efficient allocation. Analysis of efficiency in the context of resource allocation has been a central concern of economic theory from ancient times, and is an essential element of modern microeconomic theory. The ends of economic action are seen to be the satisfaction of human wants through the provision of goods and services. These are supplied by production and exchange and limited by scarcity of resources and technology. In this context efficiency means going as far as possible in the satisfaction of wants within resource and technological constraints. This is expressed by the concept of Pareto optimality, which can be stated informally as follows: a state of affairs is Pareto optimal if it is within the given constraints and it is not the case that everyone can be made better off in his own view by changing to another state of affairs that satisfies the applicable constraints.

Because knowledge about wants, resources and technology is dispersed, efficient outcomes can be achieved only by coordination of economic activity. Hayek (1945) pointed out the role of knowledge or information, particularly in the context of prices and markets, in coordinating economic activity. Acquiring, processing and transmitting information are costly activities themselves subject to constraints imposed by technological and resource limitations. Hayek pointed out that the institutions of markets and prices function to communicate information dispersed among economic agents so as to bring about coordinated economic action. He also drew attention to motivational properties of those institutions, or incentives. In this context, the concept of efficiency takes account of the organizational constraints on information processing and transmission in addition to those on production of ordinary goods and services. The magnitude of resources devoted to business or governmental bureaucracies, and to some of the functions performed by industrial salesmen, attests to the importance of these constraints. Economic analysis of efficient allocation has formally imposed only the constraints on production and exchange, and until recently recognized organizational constraints only in an informal way. But it is these constraints that motivate the pervasive and enduring interest in decentralized modes of economic organization, particularly the competitive mechanism.

It is necessary to limit the scope of this essay so that it is not coextensive with microeconomic theory. The main limitation imposed here is to confine attention to models in which either the role of information is ignored, or in which agents do not behave strategically on the basis of private information. In so doing, a large and important class of models involving problems of efficient allocation in the presence of incentive constraints is excluded.

The main ideas of efficient resource allocation are present in their simplest form in the linear activity analysis model of production. We begin with that model.

EFFICIENCY OF PRODUCTION: LINEAR ACTIVITY ANALYSIS

The analysis of production can to some extent be separated from that of other economic activity. The concept of efficiency appropriate to this analysis descends from that of Pareto optimality, which refers to both productive and allocative efficiency in the full economy in which production is embedded. It is useful to begin with a model in which technological possibilities afford constant returns to scale, that is, with the (linear) activity analysis model of production pioneered by Koopmans (1951a, 1951b, 1957), and closely related to the development of linear programming associated with Dantzig (1951a, 1951b) and independently with the Russian mathematician Kantorovitch (1939, 1942) and Kantorovitch and Gavurin (1949).

The two primitive concepts of the model are *commodity* and *activity*. A list of n commodities is postulated; a commodity *bundle* is given by specifying a sequence of n numbers $a_1, a_2, \ldots, a_n$. Technological possibilities are thought of as knowledge of how to transform commodities. Such knowledge may be described in terms of collections of activities called *processes*, much as knowledge of how to prepare food is described by recipes. A recipe commonly has two parts, a list of ingredients or inputs and of the output(s) of the recipe, and a description of how the ingredients are to be combined to produce the output(s). In the activity analysis model the description of productive activity is suppressed. Only the specification of inputs and outputs is retained; this defines the production process.

Commodities are classified into 'desired', 'primary' and 'intermediate' commodities. Desired commodities are those whose consumption or availability is the recognized goal of production; they satisfy wants. Primary commodities are those available from nature. (A primary commodity that is also desired is listed separately among the desired commodities and must be transformed by an act of production into its desired form.) Intermediate commodities are those that merely pass from one stage of production to another. Each commodity can exist in any non-negative amount (*divisibility*). Addition and subtraction of the numbers measuring the amount of a commodity represent joining and separating corresponding amounts of the commodity.

An activity is characterized by a *net output number* for each commodity, which is positive if the commodity is a net output, negative if it is a net input and zero if it is neither. The term *input–output vector* is also used for this ordered array of numbers. Activity analysis postulates a finite number of basic activities from which all technologically possible activities can be generated by suitable combination. Allowable combinations are as follows. If two activities are known to be possible, then the activity given by their algebraic sum is also possible, i.e. if $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, then $a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ is also possible. Thus, additivity embodies an assumption of non-interaction between productive activities, at least at the level of knowledge. Furthermore, if an activity is possible, then so is every non-negative multiple of it (*proportionality*), i.e. if $a = (a_1, a_2, \ldots, a_n)$ is possible, then so is $\mu a = (\mu a_1, \mu a_2, \ldots, \mu a_n)$ for any non-negative real number μ. This expresses the assumption of constant returns to scale. The family of activities consisting of all non-negative multiples of a given one forms a process. Since there is a finite number of basic activities, there is also a finite number of basic processes, each intended to describe a basic method of production capable of being carried out at different levels, or intensities.

