + \phi_T \frac{(D_i - D_j)(L_\ell^T - L_h^T - L^* + L_h^T) + (E_j - E_i)(L^* - L_h^T)}{(D_i + E_i + F_i)^2}$$

which equals (17). The same argument (in the online appendix) yields the same expression for $\partial u_i/\partial D_i - \partial u_j/\partial D_j$. □

Proof of Theorem 3. By Lemma 4 it is without loss to restrict attention to Inverting and Non-inverting equilibria. Consider any Non-inverting equilibrium profile p. Observe that for any i, j, either $(p_{i\ell}, p_{ih}) \geq (p_{j\ell}, p_{jh})$ or $(p_{i\ell}, p_{ih}) \leq (p_{j\ell}, p_{jh})$.

Suppose $(p_{i\ell}, p_{ih}) \lneq (p_{j\ell}, p_{jh})$, i.e. using the above notation suppose $D_j \geq D_i$ and $E_j \geq E_i$ with at least one inequality being strict. By (17) i has a greater marginal incentive to send recipients (of either type) to T than j does. This implies either that i has the strict incentive to (feasibly) increase p_i or that j has the strict incentive to (feasibly) strictly decrease p_j, contradicting the equilibrium assumption. A parallel argument applies to Inversion equilibria. □

6.2.3 NI-candidate existence and uniqueness

To prove Proposition 2 we write the partial derivatives of payoffs u_i also as a function of k. For any (symmetric, non-wasteful) non-inversion strategy profile and k, define

δ^W and δ^N by evaluating (15) and (16) at such profiles.

$$\delta^W(E,k) \equiv \frac{\partial u}{\partial E_i}\bigg|_{\forall j\, D_j=0,\, E_j=E} \tag{19}$$
$$= (L_h^T - L_h^N) - \phi_N \frac{r_\ell(L_\ell^N - L_h^N)}{n(r_\ell + r_h - nE)^2} - \phi_N \frac{(n-1)(L^* - L_h^N)}{n(r_\ell + r_h - nE)} + \phi_T \frac{(n-1)(L^* - L_h^T)}{n^2 E}$$
$$\delta^N(D,k) \equiv \frac{\partial u}{\partial D_i}\bigg|_{\forall j\, D_j=D,\, E_j=\frac{r_h}{n}} \tag{20}$$
$$= (L_\ell^T - L_\ell^N) - \phi_N \frac{(n-1)(L^* - L_\ell^N)}{n(r_\ell - nD)} - \phi_T \frac{r_h(L_\ell^T - L_h^T)}{n(nD + r_h)^2} + \phi_T \frac{(n-1)(L^* - L_\ell^T)}{n(nD + r_h)}$$

With this notation we rewrite Definition 1 as follows.

Definition (NI-candidate)**.** A symmetric profile p^* (inducing strategies $D_i = p_{i\ell}^* r_\ell/n$, $E_i = p_{ih}^* r_h/n$) is an NI-candidate if

- (1-NI) $p_{i\ell}^* \equiv 0$ and $\delta^W(E,k) = 0$; or
- (1-NI corner solution) $p_{i\ell}^* \equiv 0$, $nE_i = k\phi$, and $\delta^W(E,k) \leq 0$; or
- (3-NI) $p_{ih}^* \equiv 1$ and $\delta^N(D,k) = 0$; or
- (3-NI corner solution) $p_{ih}^* \equiv 1$, $r_\ell - nD_i = (1-k)\phi$, and $\delta^N(D,k) \geq 0$; or
- (2-NI) $p_{i\ell}^* \equiv 0$, $p_{ih}^* \equiv 1$, $\delta^W(E,k) \geq 0$, and $\delta^N(D,k) \leq 0$.

The following lemma conveys the intuition that the benefit of assigning more recipients to T increases in k and decreases in the total mass of recipients assigned to T. This intuition is always true for high type-recipients but requires mild assumptions for low-type recipients since they congest their agent's high-type recipients.

Lemma 7 (Properties of δ^N, δ^W)**.**

(i) $\delta^W(E,k)$ *is linearly increasing in* $k \in [0,1]$ *and decreasing in* $E \in [0, r_h/n]$.

(ii) If Assumption 1 holds, $n \geq 2$ *implies* $\delta^N(D,k)$ *is linearly increasing in* $k \in [0,1]$*, and* $n \geq 3$ *implies* $\delta^N(D,k)$ *is decreasing in* $D \in [0, r_\ell/n]$.

Proof. To prove the first claim, note that δ^W is continuous and differentiable. Differentiating $\delta^W(E,k)$ with respect to k yields

$$\frac{\partial \delta^W}{\partial k} = \phi \frac{r_\ell(L_\ell^N - L_h^N)}{n(r_\ell + r_h - nE)^2} + \phi \frac{(n-1)(L^* - L_h^N)}{n(r_\ell + r_h - nE)} + \phi \frac{(n-1)(L^* - L_h^T)}{n^2 E} > 0 \tag{21}$$

which is a sum of positive terms independent of k; so δ^W is linearly increasing in k. Likewise,

$$\frac{\partial \delta^W}{\partial E} = -\phi_N \frac{2r_\ell(L_\ell^N - L_h^N)}{(r_\ell + r_h - nE)^3} - \phi_N \frac{(n-1)(L^* - L_h^N)}{(r_\ell + r_h - nE)^2} - \phi_T \frac{(n-1)(L^* - L_h^T)}{(nE)^2}$$

which for any $E \in (0, r_h/n]$ is a sum of three strictly negative terms; so δ^W is decreasing in E.

Analogously for the second claim,

$$\frac{\partial \delta^N}{\partial k} = \phi \frac{(n-1)(L^* - L_\ell^N)}{n(r_\ell - nD)} + \phi \frac{-r_h(L_\ell^T - L_h^T) + (n-1)(r_h + nD)(L^* - L_\ell^T)}{n(r_h + nD)^2} \tag{22}$$

If $n \geq 2$ and Assumption 1 holds, then the second term is strictly positive. Since the first term is positive, δ^N is linearly increasing in k. Likewise

$$\frac{\partial \delta^N}{\partial D} = \phi_N \frac{-(n-1)(L^* - L_\ell^N)}{(r_\ell - nD)^2} + \phi_T \frac{2r_h(L_\ell^T - L_h^T)}{(nD + r_h)^3} - \phi_T \frac{(n-1)(L^* - L_\ell^T)}{(nD + r_h)^2}$$

Since $r_h/(nD + r_h) < 1$,

$$\frac{\partial \delta^N}{\partial D} < \phi_N \frac{-(n-1)(L^* - L_\ell^N)}{(r_\ell - nD)^2} + \phi_T \frac{2(L_\ell^T - L_h^T)}{(nD + r_h)^2} - \phi_T \frac{(n-1)(L^* - L_\ell^T)}{(nD + r_h)^2}$$

If $n \geq 3$ and Assumption 1 holds, then the magnitude of the third term exceeds that of the second term; so δ^N is decreasing in D. $\square$

Lemma 8 (Region 2-NI)**.** *Fix k and let p^s denote the "NI-separating" profile, $p_{i\ell}^s \equiv 0$ and $p_{ih}^s \equiv 1$. Then p^s is an NI-candidate for k if and only if $k_n' \leq k \leq k_n^*$, where $k_n' < k_n^*$ are defined by* (11) *and* (12)*.*

Proof. Fixing k, p^s is an NI-candidate if and only if an agent has no incentive to decrease E from its value r_h/n and has no incentive to increase D above 0.

