Trade Barriers in Activity Analysis[1]

I. INTRODUCTION

Economists have for a long time been interested in the analysis of technological structures in production, i.e. modes of combining various techniques of production to form producing units, and particularly in the relationships between technological structures and economic organisation. Thus, discussion of the technological structure of firms arose from the desire to find an unambiguous definition of the supply curve for an industry under conditions of perfect competition.

The recent development by Koopmans, Dantzig, and others [1], [2], of the activity analysis model of production and allocation sets out the problem in a new form and at the same time opens a new approach to it. Welfare analysis of production and allocation has been presented, by Koopmans particularly, in terms of a " game " in which economic functions are performed by persons whose behaviour is presented by the " rules of the game." Precisely which variables are to be controlled by a single decision-maker is left vague, and, while in static models this gives rise to no trouble, it is to be expected that in dynamic circumstances the existence of some technological structures which are better than others would lead to a narrowing of the class of desirable methods of organising production.

The activity analysis model seems particularly adapted to the study of technological structures in production, because particular pieces of technology (activities) are explicit in the model, and specific decision variables (activity levels) are associated with them. While the existing formulation of the model does not permit distinctions between different groupings of the same set of activities, a minor modification which makes the model additive only over sectors rather than over the whole yields the desired result. This paper will be limited to the formulation of a model additive only over sectors (Section III), and the presentation of a simple example illustrating its application (Section IV).

II. STATIC AND LINEAR ACTIVITY ANALYSIS MODEL[2]

The linear activity analysis model of production describes production possibilities by the transformation :

$$(1.0) \qquad y = Ax \qquad x \geq 0$$

where x is a K dimensional vector whose k^{th} component denotes the level of the k^{th} activity, y is an N dimensional vector whose n^{th} component denotes the total net rate of flow of the n^{th} commodity, and A is an NxK matrix whose general element a_{nk} denotes the flow of the n^{th} commodity involved in the k^{th} activity when carried out at the unit level. As (1.0) implies, it is assumed that (1) commodity flows are perfectly divisible ; (2) commodity flows are completely additive, that is, the total net flow of the n^{th} commodity is the algebraic sum of the flows of that commodity in each activity ; and (3) each activity yields constant returns to scale.

[1] I am indebted to T. C. Koopmans, Cowles Commission for Research in Economics, and especially to L. Hurwicz, University of Minnesota, for comments and criticism. This paper is a result of research undertaken at the Cowles Commission under a contract with the RAND Corporation and subsequently completed at Stanford University with the support of the Office of Naval Research. (To be reprinted as a Cowles Commission New Series Paper.)

[2] This exposition follows that of Koopmans, [1], p. 33 ff.

Further concepts of use here are, first, the idea of partitioning the commodity vector as follows :

$$(1.1) \qquad y \equiv \begin{bmatrix} y_F \\ y_I \\ y_P \end{bmatrix}$$

where y_F is a vector whose components denote flows of final commodities (consumers' goods), y_I a vector whose components denote flows of intermediate commodities, i.e. commodities not in themselves desired, but used in the production of final commodities, and y_P is the vector whose components denote flows of primary inputs, i.e. those commodities coming into the economy from " Nature " or households.[1]

Secondly, it is required that :

$$(1.2) \qquad y_P \geqq \eta_P$$

where the components of y_P, η_P are the constants limiting the inflows of primary inputs.

Thirdly, to prevent endless accumulation of inventories of intermediate products, it is required that :

$$(1.3) \qquad 0 = y_I.$$

Corresponding to the partitioning of the commodity vector, there is a partitioning of the technology matrix A, so that :

$$(1.4) \qquad A \equiv \begin{bmatrix} A_F \\ A_I \\ A_P \end{bmatrix}$$

A point y satisfying (1.0), (1.2), (1.3) is called *attainable.* An attainable point y^0 is called ***efficient*** if there is no other attainable y with at least one final or primary commodity co-ordinate greater than and no co-ordinate less than the corresponding co-ordinates of y^0.

III. A MODEL ADDITIVE ONLY OVER SECTORS

Different *sectors* or *plants* comprising groups of activities are distinguished. Assume there are J sectors, in each of which the technology satisfies the postulates of the linear activity analysis model. The technological relations between sectors will be described by transfer activities expressing the possibility of transferring flows of specified commodities from one sector to another. To express this, the following notation is used.

The net output of the n^{th} commodity ($n = 1, \ldots, N$) in the j^{th} sector ($j = 1, \ldots, J$) will be denoted by $y_n(j)$. Correspondingly, the vector of commodity flows in the j^{th} sector is $y(j)$.