The assumptions of additivity and proportionality determine a linear model of technology that can be given the following form. Let A be an n by k matrix whose jth column is the input–output vector representing the basic activity that defines the jth basic process, and let $x = (x_1, x_2, \ldots, x_n)$ be the vector whose jth component x_j is the scale (level or intensity) of the jth basic process. Let $y = (y_1, y_2, \ldots, y_n)$ be the vector of commodities. Technology is represented by a linear transformation mapping the space of activity levels into the commodity space, i.e.

$$y = Ax \qquad x \geqslant 0.$$

With the properties assumed, a process can be represented geometrically in the commodity space by a halfline from the origin including all non-negative multiples of some activity in that process. The finite number of halflines representing basic processes generate a convex polyhedral cone consisting of all activities that can be expressed as sums of activities in the basic processes, or equivalently, as non-negative linear combinations of the basic activities, sometimes called a *bundle of basic activities*. This cone is called the *production set*, or set of *possible productions*.

Two other assumptions are made about the production set itself, rather than just the individual activities. First, there is no activity, whether basic or derived, in the production set with a positive net output of some commodity and non-negative net outputs of all commodities. This excludes the possibility of producing something from nothing, whether directly or indirectly. Second, it is assumed that the production set contains at least one activity with a positive net output of some commodity.

If the availability of primary commodities is subject to a bound, the technologically possible productions described by the production set are subject to another restriction; only those possible productions that do not require primary inputs in amounts exceeding the given bounds can be produced. Furthermore, because intermediate commodities are not desired in themselves, their net output is required to be zero. (Strictly speaking, the technological constraint on intermediate commodities is that their net output be non-negative. The requirement that they be zero can be viewed as one of elementary efficiency, excluding accumulation or necessity to dispose of unwanted goods.) With these restrictions the model can be written

$$y = Ax, \ x \geqslant 0, \ y_i = 0$$

if i is an intermediate commodity, and

$$y_i \geqslant r_i \quad \text{if } i \text{ is a primary commodity,}$$

where r_i is the (non-positive) limit on the availability of primary commodity i. This leads to the concept of an *attainable* activity.

A bundle of basic activities is *attainable* if the resulting net outputs are non-negative for all desired commodities, zero for intermediate commodities and non-positive for primary commodities, and if the total inputs of primary commodities do not exceed (in absolute amount) the prescribed bounds of availability of those commodities. The set of activities satisfying these conditions is a truncated convex polyhedral cone in the commodity space called the *set of attainable productions*.

The concept of productive efficiency in this model is as follows. An activity (a bundle of basic activities) is *efficient* if it is attainable and if every activity that provides more of some desired commodity and no less of any other is not attainable.

This concept can be seen to be a specialization of Pareto optimality. If for each desired commodity there is at least one consumer who is not satiated in that commodity, at least in the range of production attainable within the given resource limitations, then increasing the amount of any desired commodity without decreasing any other can improve the state of some non-satiated consumer without worsening that of any other.

CHARACTERIZING EFFICIENT PRODUCTION IN TERMS OF PRICES

Efficient production can be characterized in terms of *implicit prices*, also called *shadow prices*, or in the context of linear programming, *dual variables*. Efficient activities are precisely those that maximize profit for suitably chosen prices. The profit returned by a process carried out at the level x is

$$x \sum p_i a_i,$$

where the prices are $p = (p_1, \ldots, p_n)$, and $a = (a_1, \ldots, a_n)$ is the basic activity defining the process; the profit on the bundle of activities Ax at prices p is given by the inner product $py = pAx$.

This characterization is the economic expression of an important mathematical fact about convex sets in n-dimensional Euclidean space, namely that through every point of the space not interior to the convex set in question there passes a hyperplane that contains the set in one of its two halfspaces (Fenchel, 1950; Nikaido, 1969, 1970). (A hyperplane in n dimensional space is a level set of a linear function of n variables, and thus is a translate of an $n - 1$ dimensional linear subspace. A hyperplane is given by an equation of the form $c_1x_1 + c_2x_2 + \cdots + c_nx_n = k$, where the x's are variables, the c's are coefficients defining the linear function and k is a constant identifying the level set. A hyperplane divides the space into two halfspaces corresponding to the two inequalities $c_1x_1 + c_2x_2 + \cdots + c_nx_n \gtrless k$ respectively.) It can also be seen that a point of a convex set is a boundary point if and only if it maximizes a linear function on the (closure of the) set. These facts can be used to characterize efficient production because the attainable production set is convex and efficient activities are boundary points of it. Because the efficient points are those, roughly speaking, on the 'north-east' frontier of the set, the linear functions associated with them have non-negative coefficients, interpreted as prices. On the other hand, if a point of the attainable set maximizes a linear function with strictly positive coefficients (prices), then it is on the 'north-east' frontier of the set.

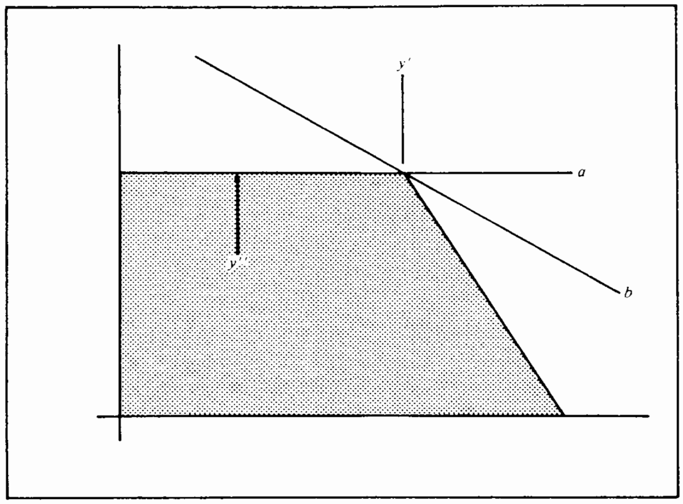

Figure 1

In Figure 1 the set enclosed by the broken line and the axes is the projection of the attainable set on the output coordinates; inputs are not shown. The point y' in the figure is efficient; the point y'' is not; both y' and y'' maximize a linear function with non-negative coefficients (the level set containing y' is labelled a and also contains y''). However, y' maximizes a linear function with positive coefficients (one such, whose level set through y' is labelled b, is shown), while y'' does not.

