Econometrica, Vol. 70, No. 5 (September, 2002), 1841–1863

THE OPTIMALITY OF A SIMPLE MARKET MECHANISM

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Strategic behavior in a finite market can cause inefficiency in the allocation, and market mechanisms differ in how successfully they limit this inefficiency. A method for ranking algorithms in computer science is adapted here to rank market mechanisms according to how quickly inefficiency diminishes as the size of the market increases. It is shown that trade at a single market-clearing price in the k-double auction is *worst-case asymptotic optimal* among all plausible mechanisms: evaluating mechanisms in their least favorable trading environments for each possible size of the market, the k-double auction is shown to force the worst-case inefficiency to zero at the fastest possible rate.

KEYWORDS: Market mechanism, double auction, rate of convergence.

1. INTRODUCTION

THE RULES THAT DETERMINE how trading proceeds within a market can be regarded as an algorithm for solving the problem of the market, which is to allocate units from the traders who initially own them to those who value them most highly. Market mechanisms and computer algorithms are more than just analogous. Almost every financial exchange in the world is developing a computerized trading system to complement or even substitute for its floor trading system. In the field of experimental economics, trade is commonly studied using a computer network to organize exchange among subjects. A market mechanism in each of these cases is explicitly a computer algorithm that specifies how traders may communicate with one another and how their messages determine the terms of trade.

The relationship between market mechanisms and algorithms is used in this paper to show that a family of common market mechanisms is optimal in a sense motivated by the study of algorithms in computer science. This is part of an effort to develop a theory of market mechanisms that is analogous to the theory of auctions. Currently, economic theory provides little guidance to financial exchanges in the selection of computer algorithms and floor procedures for trading. A theory of market mechanisms would provide such guidance and also complement the rich literature on markets in experimental economics, which is currently the main source of guidance in the design of market mechanisms.

[1] We thank Henryk Wozniakowski and Sunil Chopra for suggestions concerning the asymptotic analysis of algorithms. Roy Radner provided helpful comments concerning the maxmin approach. Finally, we thank seminar participants at Columbia University, the Decentralization Conference, the Games 2000 Conference of the Game Theory Society, IDEI Toulouse, the Midwest and the Southeast Theory and International Trade Meetings, Northwestern University, Pennsylvania State University, the Summer Meetings of the Econometric Society, the Technion, Tel Aviv University, University of Arizona, and University of Western Ontario for their comments.

The family of *k-double auction* (or k-DA) mechanisms is proven to be optimal here, where each choice of the index $k \in [0, 1]$ determines a different mechanism in the family. The k-DA operates as follows. Bids and offers are simultaneously submitted by the traders and then aggregated to form demand and supply curves. Using the weight k, a market-clearing price $p = (1-k)a + kb$ is selected from the interval $[a, b]$ of all possible market-clearing prices. Buyers whose bids are above p then trade with sellers whose offers are below p. The k-DA institutionalizes Marshall's model of demand and supply as a market mechanism. It is a practical method for organizing trade that is well-grounded in classical microeconomic thought.[2]

If the market is perfectly competitive, then the k-DA solves the problem of the market exactly in the sense that its allocation is efficient. If the market has only a finite number of traders, however, then the k-DA's allocation may be inefficient, i.e., there may be "error" in this algorithm's solution. This may occur because traders in a finite market who privately know their preferences need not act as price-takers and their strategic efforts to influence price in their favor can cause inefficiency in the allocation. Such inefficiency is common among market mechanisms, for every such mechanism must manage the strategic behavior of the traders as each attempts to manipulate the market's outcome in his favor. Interpreted as an algorithm, the error of a market mechanism in computing the gains from trade is the fraction of the expected potential gains from trade that the traders inefficiently fail to achieve because of their strategic behavior. Market mechanisms differ in how successfully they limit this error. Reflecting the theory of perfect competition, however, the error in any reasonable mechanism should converge to zero as the number of traders on each side of the market increases to infinity.

It is common in computer science to evaluate an algorithm by bounding its error with a function of some measure m of a binding constraint on the operation of the algorithm. The number m, for instance, could be the number of numerical inputs into the algorithm, the number of iterations the algorithm is permitted, or some measure of the amount of time that the algorithm is allowed to approximate the exact solution to the problem. A bound of this kind expresses the rate at which error diminishes as the constraining measure m is relaxed, with error converging to zero as m goes to infinity. An algorithm with a faster rate of convergence is deemed superior to an algorithm with a slower rate of convergence because it approximates the exact solution of the problem more accurately than the slower algorithm when m is sufficiently large.

We adapt this methodology from computer science to rank the k-DA relative to other market mechanisms in a simple model of trading. The trading model is as follows. There are m buyers, each of whom wishes to purchase at most one unit of an indivisible, homogeneous good, and m sellers, each of whom has one

[2] There are a multitude of other market mechanisms that are used in practice, studied theoretically, and tested in experiments. See Friedman (1993) for a survey of these mechanisms, including the k-DA.

unit of the good to sell. The number m is the *size* of the market. Each buyer i and each seller j privately knows the value v_i or cost c_j that he places on a unit. Buyer i receives a payoff of $v_i - x_i$ when he purchases a unit and pays x_i while seller j receives a payoff of $y_j - c_j$ when he sells his unit and receives a payment of y_j. A trader who does not trade receives zero as his payoff. Each trader regards the values of all buyers as independent draws from the distribution $G(\cdot)$ and the costs of all sellers as independent draws from the distribution $F(\cdot)$. All of the above is common knowledge among the traders. A pair (G, F) is an *environment*.[3] An independent private values model with quasilinear utility is thus assumed here and the Bayesian game approach of Harsanyi (1967–68) is used to analyze the strategic behavior of traders.

A *maxmin* approach is used to evaluate mechanisms. For each m, the *error* of a mechanism in a particular environment (G, F) is its *relative inefficiency*, i.e., the fraction of the expected potential gains from trade (calculated with respect to G and F) that the mechanism inefficiently fails to achieve in equilibrium. The worst-case error of the mechanism is computed by maximizing this error over a set of possible environments. The rate at which worst-case error converges to zero is then used to compare mechanisms.

The k-DA is compared in this paper to mechanisms that satisfy for each m and each environment both *interim individual rationality* (i.e., each trader's conditional expected payoff as a function of his value/cost is nonnegative) and *ex ante budget balance* (the expected sum of the transfers among the traders is nonnegative, so that the mechanism on average does not require a subsidy to operate). These rather weak restrictions are satisfied by most common mechanisms for trading.[4] Applying a term from computer science, the main result of this paper is that the k-DA is *worst-case asymptotic optimal* among all mechanisms for organizing trade that satisfy these two constraints. "Asymptotic" refers here to the ranking of mechanisms using rates of convergence and "worst-case" refers to the evaluation of each mechanism in its least favorable environment for each value of m. Stated simply, this result means that the k-DA's worst-case error over a set of environments converges to zero at the fastest possible rate among all interim individually rational and ex ante budget balanced mechanisms.[5]

[3] A model of Telser (1978) explains (G, F) as the demand and supply of the limiting continuum market: $1 - G(p)$ is the mass of buyers and $F(p)$ the mass of sellers in a continuum market who can profitably trade at the price p. The finite market that we consider is obtained by independently sampling m buyers from the demand curve $1 - G$ and m sellers from the supply curve F. Common knowledge and symmetry of beliefs in the finite market both follow from common knowledge of demand and supply in the continuum market together with common knowledge of the sampling process.

[4] Most common mechanisms (such as the k-DA) satisfy the stricter constraints of *ex post budget balance* (the transfers among the traders balance for every sample of values/costs) and *ex post individual rationality* (a trader is never forced to accept an unprofitable trade). We use the weaker constraints of interim individual rationality and ex ante budget balance here because optimizing over this larger class of mechanisms strengthens the sense in which the k-DA is deemed optimal.

[5] This result complements an earlier result of Wilson (1985), which showed that the k-DA is interim incentive efficient in the Holmström-Myerson (1983) sense when m is sufficiently larger.

