ESTIMATES OF BOUNDED RELATIVE ERROR FOR THE RATIO OF VARIANCES OF NORMAL DISTRIBUTIONS

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STATISTICAL methods for comparing the variability of two competing industrial processes are often just as important as statistical methods for comparing location parameters. For example, a manufacturer of half-inch ball bearings is not likely to be interested in a proposed modification of his manufacturing process which promises to produce bearings of larger average diameter, though he may welcome a change which can be expected to produce bearings which are substantially less variable in size.

In this paper we assume that the characteristic whose variability is in question behaves like a random variable which is normally distributed with variance σ_1^2 for the first process and variance σ_2^2 for the second. We assume further that the σ^2s are unknown but that we can obtain appropriate data on the output of each process sufficient to enable us to make an unbiased estimate $\hat{\sigma}_1^2$ of σ_1^2 and an unbiased estimate $\hat{\sigma}_2^2$ of σ_2^2 such that $n_1\hat{\sigma}_1^2/\sigma_1^2$ and $n_2\hat{\sigma}_2^2/\sigma_2^2$ are independently distributed like chi-square with n_1 and n_2 degrees of freedom respectively. We then wish to estimate the ratio σ_1^2/σ_2^2.[1]

Under these assumptions it is common practice to use $\hat{\sigma}_1^2/\hat{\sigma}_2^2$ as the desired estimate of σ_1^2/σ_2^2, and to describe the precision of the estimate by using tables of the F distribution to determine numbers h_α and H_α such that for a preassigned confidence level α we have

$$P_r\left\{\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} < h_\alpha \frac{\hat{\sigma}_1^2}{\sigma_2^2}\right\} = P\left\{\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} > H_\alpha \frac{\sigma_1^2}{\sigma_2^2}\right\} = \frac{1-\alpha}{2}\cdot$$

Experience has shown, however, that this solution to the problem is a rather awkward one from the standpoint of practical application, particularly when the estimate of σ_1^2/σ_2^2 is to be used as one of the terms in a chain of calculations. In this case the engineer will often prefer an estimate about which he can say, for example, "The chances are 19 in 20 that the ratio of this estimate to its true value will lie between 1.1 and 1/1.1".

In the terminology of Blackwell and Girshick an estimate with the last mentioned property is said to be of bounded relative error. It has been shown[2] that a minimax estimate, which is of bounded relative

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[1] E. J. G. Pitman, "Tests of hypotheses concerning location and scale parameters", *Biometrika*, 31 (1939), 200–15, for a more general statement of this problem.

[2] O. Blackwell and M. A. Girshick, *Theory of Games and Statistical Decisions*. (New York: John Wiley and Sons, 1954), pp. 316–23.

error with a confidence coefficient which is independent of the parameter, always exists if one is dealing with a distribution in which the only unknown parameter is a scale parameter. What follows is a straightforward application of the general theory given in the present case. A similar application of the theory has been given by Girshick et al., where the problem of finding an optimum estimate of bounded relative error for the variance of a single normal distribution is discussed.[3] The advantages of estimates of bounded relative error in this latter case are pointed out,[4] without, however, the development of a theory for obtaining optimum estimates of this type.

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Writing $\tau=\sigma_1^2/\sigma_2^2$, we shall consider estimates $\hat{\tau}$ of the form

$$\hat{\tau} = \beta\cdot\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{1}{b}\,\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}\, b > 0$$

where we assume that $n_1\hat{\sigma}_1^2/\sigma_2^2$ and $n_2\hat{\sigma}_2^2/\sigma_2^2$ are independently distributed like chi-square with n_1 and n_2 degrees of freedom respectively. Under these assumptions it is well known that the quantity $b\hat{\tau}/\tau$ has the F distribution with n_1 and n_2 degrees of freedom. The condition that $\hat{\tau}$ be of bounded relative error with confidence α may be expressed by the relation

$$P\left\{\frac{1}{K} \leqq \frac{\tau}{\tau} \leqq K\right\} = \alpha(K)$$

for $K>0$, where $\alpha(K)$ does not depend on τ. But this is the same as

$$P\left\{\frac{b}{K} \leqq \frac{b\hat{\tau}}{\tau} \leqq bK\right\} = \int_{b/K}^{bK} \phi(x, n_1, n_2)dx = \psi_K(b, n_1, n_2) \qquad (1.1)$$

where $\phi(x, n_1, n_2)$ is the F density with n_1 and n_2 degrees of freedom.