These implicit, or efficiency prices arise from the logic of efficiency or maximization when the relevant sets are convex, not from any institutions such as markets or exchange. An important reason for interest in them is the possibility of achieving efficient performance by decentralized methods. As described above, under the assumptions of additivity and constant returns to scale the production set can be seen to be generated by a finite number of basic processes, each of which consists of the activities that are non-negative multiples of a basic activity, the multiple being the scale (level, or intensity) at which the process is operated. Following the presentation of Koopmans (1957), each basic process is controlled by a manager, who decides on its level. The manager of a process is assumed to know only the input–output coefficients of his process. Each primary resource is in the charge of a resource holder, who knows the limit of its availability. Efficiency prices are used to guide the choices of managers and resource holders. (Under constant returns to scale, if an activity yields positive profit at a given system of prices, then increasing the scale of the process containing that activity increases the profit. Since the scale can be increased without bound, if the profitability of a process is not zero or negative, then, in the eyes of its manager, who does not know the aggregate resource constraints, it can be made infinite. Therefore, the systems of prices that can be considered for the role of efficiency prices must be restricted to those *compatible with the given* *technology*, namely prices such that no process is profitable and at least one process breaks even.) Two propositions characterize efficient production by prices and provide the basis for an interpretation in terms of decentralized control of production.

> In a given linear activity analysis model, if there is a given system of prices compatible with the technology, in which the prices of all desired commodities are positive, then any attainable bundle of basic activities selected only from processes that break even and which utilizes all positively priced primary commodities to the limit of their availability and does not use negative priced primary commodities at all, is an efficient bundle of activities.

> In a given linear activity analysis model, each efficient bundle of activities has associated with it at least one system of prices compatible with the technology such that every activity in that bundle breaks even and such that prices of desired commodities are positive, and the price of a primary commodity is non-negative, zero or non-positive, according as its available supply is full, partly, or not used at all (Koopmans, 1957).

These propositions are stated in a static form. There is no reference to managers raising or lowering the levels of the processes they control, or to resource holders adjusting prices. A dynamic counterpart of these propositions would be of interest, but because of the linearity of the model such dynamic adjustments are unstable (Samuelson, 1949).

It should also be noted that the concept of decentralization is not explicitly defined in this literature; the interpretation is by analogy with the competitive mechanism. Nevertheless, the interest in characterizing efficiency by prices and their interpretation in terms of decentralization is an important theme in the study of efficient resource allocation.

The linear activity analysis model has been generalized in several directions. These include dropping the assumption of proportionality, dropping the restriction to a finite number of basic activities, dropping the restriction to a finite number of commodities and dropping the restriction to a finite number of agents. Perhaps the most directly related generalization is to the nonlinear activity analysis, or nonlinear programming, model.

EFFICIENCY OF PRODUCTION: NONLINEAR PROGRAMMING

In the nonlinear programming model there is, as in the linear model, a finite number of basic processes. Their levels are represented by a vector $x = (x_1, x_2, \ldots, x_k)$, where k is the number of basic processes. Technology is represented by a nonlinear transformation from the space of process levels to the commodity space (still assumed to be finite dimensional), written

$$y = F(x), x \geqslant 0.$$

The production set in this model is the image in the commodity space of the non-negative orthant of the space of process levels. Under the assumptions usually made about F, the production set is convex, though, of course, not a polyhedral cone.

In this model as in the linear activity analysis model a central result is the characterization of efficient production in terms of prices. The simplest case to begin with is that of one desired commodity, say, one output, with perhaps several inputs. In this case the (vector-valued) function F can be written

$$F(x) = [f(x), g_1(x), g_2(x), \ldots, g_m(x)],$$

where the value of f is the output, and $g_1, \ldots, g_m$ correspond to the various inputs. Resource constraints are expressed by the conditions

$$g_j(x) \geqslant 0, \quad \text{for} \quad j = 1, 2, \ldots, m,$$

and non-negativity of process levels by the condition, $x \geqslant 0$. (Here the resource constraints $r_j \leqslant h_j(x) \leqslant 0$ are written more compactly as $h_j(x) - r_j = g_j(x) \geqslant 0$.)

In this model the definition of efficient production given in the linear model amounts to maximizing the value of f subject to the resource and non-negativity constraints just mentioned.

Problems of constrained maximization are intimately related to saddle-point problems. Let L be a real valued function defined on the set $X \times Y$ in R^n. A point (x^*, y^*) in $X \times Y$ is a *saddle point* of L if

$$L(x, y^*) \leqslant L(x^*, y^*) \leqslant L(x^*, y), \quad \text{for all } x \text{ in } X \text{ and all } y \text{ in } Y.$$

The concept of a concave function is also needed. A real valued function f defined on a convex set X in R^n is a *concave function* if for all x and y in X and all real numbers $0 \leqslant a \leqslant 1$

$$f(ax + (1 - a)y) \geqslant af(x) + (1 - a)f(y).$$

The following mathematical theorem is fundamental.