Our approach is as follows. Rustichini, Satterthwaite, and Williams (1994) proved that error in a k-DA is at most κ/m^2 for some constant $\kappa \in \mathbb{R}^+$ determined by the set of environments. The issue in this paper is whether the error in any mechanism can exhibit a faster rate of convergence than this quadratic rate exhibited by the k-DA. This is addressed by considering the constrained efficient mechanism, which is a mechanism that minimizes error over all mechanisms in each environment subject to the constraints of incentive compatibility, interim individual rationality, and ex ante budget balance. The main technical result of this paper bounds below the error in the constrained efficient mechanism by γ/m^2 for some $\gamma \in \mathbb{R}^+$. For reasons of tractability, this is done only in the case in which both G and F are uniform. Because the worst-case error of any mechanism is at least as large as the worst-case error in the constrained efficient mechanism, and because worst-case error of the constrained efficient mechanism is at least as large as its error in the uniform environment, it follows that worst-case error in any mechanism is at least γ/m^2. The quadratic rate of convergence of worst-case error in the k-DA is thus the fastest rate possible.

It is noteworthy that our optimality result concerns a property of this simple, well-motivated mechanism across a range of possible environments and sizes of the market, rather than simply for a single, fixed environment and size of market. Our result thus responds to the *Wilson Critique* (Wilson (1987)) of mechanism design. Wilson criticized this field for focusing upon the problem of designing a mechanism explicitly for each specific problem (e.g., as determined here by the specification of an environment and a market size). An economic consultant asked for advice on the selection of a mechanism may not know all the parameters that specify the problem, and the parameters may change over time; theoretical results that describe how the mechanism should be chosen assuming detailed knowledge of the problem may thus have little value to the consultant. A more meaningful task for mechanism design is to establish the sense in which a simple mechanism performs reasonably well across the variety of problems that might be encountered in practice, which is the nature of our results.

There are obviously stronger senses in which a mechanism may be deemed optimal than worst case asymptotic optimality, and so our adoption of this standard needs explanation. The worst case asymptotic approach reduces the evaluation of algorithms over a range of problems to a single case (the worst one) and a single statistic (the rate of convergence of error to zero). This approach is justified by pragmatism, for it simplifies the problem of comparing algorithms and allows progress to be made. Nevertheless any statistic is only a summary that cannot perfectly capture every sample. Therefore, before this approach is accepted, two objections must be addressed: (I) a mechanism's asymptotic rate of convergence may not reflect its absolute performance in small and moderate sized markets, and (II) the worst case that determines a mechanism's ranking may be irrelevant to its performance in more plausible environments.[6]

[6] See Russell (1997, particularly pp. 65–68) for an insightful discussion of the usefulness of the worst case asymptotic optimality criterion within artificial intelligence research.

With respect to objection (I), numerical experiments that we have done indicate that the asymptotic rate at which a mechanism's error converges to zero is a good predictor of the mechanism's performance in small and moderate sized markets. In particular, for a variety of different environments, calculation of equilibria for the k-DA, the constrained efficient mechanism, and a variety of other mechanisms suggests the following:[7]

(i) The asymptotic convergence rate of a mechanism's error is exhibited in small and moderate sized markets, i.e., as m increases from 2 to 4 and then to 8, the mechanism's convergence rate is approximately the same as its asymptotic rate.

(ii) A disparity in the relative rates of convergence of two market mechanisms' errors mirrors the relative sizes of the errors they exhibit in moderate sized markets.

(iii) Across a range of environments, error in the k-DA is nearly indistinguishable from the error in the constrained efficient mechanism once m reaches 8.

The last point is particularly significant because it suggests that, for $m \geq 8$, the k-DA's error is essentially the same as the constrained efficient mechanism's error. Therefore, while far from comprehensive, our calculations support the use of the asymptotic rate of convergence to efficiency as a statistic for measuring the performance of market mechanisms.

Turning to objection (II), the worst-case approach is commonly taken when a probability distribution over the set of problems is either unclear or else exceedingly difficult to address, thereby making a ranking based upon expected performance infeasible. This is surely the case in the problem of selecting a market mechanism over a set of environments. Savage (1972, p. 168–169) deemed the worst-case approach meaningful if (a) the error in the worst case is reasonably small and (b) the worst case occurs in a state that is not unusual. Our results satisfy these conditions reasonably well. Specifically, with respect to (a), for all environments that we have tested numerically, the k-DA's error has been the same order of magnitude as the constrained efficient mechanism's error. With respect to (b), we derive our result that the constrained efficient mechanism's worst case error cannot be faster than quadratic in the uniform environment. This

[7] These statements are based upon calculations presented in the following papers. Drawing upon calculations from Gresik and Satterthwaite (1989), Table 5.1 of Satterthwaite and Williams (1989b) compares the 1-DA to the constrained efficient mechanism in the uniform environment for m between 2 and 12. Table III of Rustichini, Satterthwaite, and Williams (1994) compares, for the uniform environment and m between 1 and 8, the 0.5-DA to the constrained efficient mechanism and to the fixed price and dual price mechanisms that are discussed below. Finally, Satterthwaite and Williams (1999, Table 2) compare the 0.5-DA and the constrained efficient mechanism across three nonuniform environments and m between 2 and 8. The calculations in the nonuniform cases support the contention that the uniform environment is not unusual or unique in any respect. This last paper can be found online at http://www.kellogg.northwestern.edu/research/math/.

is not an odd environment concocted for the sake of establishing the worst-case result.[8]

Contrasting the rate of convergence in the k-DA mechanism with that of other mechanisms that have appeared in the literature gives further understanding of the usefulness of this approach to ranking mechanisms. Consider Hagerty and Rogerson's fixed-price mechanism (1985). Gresik and Satterthwaite (1989, p. 319) show that if it is generalized to markets of arbitrary size, then its error is at least $\beta/\sqrt{m}$ for some $\beta \in \mathbb{R}^+$. The asymptotic approach ranks the fixed-price mechanism as inferior to the market-clearing price in the k-DA, which supports common economic intuition. We know of no other formal criterion that quantifies a sense in which trade at a fixed price is inferior to trade at a market-clearing price in the k-DA.

There are two other mechanisms besides the k-DA that are known to be worst-case asymptotic optimal. First, the constrained efficient mechanism is itself worst-case asymptotic optimal. Second, McAfee (1992) designed an interim individually rational mechanism that generates a monetary surplus through the use of a different price for buyers than for sellers. Ex ante payments can be devised to return the expected surplus to the traders and thereby insure that the ex ante budget constraint is satisfied with equality. If such payments are included as part of the mechanism, then it too is worst-case asymptotic optimal. Such ex ante payments, however, must vary with the environment. Altered in this way, McAfee's mechanism shares the flaw of the constrained efficient mechanism of failing the Wilson Critique in the sense that its rules depend upon the environment. This flaw renders a market mechanism implausible for actual use. If such payments are disallowed in McAfee's mechanism and the surplus is instead regarded as a cost of arranging trade,[9] then its worst-case error is at least δ/m for some $\delta \in \mathbb{R}^+$ (Rustichini, Satterthwaite, and Williams (1992)). McAfee's mechanism

[8] More formally, Gilboa and Schmeidler (1989) developed an axiomatic justification for the worst case approach. Let C be a set of choices, X a set of states, and $u(c,x)$ the ex post utility of choice $c \in C$ given the state $x \in X$. In our setting, c is a market mechanism, x is a vector of buyers' values and sellers' costs, and $u(c,x)$ is the ex post gains from trade that mechanism c realizes in state x. Gilboa and Schmeidler modified the Savage axioms and showed that an ordering of C satisfying their axioms necessarily ranks choices c according to the values of

$$\min_{F \in E} \int u(c,x)\, dF$$

for some set E of prior distributions over X. An ordering thus ranks choices according to their worst case expected utilities over E, and a most preferred choice is therefore a maxmin choice. While there are substantial issues that must be addressed before the Gilboa-Schmeidler approach could be formally applied to the mechanism choice problem that we consider, their results do suggest a rationale for our use of the worst case in ranking market mechanisms. For a similar purpose, the Gilboa-Schmeidler theory has been cited recently by macroeconomists to support the use of the maxmin criterion in model selection. See the symposium, "Robustness to the Uncertainty," in the *American Economic Review* of May, 2001, which includes papers by Epstein (2001), Sims (2001), Chamberlain (2001), and Hansen and Sargent (2001).

[9] This perspective can be defended on the grounds that the surplus consists of gains from trade generated by the preferences of the traders that they sacrifice in order to achieve for themselves a

without payments is thus inferior to the k-DA in a worst-case asymptotic analysis. Though we suspect that other mechanisms besides the k-DA can be both worst-case asymptotic optimal and robust in the sense that their outcome functions do not vary with the environment, examples of such mechanisms have not yet been found.[10]

We discuss in Sections 2, 3, and 4 the model, prior results on the k-DA, and a formal statement of our main result. The task of bounding error in the constrained efficient mechanism for all sizes of markets is carried out in Sections 5 and 6, which is then followed by a brief conclusion.