We observe that $\psi_K(b, n_1, n_2)$ does not depend on τ. Moreover, for fixed n_1, n_2 and K we obtain the best estimate $\hat{\tau}$ by choosing that value b^* of b which maximizes

$$\psi_K(b, n_1, n_2) = P\left\{1/K \leqq \frac{\hat{\tau}}{\tau} \leqq K\right\}.$$

Differentiating (1.1) with respect to b and putting the derivative equal to zero, we obtain the condition

$$K\phi(bK, n_1, n_2) = \frac{1}{K}\phi(b/K, n_1, n_2). \qquad (1.2)$$

[3] M. A. Girshick, H. Rubin, and R. Sitgreaves, "Estimates of bounded relative error in particle counting", *Annals of Mathematical Statistics*, 26 (1955), 276–85.

[4] J. A. Greenwood, and M. M. Sandomire, "Sample size required for estimating the standard deviation of a per cent of its true value," *Journal of the American Statistical Association*, 45 (1950), 257–60.

By making use of the explicit form

$$\phi(x, n_1, n_2) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \left(\frac{n_1}{n_1}\right)^{n_1/2} x^{(n_1/2)-1}\left(1 + \frac{n_1}{n_2}x\right)^{-(n_1+n_2)/2} \quad (1.3)$$

of the F density we then solve (1.2) for the maximizing value b^* of b, and find

$$b^* = \frac{K^{n_1/n_1+n_2} - K^{-n_1/n_1+n_2}}{K^{n_1/n_1+n_2} - K^{-n_2/n_1-n_2}} \cdot \frac{n_2}{n_1}. \quad (1.4)$$

Hence the optimum estimate $\hat{\tau}^*$ of σ_1^2/σ_2^2 is

$$\hat{\tau}^* = \frac{K^{1/1+\theta} - K^{-1/1+\theta}}{K^{\theta/1+\theta} - K^{-1/1+\theta}} \cdot \theta \cdot \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \beta^* \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} \quad (1.5)$$

where we have written θ for the ratio n_1/n_2. The associated confidence coefficient is

$$\psi_K(b^*, n_1, n_2) = P\left\{\frac{1}{K} \leqq \frac{\tau}{\tau} \leqq K\right\} = \int_{b^*/K}^{b^*K} \phi(x, n_1, n_2)dx. \quad (1.6)$$

Several interesting points may now be noted. In the first place, the derivation of (1.5) and (1.6) makes no assumptions about $\hat{\sigma}_1$ and $\hat{\sigma}_2$ other than that $n_1\hat{\sigma}_1^2/\sigma_1^2$ and $n_2\hat{\sigma}_2^2/\sigma_2^2$ be independently distributed like chi-square with n_1 and n_2 degrees of freedom. Thus the result if equally applicable to cases in which the quantities σ_1^2 and σ_2^2 are, for example, conditional variances or variances about a regression function, and where the estimates $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ need not be the conventional mean square deviations from the sample averages. Secondly, it can be seen from (1.5) that, for fixed K, the optimal value of the factor β^* which multiples $\hat{\sigma}_1^2/\hat{\sigma}_2^2$ depends only on the ratio of the sample sizes and is unity when $n_1 = n_2$. Values of β^* for selected values of K and selected values of $\theta = n_1/n_2$ are given in Table 1. Values of the confidence coefficient $\alpha = \psi_K(b^*, n_1, n_2)$, which depends in a more complicated way on K, n_1 and n_2, may be obtained from the charts in the Appendix. Finally it follows from (1.1) and (1.3) that when $n_1 = n_2$ we have

$$P\left\{\frac{\hat{\tau}^*}{\tau} < \frac{1}{K}\right\} = P\left\{\frac{\hat{\tau}^*}{\tau} > K\right\} = \frac{1-\alpha}{2}$$

so that when the sample sizes are equal the classical estimating procedure using equal tails of the F distribution is equivalent to the procedure using $\hat{\tau}^*$.

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There are at least two types of situations in which Table 1 and the charts in the appendix may prove useful. In the first situation the ex-

TABLE 1

VALUES OF β^* TO BE USED WITH THE ESTIMATE $\hat{\tau} = \beta^* \hat{\sigma}_1^2/\sigma_2^2$, FOR SELECTED VALUES OF $\theta = n_1/n_2$ AND FOR SELECTED VALUES OF THE RELATIVE ERROR BOUND K