Theorem (Kuhn and Tucker, 1951; Uzawa, 1958): Let f and g_1, $g_2, \ldots, g_m$ be real valued concave functions defined on a convex set X in R^n. If f achieves a maximum on X subject to $g_j(x) \geqslant 0$, $j = 1, 2, \ldots, m$ at the point x^* in X, then there exist non-negative numbers $p_0^*, p_0^*, \ldots, p_m^*$, not all zero, such that $p_0^* f(x) + p^* g(x) \leqslant p_0^* f(x^*)$ for all x in X, and furthermore, $p^* g(x^*) = 0$. (Here the vectors $p^* = (p_1^*, p_2^*, \ldots, p_m^*)$, and $g(x) = [g_1(x), g_2(x), \ldots, g_m(x)]$.) The vector p^* may be chosen so that

$$\sum_0^m p_j^* = 1.$$

An additional condition (Slater, 1950) is important. (It ensures that the coefficient p_0 of f is not zero.)

Slater's Condition: There is a point x' in X at which $g_j(x') > 0$ for all $j = 1, 2, \ldots, m$.

If attention is restricted to concave functions, as in the Kuhn–Tucker, Uzawa Theorem, the relation between constrained maxima and saddle points can be summarized in the following theorem.

Theorem: If f and g_j, $j = 1, 2, \ldots, m$ are concave functions defined on a convex subset X in R^n, and if Slater's Condition is satisfied, then x^* in X maximizes f subject to $g_j(x) \geqslant 0$, $j = 1, 2, \ldots, m$, if and only if there exists $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$, $\lambda_j^* \geqslant 0$ for $j = 1, 2, \ldots, m$, such that (x^*, λ^*) is a saddle point of $L(x, \lambda) = f(x) + \lambda g(x)$ on $X \times R_+^n$.

This theorem is easily seen to cover the case where some constraints are equalities, as in the case of intermediate commodities. The sufficiency half of this theorem holds for functions that are not concave.

The auxiliary variables $\lambda_1, \lambda_2, \ldots, \lambda_m$, called *Lagrange multipliers*, play the role of efficiency prices, or shadow prices; they evaluate the resources constrained by the condition $g(x) \geqslant 0$. The maximum characterized by the theorem is a global one, as in the case of linear activity analysis.

If the functions involved are differentiable, a saddle point of the Lagrangian can be studied in terms of first-order conditions. The first-order conditions are necessary conditions for a saddle point of L. If the functions f and the g's are concave on a convex set X, then the first-order conditions at a point

(x^*, λ^*) are also sufficient; that is, they imply that (x^*, λ^*) is a saddle point of L. Thus,

Theorem: If $f, g_1, g_2, \ldots, g_m$ are concave and differentiable on an open convex set X in R^n, and if Slater's Condition is satisfied, then x^* maximizes f subject to $g_j(x) \geqslant 0$ for $j = 1, 2, \ldots, m$ if and only if there exists numbers $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ such that the first-order conditions for a saddle point of $L(x, \lambda) = f(x) + \lambda g(x)$ are satisfied at (x^*, λ^*).

If there are non-negativity conditions on the x's,

$$g_j(x) \geqslant 0, \quad x \geqslant 0, \quad x \text{ in } R^n$$

and the first-order conditions can be written

$$f_x^* + \lambda^* g_x^* \leqslant 0, (f_x^* + \lambda^* g_x^*)x^* = 0,$$
$$\lambda^* g(x^*) = 0 g(x^*) \geqslant 0, g(x^*) \geqslant 0,$$
$$\lambda^* \geqslant 0 \quad \text{and} \quad \lambda^* g(x^*) = 0,$$

where f_x^* denotes the derivative of f evaluated at x^*. In more explicit notation, the conditions $f_x^* + \lambda^* g_x^* = 0$ can be written as

$$\partial f/\partial x_i + \sum_{j=1}^{m} \lambda_j^* \partial g_j/\partial x_i = 0, \qquad i = 1, 2, \ldots, n$$

When the assumption of concavity is dropped, it is no longer possible to ensure that local maximum is also a global one. However, it is still possible to analyse local constrained maxima in terms of local saddle-point conditions. In this case a condition is needed to ensure that the first-order conditions for a saddle point are indeed necessary conditions. The Kuhn–Tucker Constraint Qualification is such a condition. Arrow, Hurwicz and Uzawa (1961) have found a number of conditions, more useful in application to economic models, that imply the Constraint Qualification.

The case of more than one desired commodity leads to what is called the *vector maximum problem*, Kuhn and Tucker (1951). This may be defined as follows. Let $f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_m$ be real valued functions defined on a set X in R^n. We say x^* in X achieves a (global) *vector maximum* of $f = (f_1, f_2, \ldots, f_k)$ subject to $g_j(x) \geqslant 0$, $j = 1, 2, \ldots, m$ if,

(I) $g_j(x^*) \geqslant 0$, $j = 1, 2, \ldots, m$,

(II) there does not exist x' in X satisfying $f_i(x') \geqslant f_i(x^*)$ for $i = 1, 2, \ldots, k$ with $f_i(x') > f_i(x^*)$ for some value of i, and $g_j(x') \geqslant 0$ for $j = 1, 2, \ldots, m$.

This is just the concept of an efficient point expressed in the present notation.

A vector maximum has a saddle-point characterization similar to that for a scalar valued function.