In each sector the resource limitations are expressed by constants $\eta_n(j)$ (n refers to a primary commodity) and the requirement :

$$(2.0) \qquad y_n(j) \geq \eta_n(j)$$

and similarly, for the intermediate commodities in each sector :

$$(2.1) \qquad 0 = y_I(j).$$

[1] The total net output of a primary commodity, $\bar{y}_n$, is non-positive. It will be convenient to regard flows of primary commodities as inputs, defining $\hat{y}_n = -\bar{y}_n$.

The technology matrix for the economy comprising several sectors has the form $[A^* : B]$ where :

$$(2.2) \qquad A^* \equiv \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ & & \cdot & \\ & & & \cdot \\ & & & \cdot \\ 0 & 0 & & A \end{bmatrix}$$

and B is a matrix of transfer activities expressing the technological relations between sectors. These transfer activities have the following form. If the activity which transfers commodity n from sector j to sector j' is carried out at the level $z_n(j,j') = 1$, it requires a unit flow of the n^{th} commodity in sector j and produces a unit flow of the n^{th} commodity in sector j'. The activity may also involve other inputs, reflecting costs of transfer in terms of resources. Assuming that there is at most one way of transferring a commodity between sectors, and that there are N commodities, it follows that there are $(J^2 - J)$ ordered pairs of sectors and, therefore, at most $N\ (J^2 - J)$ transfer activities. In general, the matrix B of available transfer activities will have less than $N\ (J^2 - J)$ columns, reflecting the fact that there are barriers preventing the transfer of some commodities between sectors. This matrix is assumed to be given, as is A^*.

The existence of barriers to the transfer of commodities between sectors may be expected to give effect to technical interdependences among activities and thus lead to preferred groupings of activities into plants or sectors. While these technical interdependences (expressed by relations among the technical coefficients of the various activities) are not ruled out by the postulates of the linear activity model, the postulate of complete additivity of commodity flows precludes their having any effect on the technical structure of production. Explicit introduction of barriers to transfer into the activity analysis model permits analysis of efficient groupings or structures of activities. The example in the following section illustrates this. Another interpretation of this model is possible. If the sectors are regarded as countries, then the problem as formulated is to determine efficient specialisation induced by barriers to trade in various commodities and to examine the enlargement of the set of achievable (or efficient) points resulting from the elimination of a set of barriers.

IV. TECHNOLOGICAL STRUCTURE OF PRODUCTION : AN EXAMPLE

Consider an economy having available the activities listed in Table I.

TABLE I

Activities / Commodities:	x_1	x_2	x_3	x_4	x_5
y_1	1	0	0	0	0
y_2	0	1	0	0	0
y_3	$-a_{31}$	$-a_{32}$	a_{33}	-1	0
y_4	$-a_{41}$	$-a_{42}$	a_{43}	0	-1
$-\bar{y}_5$	0	0	-1	0	0

Commodities 1 and 2 are final products ; commodities 3 and 4 are intermediate products ; commodity 5 is a primary input. Activity 1 states that flows of a_{31} units of commodity 3 and a_{41} units of commodity 4 are required to produce a unit flow of commodity 1. Activities 2 and 3 are read in similar fashion. Activities 4 and 5 are disposal activities and reflect the assumption that disposal of intermediate commodity flows is costless. I shall assume for definiteness that :

$$\frac{a_{31}}{a_{41}} > \frac{a_{33}}{a_{43}} > \frac{a_{32}}{a_{42}} . \tag{3.0}$$

Suppose that there are two sectors (plants) in this economy, $j = 1, 2$, and that costless transfer of final and primary commodities is possible between them, but that transfer of intermediate commodities is not possible. This distinction between sectors can then be ignored in so far as final and primary commodities are concerned. In particular, both sectors can be regarded as drawing on a common pool of the primary input, the inflow of which is assumed to be limited by a constant, η_5.

I shall compare two structures of production in this economy. One alternative is to have each final commodity produced in its own plant, the plants being separated by the barrier preventing flows of intermediate products between them. Thus, in Plant 1, activities 1 and 3 (and disposal activities, if necessary) will be carried out, while activities 2 and 3 comprise the structure of Plant 2.[1] The other possibility will be to combine production of both final commodities in a single plant comprising the five activities of Table I.

The technological structure of this economy in the case of separated production is given in Table II.