K \ θ	1.0	1.5	2.0	5.0	10.0	100.0
1.01	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.02	1.0000	1.0000	1.0000	1.0000	.9999	.9999
1.05	1.0000	.9999	.9999	.9997	.9997	.9996
1.10	1.0000	.9997	.9995	.9990	.9988	.9985
1.25	1.0000	.9983	.9972	.9945	.9932	.9919
1.30	1.0000	.9977	.9962	.9924	.9907	.9888
1.40	1.0000	.9962	.9937	.9875	.9847	.9817
1.50	1.0000	.9946	.9909	.9820	.9779	.9736

periment has already been performed, and the statistician is asked to do the best he can with the data. In this case the statistician can enter Table 1 with the ratio $n_1/n_2 = \theta$, and for a suitable value of K determine β^* and thus obtain his best estimate $\hat{\tau}^*$. The confidence level associated with the resulting estimate can then be read from the chart in the appendix corresponding to the selected value of K by noting the position of the point (n_1, n_2) relative to the curves on the chart. In the second situation the statistician may be asked to determine, before any observations are made, appropriate values of n_1 and n_2 in such a way as to yield a prescribed level of accuracy at minimum cost. In this case the statistician should go first to the chart in the appendix corresponding to the selected value of K, and find the curve corresponding to the confidence level. If $C(n_1, n_2)$ denotes the cost of getting estimates $\hat{\sigma}_1$ and $\hat{\sigma}_2$ with n_1 and n_2 degrees of freedom respectively, he can select from the curve several pairs of values (n_1, n_2), and by substituting them in the function $C(n_1 n_2)$ determine the approximate minimum and the corresponding values of n_1 and n_2. Or he can plot several level curves $C(n_1, n_2) = \text{constant}$ to the same scale as the chart, and thereby determine a value c_o for the constant such that the curve $(C(n_1, n_2) = c_o$ is tangent to the chosen curve in the appendix. The point of tangency will then yield the desired values of n_1 and n_2.

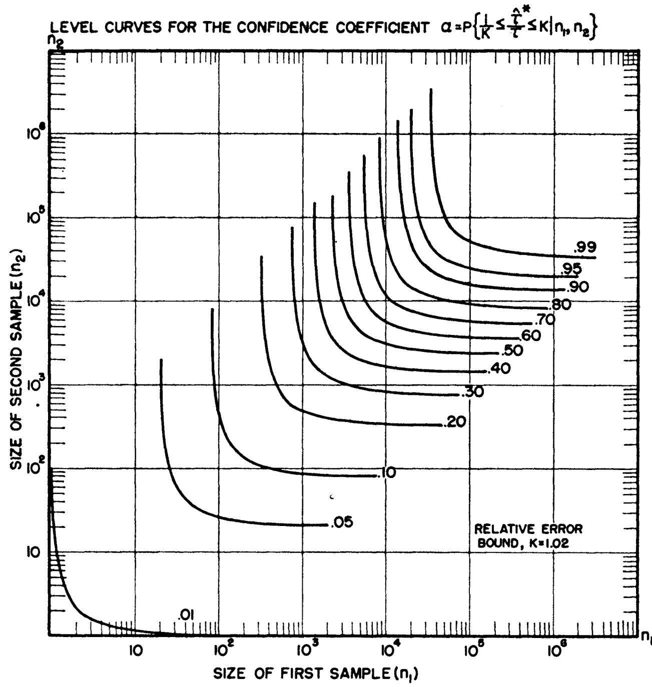

LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \le \frac{\hat{\tau}^*}{\tau} \le K \mid n_1, n_2\right\}$
n_2
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
n_1
10
10^2
10^3
10^4
10^5
10^6
.01
.05
.10
.20
.30
.40
.50
.60
.70
.80
.90
.95
.99
RELATIVE ERROR
BOUND, K=1.02

FIG. 1

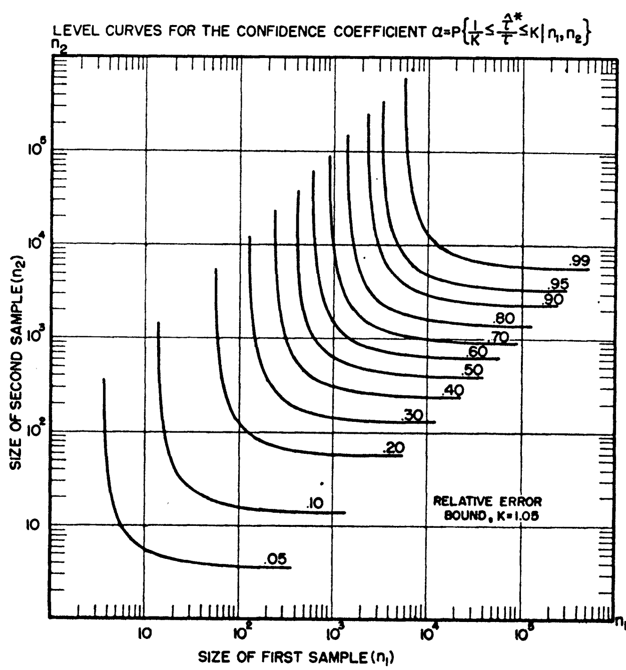

LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \le \frac{\hat{\tau}^*}{\tau} \le K \mid n_1, n_2\right\}$
n_2
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
n_1
10
10^2
10^3
10^4
10^5
.05
.10
.20
.30
.40
.50
.60
.70
.80
.90
.95
.99
RELATIVE ERROR
BOUND, K=1.05