Theorem: Let $f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_m$ be real valued concave functions defined on a convex X set in R^n. Suppose there is x^0 in X such that $g_j(x^0) > 0$, $j = 1, 2, \ldots, m$ (Slater's Condition). If x^* achieves a vector maximum of f subject to $g(x) \geqslant 0$ then there exist $a = (a_1, a_2, \ldots, a_k)$ and $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ with $a_j \geqslant 0$ for all j, $a \neq 0$ and $\lambda \geqslant 0$ such that (x^*, λ^*) is a saddle point of the Lagrangean $L(x, \lambda) = af(x) + \lambda g(x)$.

Several different 'converses', to this theorem are known. One states that if x^* maximizes $L(x, \lambda^*)$ for some strictly positive vector a and non-negative λ^*, and if $\lambda^* g(x^*) = 0$ and $g(x^*) \geqslant 0$, then x^* gives a vector maximum of f subject to $g(x) \geqslant 0$, and x in X. Another, parallel to the result for the case of one desired commodity, is the following.

Theorem: Let f and g be functions as in the theorem above. If there are positive real numbers $a_1, a_2, \ldots, a_k$ and if (z^*, λ^*) is a saddle point of the Lagrangean L (defined as above) then (I) x^* achieves a maximum of f subject to $g(x) \geqslant 0$ on X, and (II) $\lambda^* g(x^*) = 0$.

The positive numbers $a_1, \ldots, a_k$ are interpreted as prices of desired commodities, and the non-negative numbers λ_j^* are prices of the remaining commodities. The condition $\lambda^* g(x^*) = 0$ which arises in these theorems states that the value of unused resources at the efficiency prices λ^* is zero; that is, resources not fully utilized at a vector maximum have a zero price.

The connection between vector maxima and Pareto optima is as follows. Because a vector maximum is an efficient point (for the vectorial ordering of the commodity space), it is a Pareto optimum for appropriately specified (non-satiated) utility functions, as was already pointed out in the case of the linear activity analysis model. Furthermore, if the functions $f_1, \ldots, f_k$ are themselves utility functions, and the variable x denotes allocations, with the constraints g defining feasibility, then a vector maximum of f subject to the constraints $g(x) \geqslant 0$ and x in X is a Pareto optimum, and vice versa. Hence the saddle-point theorems give a characterization of Pareto optima by prices. The interpretation of prices in terms of decentralized resource allocation described in the linear activity analysis model also applies in this nonlinear model. The proofs of these theorems reveal an important logical role played by the principle of marginal cost pricing.

The basic theorems of nonlinear programming, especially the Kuhn–Tucker–Uzawa Theorem in the setting of the vector maximum problem, have been extended to the case of infinitely many commodities. (Hurwicz, 1958, first obtained the basic results in this field.) Technicalities aside, the theorems carry over to certain infinite dimensional spaces, namely linear topological spaces, or in the case of first-order conditions, Banach spaces.

Dropping the restriction to a finite number of basic processes leads to classical production or transformation function models of production, whose properties depend on the detailed specifications made.

Samuelson (1947) used Lagrangian methods to analyse interior maxima subject to equality constraints in the context of production function models, as well as that of optimization by consumers. He also gave the interpretation of Lagrange multipliers as shadow prices.

EFFICIENT ALLOCATION IN AN ECONOMY WITH CONSUMERS AND PRODUCERS

In an economy with both consumption and production decisions, efficiency is concerned with distribution as well as production. Data about restrictions on consumption and the wants of consumers must be specified in addition to the data about production. The elements of the models are as follows.

The commodity space is denoted X; it might be l-dimensional Euclidean space, or a more abstract space such as an additive group in which, for example, some coordinates are restricted to have integer values. There is a (finite) list of consumers, $1, 2, \ldots, n$, and a similar list of producers, $1, 2, \ldots, m$. A *state* of the economy is an array consisting of a commodity bundle for each agent in the economy, consumer or producer. This may be written $(\langle x^i \rangle, \langle y^i \rangle)$, where $\langle x^i \rangle = (x^1, x^2, \ldots, x^n)$ and $\langle y^j \rangle = (y^1, y^2, \ldots, y^m)$ and x^i and y^j are commodity bundles. Absolute constraints on consumption are expressed by requiring that the allocation $\langle x^i \rangle$ belong to a specified subset X of the space X^n of allocations.

Examples of such constraints are:

1. The requirement that the quantity of a certain commodity be non-negative.
2. The requirement that a consumer requires certain minimum quantities of commodities in order to survive.

Each consumer i has a preference relation, denoted $\succsim_i$, defined on X. This formulation admits externalities in consumption, including physical externalities and externalities in preferences; for example, preferences that depend on the consumption of other agents, termed non-selfish preferences. The consumption set of the ith consumer is the projection X^i of X onto the space of commodity bundles whose coordinates refer to the holdings of the ith consumer.

Technology is specified by a production set Y, a subset of X^m, consisting of those arrays $\langle y^j \rangle$ of input–output vectors that are jointly feasible for all producers. The production set of the jth producer, denoted Y^j, is the projection of Y onto the subspace of X^m whose coordinates refer to the jth producer.

The (aggregate) initial endowment of the economy is denoted by w, a commodity bundle in X.