2. THE MODEL

The analysis of this paper is carried out over any set E of environments (G, F) with the following properties:

E1: G and F are C^1 functions with support $[0, 1]$;

E2: for some $\underline{q}, \bar{q}$ satisfying $0 < \underline{q} \le 1 \le \bar{q}$, the densities g and f of the distributions G and F satisfy the bounds

$$(1) \qquad 0 < \underline{q} \le g, f \le \bar{q};$$

E3: E contains the *uniform environment* (G^u, F^u) in which both G and F are uniform on $[0, 1]$.

The bound (1) permits a worst-case analysis and assumption E3 insures that the uniform case is available for comparison with the worst-case. This will be useful for the sake of tractability.

A *market game ϕ_m of size m over E* consists of:

M1: a strategy set A_i for each of the $2m$ traders;

M2: an outcome mapping ζ_m: $(\prod_{i=1}^{2m} A_i) \times E \to ([0, 1] \times \mathbb{R})^{2m}$ that specifies for each trader his probability of receiving a unit along with a monetary transfer as functions of the profile of strategies and the environment;

M3: the selection of a Bayesian-Nash equilibrium in the game defined by M1 and M2 for each environment $(G, F) \in E$.[11]

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portion of the potential gains from trade. In other words, the surplus is a cost of arranging trade. This, however, ignores the welfare of an intermediary who receives the surplus.

[10] More recently, Yoon (2001) modified the Vickrey (or two-price) double auction by (i) collecting taxes from the agents to fund this mechanism's deficit, and (ii) allowing a trader to opt out of the mechanism after learning his value/cost and thereby avoid his tax. Given the environment (G, F), Yoon showed that the taxes can be chosen with (i) and (ii) in mind so that this modified Vickrey double auction satisfies ex ante budget balance and interim individual rationality while retaining the dominant strategy incentive compatibility of the Vickrey mechanism. Yoon then showed that error in this modified mechanism is at most λ/m for some constant $\lambda \in \mathbb{R}^+$. His numerical calculations suggest that error converges to zero more slowly in this mechanism than the quadratic rate of the k-DA. If his calculations are representative, then this modified Vickrey double auction is also inferior to the k-DA in the worst-case asymptotic sense.

[11] Our definition of a market game is unusual in that (i) the outcome mapping ζ_m can depend upon the environment and (ii) an equilibrium is specified for each environment. Property (i) allows the

A *market mechanism over* E is a sequence $\Phi \equiv (\phi_m)_{m\in\mathbb{N}}$ in which ϕ_m is a market game of size m over E.

Efficiency dictates that in each sample of $2m$ values/costs the m units must be allocated to the traders with the m highest values/costs. In the efficient allocation, buyers whose values are among the top m values/costs purchase units from sellers whose costs are among the m smallest values/costs. Let $\Gamma_m(G,F)$ denote the expected potential gains from trade among the $2m$ traders, computed with respect to the joint distribution of their $2m$ values/costs. The value $\phi_m(G,F)$ denotes the expected gains from trade achieved by the $2m$ traders in the selected equilibrium of the market game ϕ_m when (G,F) is the environment. Our measure of error in a market game is *relative inefficiency* $e(\phi_m, G, F)$, which is the fraction of the expected potential gains from trade in the environment (G,F) that is inefficiently not achieved in the selected equilibrium of ϕ_m:

$$e(\phi_m, G, F) \equiv \frac{\Gamma_m(G,F) - \phi_m(G,F)}{\Gamma_m(G,F)}. \tag{2}$$

Myerson and Satterthwaite (1983) showed in the case of bilateral trade ($m = 1$) that $e(\phi_m, G, F) > 0$ in any market game ϕ_m satisfying interim individual rationality and ex ante budget balance. This result was later extended to arbitrary values of m by Williams (1999, Theorem 4). These results imply that an interim individually rational and ex ante budget balanced mechanism Φ is necessarily inefficient, regardless of the size of the market m.

3. RESULTS ON THE k-DA

A k-DA *mechanism* $\Phi^{k\text{-DA}} \equiv (\phi_m^{k\text{-DA}})_{m\in\mathbb{N}}$ is the sequence of market games described at the beginning of this paper[12] together with the selection for each m of an equilibrium for the market of size m in the environment (G,F) that has the following three properties:

Symmetry: Each buyer uses the same function $B_m(\cdot)$ and each seller uses the same function $S_m(\cdot)$ to select his bid/ask as functions of his value/cost.

Nondominated Strategies: At every $v_i, c_j \in [0,1]$, $B_m(v_i) \leq v_i$ and $S_m(c_j) \geq c_j$.

Nontriviality: The sets $\{v_i | B_m(v_i) > 0\}$ and $\{c_j | S_m(c_j) < 1\}$ have positive measure, which implies that trade occurs with positive probability.

An equilibrium satisfying these three properties is denoted $\langle B_m, S_m \rangle$. Existence of an equilibrium with these properties is proven by Jackson and

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market game to be chosen optimally for each environment, which is a central theme in mechanism design. Property (ii) is part of the definition of a market game purely because this simplifies the discussion.

[12] The rules of the k-DA are defined in detail in Rustichini, Satterthwaite, and Wiliams (1994, p. 1045). We will not be analyzing the operation of this mechanism here, relying instead on results drawn from this earlier paper. It is thus sufficient to understand that the market game $\phi_m^{k\text{-DA}}$ operates as described in the Introduction.

Swinkels (1999).[13,14,15] With our definition of a mechanism, each rule for selecting an equilibrium defines a different k-DA. The precise rule for choosing an equilibrium, however, is immaterial for our purposes as long as the selected equilibrium satisfies these three properties. The requirement that strategies be nondominated insures that each equilibrium $\langle B_m, S_m \rangle$ satisfies interim individual rationality, and the rule that all trades in the k-DA are consummated at a market-clearing price insures that every equilibrium satisfies ex ante budget balance. Any k-DA mechanism thus satisfies these two constraints.

The following theorem concerning the rate at which the relative inefficiency of the k-DA mechanism converges to zero is the main result on the k-DA that is needed in this paper.

THEOREM 1 (Rustichini, Satterthwaite, and Williams (1994)): *There exists a continuous function* $\kappa : \mathbb{R}^{+^2} \to \mathbb{R}^{+}$ *such that*

$$e(\phi_m^{k\text{-}DA}, G, F) = \frac{\kappa(\underline{q}, \bar{q})}{m^2} \tag{3}$$

in any environment (G, F) *satisfying* $0 < \underline{q} \leq g,\ f \leq \bar{q}$.

The bound (3) thus holds for all $(G, F) \in E$ and all $k \in [0, 1]$. Because $\underline{q}$ and $\bar{q}$ are not varied in the remainder of this paper, for simplicity they will be omitted as variables in κ when (3) is applied below. This theorem follows from Theorem 3.2 of Rustichini, Satterthwaite, and Williams (1994), which states that $e(\phi_m^{k\text{-DA}}, G, F)$ is bounded above by ξ/m^2 for some function ξ of G, F, and k. In (3) we have replaced ξ with a bound that holds for all $k \in [0, 1]$ and that expresses the dependence of the bound on G and F explicitly in terms of the bounds $\underline{q}$ and

[13] Theorem 6 and Corollary 7 of Jackson and Swinkels (1999) establish the existence of an equilibrium in *distributional* strategies that satisfies *nondominated strategies* and *nontriviality*. The proofs of these results can be adapted to insure that *symmetry* also holds using the approach of their Theorem 3, which concerns the existence of symmetric equilibria in symmetric Bayesian games. A distributional strategy allows a trader to employ a different mixed strategy for each of his possible values/costs. As Jackson and Swinkels (1999, p. 6) point out, the monotonicity of equilibrium strategies implies that an equilibrium in distributional strategies can be altered to define an equilibrium in pure strategies. Theorem 2.1 of Rustichini, Satterthwaite, and Williams (1994) provides the requisite monotonicity argument in the case of the k-DA.