TABLE II[2]

Commodities	Plant 1 $x_1^{(1)}$	$x_3^{(1)}$	$x_4^{(1)}$	$x_5^{(1)}$	Plant 2 $x_2^{(2)}$	$x_3^{(2)}$	$x_4^{(2)}$	$x_5^{(2)}$
y_1	1	0	0	0	0	0	0	0
y_2	0	0	0	0	1	0	0	0
$y_3^{(1)}$	$-a_{31}$	a_{33}	-1	0	0	0	0	0
$y_4^{(1)}$	$-a_{41}$	a_{43}	0	-1	0	0	0	0
$y_3^{(2)}$	0	0	0	0	$-a_{32}$	a_{33}	-1	0
$y_4^{(2)}$	0	0	0	0	$-a_{42}$	a_{43}	0	-1
y_5	0	-1	0	0	0	-1	0	0

[1] This does not really violate the assumption that both sectors have the same technology, for if activity 1 required an additional input available in Sector 1, but not in Sector 2, and similarly activity 2 required another input available in Sector 2, but not in Sector 1, with no transfer permitted for these inputs, the formal requirements would be satisfied without altering the example as given.

[2] Because there is little danger of ambiguity, I omit the superscripts j from the elements of the technology matrix.

Applying conditions (2.0) and (2.1) and defining $y_5 = -\bar{y}_5$, and $-\bar{\eta}_5 = \eta_5$, the net output equations for this structure are :

(3.1) $y_1 = x_1^{(1)}$

(3.2) $y_2 = x_2^{(2)}$

(3.3) $0 = -a_{31}\, x_1^{(1)} + a_{33}\, x_3^{(1)} - x_4^{(1)}$

(3.4) $0 = -a_{41}\, x_1^{(1)} + a_{43}\, x_3^{(1)} - x_5^{(1)}$

(3.5) $0 = -a_{42}\, x_2^{(2)} + a_{33}\, x_3^{(2)} - x_4^{(2)}$

(3.6) $0 = -a_{42}\, x_2^{(2)} + a_{43}\, x_3^{(2)} - x_5^{(2)}$

(3.7) $-\bar{\eta}_5 \geq -\bar{y}_5 = -x_3^{(1)} - x_3^{(2)}$

On elimination of $x_3^{(1)}$ in (3.3)–(3.7), and by applying the definition of efficiency, the requirement $x_k > 0$ and assumption (3.0) we find :

(3.8) $0 = x_4^{(1)} = a_{31}\, x_1^{(1)} - a_{33}\, (\bar{\eta}_5 - x_3^{(2)})$

(3.9) $x_5^{(1)} = a_{41}\, x_1^{(1)} - a_{43}\, \bar{\eta}_5 - x_3^{(2)})$

(4.0) $x_4^{(2)} = a_{32}\, x_2^{(2)} - a_{33}\, x_3^{(2)}$

(4.1) $0 = x_5^{(2)} = a_{43}\, x_2^{(2)} - a_{43}\, x_3^{(2)}$

Any point satisfying (3.8) and (4.1) will also satisfy (3.9) and (4.0). Using (3.8) and (4.1) and substituting for $x_1^{(1)}$ and $x_2^{(2)}$ from (3.1) and (3.2) gives (4.2), the equation of the efficient point set in the commodity space. This is :

(4.2) $$y_2 = \frac{a_{43}}{a_{42}} \left[\bar{\eta}_5 - \frac{a_{31}}{a_{33}} y_1 \right]$$

The difference between this structure and the alternative is that in the alternative structure the barrier preventing transfer of intermediate commodities is removed. Thus the technology matrix for this structure (Table III) is obtained by joining to the activities of Table II transfer activities whose levels are $F_n^{(j,j')}$ for the intermediate products.

TABLE III

	Activities											
Commodities :	x_1	x_3	x_4	x_5	x_2	x_3	x_4	x_5	$z_3^{(1,2)}$	$z_4^{(1,2)}$	$z_3^{(2,1)}$	$z_4^{(2,1)}$
y_1 ..	1	0	0	0	0	0	0	0	0	0	0	0
y_2 ..	0	0	0	0	1	0	0	0	0	0	0	0
$y_3^{(1)}$ ..	$-a_{31}$	a_{33}	-1	0	0	0	0	0	-1	0	1	0
$y_4^{(1)}$ ..	$-a_{41}$	a_{43}	0	-1	0	0	0	0	0	-1	0	1
$y_3^{(2)}$ ..	0	0	0	0	$-a_{32}$	a_{33}	-1	0	1	0	-1	0
$y_4^{(2)}$ ..	0	0	0	0	$-a_{42}$	a_{43}	0	-1	0	1	0	-1
y_5 ..	0	-1	0	0	0	-1	0	0	0	0	0	0

In the presence of these transfer activities the distinction between the two sectors (plants) vanishes. Performing the obvious reduction of the technology matrix of Table III, Table I is obtained.[1] This is the technology matrix of the structure in which all activities are carried out in one sector or plant. The superscript j may now be dropped from all symbols.