The former requirement is $\delta^W(r_h/n, k) \geq 0$ which, by Equation 19, is

$$(L_h^T - L_h^N) - \frac{\phi_N}{r_\ell}\frac{1}{n}(L_\ell^N - nL_h^N + (n-1)L^*) + \frac{\phi_T}{r_h}\frac{n-1}{n}(L^* - L_h^T) \geq 0$$

Substituting for $\phi_N = (1-k)\phi$ and $\phi_T = k\phi$ this inequality holds when

$$k \geq \frac{-(L_h^T - L_h^N) + \frac{\phi}{r_\ell}\frac{1}{n}\Big((n-1)L^* + L_\ell^N - nL_h^N\Big)}{\frac{\phi}{r_\ell}\frac{1}{n}\left((n-1)L^* + L_\ell^N - nL_h^N\right) + \frac{\phi}{r_h}\frac{n-1}{n}(L^* - L_h^T)} \equiv k_n'$$

establishing (11).

The latter requirement is $\delta^N(0,k) \leq 0$ which, by Equation 20, is

$$(L_\ell^T - L_\ell^N) - \phi_N \frac{\frac{n-1}{n}(L^* - L_\ell^N)}{r_\ell} + \phi_T \frac{-\frac{1}{n}(L_\ell^T - L_h^T) + \frac{n-1}{n}(L^* - L_\ell^T)}{r_h} \leq 0$$

which holds when

$$\begin{aligned} k\alpha &\leq (L_\ell^N - L_\ell^T) + \frac{\phi}{r_\ell}\frac{n-1}{n}(L^* - L_\ell^N) \\ \text{where } \alpha &= \left[\frac{\phi}{r_\ell}\frac{n-1}{n}(L^* - L_\ell^N) + \frac{\phi}{r_h}\left[-\frac{1}{n}(L_\ell^T - L_h^T) + \frac{n-1}{n}(L^* - L_\ell^T)\right]\right] \end{aligned}$$

Assumption 1 and the assumption that $n \geq 3$ imply $\alpha > 0$. Dividing both sides of the inequality by α yields $k \leq k_n^*$ as defined in (12). □

The following implies that separation (2-NI) occurs for non-degenerate values of k.

Lemma 9. *For k_n', k_n^* defined in (11)–(12), $k_n' < k_n^*$ and $\frac{r_h}{r_\ell + r_h} \equiv \bar{k} < k_n^*$.*

Proof. It is clear from (11) and (12) that $k_n' < 1$ and $k_n^* > 0$. Hence if $k_n' = 0$ or $k_n^* = 1$ the conclusion is immediate.

Suppose $k_n' > 0$ and $k_n^* < 1$, hence $\delta^W(r_h/n, k_n') = 0$ and $\delta^N(0, k_n^*) = 0$. Since δ^W is increasing in k, we prove the result by showing $\delta^W(r_h/n, k_n^*) > 0 = \delta^N(0, k_n^*)$. We do this by showing that (i) $\delta^W(r_h/n, k) - \delta^N(0,k)$ increases in k, and (ii) $\delta^W(r_h/n, \bar{k}) > \delta^N(0, \bar{k})$ at the "proportional" value $\bar{k} \equiv \frac{r_h}{r_\ell + r_h} < k_n^*$.

To show (i) we evaluate (21)–(22) at $(r_h/n, k)$ and $(0,k)$ (reordering the first two terms of the first expression).

$$\begin{aligned} \frac{\partial \delta^W}{\partial k}(r_h/n, k) &= \phi\frac{(n-1)(L^* - L_h^N)}{nr_\ell} + \phi\frac{(L_\ell^N - L_h^N)}{nr_\ell} + \phi\frac{(n-1)(L^* - L_h^T)}{nr_h} \\ \frac{\partial \delta^N}{\partial k}(0, k) &= \phi\frac{(n-1)(L^* - L_\ell^N)}{nr_\ell} + \phi\frac{-(L_\ell^T - L_h^T)}{nr_h} + \phi\frac{(n-1)(L^* - L_\ell^T)}{nr_h} \end{aligned}$$

It is easy to see that the three terms in the first expression are greater than the respective terms in the second expression, proving (i).

To prove (ii), evaluate the two derivatives at $\bar{k}$.

$$\begin{aligned}
\delta^N(0,\bar{k}) &= (L_\ell^T - L_\ell^N) - \phi\frac{r_\ell}{r_h+r_\ell}\frac{(n-1)(L^* - L_\ell^N)}{nr_\ell} + \phi\frac{r_h}{r_h+r_\ell}\frac{-(L_\ell^T - L_h^T) + (n-1)(L^* - L_\ell^T)}{nr_h} \\
&= (L_\ell^T - L_\ell^N) - \phi\frac{(n-1)(L^* - L_\ell^N)}{n(r_h+r_\ell)} + \phi\frac{-(L_\ell^T - L_h^T) + (n-1)(L^* - L_\ell^T)}{n(r_h+r_\ell)} \\
&= (L_\ell^T - L_\ell^N) - \phi\frac{(L_\ell^T - L_h^T)}{n(r_h+r_\ell)} + \phi\frac{(n-1)(L_\ell^N - L_\ell^T)}{n(r_h+r_\ell)} < 0
\end{aligned}$$

which is negative since the magnitude of the first (negative) term exceeds that of the third (positive) term. Additionally, since $\delta^N(0,k_n^*) = 0$ and is increasing in k this implies $\bar{k} < k_n^*$.

Secondly,

$$\begin{aligned}
\delta^W(r_h/n,\bar{k}) &= (L_h^T - L_h^N) - \phi\frac{r_\ell}{r_h+r_\ell}\frac{(L_\ell^N - L_h^N)}{nr_\ell} \\
&\quad - \phi\frac{r_\ell}{r_h+r_\ell}\frac{(n-1)(L^* - L_h^N)}{nr_\ell} + \phi\frac{r_h}{r_h+r_\ell}\frac{(n-1)(L^* - L_h^T)}{nr_h} \\
&= (L_h^T - L_h^N) - \phi\frac{(L_\ell^N - L_h^N)}{n(r_h+r_\ell)} - \phi\frac{(n-1)(L^* - L_h^N)}{n(r_h+r_\ell)} + \phi\frac{(n-1)(L^* - L_h^T)}{n(r_h+r_\ell)} \\
&= (L_h^T - L_h^N) - \phi\frac{(L_\ell^N - L_h^N)}{n(r_h+r_\ell)} - \phi\frac{(n-1)(L_h^T - L_h^N)}{n(r_h+r_\ell)}
\end{aligned}$$

Note that

$$\begin{aligned}
\delta^W(r_h/n,\bar{k}) - \delta^N(0,\bar{k}) &= (L_h^T - L_h^N) - (L_\ell^T - L_\ell^N) - \phi\frac{(L_\ell^N - L_h^N) - (L_\ell^T - L_h^T)}{n(r_h+r_\ell)} \\
&\quad - \phi(n-1)\frac{L_h^T - L_h^N + L_\ell^N - L_\ell^T}{n(r_h+r_\ell)} \\
&= \left[L_h^T - L_h^N - L_\ell^T + L_\ell^N\right]\left[1 - \frac{\phi}{r_h+r_\ell}\right] > 0
\end{aligned}$$

since $L_h^T > L_h^N$, $L_\ell^N > L_\ell^T$, and $\phi < r_h + r_\ell$. Therefore at $k_n^* > \bar{k}$, (i) implies

$$\delta^W(r_h/n,k_n^*) > \delta^N(0,k_n^*) = 0 = \delta^W(r_h/n,k_n')$$

implying $k_n^* > k_n'$. $\square$

Lemma 10 (Region 3-NI)**.** *If $k > k_n^*$ then there exists a unique NI-candidate. It satisfies $p_{ih} \equiv 1$.*

Proof. Let p^s be defined as in Lemma 8 and (with a slight abuse of notation) recall $\delta^N(p^s, k_n^*) = 0$ by definition of k_n^*. By Lemma 8, $k > k_n^* > k_n'$ implies $\delta^W(p^s, k) > 0$. The lemma furthermore implies $\delta^W(p, k) > 0$ for any symmetric profile satisfying $p_{i\ell} \equiv 0$, i.e. there can be no NI-candidate in region 1-NI.