[14] Similar existence theorems for Bayesian games were proven concurrently by Simon and Zame (1999) using similar methods. We cite Jackson and Swinkels (1999) above because they provide the extra step of proving the existence of an equilibrium in the k-DA in which trade occurs with positive probability. It has long been known that a symmetric equilibrium in nondominated strategies exists in the k-DA, namely, the no-trade equilibrium defined by $B_m = 0$ and $S_m = 1$. The existence of an equilibrium in which trade occurs has always been the crucial issue, which Jackson and Swinkels specifically address. See also Ye (1998) for an alternative approach for proving existence of equilibria in the k-DA.

[15] Earlier existence results include Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989a), which prove the existence of a variety of equilibria in $\phi_1^{k\text{-DA}}$ (i.e., the bilateral k-DA), and Williams (1991), which proves the existence of a unique smooth equilibrium in $\phi_m^{1\text{-DA}}$ (the 1-DA) for a generic set of environments.

$\bar{q}$ on the densities. A function $\kappa(\cdot)$ that satisfies (3) can be obtained by working through the proofs of Theorems 3.1 and 3.2 in Rustichini, Satterthwaite, and Williams (1994).

4. THE MAIN RESULT

A market game ϕ_m is evaluated over a set of environments E according to its *worst-case error* $e^{wor}(\phi_m, E)$, which is defined as

$$e^{wor}(\phi_m, E) \equiv \sup_{(G,F)\in E} e(\phi_m, G, F). \tag{4}$$

Given a set E of environments, a mechanism Φ defines a sequence of worst-case error values. A mechanism Φ is *worst-case asymptotic optimal* over E among some set M of mechanisms defined on E if the sequence of worst-case error values for any other mechanism in M does not converge to zero at a faster rate than the sequence defined by the mechanism Φ. This notion of optimality is captured by the following definition.

DEFINITION: Given a set E of environments and a set M of mechanisms defined on E, a mechanism Φ is *worst-case asymptotic optimal* over E among mechanisms in M if, for any other mechanism $\Phi^* \in M$, there exists a constant $\eta \in \mathbb{R}^+$ such that

$$e^{wor}(\phi_m, E) \leq \eta \cdot e^{wor}(\phi_m^*, E) \tag{5}$$

for all $m \in \mathbb{N}$.

The main theorem of this paper can now be stated.

THEOREM 2: *Assume that the set of environments E has the properties E1–E3 stated at the beginning of Section 2. A k-DA mechanism $\Phi^{k\text{-}DA}$ is worst-case asymptotic optimal over E among all interim individually rational and ex ante budget-balanced mechanisms defined on E.*

The strength of this result is emphasized by noting that the constraints that it imposes on a mechanism are weak enough to allow the possibility that the mechanism operates over time, consummates trades at a number of different prices, runs surpluses and deficits that cancel only in expectation, or compels traders on occasion to accept losses ex post. A great variety of market mechanisms are thus covered by Theorem 2. The theorem is also strengthened because the assumptions on the set of environments E are so modest: that E is not required to be in some sense large shows that our worst-case analysis does not require the consideration of odd environments, and conversely, that E is not restricted beyond the conditions at the beginning of Section 2 shows that our worst-case analysis does not depend upon avoiding plausible environments.

Let $\Phi^{ce} = (\phi_m^{ce})_{m\in\mathbb{N}}$ denote a mechanism with the property that, for each m and each environment (G, F) in E, ϕ_m^{ce} maximizes the achieved gains from trade in the environment (G, F) subject to the constraints of interim individual rationality and ex ante budget balance.[16] Alternatively, ϕ_m^{ce} can be described as a market game ϕ_m that solves the constrained optimization problem

$$\text{(6)} \qquad \min_{\phi_m} e(\phi_m, G, F)$$

subject to the constraints on ϕ_m of interim individual rationality and ex ante budget balance. A market game ϕ_m^{ce} that solves (6) is *constrained efficient* in the environment (G, F); ϕ_m^{ce} is constrained efficient in E if it solves (6) for every $(G, F) \in E$. A mechanism Φ^{ce} is *constrained efficient* in (G, F) (or E) if each of its market games ϕ_m^{ce} is constrained efficient in (G, F) (or E). Solving for ϕ_m^{ce} is one instance of the central problem in Bayesian mechanism design. The existence and the properties of Φ^{ce} will be discussed in Section 5.

Any interim individually rational and ex ante budget-balanced mechanism Φ defined on E satisfies

$$\text{(7)} \qquad e^{wor}(\phi_m, E) \geq e(\phi_m, G^u, F^u) \geq e(\phi_m^{ce}, G^u, F^u)$$

for each m, where the first inequality is true because $(G^u, F^u) \in E$ and the second holds because ϕ_m^{ce} solves (6). As demonstrated below, Theorem 2 is a consequence of the following theorem.

THEOREM 3: *There exists a positive number γ such that*

$$\text{(8)} \qquad e(\phi_m^{ce}, G^u, F^u) \geq \frac{\gamma}{m^2}.$$

Establishing this lower bound on the relative inefficiency of the constrained efficient mechanism in the uniform environment constitutes most of the formal analysis of the paper. This theorem is proven in Section 6. We now show how the proof of our main result follows directly from it.

PROOF OF THEOREM 2: Letting Φ denote an alternative mechanism defined on E, we need to find a positive number η that satisfies

$$e^{wor}(\phi_m^{k\text{-}DA}, E) \leq \eta(e^{wor}(\phi_m, E))$$

for all $m \in \mathbb{N}$. Inequalities (7) and (8) together imply

$$e^{wor}(\phi_m, E) \geq \frac{\gamma}{m^2} = \left(\frac{\gamma}{\kappa}\right)\left(\frac{\kappa}{m^2}\right).$$

Theorem 1 then implies

$$e^{wor}(\phi_m, E) \geq \left(\frac{\gamma}{\kappa}\right) e^{wor}(\phi_m^{k\text{-}DA}, E).$$

The proof is completed by setting $\eta = \kappa/\gamma$. *Q.E.D.*

[16] Recall that the constraint of incentive compatibility is implicit in our definition of a mechanism.

5. THE CONSTRAINED EFFICIENT MECHANISM

All that remains to be proven is Theorem 3, which bounds below by γ/m^2 the error $e(\phi^{ce}_m, G^u, F^u)$ of the constrained efficient market game ϕ^{ce}_m in the uniform environment (G^u, F^u). The purpose of this section is to establish a result concerning Φ^{ce} that is needed in the next section for the proof of Theorem 3.

The argument in this section proceeds as follows. A trader's *α-virtual utility* is defined below as a particular function of his value/cost, the environment (G, F), and a parameter $\alpha \in [0, 1]$. Intuitively, this function adjusts each trader's value/cost in a way that compensates for the privacy of his information and thereby neutralizes his incentive to misrepresent. Theorem 4 below, which illustrates a general principle of Bayesian mechanism design,[17] is a result of Gresik and Satterthwaite (1983) stating that ϕ^{ce}_m allocates the m units to those traders with the largest α-virtual utilities for a nonzero value of $\alpha = \alpha^*_m(G, F)$ that solves a particular equation. Allocation according to α-virtual utilities is efficient only when $\alpha = 0$ because the 0-virtual utility of a trader equals his true value/cost. If $\alpha > 0$, then allocation according to α-virtual utilities leads in expectation to inefficiency that is increasing in α. A key issue in the remainder of this paper is therefore to bound below the rate at which the sequence $(\alpha^*_m(G^u, F^u))_{m\in\mathbb{N}}$, which characterizes Φ^{ce}, converges to 0 as the market size m goes to infinity.

The difficulty is that $\alpha^*_m(G, F)$ is customarily characterized as the solution of a $2m$-dimensional integral equation. This equation is all but intractable except for small values of m. A technique that we invent here is to devise a particular sequence of market games that allocate the units according to the α-virtual utilities. The equation that characterizes $\alpha^*_m(G, F)$ is simply a budget balance condition on the mth market game in this sequence that can be expressed as a two-dimensional integral equation. The value of this alternative equation is then demonstrated in Lemma 2 of the next section in which the desired lower bound on $\alpha^*_m(G^u, F^u)$ is derived. While this section may thus seem at first glance to be a digression, it is in fact a large part of this paper's original analysis because it provides a tractable means for characterizing the constrained efficient market game ϕ^{ce}_m for arbitrary values of m.