The net output equations obtained from Table I, assuming $-\bar{y}_5 \leq \bar{\eta}_5$, are :

(4.3) $y_1 = x_1$

(4.4) $y_2 = x_2$

(4.5) $0 = -a_{31}x_1 - a_{32}x_2 + a_{33}x_3 - x_4$

(4.6) $0 = -a_{41}x_1 - a_{42}x_2 + a_{43}x_3 - x_5$

(4.7) $-\bar{\eta} \geq -x_3$

or :

(4.8) $-a_{31}x_1 - a_{33}x_2 + a_{33}\bar{\eta}_5 - x_4 = 0$

(4.9) $-a_{41}x_1 - a_{42}x_2 + a_{43}\bar{\eta}_5 - x_1 = 0.$

Assumption (3.0) ensures that (4.8) and (4.9) have a solution (x_1^0, x_2^0) in the positive quadrant with $x_4 = x_5 = 0$. Efficiency requires that not both x_4 and x_5 be positive. Therefore, for $x_1 \geq x_1^0$, equation (4.8) is the equation of the efficient point set ; for $x_1 < x_1^0$, equation (4.9) is the equation of the efficient point set. Substitution in (4.8) and (4.9) from (4.3) and (4.4) gives the equations for the efficient point set in the space of final commodities.

In Fig. 1 the efficient point sets of the two structures are compared graphically.

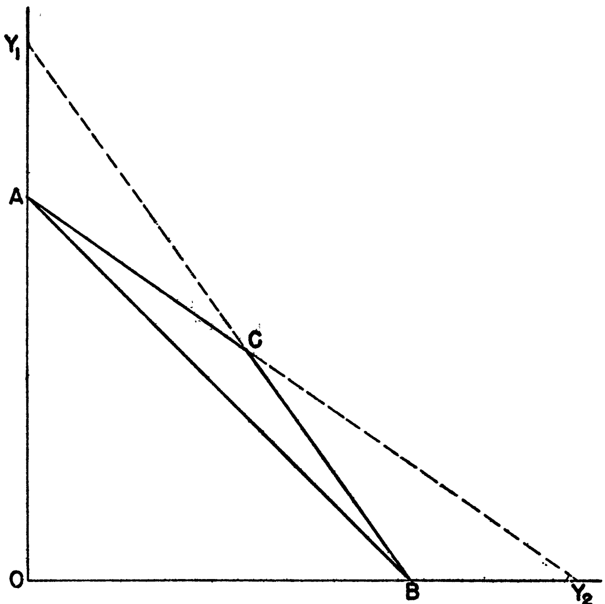

Figure I.

The line $\overline{AB}$ in Fig. 1 is the efficient point set in the case of separate production, while the broken line $\overline{ACB}$ is the efficient point set in the structure which combines production of the two final commodities. Obviously the second structure is superior to the

[1] For the concept " reduction," see [1], p. 57.

first. This is not surprising. A structure of production which permits excess flows of intermediate products from one activity to be used in another, rather than be wasted, is obviously more efficient than one which does not.

The existence of a barrier to transfer is not alone sufficient to create differences in the efficiency of alternative structures of activities. There must also be present sufficient " complexity " in the technology. More formally, the conditions created by barriers to transfer may, as far as this example is concerned, be treated as follows. Ignoring disposal activities and distinctions between sectors, the technology matrix A is of order NxK, where N is the dimension of the commodity space, and K the dimension of the activity space. If A_I is the matrix consisting of the rows of A which map x into intermediate products, we require :

$$(5.0) \qquad 0 = A_I x \qquad x \geqq 0.$$

There will be $r = K - \rho_I$ linearly independent solutions x satisfying (5.0), where ρ_I is the rank of A_I and is at most equal to the dimension of the space of intermediate products. (The number r is obviously unaffected by the restriction $x \geq 0$.) Thus, if the number of linearly independent rows of A_I is increased relative to K, the rank, ρ_I, will be in general increased relative to K and the number of linearly independent vectors x satisfying (5.0) will be decreased. If ρ_I is too large relative to K, disposal activities must be joined to A_I to permit positive levels of activity to be found without violating (5.0).

The distinction between sectors, and the introduction of a barrier, in the example just analysed have the effect of making ρ_I large relative to K ; conversely, the introduction of transfer activities for the intermediate products decrease ρ_I relative to K, consequently enlarging the set of attainable points.

It should be noted that the enlargement of the achievable point set is accomplished by introducing new facets in the boundary. This phenomenon is typical of the general case, and its analysis is the main mathematical problem which confronts a theoretical treatment of technological structures in models additive only over sectors.

Stanford, California. S. REITER.

REFERENCES

[1] Koopmans, Tjalning C., " Analysis of Production as an Efficient Combination of Activities," *Activity Analysis of Production and Allocation*, ed. T. C. Koopmans, Cowles Commission Monograph No. 13. New York : John Wiley & Sons, Inc., 1951.

[2] Dantzig, George B., " Programming of Interdependent Activities : II. Mathematical Model," op. cit.