Lemma 8 similarly implies $\delta^N(p^s, k) > 0$. By Lemma 7, $\delta^N(\cdot, k)$ continuously decreases as we increase D ($p_{i\ell}$) from zero. Either $\delta^N(D, k) = 0$ at some unique D or we have (corner solution) $\delta^N(r_\ell/n, k) > 0$. In the latter case we clearly have a unique NI-candidate. In the former (interior) case, recall by Lemma 5 (statement (ii)) that at such a profile, an agent's payoffs are either decreasing or concave in $p_{i\ell}$. Since $\delta^N(D, k) = 0$ we must have concavity with respect to $p_{i\ell}$, hence this point uniquely satisfies the local first- and second-order conditions. □

Lemma 11 (Region 1-NI)**.** *If $k < k_n'$ then there exists a unique NI-candidate. It satisfies $p_{i\ell} \equiv 0$.*

The omitted proof mirrors that of Lemma 10 with the simplification that, in reference to Lemma 5, payoffs are always concave in p_{ih}.

Proof of Proposition 2. NI-candidate existence, uniqueness, and their description follow from the above lemmas. Monotonicity of $p^*()$ follows from Lemma 7. □

6.2.4 Optimal NI-candidate

The proof of Theorem 5 relies on the following lemma, stating that in region 3-NI we have $\pi^T > \pi^N$.

Lemma 12 ($\pi^T > \pi^N$ in Region NI-3)**.** *Fix k, and suppose p^* is a NI-3 equilibrium: for all i, $p_{i\ell}^* = p_\ell^* > 0$ (and hence $p_{ih}^* = 1$). Then $k > (p_\ell^* r_\ell + r_h)/(r_\ell + r_h)$, that is, the equilibrium allocation probability is higher in T than in N: $\pi^T > \pi^N$.*

Proof. By Lemma 5, $p_{ih}^* = 1$ implies $u_i(p^*)$ is either decreasing or concave in $p_{i\ell}$. Since $p_\ell^* > 0$ it must be concave. Therefore either the partial derivative (20) is zero, or the equilibrium is at a corner (where the N-nonwastefulness constraint binds and $\pi^N = 1$). However Fact 1(ii) rules out the latter, hence (20) is zero.

Recall for NI-3 equilibria that $A = r_\ell/n - D$, $B = 0$, $C = (n-1)A$, $E = r_h/n$, $F = (n-1)(D+E)$. So $\pi^N = \phi_N/(A+B+C) = \phi_N/(r_\ell - nD)$ and $\pi^T = \phi_T/(D+E+F) = \phi_T/(r_h + nD)$. Let $\lambda = r_h/(r_h+nD)$. Since Equation 20 is zero we have

$$\begin{aligned}
&(L_\ell^N - L_\ell^T) + \phi_N \frac{(n-1)(L^* - L_\ell^N)}{n(r_\ell - nD)} \\
&\qquad = \phi_T \frac{-r_h(L_\ell^T - L_h^T) + (n-1)(nD + r_h)(L^* - L_\ell^T)}{n(nD+r_h)^2} \\
&L_\ell^N + \pi^N \frac{(n-1)}{n}(L^* - L_\ell^N) \\
&\qquad = L_\ell^T + \pi^T \frac{-r_h(L_\ell^T - L_h^T) + (n-1)(nD + r_h)(L^* - L_\ell^T)}{n(nD+r_h)}
\end{aligned}$$

Thus

$$\begin{aligned}
&(1-\pi^N)L_\ell^N + \pi^N\left(\frac{(n-1)}{n}L^* + \frac{1}{n}L_\ell^N\right) \\
&\qquad = L_\ell^T + \pi^T\left(\frac{(n-1)(L^* - L_\ell^T)}{n} + \frac{-r_h(L_\ell^T - L_h^T)}{n(nD+r_h)}\right) \\
&\qquad = L_\ell^T + \pi^T\left(\frac{n-1}{n}(L^* - L_\ell^T) + \frac{-\lambda(L_\ell^T - L_h^T)}{n}\right) \\
&\qquad = (1-\pi^T)L_\ell^T + \pi^T\left(\frac{n-1}{n}L^* + \frac{(1-\lambda)L_\ell^T + \lambda L_h^T}{n}\right) \\
&\qquad < (1-\pi^T)L_\ell^N + \pi^T\left(\frac{n-1}{n}L^* + \frac{1}{n}L_\ell^N\right)
\end{aligned}$$

Since $L^* > L_\ell^N$ we have $\pi^T > \pi^N$; equivalently $k > (p_\ell^* r_\ell + r_h)/(r_\ell + r_h)$. □

Proof of Theorem 5. For any k let $f(k)$ and $\pi^T(k)$ respectively denote the fraction of objects allocated to high types and the probability that a recipient assigned to T receives an object, under k's NI-candidate. We prove the results regarding f. The results regarding agents' total payoffs follow directly using the same arguments made in Subsection 3.2 under Perfect Competition.

It is immediate that $f()$ is increasing on $[k_n', k_n^*]$ since the strategy profile is the same for all NI-candidates on this range. The remainder of the proof consists of showing (i) f is decreasing on $[k_n^*, 1]$, and (ii) $f(k) < f(k_n^*)$ for $k \in [0, k_n']$.

Step (i). For any $k \in (k_n^*, 1]$, there is at most one symmetric profile (namely the NI-candidate $p(k)$) satisfying $\delta^N(D,k)=0$ by Lemma 10. Whenever such $p(k)$ exists (i.e. the NI-candidate is not a corner solution), let $D(k)=p(k)r_\ell/n$ denote the corresponding mass of low types each agent sends to T.

By Lemma 7 $D(k)$ is increasing in k; hence the values of $k>k_n^*$ for which such $\delta^N(D(k),k)=0$ exist are an interval (of the form $(k_n^*,x]$ by continuity). We show that $\pi^T(k)$ is decreasing in k on this interval. Since $p_{ih}(k)\equiv 1$ on this range, a decrease in $\pi^T()$ necessarily decreases $f()$, proving (i).

We implicitly differentiate $\delta^N(D(k),k)=0$ (Equation 20) w.r.t. k after substituting $\phi_N=(1-k)\phi$ and $\phi_T=k\phi$. (Write $D=D(k)$ and $D'=\partial D(k)/\partial k$, and ignore the corner case $p_{i\ell}\equiv 1$, where $nD=r_\ell$.) This yields

$$\phi\frac{(n-1)(L^*-L_\ell^N)}{n(r_\ell-nD)}-\phi(1-k)\frac{(n-1)(L^*-L_\ell^N)}{n(r_\ell-nD)^2}nD'-\phi\frac{r_h(L_\ell^T-L_h^T)}{n(nD+r_h)^2}$$
$$+2\phi k\frac{r_h(L_\ell^T-L_h^T)}{n(nD+r_h)^3}nD'+\phi\frac{(n-1)(L^*-L_\ell^T)}{n(nD+r_h)}-\phi k\frac{(n-1)(L^*-L_\ell^T)}{n(nD+r_h)^2}nD'=0$$

Denoting $r=r_\ell+r_h$ and $S=nD+r_h<r$, we obtain

$$D'=\frac{\frac{(n-1)(L^*-L_\ell^N)}{n(r-S)}+\frac{(n-1)(L^*-L_\ell^T)}{nS}-\frac{r_h(L_\ell^T-L_h^T)}{nS^2}}{(1-k)\frac{(n-1)(L^*-L_\ell^N)}{(r-S)^2}+k\frac{(n-1)(L^*-L_\ell^T)}{S^2}-2k\frac{r_h(L_\ell^T-L_h^T)}{S^3}}$$
$$=\frac{(n-1)(L^*-L_\ell^N)S^2+(n-1)(L^*-L_\ell^T)(r-S)S-r_h(L_\ell^T-L_h^T)(r-S)}{(1-k)(n-1)(L^*-L_\ell^N)S^3+k(n-1)(L^*-L_\ell^T)S(r-S)^2-2kr_h(L_\ell^T-L_h^T)(r-S)^2}\frac{(r-S)S}{n}$$

To show that the derivative of $\pi^T(k)\equiv\frac{k\phi}{nD+r_h}$ is negative, i.e. that

$$\frac{\phi}{nD+r_h}-\frac{nk\phi}{(nD+r_h)^2}D'=\frac{\phi}{S}-\frac{nk\phi}{S^2}D'<0$$

we need to show $D'>S/(nk)$. Using the derivation of D' above, this inequality becomes

$$(L^*-L_\ell^N)S^2(n-1)(kr-S)>-kr_h(L_\ell^T-L_h^T)(r-S)^2$$

Since $r>S$ this is true whenever $k\geq S/r$, i.e. whenever $\pi^T(k)\geq\pi^N(k)$, which is true by Lemma 12. Hence $\pi^T()$ and $f()$ are decreasing on $[k_n^*,1]$.