Let $\alpha \in [0, 1]$. A trader's *α-virtual utility* is defined as a function of his value/cost as follows: buyer i's α-virtual utility function is

$$\Psi^b_\alpha(v_i) \equiv v_i + \alpha \frac{G(v_i) - 1}{g(v_i)},$$

[17] Efficiency in a general mechanism design problem is a mapping that specifies an outcome as a function of the true types of the agents. The general principle is that the outcome in a constrained efficient revelation mechanism is determined by applying the efficiency mapping to the α-virtual utilities of the agents, where α is chosen so that the mechanism is ex ante budget balanced. This principle originated in Myerson (1981), which concerned the constrained efficient mechanism for auctioning an item. It is derived as a general principle in Wilson (1993) and it appears in almost all derivations of constrained efficient mechanisms.

and seller j's α-virtual utility function is

$$\Psi_\alpha^s(c_j) \equiv c_j + \alpha \frac{F(c_j)}{f(c_j)}.$$

The environment (G, F) is *regular* if, for each α, the functions $\Psi_\alpha^b(\cdot)$, $\Psi_\alpha^s(\cdot)$ are increasing on [0, 1]. The uniform environment is regular and we restrict the remainder of this section to regular environments.

For a sample of $2m$ values/costs of the traders, let $t_{(j)}$ denote the jth smallest α-virtual utility among the corresponding $2m$ α-virtual utilities:

$$t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(2m)}.$$

Recall that the definition of a market mechanism Φ specifies an equilibrium in each market game ϕ_m for each environment (G, F). For a given $\alpha \in [0, 1]$, a market game ϕ_m is an *α-market game in the environment (G, F)* if ϕ_m allocates the m units to buyers and sellers whose α-virtual utilities are among the top m values $t_{(m+1)} \leq t_{(m+2)} \leq \cdots \leq t_{(2m)}$. Trades thus occur in an α-market game between buyers whose α-virtual utilities are at least $t_{(m+1)}$ and sellers whose α-virtual utilities are no more than $t_{(m)}$. For a sequence $A = (\alpha_m)_{m \in \mathbb{N}}$, a market mechanism Φ is an *A-mechanism in the environment (G, F)* if for each m the market game ϕ_m is an α_m-market game in this environment. As mentioned above, a theorem of Gresik and Satterthwaite (1983) states that an interim individually rational, ex ante budget balanced mechanism is constrained efficient in the environment (G, F) if and only if it is an A^*-mechanism for the particular sequence $A^* \equiv (\alpha_m^*(G, F))_{m \in \mathbb{N}}$ that is characterized below.

Some notation is needed to state their theorem. Let $\sigma_m \equiv (v_1, \ldots, v_m, c_1, \ldots, c_m)$ denote a sample of $2m$ values/costs. Given σ_m and $\alpha \in [0, 1]$, for buyer i define $p_\alpha^i(\sigma_m)$ as

$$p_\alpha^i(\sigma_m) \equiv \begin{cases} 1 & \text{if } \Psi_\alpha^b(v_i) \geq t_{(m+1)}, \\ 0 & \text{if } \Psi_\alpha^b(v_i) < t_{(m+1)}, \end{cases}$$

and for seller j define $q_\alpha^j(\sigma_m)$ as

$$q_\alpha^j(\sigma_m) \equiv \begin{cases} 1 & \text{if } \Psi_\alpha^s(c_j) \leq t_{(m)}, \\ 0 & \text{if } \Psi_\alpha^s(c_j) > t_{(m)}. \end{cases}$$

These are indicator functions that equal one if and only if the trader trades in the given sample σ_m when items are allocated by an α-market game. Denote with $r_\alpha^i(\sigma_m)$ the payment that buyer i makes to the mechanism and denote with $s_\alpha^j(\sigma_m)$ the payment seller j receives as functions of σ_m. The expression

$$\text{(9)} \qquad \mathscr{E}_{\sigma_m}\left(\sum_{i=1}^m r_\alpha^i(\sigma_m) - \sum_{j=1}^m s_\alpha^j(\sigma_m)\right)$$

is the α-market game's *expected surplus*. If (9) is zero, then the market game is ex ante budget balanced. Finally, let $U_i(v_i)$ and $V_j(c_j)$ be the interim expected utilities of buyer i with value v_i and seller j with cost c_j respectively.

Define the function $\mathrm{Sur}(\alpha, m, G, F)$ by the formula

$$\mathrm{Sur}(\alpha, m, G, F) \equiv \mathscr{E}\left[\left(\sum_{i=1}^{m} \Psi_1^b(v_i) p_\alpha^i(\sigma_m)\right) - \left(\sum_{j=1}^{m} \Psi_1^s(c_j) q_\alpha^j(\sigma_m)\right)\right]. \tag{10}$$

This function is crucial because $\mathrm{Sur}(a, m, G, F) = 0$ determines the value of $\alpha_m^*(G, F)$ that characterizes the constrained efficient market game in this environment. This is the $2m$-dimensional integral equation that we mentioned at the beginning of this section and that we reinterpret below as a two-dimensional integral.

Theorem 4 combines a number of results in Gresik and Satterthwaite (1983) to characterize the constrained efficient market game.[18,19]

THEOREM 4 (Gresik and Satterthwaite (1983)): *The following statements are true in the case of a regular environment* (G, F).

(i) *For each* $m \geq 1$, *there exists a unique* $\alpha_m^*(G, F) \in (0, 1)$ *that satisfies* $\mathrm{Sur}(\alpha_m^*(G, F), m, G, F) = 0$.

(ii) *Let* $A^* \equiv (\alpha_m^*(G, F))_{m \in \mathbb{N}}$. *A constrained efficient mechanism exists in the environment* (G, F) *and is an* A^**-mechanism. Conversely, any* A^**-mechanism that satisfies interim individual rationality and ex ante budget balance is constrained efficient in this environment.*

An interpretation of formula (10) for $\mathrm{Sur}(\alpha, m, G, F)$ follows directly from Gresik and Satterthwaite's derivation of the constrained efficient mechanism. Suppose an α-market game ϕ_m satisfies $U_i(0) = 0 = V_j(1)$ for each buyer i and each seller j. The quantity $\mathrm{Sur}(\alpha, m, G, F)$ then equals this market game's expected surplus,

$$\mathrm{Sur}(\alpha, m, G, F) = \mathscr{E}_{\sigma_m}\left(\sum_{i=1}^{m} r_\alpha^i(\sigma_m) - \sum_{j=1}^{m} s_\alpha^j(\sigma_m)\right). \tag{11}$$

[18] This theorem follows from Theorems 2 and 3 of their paper together with the following three observations. First, while Gresik and Satterthwaite assume the stronger constraint of ex post budget balance, only the weaker constraint of ex ante budget balance is needed to derive Theorem 4 above. They showed that transfers in a constrained efficient mechanism can always be altered to satisfy ex post budget balance without disturbing the constrained efficiency of the mechanism. Second, Williams (1999, Thm. 4) proved that $\mathrm{Sur}(0, m, G, F) < 0$. This inequality implies both that ϕ_m^{ce} cannot be an $\alpha = 0$ market game (which is one of the possible conclusions of their theorems) and also (together with their Theorem 3) the existence of a solution to $\mathrm{Sur}(\alpha_m^*(G, F), m, G, F) = 0$. Third, any $\alpha_m^*(G, F)$ that solves this equation is shown by Gresik and Satterthwaite to define a constrained efficient market game. Because inefficiency is increasing in α, the solution $\alpha_m^*(G, F)$ must therefore be unique.

[19] Results of this kind are now standard in the derivation of constrained efficient mechanisms. Because the relevant material of Gresik and Satterthwaite (1983) is unpublished, the reader may wish to consult the derivation in Myerson and Satterthwaite (1983), which presents the main ideas of the analysis in the simplified setting of bilateral trade ($m = 1$), or the general discussion in Wilson (1993).

Depending on the value of α, it may be positive, zero, or negative. The calculation that establishes (11) is outlined in the Appendix.

Consider next the family of *two-price A-mechanisms* that we invent for the purpose of computing $\mathrm{Sur}(\alpha, m, G, F)$ using (11) instead of (10). For a given sequence $A \equiv (\alpha_m)_{m\in\mathbb{N}}$ and for $m \in \mathbb{N}$, define the market game $\Phi_m^{2,A}$ in a two-price A-mechanism $\Phi^{2,A} = (\phi_m^{2,A})_{m\in\mathbb{N}}$ by starting with values/costs as reported by the traders and computing the α_m-virtual utilities as functions of these reports. Allocate the m units to the traders whose α_m-virtual utilities are the m largest (i.e., those at or above $t_{(m+1)}$); items are allocated in the case of a tie of $t_{(m)} = t_{(m+1)}$ first by assigning items to those traders whose α_m-virtual utilities are strictly above $t_{(m+1)}$, second to buyers whose α_m-virtual utilities equal $t_{(m+1)}$, and last to sellers whose α_m-virtual utilities equal $t_{(m+1)}$, using a fair lottery whenever necessary. Each buyer who purchases a unit pays $(\Psi_{\alpha_m}^b)^{-1}(t_{(m)})$ as his price and each seller who sells his unit receives $(\Psi_{\alpha_m}^s)^{-1}(t_{(m+1)})$. Traders who fail to trade neither receive nor pay a monetary transfer. Notice that a trader who successfully trades cannot influence his price in his favor by changing his reported value/cost. As proven in the Appendix, the market game $\phi_m^{2,A}$ has the following properties.