FIG. 2

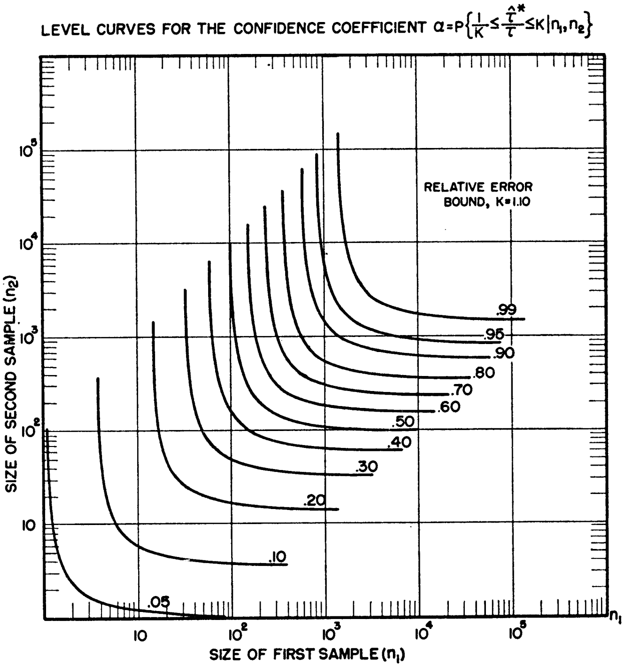

LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \leq \frac{\hat{\tau}^*}{\tau} \leq K \mid n_1, n_2\right\}$
RELATIVE ERROR BOUND, K=1.10
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
.99
.95
.90
.80
.70
.60
.50
.40
.30
.20
.10
.05

FIG. 3

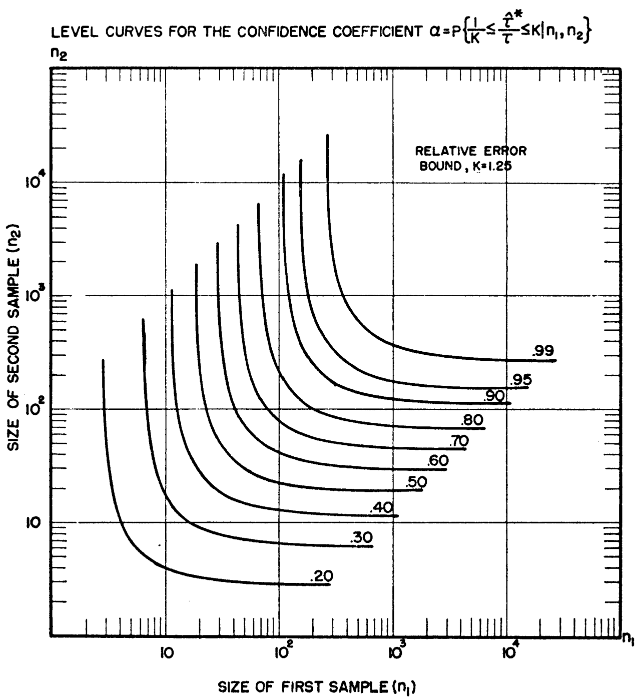

LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \leq \frac{\hat{\tau}^*}{\tau} \leq K \mid n_1, n_2\right\}$
RELATIVE ERROR BOUND, K=1.25
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
.99
.95
.90
.80
.70
.60
.50
.40
.30
.20