Step (ii): consider the case $k\in[0,k_n']$.[16] By previous arguments, NI-candidate

[16] This case is mostly symmetric to the previous one, except that the possibility that $\bar{k}<k_n'$ necessitates additional arguments.

profiles vary continuously in k; therefore $f()$ is continuous. Hence we can choose

$$\tilde{k} = \arg\max_{[0,k'_n]} f(k)$$

We show $f(\tilde{k}) < f(k^*_n) \equiv k^*_n$.

Case 1: $\pi^N(\tilde{k}) \geq \pi^T(\tilde{k})$. A low type receives an object with probability $\pi^N(\tilde{k})$, whereas a high type receives an object with a weakly lower probability of

$$(1 - p_{ih})\pi^N(\tilde{k}) + p_{ih}\pi^T(\tilde{k})$$

where $(0, p_{ih})$ is the NI-candidate for $\tilde{k}$. Since high types receive objects with lower probability than low types, they collectively receive no more than the (unconditional) object allocation rate: $f(\tilde{k}) \leq \frac{r_h}{r_\ell + r_h} < k^*_n$, where the second inequality follows from Lemma 9 ($\bar{k} < k^*_n$).

Case 2: $\pi^N(\tilde{k}) < \pi^T(\tilde{k})$. We show that f is increasing at $\tilde{k}$. This means $\tilde{k} = k'_n$, implying the desired conclusion.

Since the mass of objects allocated to *low* types is $\pi^N(\tilde{k})r_\ell$, $f(\tilde{k}) = 1 - \frac{\pi^N}{\phi} r_\ell$. To show f is increasing we show $\pi^N()$ is decreasing at $\tilde{k}$.

To show the derivative of $\pi^N(k) \equiv \frac{(1-k)\phi}{r_\ell + r_h - nE}$ is negative at $\tilde{k}$, i.e. that

$$\frac{-\phi}{r_\ell + r_h - nE} + \frac{(1-\tilde{k})\phi n E'}{(r_\ell + r_h - nE)^2} = \left(\frac{-\phi}{r_\ell + r_h - nE}\right)\left(1 - \frac{(1-\tilde{k})nE'}{r_\ell + r_h - nE}\right) \leq 0$$

we need to show

$$E'(\tilde{k}) \leq \frac{r_\ell + r_h - nE(\tilde{k})}{(1-\tilde{k})n} \tag{23}$$

We implicitly differentiate $\delta^W(E(k), k) = 0$ (Equation 19) w.r.t. k and evaluate at $\tilde{k}$. Writing $E = E(k)$ and $E' = E'(k)$ we obtain

$$\begin{aligned}
&\phi\frac{(n-1)(L^* - L_h^T)}{nE} - \phi(1-\tilde{k})\frac{(n-1)(L^* - L_h^N)}{(r_\ell + r_h - nE)^2} nE' + \phi\frac{r_\ell(L_\ell^N - L_h^N)}{(r_\ell + r_h - nE)^2} \\
&\quad - 2\phi(1-\tilde{k})\frac{r_\ell(L_\ell^N - L_h^N)}{(r_\ell + r_h - nE)^3} nE' + \phi\frac{(n-1)(L^* - L_h^N)}{(r_\ell + r_h - nE)} - \phi\tilde{k}\frac{(n-1)(L^* - L_h^T)}{(nE)^2} E'n = 0
\end{aligned}$$

Denoting $S = r_\ell + r_h - nE(\tilde{k})$ and $r = r_h + r_\ell$ this yields

$$E'(\tilde{k}) = \frac{r_\ell(L_\ell^N - L_h^N)S^{-2} + (n-1)(L^* - L_h^N)S^{-1} + (n-1)(L^* - L_h^T)(r-S)^{-1}}{2(1-\tilde{k})r_\ell(L_\ell^N - L_h^N)S^{-3}n + (1-\tilde{k})(n-1)(L^* - L_h^N)S^{-2}n + \tilde{k}(n-1)(L^* - L_h^T)(r-S)^{-2}n}$$

Therefore one can show that (23) is equivalent to

$$(n-1)(L^* - L_h^T)[(1-\tilde{k})(r-S) - \tilde{k}S] \leq r_\ell(L_\ell^N - L_h^N)(1-\tilde{k})\left(\frac{r-S}{S}\right)^2$$

Note that $\frac{\phi(1-\tilde{k})}{S} = \pi^N(\tilde{k}) < \pi^T(\tilde{k}) = \frac{\phi\tilde{k}}{r-S}$ implies that the LHS is non-positive. Since the RHS is non-negative (23) holds. □

Proof of Proposition 4. When $k_n^* < 1$ and $\phi \leq r_h$, Equation 12 takes the form

$$\begin{aligned} k_n^* &= \frac{(L_\ell^N - L_\ell^T) + \frac{\phi}{r_\ell}\frac{n-1}{n}(L^* - L_\ell^N)}{\frac{\phi}{r_\ell}\frac{n-1}{n}(L^* - L_\ell^N) + \frac{\phi}{r_h}\left[\frac{n-1}{n}L^* + \frac{1}{n}L_h^T - L_\ell^T\right]} \\ &= \frac{\Delta_\ell + \frac{\phi}{r_\ell}\frac{n-1}{n}\Delta_*}{\frac{\phi}{r_\ell}\frac{n-1}{n}\Delta_* + \frac{\phi}{r_h}\left[\frac{n-1}{n}(\Delta_* + \Delta_\ell) - \frac{1}{n}\Delta_T\right]} \\ &= \frac{\Delta_\ell + \frac{\phi}{r_\ell}\frac{n-1}{n}\Delta_*}{\frac{\phi}{r_h}\frac{n-1}{n}\Delta_\ell + \left(\frac{\phi}{r_\ell} + \frac{\phi}{r_h}\right)\frac{n-1}{n}\Delta_* - \frac{\phi}{r_h}\frac{1}{n}\Delta_T} = \frac{a\Delta_\ell + b\Delta_*}{a'\Delta_\ell + b'\Delta_* + c\Delta_T} \end{aligned}$$

where $a > a'$, $b < b'$, and $c < 0$. It is clearly decreasing in ϕ and increasing in Δ_T. Differentiating the last expression yields the remaining results since $k_n^* < 1$. □

6.2.5 Non-inversion equilibrium existence

The proof of Theorem 4 is presented last as it makes use of Proposition 2. We prove a (technically) stronger result since (8) implies $\bar{k} \geq k'$ (proven in the Online Appendix).

Theorem. *Suppose $n \geq 3$ and that Assumption 1 holds. If $k \geq \max\{k'_n, \bar{k}\}$ and (9) holds then there exists a unique Non-inversion equilibrium.*

Proof. Make the assumptions of the theorem and let p^* be the unique NI-candidate for k. We first show that agent i's best response to p^*_{-i} must be a Non-inversion strategy, then show p^*_i is optimal among all such strategies.

Claim 1: any best response to p^*_{-i} satisfies $p_{i\ell} = 0$ or $p_{ih} = 1$.