LEMMA 1: *If the environment (G, F) is regular, then the following statements hold for a market game $\phi_m^{2,A}$ in a two-price A-mechanism.*

(i) *Honestly reporting one's value/cost is the unique dominant strategy of each trader.*

(ii) *The m items are allocated in the dominant strategy equilibrium to traders whose α_m-virtual utilities are among the m largest.*

(iii) *The dominant strategy equilibrium is interim individually rational, and a buyer with value $v_i = 0$ or a seller with cost $c_j = 1$ has an interim expected payoff equal to* 0.

The selection of the dominant strategy equilibrium for each m and each (G, F) completes the definition of the two-price A-mechanism. Property (ii) insures that it is indeed an A-mechanism. Ties among the α_m-virtual utilities occur with probability zero in the dominant strategy equilibrium. Consequently, we ignore them in the remainder of this paper. Property (iii) implies that the expected surplus in $\phi_m^{2,A}$ is $\mathrm{Sur}(\alpha_m, m, G, F)$.

We now can realize the goal of this section, which is to derive a tractable formula for $\mathrm{Sur}(\alpha_m, m, G, F)$ using the properties of the two-price A-mechanism. Let $H(t_{(m)}, t_{(m+1)})$ denote the expected number of trades conditional on the values of the mth and the $(m+1)$st α_m-virtual utilities $t_{(m)}$ and $t_{(m+1)}$. The expected surplus conditional on $t_{(m)}$ and $t_{(m+1)}$ is

$$H(t_{(m)}, t_{(m+1)})\big((\Psi_{\alpha_m}^b)^{-1}(t_{(m)}) - (\Psi_{\alpha_m}^s)^{-1}(t_{(m+1)})\big), \tag{12}$$

because $(\Psi_{\alpha_m}^b)^{-1}(t_{(m)})$ is the price that buyers pay and $(\Psi_{\alpha_m}^s)^{-1}(t_{(m+1)})$ is the price that sellers receive. Taking expectations with respect to the joint distribution of $t_{(m)}$ and $t_{(m+1)}$ and replacing α_m with the generic parameter α produces the desired formula for $\mathrm{Sur}(\alpha, m, G, F)$.

THEOREM 5: *If the environment* (G, F) *is regular, then*

$$\mathrm{Sur}(\alpha, m, G, F) = \mathscr{E}\big[H(t_{(m)}, t_{(m+1)})\big((\Psi_\alpha^b)^{-1}(t_{(m)}) - (\Psi_\alpha^s)^{-1}(t_{(m+1)})\big)\big], \tag{13}$$

where the second expression is defined in (12) *using the two-price A-mechanism.*

Equation (13) clearly differs from the standard formula for $\mathrm{Sur}(\alpha, m, G, F)$ in (10) in its dependence upon both the transfers and the allocation rule in the two-price A-mechanism. Most importantly for our purposes, (13) is a two-dimensional integral instead of the $2m$-dimensional integral in (10).

6. A LOWER BOUND ON THE INEFFICIENCY OF THE CONSTRAINED EFFICIENT MECHANISM IN THE UNIFORM ENVIRONMENT

The alternative formula (13) for the expected surplus is valuable because it makes the equation $\mathrm{Sur}(\alpha_m^*(G^u, F^u), m, G^u, F^u) = 0$ solvable for a lower bound on $\alpha_m^*(G^u, F^u)$. This bound is derived below in Lemma 2. The bound is then used in Theorem 3 to establish the desired lower bound on $e(\phi_m^{ce}, G^u, F^u)$. Uniformity is used to draw several conclusions in the proofs of these results concerning the distributions and the expected differences of order statistics for the sake of reducing (13).[20] Because these two results concern only the uniform environment (G^u, F^u) and a fixed market size m, "$\alpha_m^*(G^u, F^u)$" is replaced in this section by the generic parameter "α", except in the statement of the lemma.

LEMMA 2: *There exists a constant* $\tau \in \mathbb{R}^+$ *such that the value* $\alpha_m^*(G^u, F^u)$, *which characterizes the constrained efficient market game* ϕ_m^{ce} *in the uniform environment, is at least* τ/m *for all* m.

The proof of Lemma 2 is in the Appendix.

PROOF OF THEOREM 3: Let $s_{(j)}$ denote the jth smallest value/cost in a sample of $2m$ buyers' values and sellers' costs in the uniform environment. A lower bound on the expected value of the unrealized gains from trade will be computed by bounding a portion of the losses in the event D that is defined by the following two conditions:

(i) $s_{(m)}$ is a seller's cost and $s_{(m+1)}$ is a buyer's value;

(ii) $\psi_\alpha^b(s_{(m+1)}) < \psi_\alpha^s(s_{(m)}) \Leftrightarrow s_{(m+1)} - s_{(m)} < \alpha/(\alpha+1)$.

Recall that efficiency requires that the m items be assigned to the traders with the m highest values/costs while a constrained efficient market game ϕ_m^{ce} assigns the items to the traders with m highest α-virtual utilities. Condition (i) implies that both the buyer with value $s_{(m+1)}$ and the seller with cost $s_{(m)}$ should

[20] While the proofs of Lemma 2 and Theorem 3 are greatly simplified by the assumption of the uniform environment, intuition suggests that this environment is merely an expedient in these proofs, i.e., the bounds in these results do not fundamentally depend upon uniformity. Recent work by Tatur (2001) in fact suggests that these results extend to nonuniform environments.

trade for the sake of efficiency, either with each other or with others. Because $\psi_\alpha^b(\cdot)$ is increasing and $c_j \le \psi_\alpha^s(c_j)$, the $m-1$ α-virtual utilities of traders whose values/costs exceed $s_{(m+1)}$ are above $\psi_\alpha^b(s_{(m+1)})$. Condition (ii) implies that there is an additional α-virtual utility above $\psi_\alpha^b(s_{(m+1)})$, for a total of at least m. The buyer whose value equals $s_{(m+1)}$ thus does not trade in ϕ_m^{ce}. A similar argument shows that the seller with cost $s_{(m)}$ also does not trade. The unrealized gains from trade are therefore at least $s_{(m+1)} - s_{(m)}$ in event D.

A lower bound on $\Gamma_m(G^u, F^u) - \phi_m^{ce}(G^u, F^u)$ will now be computed by integrating $s_{(m+1)} - s_{(m)}$ over event D. Define $w \equiv s_{(m+1)} - s_{(m)}$ and let $\rho(w; m)$ denote its density function. Notice first that, for any given value of w, the probability that $s_{(m+1)}$ is a buyer's value and $s_{(m)}$ is a seller's cost equals 1/4. This is true because each trader's value/cost is independently drawn from the same distribution. A lower bound on $\Gamma_m(G^u, F^u) - \phi_m^{ce}(G^u, F^u)$ is thus given by (14):

$$\Gamma_m(G^u, F^u) - \phi_m^{ce}(G^u, F^u) > \frac{1}{4}\int_0^{\frac{\alpha}{\alpha+1}} w\rho(w; m)\, dw \tag{14}$$

$$> \frac{1}{4}\int_0^{\frac{\tau}{2m}} w\rho(w; m)\, dw. \tag{15}$$

Because $\tau/m < \alpha \in [0, 1]$, it follows that $\alpha/(\alpha+1) \ge \alpha/2 > \tau/2m$. This implies (15).