These specifications define an *environment*, a term introduced by Hurwicz (1960) in this usage and according to him suggested by Jacob Marschak. This term refers to the primitive or given data from which analysis begins. Each environment determines a set of *feasible* states. These are the states $(\langle x^i \rangle, \langle y^j \rangle)$ such that $\langle x^i \rangle$ is in X, $\langle y^j \rangle$ is in Y and $\Sigma x^i - \Sigma y^j \leqslant w$.

An environment determines the set of states that are Pareto optimal for that environment. Explicitly, they are the states $(\langle x^{*i} \rangle, \langle y^{*j} \rangle)$ that are feasible in the given environment, and such that if any other state $(\langle x^i \rangle, \langle y^j \rangle)$ has the property that $\langle x^i \rangle \succsim_i \langle x^{*i} \rangle$ for all i with $\langle x^{i'} \rangle \succ_{i'} \langle x^{*i'} \rangle$ for some i', then $(\langle x^i \rangle, \langle y^j \rangle)$ is not feasible in the given environment.

It is important to note that the set of feasible states and the set of Pareto optimal states are completely determined by the environment; specification of economic organization is not involved.

At this level of generality, where externalities in consumption and production are admitted as possibilities, and where commodities may be indivisible, no general characterization of Pareto optima in terms of prices is possible. (Indeed, Pareto optima may not exist. Conditions that make the set of feasible allocations non-empty and compact and preferences continuous suffice to ensure the existence of Pareto optima.) In environments with externalities, or other non-neoclassical features, Pareto optima are generally not attainable by decentralized processes (Hurwicz, 1966).

If the class of environments under consideration is restricted to the neoclassical environments, the fundamental theorems of welfare economics provide a characterization of Pareto optimal states via efficiency prices. That characterization has a natural interpretation in terms of a decentralized mechanism for allocation of resources.

The framework for these results is obtained by restricting the class of environments specified above as follows. The commodity space is to be Euclidean space of l dimensions, i.e. $X = R^l$. The consumption set for the economy is to be the product of its projections, i.e. $X = X^1 \times X^2 \times \cdots \times X^n$. This expresses the fact that if each agent's consumption is feasible for him, the total array is jointly feasible. Furthermore, each agent is restricted to have selfish preferences; that is, agent i's preference relation depends only on the coordinates of the allocation that refer to his holdings. In that case the preference relation $\succsim_i$ may be defined only on X^i, for each i. Similarly, externalities are ruled out in production, i.e. $Y = Y^1 \times Y^2 \times \cdots \times Y^m$.

The concept of an *equilibrium relative to a price system* (Debreu, 1959) serves to characterize Pareto optima by prices. A price system, denoted p, is an element of R^l; the environment $e = [(X^i), (\succsim_i), (Y^j), w]$ is of the restricted type specified above (free of externalities and indivisibilities).

A state $[(x^{*i}), (y^{*j})]$ of e is an *equilibrium relative to price system p* if:

1. For every consumer i, x^{*i} maximizes preference $\succsim_i$ on the set of consumption bundles whose value at the prices p does not exceed the value of x^{*i} at those prices, i.e., if x^i is in $\{x^i \text{ in } X^i : px^i \leqslant px^{*i}\}$ then $x^i \precsim_i x^{*i}$.
2. For every producer j, y^{*j} maximizes profit py^j on Y^j.
3. Aggregate supply and demand balance, i.e.

$$\sum_i x^{*i} - \sum_j y^{*j} = w.$$

An equilibrium relative to a price system differs from a competitive equilibrium (see below) in that the former does not involve the budget constraints applying to consumers in the latter concept. In an equilibrium relative to a price system the distribution of initial endowment and of the profits of firms among consumers need not be specified.

The first theorem of neoclassical welfare economics states, subject only to the exclusion of externalities and a mild condition that excludes preferences with thick indifference sets, that a state of an environment e that is an equilibrium relative to a price system p is a Pareto optimum of e (Koopmans, 1957).

The second welfare theorem is deeper and holds only on a smaller class of environments, sometimes referred to in the literature as the *classical environments* (called neoclassical above). One version of this theorem is as follows. Let $e = [(X^i), (\succsim_i), (Y^i), w]$ be an environment such that for each i

1. X^i is convex.
2. The preference relation $\succsim_i$ is continuous.
3. The preference relation $\succsim_i$ is convex.
4. The set $\Sigma_j Y^j$ is convex.

Let $[(x^{*i}), (y^{*j})]$ be a Pareto optimum of e such that there is at least one consumer who is not satiated at x^{*i}. Then there is a price system p, with not all components equal to 0, such that – except for Arrow's (1951) 'exceptional case', where p is such that for some i the expenditure px^{*i} is a minimum on the consumption set X^i – the state $[(x^{*i}), (y^{*j})]$ is an equilibrium relative to p.

(The condition that preferences are convex and not satiated is sufficient to exclude 'thick' indifference sets. A preference relation on X^i is convex if whenever x' and x'' are points of X^i with x' strictly preferred to x'' then the line segment connecting them (not including the point x'') is strictly preferred to x'. The consumption set X^i must be convex for this property to make sense. A preference relation is not satiated if there is no consumption preferred to all others.)

Hurwicz (1960) has given an alternative formalization of the competitive mechanism in which Arrow's exceptional case presents no difficulties.

If the exceptional case is not excluded, then it can still be said that:

1. x^{*i} minimizes expenditure at prices p on the upper contour set of x^{*i}, for every i, and
2. y^{*j} maximizes 'profit' py^j on the production set Y^j, for every j.