To prove Claim 1 it is sufficient to show that (13) is positive for any profile (p_i, p^*_{-i}). Since p^*_{-i} is fixed throughout let $\pi^N(p_i)$ and $\pi^T(p_i)$ denote the allocation probabilities when i uses strategy p_i. We want to show that for any p_i,

$$(1-\pi^N(p_i))(\Delta_h+\Delta_T+\Delta_\ell)-(1-\pi^T(p_i))\Delta_T>0$$

If $\pi^N(p_i)\le\pi^T(p_i)$ the inequality is immediate; if $\pi^N(p_i)>\pi^T(p_i)$ we must show

$$\frac{\Delta_T}{\Delta_h+\Delta_\ell}<\frac{1-\pi^N(p_i)}{\pi^N(p_i)-\pi^T(p_i)} \tag{24}$$

Since $k\ge k'_n$ implies $p^*_{jh}=1$ for all $j\ne i$, for all p_i we have

$$\pi^N(p_i)=\frac{(1-k)\phi}{\frac{n-1}{n}(1-p^*_{j\ell})r_\ell+(1-p_{i\ell})\frac{r_\ell}{n}+(1-p_{ih})\frac{r_h}{n}}\le\frac{(1-k)\phi}{\frac{n-1}{n}(1-p^*_{j\ell})r_\ell} \tag{25}$$

Separately, $k\ge\max\{k'_n,\bar{k}\}$ implies

$$\frac{\phi(1-k)}{(1-p^*_{i\ell})r_\ell}=\pi^N(p^*_i)<\pi^T(p^*_i)=\frac{\phi k}{r_h+p^*_{i\ell}r_\ell}$$

since either $k\ge k^*_n$ (in which case Lemma 12 applies) or $k\in[k'_n,k^*_n]$ (in which case $p^*_{i\ell}(k)=0$, $p^*_{ih}(k)=1$, and $k\ge\bar{k}$ imply the inequality). The inequality can be rewritten as

$$p^*_{i\ell}<\frac{kr_\ell-(1-k)r_h}{r_\ell}$$

With (25) this means that for any p_i,

$$\pi^N(p_i)\le\frac{(1-k)\phi}{\frac{n-1}{n}(1-p^*_{i\ell})r_\ell}<\frac{(1-k)\phi}{\frac{n-1}{n}\left(1-\frac{kr_\ell-(1-k)r_h}{r_\ell}\right)r_\ell}=\frac{\phi}{\frac{n-1}{n}(r_\ell+r_h)}$$

Hence (9) implies $1-\pi^N(p_i)>\Delta_T/(\Delta_H+\Delta_\ell)$, implying (24) and the claim.

Claim 2: p^*_i is a best response to p^*_{-i}.

By Claim 1 it suffices to compare p^*_i only to other Non-inversion strategies. We show that $u_i(\cdot,p^*_{-i})$ is concave across the entire range of such (non-wasteful) strategies, proving the result (since p^*_i is a local maximizer).

Lemma 5(i) implies $u_i(p_i,p^*_{-i})$ is concave over the range where $p_{i\ell}=0$ and $p_{ih}\in[0,1]$. Lemma 5(iii) implies $u_i(p_i,p^*_{-i})$ is concave over the range where $p_{i\ell}\in[0,1]$ and

$p_{ih} = 1$. Consider their intersection, $p'_i = (0,1)$. At profile (p'_i, p^*_{-i}),

$$\begin{aligned}
\frac{\partial u_i}{\partial E_i} - \frac{\partial u_i}{\partial D_i} &= (L^N_\ell - L^N_h)\left(1 - \frac{\phi_N}{\frac{r_\ell}{n} + C_i}\right) - (L^T_\ell - L^T_h)\left(1 - \frac{\phi_T}{\frac{r_h}{n} + F_i}\right) \\
&= (L^N_\ell - L^N_h)(1 - \pi^N(p'_i)) - (L^T_\ell - L^T_h)(1 - \pi^T(p'_i)) \qquad (26)
\end{aligned}$$

Note also that

$$\pi^T(p'_i) = \frac{\phi_T}{r_h + \frac{n-1}{n} p^*_{j\ell} r_\ell} > \frac{\phi_T}{r_h + p^*_{j\ell} r_\ell} = \pi_T(p^*_i)$$
$$\pi^N(p'_i) = \frac{\phi_N}{\frac{1}{n} r_\ell + \frac{n-1}{n}(1 - p^*_{j\ell}) r_\ell} < \frac{\phi_N}{(1 - p^*_{j\ell}) r_\ell} = \pi^N(p^*_i)$$

Lemma 12 implies $\pi^T(p^*_i) > \pi^N(p^*_i)$, $\pi^T(p'_i) > \pi^N(p'_i)$. Combining with (26) we have $\frac{\partial u_i}{\partial E} > \frac{\partial u_i}{\partial D}$, i.e. $u_i(\cdot, p^*_{-i})$ is concave at p'_i. □

A Online Appendix

(This online appendix will ultimately be separated from the main paper and given its own title.)

A.1 Payoffs are nowhere-concave

A necessary condition for (weak) concavity of U_i is for the determinant of the Hessian matrix to be negative. Firstly, one can confirm that

$$\begin{aligned}
\frac{\partial^2 U_i}{\partial D \partial E} &= \phi_N \frac{-(r_\ell + r_h - D - E - F)(L_\ell^N - L_h^N) + 2[(r_h/n - E)(L_\ell^N - L_h^N) - ((n-1)(r_\ell/n + r_h/n) - F)(L^* - L_\ell^N)]}{(r_\ell + r_h - D - E - F)^3} \\
&\quad - \phi_T \frac{(D+E+F)(L_\ell^T - L_h^T) + 2(-E(L_\ell^T - L_h^T) + F(L^* - L_\ell^T))}{(D+E+F)^3} \\
\frac{\partial^2 U_i}{\partial E^2} &= -2\phi_N \frac{(r_\ell/n - D)(L_\ell^N - L_h^N) + [(n-1)(r_\ell/n + r_h/n) - F](L^* - L_h^N)}{(r_\ell + r_h - D - E - F)^3} - 2\phi_T \frac{D(L_\ell^T - L_h^T) + F(L^* - L_h^T)}{(D+E+F)^3} \\
\frac{\partial^2 U_i}{\partial D^2} &= 2\phi_N \frac{(r_h/n - E)(L_\ell^N - L_h^N) - [(n-1)(r_\ell/n + r_h/n) - F](L^* - L_\ell^N)}{(r_\ell + r_h - D - E - F)^3} - 2\phi_T \frac{-E(L_\ell^T - L_h^T) + F(L^* - L_\ell^T)}{(D+E+F)^3}
\end{aligned}$$

The determinant is $\frac{\partial^2 U_i}{\partial E^2} \cdot \frac{\partial^2 U_i}{\partial D^2} - \left(\frac{\partial^2 U_i}{\partial D \partial E}\right)^2$. Substituting with the above expressions and rearranging terms, the determinant can be written as

$$\frac{-S^2}{(D+E+F)^4 (r_h + r_\ell - (D+E+F))^4}$$

where (letting $r = r_\ell + r_h$)

$$S = -\phi_T (L_\ell^T - L_h^T) \left(r - (D+E+F)\right)^2 - \phi_N (L_\ell^N - L_h^N)(D+E+F)^2$$

which is always negative. Therefore the determinant is negative for any parameter values (setting aside the two degenerate combinations where $D + E + F = 0 = k$ and where $(D+E+F)/r = 1 = k$), so U_i is not concave (nor convex) at any point.