The integral in (15) is straightforward to evaluate given that buyers' values and sellers' costs are distributed uniformly on [0, 1]. Equation (2.3.1) in David (1981, p. 11) implies that

$$\rho(w; m) = \frac{2m!}{((m-1)!)^2}\int_0^{1-w} x^{m-1}(1-x-w)^{m-1}\, dx. \tag{16}$$

Integration by parts implies that for $j, k \ge 1$,

$$\int_0^{1-w} x^j(1-x-w)^k\, dx = \int_0^{1-w} \frac{j}{k+1} x^{j-1}(1-x-w)^{k+1}\, dx.$$

Applying this formula to (16) a total of $m-1$ times and then simplifying produces $\rho(w; m) = 2m(1-w)^{2m-1}$. This formula allows us to evaluate the integral in (15):

$$\int_0^{\frac{\tau}{2m}} w\rho(w; m)\, dw = \frac{1-(1+\tau)(1-\frac{\tau}{2m})^{2m}}{2m+1}. \tag{17}$$

The term $(1-(\tau/2m))^{2m}$ in (17) is positive and decreasing in m to $\lim_{m\to\infty}(1-(\tau/2m))^{2m} = e^{-\tau}$. Substitution into (14)–(15) thus implies

$$\Gamma_m(G^u, F^u) - \phi_m^{ce}(G^u, F^u) > \frac{1}{4} \times \frac{1-(1+\tau)e^{-\tau}}{2m+1} > \frac{\gamma}{m},$$

where

$$\gamma \equiv \frac{1}{4} \times \frac{1-(1+\tau)e^{-\tau}}{3}.$$

To show that $\gamma > 0$, regard τ as a variable and note that: (i) $1-(1+\tau)e^{-\tau}=0$ at $\tau=0$; (ii) $d/d\tau[1-(1+\tau)e^{-\tau}]=\tau e^{-\tau}>0$ for $\tau>0$. It follows that $\gamma>0$ for the positive value of τ given by Lemma 2.

Turning finally to $e(\phi_m^{ce}, G^u, F^u)$, we have

$$e(\phi_m^{ce}, G^u, F^u) \equiv \frac{\Gamma_m(G^u, F^u)-\phi_m^{ce}(G^u, F^u)}{\Gamma_m(G^u, F^u)} > \frac{\gamma}{m\Gamma_m(G^u, F^u)}.$$

The expected potential gains from trade $\Gamma_m(G^u, F^u)$ are at most m because at most m trades can be made, each of value one or less. Therefore, $e(\phi_m^{ce}, G^u F^u) > \gamma/m^2$. *Q.E.D.*

7. CONCLUSION

We have shown in this paper that the k-DA is worst-case asymptotic optimal. The demonstration that the rate at which its worst case error converges to zero is as fast as possible quantifies a sense in which trade at a market-clearing price is superior to or equal to trade using other mechanisms. This formalizes common economic intuition.

The rate at which worst-case error converges to zero is a coarse measure of a market mechanism, first because it reduces performance over all environments to a worst case, and second because it summarizes performance over all sizes of markets with a single rate as the sole statistic. Our defense of this measure is that (i) it successfully distinguishes some mechanisms as optimal and some as inferior, and (ii) it accurately mirrors the performance of mechanisms in the small markets and the variety of environments that we have numerically investigated. The analysis of algorithms in computer science suggests that this measure may not be sufficiently fine for all purposes of selecting a market mechanism. It is, however, a useful first cut that distinguishes the plausible from the inferior, a step that has not been taken for market mechanisms by other formal analyses.

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Manuscript received December, 1999; final revision received January, 2002.

APPENDIX

VERIFICATION OF FORMULA (11) FOR SUR$(\cdot)$. Let T denote the expected surplus (9) of the α-market game ϕ_m. Formula (11) is proven by equating two formulas for the ex ante expected sum of the utilities of the $2m$ traders in this game, solving this equation for T, and then simplifying. The first formula is simply

$$\mathscr{E}_{\sigma_m}\left(\sum_{i=1}^{m} p_\alpha^i(\sigma_m)v_i - \sum_{j=1}^{m} q_\alpha^j(\sigma_m)c_j\right) - T. \tag{18}$$

The second formula is derived from incentive compatibility. A standard argument in mechanism design applies this constraint to represent buyer i's interim expected utility as

$$U_i(v_i) = U_i(0) + \int_0^{v_i} \bar{p}^i_\alpha(t)\,dt,$$

and seller j's interim expected utility as

$$V_j(c_j) = V_j(1) + \int_{c_j}^1 \bar{q}^j_\alpha(t)\,dt,$$

where $\bar{p}^i_\alpha(t)$ or $\bar{q}^j_\alpha(t)$ is the trader's interim expected probability of trading given that his value/cost is t. Myerson and Satterthwaite (1983, p. 269–270) illustrate the derivation of these formulas. If $U_i(0) = 0 = V_j(1)$ for each buyer i and each seller j (as hypothesized in this calculation), then a second formula for the ex ante expected sum of the utilities is

$$\begin{aligned}(19)\qquad & \sum_{i=1}^m \mathscr{E}_{v_i}[U_i(v_i)] + \sum_{j=1}^m \mathscr{E}_{c_j}[V_j(c_j)] \\ &= \sum_{i=1}^m \int_0^1 \int_0^{v_i} \bar{p}^i_\alpha(t) g(v_i)\,dt\,dv_i + \sum_{j=1}^m \int_0^1 \int_{c_j}^1 \bar{q}^j_\alpha(t) f(c_j)\,dt\,dc_j \\ &= \sum_{i=1}^m \mathscr{E}_{\sigma_m}\left(\frac{(1-G(v_i))}{g(v_i)} p^i_\alpha(\sigma_m)\right) + \sum_{j=1}^m \mathscr{E}_{\sigma_m}\left(\frac{F(c_j)}{f(c_j)} q^j_\alpha(\sigma_m)\right).\end{aligned}$$

The third line follows from the second using exactly the same manipulations that Myerson and Satterthwaite (1983) use in the $m = 1$ case in the long equation between equations (5) and (6) in their paper. Equating (18) and (19), solving for T, and then simplifying shows that the expected surplus T equals (10), the standard formula for Sur$(\cdot)$. *Q.E.D.*

PROOF OF LEMMA 1: Consider a buyer with value v who reports v^*. Let $u_{(m)}$ denote the mth smallest α_m-virtual utility among the $2m-1$ values computed using the reports of the other traders. The selected buyer's ex post payoff is

$$(20)\qquad \begin{array}{lll} v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)}) & \text{if} & \Psi^b_{\alpha_m}(v^*) > u_{(m)}; \\ \pi(v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)})) & \text{if} & \Psi^b_{\alpha_m}(v^*) = u_{(m)}; \\ 0 & \text{if} & \Psi^b_{\alpha_m}(v^*) < u_{(m)}. \end{array}$$

In (20) π represents the probability that the selected buyer receives an item if randomization is needed to complete the allocation. The value of π depends only upon the values and costs reported by the $2m-1$ other traders. It is clear from (20) that the selected buyer maximizes his ex post payoff through his choice of v^* if he receives $v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)})$ when it is positive and zero when it is not. Regularity implies that

$$v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)}) > 0 \Leftrightarrow \Psi^b_{\alpha_m}(v) > u_{(m)}.$$

This equivalence implies that $v^* = v$ is the unique report that guarantees the selected buyer receives $v - (\Psi^b_{\alpha_m})^{-1}(u_{(m)})$ exactly when it is positive. Similar to the second-price Vickrey auction, it follows that $v^* = v$ is the selected buyer's unique dominant strategy. A similar argument establishes that honestly reporting his cost is the unique dominant strategy of every seller. Property (i) in the theorem is thus established.