FIG. 4

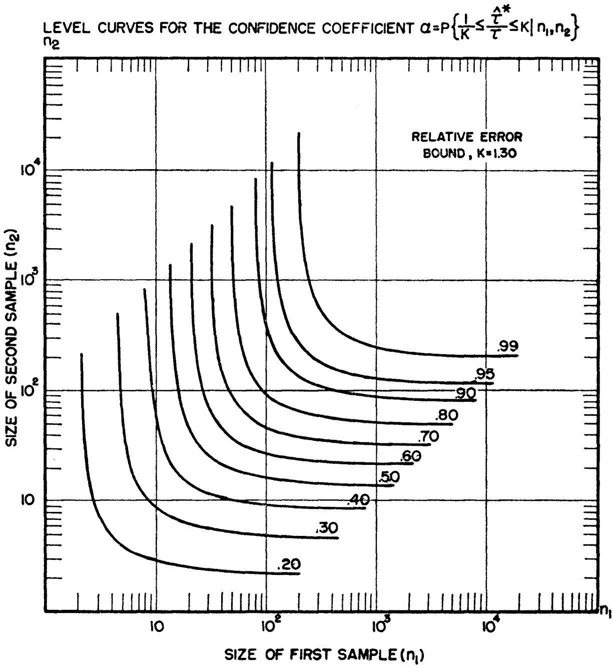

LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \le \frac{\hat{\tau}^*}{\tau} \le K \mid n_1, n_2\right\}$
n_2
RELATIVE ERROR
BOUND, K=1.30
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
n_1
10
10^2
10^3
10^4
.99
.95
.90
.80
.70
.60
.50
.40
.30
.20

FIG. 5

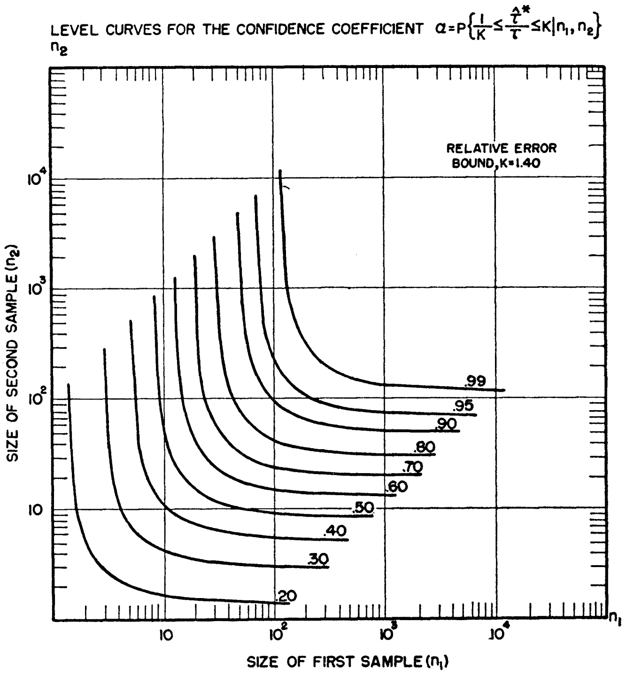
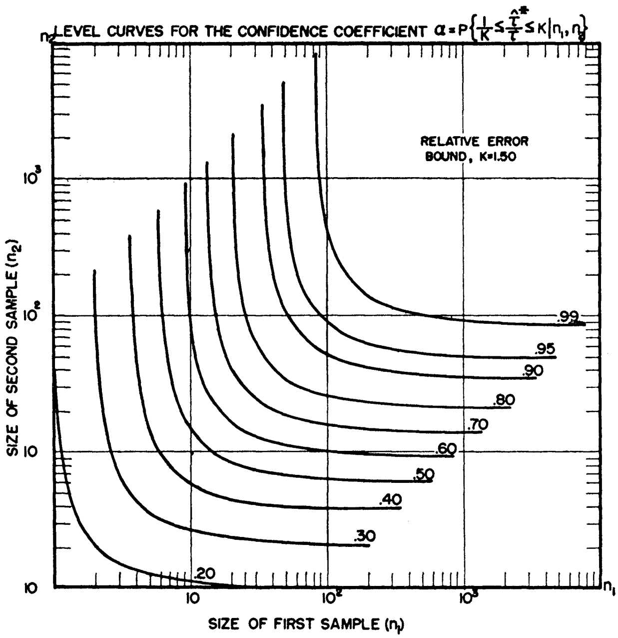

LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \le \frac{\hat{\tau}^*}{\tau} \le K \mid n_1, n_2\right\}$
n_2
RELATIVE ERROR
BOUND, K=1.40
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
n_1
10
10^2
10^3
10^4
.99
.95
.90
.80
.70
.60
.50
.40
.30
.20

FIG. 6



LEVEL CURVES FOR THE CONFIDENCE COEFFICIENT $\alpha = P\left\{\frac{1}{K} \leq \frac{\hat{\tau}^*}{\tau} \leq K \mid n_1, n_2\right\}$
RELATIVE ERROR BOUND, K=1.50
SIZE OF SECOND SAMPLE (n_2)
SIZE OF FIRST SAMPLE (n_1)
.99
.95
.90
.80
.70
.60
.50
.40
.30
.20

FIG. 7