A.2 Omitted argument in proof of Lemma 6

The claim is made that "a parallel argument shows that $\partial u_i / \partial D_i - \partial u_j / \partial D_j$ equals" the expression given in the proof. To formalize this argument here, recall $A_i = r_\ell/n - D_i$, $B_i = r_h/n - E_i$, and $C_i = (n-1)(r_\ell/n + r_h/n) - F_i$. The derivative of

(14) with respect to D_i is

$$\begin{aligned}
&(L_\ell^T - L_\ell^N) + \frac{\phi_N}{A+B+C}\left(\frac{B}{A+B+C}(L_\ell^N - L_h^N) - \frac{C}{A+B+C}(L^* - L_\ell^N)\right) \\
&+ \frac{\phi_T}{D+E+F}\left(\frac{E}{D+E+F}(L_h^T - L_\ell^T) + \frac{F}{D+E+F}(L^* - L_\ell^T)\right) \\
&= (L_\ell^T - L_\ell^N) + \phi_N \frac{(r_h/n - E_i)(L_\ell^N - L_h^N) - ((n-1)(r_\ell/n + r_h/n) - F_i)(L^* - L_\ell^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} \\
&\quad + \phi_T \frac{-E_i(L_\ell^T - L_h^T) + F_i(L^* - L_\ell^T)}{(D_i + E_i + F_i)^2} \qquad (27)
\end{aligned}$$

An analogous expression holds for j,

$$\begin{aligned}
\frac{\partial u_j}{\partial D_j} = (L_\ell^T - L_\ell^N) + \phi_N &\frac{(r_h/n - E_j)(L_\ell^N - L_h^N) - ((n-1)(r_\ell/n + r_h/n) - F_j)(L^* - L_\ell^N)}{(r_\ell + r_h - D_j - E_j - F_j)^2} \\
&+ \phi_T \frac{-E_j(L_\ell^T - L_h^T) + F_j(L^* - L_\ell^T)}{(D_j + E_j + F_j)^2}
\end{aligned}$$

Again since $D_i + E_i + F_i = D_j + E_j + F_j$ this yields

$$\begin{aligned}
\frac{\partial u_i}{\partial D_i} - \frac{\partial u_j}{\partial D_j} &= \phi_N \frac{(E_j - E_i)(L_\ell^N - L_h^N) + (F_i - F_j)(L^* - L_\ell^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} \\
&\quad + \phi_T \frac{(E_j - E_i)(L_\ell^T - L_h^T) + (F_i - F_j)(L^* - L_\ell^T)}{(D_i + E_i + F_i)^2} \\
&= \phi_N \frac{(E_j - E_i)(L^* - L_h^N) + (D_j - D_i)(L^* - L_\ell^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} \\
&\quad + \phi_T \frac{(E_j - E_i)(L^* - L_h^T) + (D_j - D_i)(L^* - L_\ell^T)}{(D_i + E_i + F_i)^2} \\
&= (E_j - E_i)\left[\frac{\phi_N(L^* - L_h^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} + \frac{\phi_T(L^* - L_h^T)}{(D_i + E_i + F_i)^2}\right] \\
&\quad + (D_j - D_i)\left[\frac{\phi_N(L^* - L_\ell^N)}{(r_\ell + r_h - D_i - E_i - F_i)^2} + \frac{\phi_T(L^* - L_\ell^T)}{(D_i + E_i + F_i)^2}\right]
\end{aligned}$$

as desired.

A.3 Assumption 1 implies $X < X'$ in proof of Lemma 5.

$$
\begin{aligned}
&X - X' = B(L_\ell^N - L_h^N) - C(L^* - L_\ell^N) + E(L_\ell^T - L_h^T) - F(L^* - L_\ell^T) \\
&= \left(\frac{r_h}{n} - E\right)(L_\ell^N - L_h^N) - \left(\frac{n-1}{n}(r_\ell + r_h) - F\right)(L^* - L_\ell^N) + E(L_\ell^T - L_h^T) - F(L^* - L_\ell^T) \\
&= \frac{r_h}{n}(L_\ell^N - L_h^N) - \frac{n-1}{n}(r_\ell + r_h)(L^* - L_\ell^N) + F(L^* - L_\ell^N - L^* + L_\ell^T) - E(L_\ell^N - L_h^N - L_\ell^T + L_h^T) \\
&< \left[\frac{r_h}{n}(L_\ell^N - L_h^N) - \frac{n-1}{n}(r_\ell + r_h)(L_\ell^N - L_h^N)\right] + F(L_\ell^T - L_\ell^N) + E((L_\ell^T - L_h^T) - (L_\ell^N - L_h^N))
\end{aligned}
$$

The inequality follows Assumption 1, and the final expression is the sum of three negative terms.

A.4 Inequality $\bar{k} \geq k_n'$

Rewriting the expression for k_n' (Equation 11) using the definitions in Equation 10, the inequality $\bar{k} > k_n'$ becomes

$$
\frac{r_h}{r_\ell + r_h} \geq \frac{-\Delta_h + \frac{\phi}{r_\ell}\left(\Delta - \frac{1}{n}\Delta_*\right)}{\frac{\phi}{r_\ell}\left(\Delta - \frac{1}{n}\Delta_*\right) + \frac{\phi}{r_h}\frac{n-1}{n}(\Delta - \Delta_h)}
$$

which is equivalent to

$$
\phi\frac{n-1}{n}\Delta - \phi\left(\Delta - \frac{1}{n}\Delta_*\right) \geq \phi\frac{n-1}{n}\Delta_h - \Delta_h(r_\ell + r_h)
$$

Simplifying and rearranging this expression leads to Equation 8.

A.5 Computational analysis: details

Here we summarize methodology and details of the computational analysis described in Subsection 4.3. The code is available at `http://www.kellogg.northwestern.edu/faculty/schummer/ftp/research/RTC/RTC-code.zip`.

The computations were performed on a workstation equipped with dual Intel Xeon Gold 5220R processors, each operating at 2.20 GHz, 256 GB of RAM, and two NVIDIA GV100 GPUs, each with 32 GB memory.

Fixing primitives that satisfy our assumptions, the goal of the computational exercises is to identify the equilibria that exist for various values of k, identify their structure, and compare welfare among all of them. The primitives are the recipient masses (normalized to $r_\ell + r_h = 1$), object mass $\phi < 1$, recipient welfare levels $L* > L_\ell^N > L_\ell^T > L_h^T > L_h^N$, and number of agents n. First, to find equilibria for a given k we find (approximate) best response functions in a discretized strategy space and their (approximate) fixed points, leveraging some analytical results to simplify the search. Second, to find the planner's optimal ration k, we compare welfare under any Inversion equilibrium found in the previous step to welfare obtained at the Non-inversion equilibrium obtained when $k = k_n^*$, ignoring all other NI equilibria by Theorem 5.

The following summary of details gives an overview of the code structure and additional technical details including how primitives and k are discretized. The result is one of the "prisms" found in Figure 2.

Step 1: Compute agent i's best responses (Python/Jax)

- Configure GPUs; Set precision to float64.
- Initialize set of economies.
  - Fix n, r_ℓ, $r_h = 1 - r_\ell$, $\phi < 1$.
  - Normalize $L_h^N = 0$, $L_\ell^N = 0.5$, $L^* \geq 1$ (Assumption 1).
  - Consider values $0 < L_h^T < L_\ell^T < 0.5$ in increments of 0.02.
  - Consider values $0 \leq k \leq 1$ in increments of 0.01.
- Create grids for best response calculation.
  - Determine 'admissible' range of profiles p_{-i} (specifically, F_i in the notation of the paper's appendix) that gives i a non-empty set of non-wasteful strategies.
  - Discretize domain of F_i (brmesh=100).
- [Parallelized] For each p_{-i} (F_i) and each of the four edges of $[0,1]^2$ (Lemma 3) determine an edge-constrained best response. For each edge:
  - Find feasibility (non-wastefulness) constraints.
  - Find edge-constrained best responses (if edge is feasible).
    - Optimization method: jaxopt.LBFGSB
    - Tolerance: 10^{-10}; Max Iterations: 1000
- Choose edge-constrained best response(s) yielding the largest payoff.