Property (ii) is immediate from the definition of the market game. Turning to property (iii), a buyer with value v_i trades only if $\Psi^b_{\alpha_m}(v_i) \geq t_{(m)}$, which, by the assumption of regularity, implies that $v_i \geq (\Psi^b_{\alpha_m})^{-1}(t_{(m)})$. The price $(\Psi^b_{\alpha_m})^{-1}(t_{(m)})$ he pays when he buys is thus no more than his value v_i. A similar argument shows that the price received by a seller is at least as large as his cost, which establishes interim individual rationality. A buyer with value $v_i = 0$ or a seller with cost $c_j = 1$ has an expected payoff equal to zero because a trader with this value or cost never trades. If buyer i, for instance, has value $v_i = 0$, then his virtual utility is $\Psi^b_{\alpha_m}(0) < 0$. The α_m-virtual utility of a seller with cost $c_j = 0$ is $\Psi^b_{\alpha_m}(0) = 0$. Regularity implies that the α_m-virtual utilities of the m sellers surely exceed the α_m-virtual utility of the ith buyer, who therefore never trades. *Q.E.D.*

PROOF OF LEMMA 2: We begin by reducing $\mathrm{Sur}(\alpha, m, G^u, F^u) = 0$. Uniformity implies that

$$\Psi^b_\alpha(v_i) = (1+\alpha)v_i - \alpha \Leftrightarrow (\Psi^b_\alpha)^{-1}(t_{(m)}) = \frac{t_{(m)} + \alpha}{1+\alpha},$$

$$\Psi^s_\alpha(c_j) = (1+\alpha)c_j \Leftrightarrow (\Psi^s_\alpha)^{-1}(t_{(m+1)}) = \frac{t_{(m+1)}}{1+\alpha}.$$

Substitution of the above formulas for $(\Psi^b_\alpha)^{-1}(t_{(m)})$ and $(\Psi^s_\alpha)^{-1}(t_{(m+1)})$ into (13) implies

$$\mathrm{Sur}(\alpha, m, G^u, F^u) = \mathscr{E}\left[H(t_{(m)}, t_{(m+1)})\left(\frac{t_{(m)} + \alpha}{1+\alpha} - \frac{t_{(m+1)}}{1+\alpha}\right)\right].$$

The equation $\mathrm{Sur}(\alpha, m, G^u, F^u) = 0$ can be then solved for α:

$$\alpha = \frac{\mathscr{E}[H(t_{(m)}, t_{(m+1)})(t_{(m+1)} - t_{(m)})]}{\mathscr{E}[H(t_{(m)}, t_{(m+1)})]}.$$

The expected number of trades $H(t_{(m)}, t_{(m+1)})$ given $t_{(m)}$ and $t_{(m+1)}$ is clearly no more than m, which implies

$$\alpha \geq \frac{\mathscr{E}[H(t_{(m)}, t_{(m+1)})(t_{(m+1)} - t_{(m)})]}{m}. \tag{21}$$

The right side of (21) still depends upon α because its value affects the distributions of $t_{(m)}$ and $t_{(m+1)}$.

Starting from (21) it is sufficient to show that there exists a constant τ, such that

$$\mathscr{E}[H(t_{(m+1)}, t_{(m)})(t_{(m+1)} - t_{(m)})] \geq \tau. \tag{22}$$

Because this proof concerns the distributions of the traders' α-virtual utilities, it is helpful to note that buyers' and sellers' α-virtual utilities are independently and uniformly distributed on $[-\alpha, 1]$ and $[0, 1+\alpha]$, respectively. Trade occurs only among buyers and sellers whose α-virtual utilities are in $[0, 1]$, for an α-virtual utility of a buyer that is in $[-\alpha, 0)$ is surely below those of all sellers and the α-virtual utility of a seller in $(1, 1+\alpha]$ is surely above those of all buyers.

The left side of (22) is calculated by summing over the m^2 events distinguished by the number of α-virtual utilities from each of the two sides of the market that lie within $[0, 1]$. For $1 \leq i, j \leq m$, define $A_{i,j}$ as the event in which exactly i buyers' α-virtual utilities and j sellers' α-virtual utilities lie in $[0, 1]$. We have

$$\mathscr{E}[(t_{(m+1)} - t_{(m)})H(t_{(m+1)}, t_{(m)})] = \sum_{1 \leq i, j \leq m} \mathscr{E}[(t_{(m+1)} - t_{(m)})H(t_{(m+1)}, t_{(m)}) \mid A_{i,j}] \cdot \Pr(A_{i,j}), \tag{23}$$

where the events in which either $i = 0$ or $j = 0$ are omitted because no trades occur in these cases (i.e., $H(t_{(m+1)}, t_{(m)}) = 0$).

We next simplify three terms in (23) in the event $A_{i,j}$. First, observe that

$$\Pr(A_{i,j}) = \binom{m}{i}\binom{m}{j}\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)}. \tag{24}$$

This follows from the distributions of buyers' and sellers' α-virtual utilities: for either a buyer or a seller, $1/(1+\alpha)$ is the probability that his α-virtual utility is in $[0, 1]$ and $\alpha/(1+\alpha)$ is the probability that it is outside this interval. Second, consider $H(t_{(m+1)}, t_{(m)})$. In event $A_{i,j}$, the α-virtual utilities of exactly i buyers' and j sellers' are independently and uniformly distributed on $[0,1]$. Consequently, there are exactly $m-i$ α-virtual utilities of buyers below 0. The values $t_{(m)}$ and $t_{(m+1)}$ in the entire sample of $2m$ α-virtual utilities are thus respectively the ith and the $(i+1)$st among those within $[0,1]$. The expected number of trades $H(t_{(m+1)}, t_{(m)})$ in event $A_{i,j}$ given the values of $t_{(m)}$ and $t_{(m+1)}$ therefore equals the expected number of the i buyers' α-virtual utilities that are among the j largest in this sample of $i+j$ α-virtual utilities from the uniform distribution on $[0,1]$. In such a sample, $j/(i+j)$ is the probability that the α-virtual utility of any one of these $i+j$ traders is among the j largest. It follows that $ij/(i+j)$ is the expected number of buyers whose α-virtual utilities are among the j largest, and so

$$H(t_{(m+1)}, t_{(m)}) = \frac{ij}{i+j} \quad \text{in the event } A_{i,j}. \tag{25}$$

Third, $t_{(m+1)} - t_{(m)}$ is the difference between the ith and $(i+1)$st values in this sample of $i+j$ α-virtual utilities that are independently and uniformly distributed on $[0,1]$. It follows from David ((1981), Ex. 3.1.1, p. 35) that

$$\mathscr{E}[t_{(m+1)} - t_{(m)} \mid A_{i,j}] = \frac{1}{i+j+1}. \tag{26}$$

Substituting (24), (25), and (26) into (23) produces

$$\begin{aligned} &\mathscr{E}[(t_{(m+1)} - t_{(m)})H(t_{(m+1)}, t_{(m)})] \\ &\quad = \sum_{1\le i,j\le m} \binom{m}{i}\binom{m}{j}\left(\frac{ij}{i+j}\right)\left(\frac{1}{i+j+1}\right)\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)}. \end{aligned} \tag{27}$$

The remainder of this proof is a calculation that bounds (27). It follows from the definition of a binomial coefficient that

$$\binom{m}{i}\binom{m}{j}\left(\frac{ij}{i+j}\right)\left(\frac{1}{i+j+1}\right) = \binom{m-1}{i-1}\binom{m-1}{j-1}(m^2)\left(\frac{1}{i+j}\right)\left(\frac{1}{i+j+1}\right).$$

Recall that $1 \le i, j \le m$, which implies that

$$m^2\left(\frac{1}{i+j}\right)\left(\frac{1}{i+j+1}\right) \ge \frac{m^2}{2m(2m+1)} = \frac{1}{4+\frac{2}{m}} \ge \frac{1}{6}.$$

The expression in (27) is thus at least

$$\begin{aligned} &\frac{1}{6}\sum_{1\le i,j\le m}\binom{m-1}{i-1}\binom{m-1}{j-1}\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)} \\ &\quad = \frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2 \sum_{1\le i,j\le m}\binom{m-1}{i-1}\binom{m-1}{j-1}\left(\frac{1}{1+\alpha}\right)^{i+j-2}\left(\frac{\alpha}{1+\alpha}\right)^{2m-(i+j)}, \end{aligned}$$

which after replacing i with $i+1$ and j with $j+1$ becomes

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2 \times \sum_{1\le i,j\le m-1}\binom{m-1}{i}\binom{m-1}{j}\left(\frac{1}{1+\alpha}\right)^{i+j}\left(\frac{\alpha}{1+\alpha}\right)^{2(m-1)-(i+j)}.$$

This expression factors as

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2\left(\sum_{i=0}^{m-1}\binom{m-1}{i}\left(\frac{1}{1+\alpha}\right)^{i}\left(\frac{\alpha}{1+\alpha}\right)^{m-1-i}\right)^2.$$

Applying the binomial expansion, this equals

$$\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2\left(\left(\frac{1}{1+\alpha}+\frac{\alpha}{1+\alpha}\right)^{m-1}\right)^2=\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2.$$

We thus have

$$\mathscr{E}[(t_{(m+1)}-t_{(m)})H(t_{(m+1)},t_{(m)})]\geq\frac{1}{6}\left(\frac{1}{1+\alpha}\right)^2\geq\frac{1}{24},$$

where the last inequality holds because $\alpha\in[0,1]$. *Q.E.D.*

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