The state (x^*, y^*) together with the prices p, constitute a *valuation equilibrium* (Debreu, 1954).

As in the case of efficiency prices in pure production models, these prices have in themselves no institutional significance. They are, however, in the same way as other efficiency prices, suggestive of an interpretation in terms of decentralization.

If, in addition to the restriction to classical environments, the economic organization is specified to be that of a system of markets in a private ownership economy, and if agents are assumed to take prices as given, then the welfare theorems can translate into the assertion that the set of Pareto optima of an environment e and the set of competitive equilibria for e (subject to the possible redistribution of initial endowment and ownership shares) are identical. More precisely, the specification of the environment given above is augmented by giving each consumer a bundle of commodities, his initial endowment, denoted w^i. The total endowment is $w = \Sigma_i w^i$. Furthermore, each consumer has a claim to a share of the profits of each firm; the claims for the profit of each firm are assumed to add up to the entire profit. When prices and the production decisions of the firms are given, the profits of the firms are determined and so is the value of each consumer's initial endowment. Therefore, the income of each consumer is determined. Hence, the set of commodity bundles a consumer can afford to buy at the given prices, called his *budget set*, is determined; this consists of all bundles in his consumption set whose value at the given prices does not exceed his income at the given prices. Competitive behaviour of consumers means that each consumer treats the prices as given constants and chooses a bundle in his budget set that maximizes his preference; that is, a bundle x^i in his budget set and such that if any other bundle x'^i is preferred to it, then x'^i is not in his budget set.

Competitive behaviour of firms is to maximize profits computed at the given prices p, regarded by the firms as constants; that is, a firm chooses a production vector y^j in its production set with the property that any other vector affording higher profits than py^j is not in the production set of firm j.

A *competitive equilibrium* is a specification of a commodity bundle for each consumer, a production vector for each firm, and a price system, together denoted $[(x^{*i}), (y^{*j}), p^*]$, where p^* has no negative components, satisfying the following conditions:

1. For each consumer i the bundle x^{*i} maximizes preference on the budget set of i.
2. For each firm j the production vector y^{*j} maximizes profit $p^* y^j$ on the production set Y^j.
3. For each commodity, the total consumption does not exceed the net total output of all firms plus the total initial endowment, i.e. $\Sigma_i x^{*i} - \Sigma_j y^{*j} \leqslant w = \Sigma_i w^i$;
4. For those commodities k for which the inequality in 3 is strict; that is, the total consumption is less than initial endowment plus net output, the price p^*_k is zero.

The welfare theorems stated in terms of equilibrium relative to a price system translate directly into theorems stated in terms of competitive equilibrium. Briefly, every competitive equilibrium allocation in a given classical environment is Pareto optimal in that environment, and every Pareto optimal allocation in a given classical environment can be made a competitive equilibrium allocation of an environment that differs from the given one only in the distribution of the initial endowment. (Arrow (1951), Koopmans (1957), Debreu (1959) and Arrow and Hahn (1971) give modern and definitive treatment of the classical welfare theorems.)

It should be noted that the equilibria involved must exist for these theorems to have content. Sufficient conditions for existence of competitive equilibrium, which, since a competitive equilibrium is automatically an equilibrium relative to a price system, are also sufficient for existence of an equilibrium relative to a price system, include convexity and and continuity of consumption sets and preferences and of production sets, as well as some assumptions which apply to the environment as a whole, restricting the ways in which individual agents may fit together to form an environment (Arrow and Debreu, 1954; Debreu, 1959; McKenzie, 1959).

The second welfare theorem involves redistribution of initial endowment. This is essential because the set of competitive equilibria from a given initial endowment is small (essentially finite) (Debreu, 1970), while the set of Pareto optima is generally a continuum. The set of Pareto optima cannot in general be generated as competitive allocations without varying the initial point. If redistribution is done by an economic mechanism, then it should be a decentralized one to support the interpretation given of the second welfare theorem. No such mechanism has been put forward as yet. Redistribution of initial endowment by lump-sum taxes and transfers has been discussed. A customary interpretation views these as brought about by a process outside economics, perhaps by a political process; no claim is made that such processes are decentralized. Some economists consider dependence on redistribution unsatisfactory because information about initial endowment is private; only the individual agent knows his own endowment. Consequently the expression of that information through political or other action can be expected to be strategic. The theory of second-best allocations has been proposed in this context. Redistribution of endowment is excluded, and the mechanism is restricted to be a price mechanism, but the price system faced by consumers is allowed to be different from that faced by producers; all agents behave according to the rules of the (static) competitive mechanism. The allocations that satisfy these conditions, when the price systems are variable, are maximal allocations in the sense that they are Pareto optimal within the restricted class just defined. These are so-called *second-best* allocations. This analysis was pioneered by Lipsey and Lancaster (1956) and Diamond and Mirrlees (1971).

EFFICIENT ALLOCATION IN NON(NEO)CLASSICAL ENVIRONMENTS

The term *nonclassical* refers to those environments that fail to have the properties of classical ones; there may be indivisible commodities, nonconvexities in consumption sets, preferences or production sets, or externalities in production or consumption. An example of nonconvex preference would arise if a consumer preferred living in either Los Angeles or New York to living half the time in each city, or living half-way between them, depending on the way the commodity involved is specified. A production set representing a process that affords increasing returns to scale is an example of nonconvexity in production. A large investment project such as a road system is an example of a significant indivisibility. Phenomena of air or water pollution provide many examples of externalities in consumption and production.