Step 2: Construct a best response function (Mathematica)

- Round numerically indistinguishable zeros and ones using a tolerance of Machine epsilon ($\approx 2.22e^{-16}$). Filter out repeated edge-constrained best responses.
- For a relatively small number of instances with multiple best response edges:

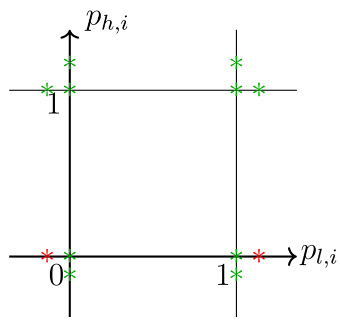

Figure 5: The figure depicts all the possible types of violations of probability constraints allowed by the optimization algorithm. Obviously, since we find one optimum per edge, at most one violation per edge is possible.

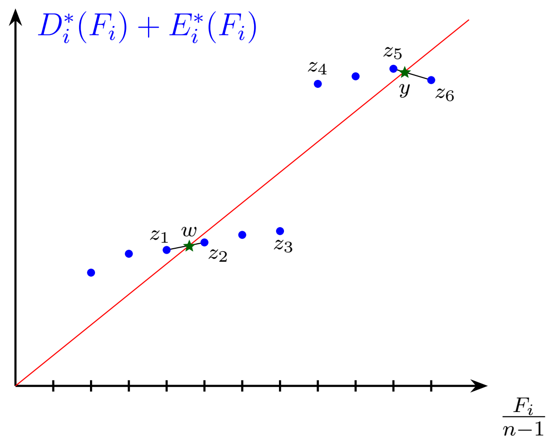

Figure 6: Masses

- Check for violations of probability constraints (expected in numerical optimization) and correct using analytical properties of the problem. Violation types are depicted in Figure 5.
  - Green violations can be removed by limited concavity (Lemma 5).
  - Red violations: if origin belongs to best responses, ignore the violation and delete it.
- For all the remaining non-singletons, both best responses in the $(p_{\ell,i}, p_{h,i})$ space yield the same mass $D + E$, so preserve both.

Step 3: Find and classify equilibria (Mathematica)

- To find BR fixed points, find all critical pairs: pairs of consecutive points which are at opposite sides of the 45° line. For example, pairs (z_1, z_2), (z_3, z_4), (z_5, z_6) in Figure 6) which illustrates i's best response to the per-capita mass of recipients i's opponents send to T $(F_i/(n-1))$.

- If both members of a critical pair yield the same (Inversion/Non-inversion) type of BR, interpolate a fixed point. E.g. if i's best response is Non-inverting at both z_1, z_2, label w as a fixed point, and hence a Non-inversion equilibrium. Otherwise classify as a jump discard the pair (e.g. (z_3, z_4)).
- If no equilibrium is found, refine the best response within the critical pair found.

Step 4: Welfare comparisons (Mathematica)

- Compute welfare for all Inversion equilibria under all considered values of k.
- Explicitly calculate k_n^* (Equation 12) and compute welfare under corresponding Non-inversion equilibrium.
- Find the welfare-optimal choice of k (and equilibrium) among these candidates (Theorem 5).

A.6 Additional computational results

As L^* increases (fixing other parameters as in the main text), Inversion equilibria are replaced by Non-inversion equilibria.

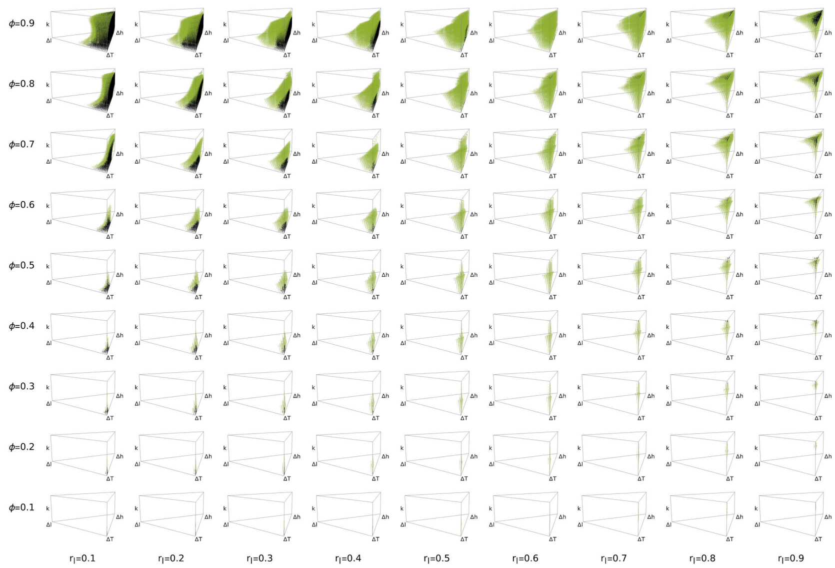

Figure 7: $L^* = 1.0$

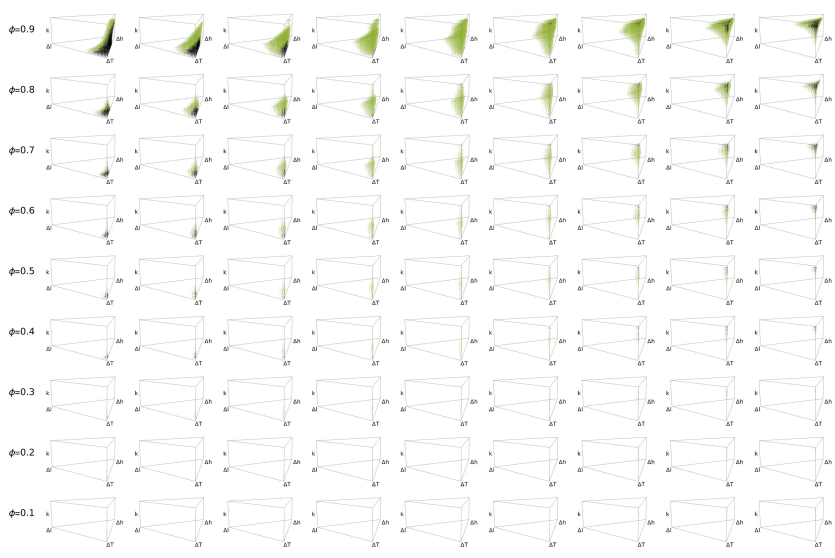

Figure 8: $L^* = 2.0$

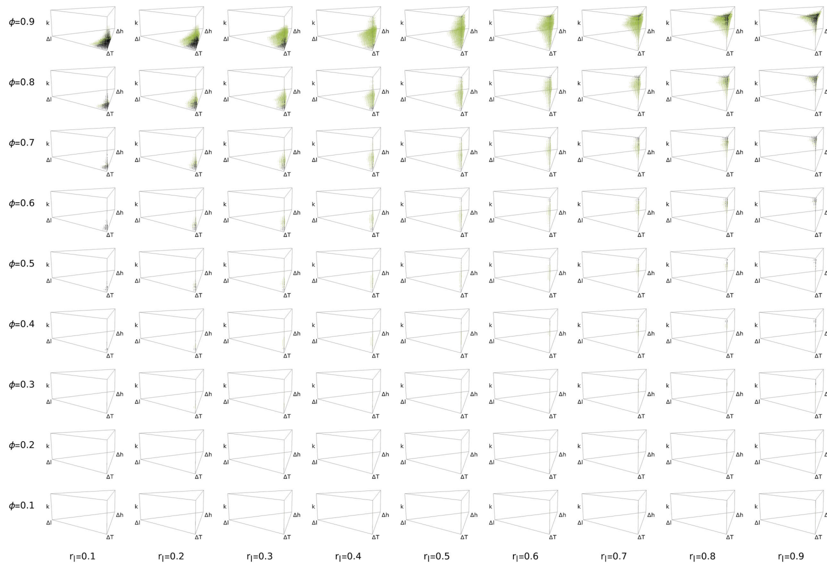

Figure 9: $L^* = 1.0$

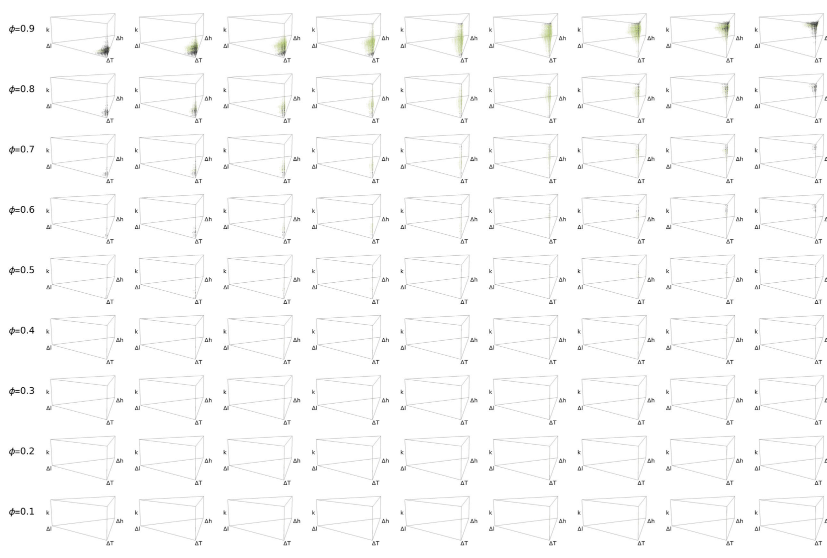

Figure 10: $L^* = 2.0$

As n increases (fixing other parameters as in the main text), Inversion equilibria are replaced by Non-inversion equilibria.

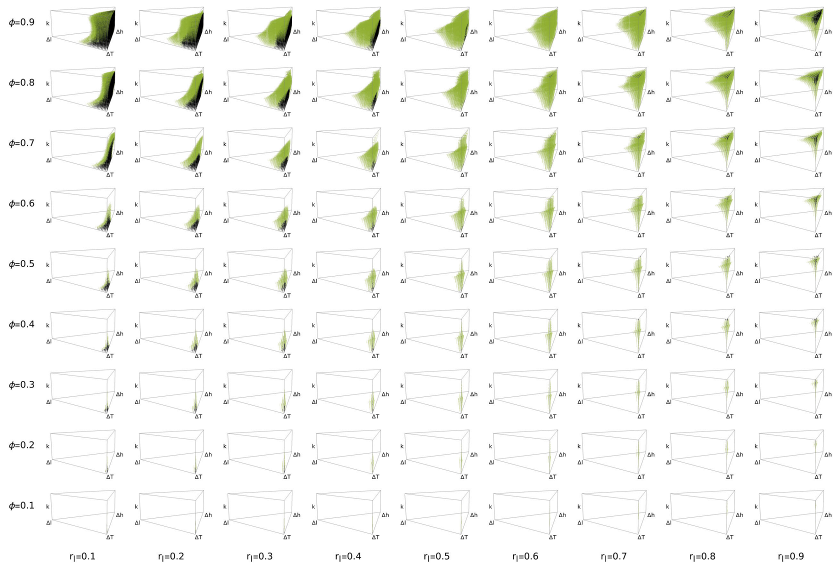

Figure 11: $n = 3$

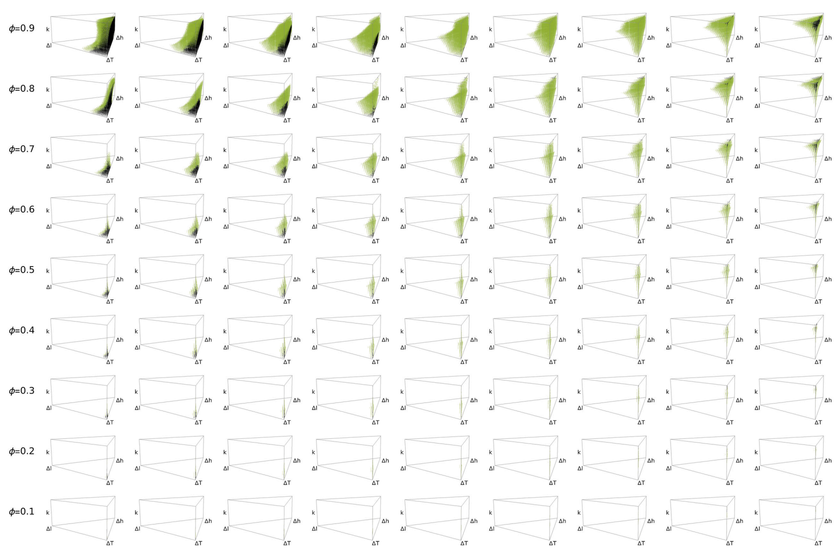

Figure 12: $n = 4$

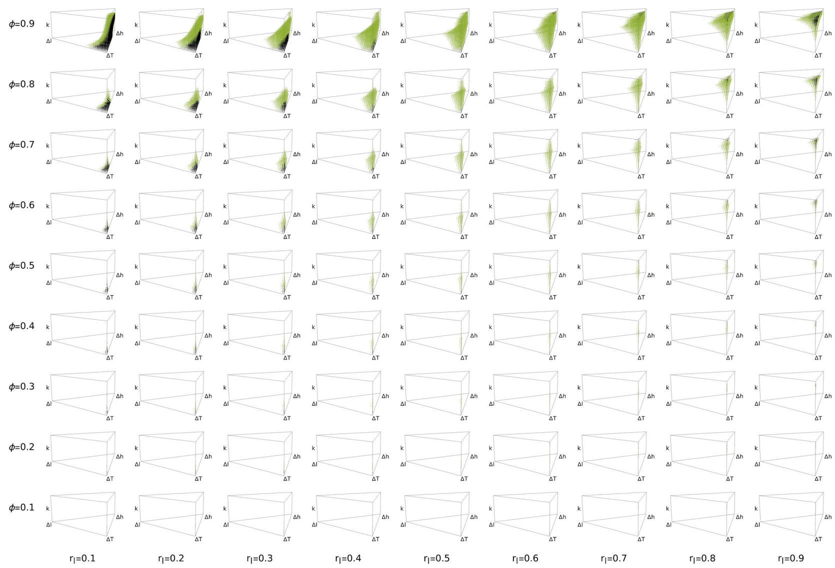

Figure 13: $n = 6$

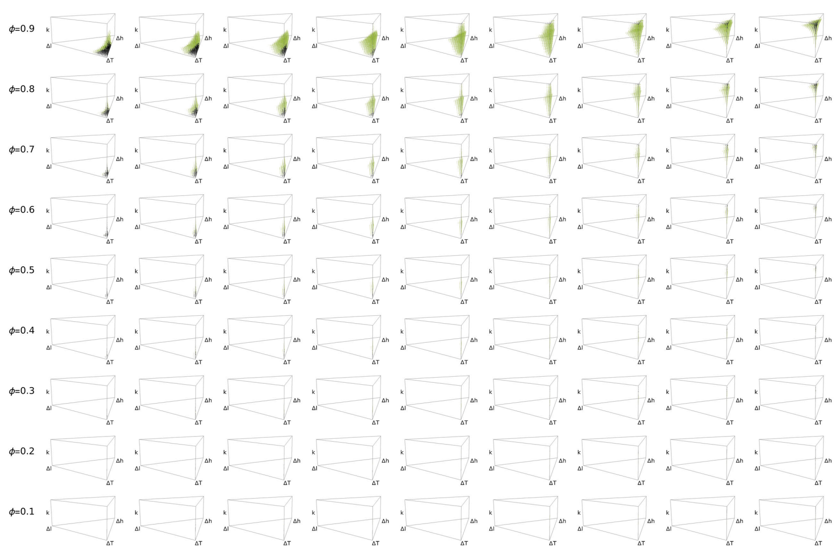

Figure 14: $n = 10$