The characterization of optimal allocation in terms of prices provided by the classical welfare theorems does not extend to nonclassical environments. If there are indivisibilities, equilibrium prices may fail to exist. Lerner (1934, 1947) has proposed a way of optimally allocating resources in the presence of indivisibilities. It would typically require adding up consumers' and producers' surplus.

Increasing returns to scale in production generally results in non-existence of competitive equilibrium, because of unbounded profit when prices are treated as given. Nash equilibrium, a concept from the theory of games, can exist even in cases of increasing returns. The difficulty is that such equilibria need not be optimal. Similar difficulties occur in cases of externalities.

Failure of the competitive price mechanism to extend the properties summarized in the classical welfare theorems to nonclassical environments has led economists to look for alternative ways of achieving optimal allocation in such cases. Such attempts have for the most part sought institutional arrangements that can be shown to result in optimal allocation. Ledyard (1968, 1971) analysed a mechanism for achieving Pareto optimal performance in environments with externalities. The use of taxes and subsidies advocated by Pigou (1932) to achieve Pareto optimal outcomes in cases of externalities is such an example. In a similar spirit Davis and Whinston (1962) distinguish externalities in production that leave marginal costs unaffected from those that do change marginal costs. In the former case they propose a pricing scheme, but one that involves lump-sum transfers. Marginal cost pricing, including lump-sum transfers to compensate for losses, which was extensively discussed as a device to achieve optimal allocation in the presence of increasing returns (Lerner, 1947; Hotelling, 1938; and many others) is another example of a scheme to realize optimal outcomes in nonclassical environments in a way that seeks to capture the benefits associated with decentralized resource allocation. In the case of production under conditions of increasing returns, the use of nonlinear prices has been suggested in an effort to achieve optimality with at least some of the benefits of decentralization. (See Arrow and Hurwicz, 1960; Heal, 1971; Brown and Heal, 1982; Brown, Heal, Khan and Vohra, 1985; Jennergren, 1971; Guesnerie, 1975.)

In the case of indivisibilities, and in the context of productive efficiency, integer programming algorithms exist for finding optima in specific problems, but a general characterization in terms of prices such as exists for the classical environments is not available. A decentralized process, involving the use of randomization, whose equilibria coincide with the set of Pareto optima has been put forward by Hurwicz, Radner and Reiter (1975). This process has the property that the counterparts of the classical welfare theorems hold for environments in which all commodities are indivisible, and the set of feasible allocations is finite, or in which there are no indivisible commodities, or externalities, but there may be nonconvexities in production or consumption sets, or in preferences. This, of course, includes the possibility of increasing returns to scale in production.

The schemes and processes that have been proposed, including many not described here, are quite different from one another. If attention is confined to pricing schemes without additional elements, such as lump-sum transfers, it may be satisfactory to proceed on the basis of an informal intuitive notion of decentralization. This amounts in effect to identifying decentralization with the competitive mechanism, or more generally with price or market mechanisms. If a broader class of processes is to be considered, including some already mentioned in this discussion, then a formal concept of decentralized resource allocation process is needed.

EFFICIENT ALLOCATION THROUGH INFORMATIONALLY DECENTRALIZED PROCESSES

A formal definition of a concept of *allocation process* was first given by Hurwicz (1960). He also gave a definition of *informational decentralization* applying to a broad class of allocation mechanisms, based in part on a discussion by Hayek (1945) of the advantages of the competitive market mechanism for communicating knowledge initially dispersed among economic agents so that it can be brought to bear on the decisions that determine the allocation of resources. Hurwicz's formulation is as follows.

There is an initial dispersion of information about the environment; each agent is assumed to observe directly his own characteristic, e^i, but to know nothing directly about the characteristics of any other agent. In the absence of externalities, specifying the array of individual characteristics specifies the environment, i.e. $e = (e^1, \ldots, e^n)$. When there are externalities, an array of individual characteristics, each component of which corresponds to a possible environment, may not together constitute a possible environment. In more technical language, when there are externalities the set of environments is not the Cartesian product of its projections onto the sets of individual characteristics.

The goal of economic activity, whether efficiency, Pareto optimality or some other desideratum such as fairness, can be represented by a relation between the set of environments and the set of allocations, or outcomes. This relation assigns to each environment the set of allocations that meet the criterion of desirability. In the case of the Pareto criterion, the set of allocations that are Pareto optimal in a given environment is assigned to that environment. Formally, this relation is a correspondence (a set-valued function) from the set of environments to the set of allocations.

An allocation process, or mechanism, is modelled as an explicitly dynamic process of communication, leading to the determination of an outcome. In formal organizations standardized forms are frequently used for communication; in organized markets like the Stock Exchange, these include such things as order forms; in a business, forms on which weekly sales are reported; in the case of the Internal Revenue Service, income tax forms. A form consists of entries or blanks to be filled in a specified way. Thus, a form can be regarded as an ordered array of variables whose values come from specified sets. In the Hurwicz model, each agent is assumed to have a *language*, denoted M^i for the ith agent, from which his (possibly multi-dimensional) *message*, m^i, is chosen. The *joint message* of all the agents, $m = (m^1, \ldots, m^n)$ is in the *message space* $M = M^1 \times \cdots \times M^n$. Communication takes place in time, which is discrete; the message $m_t = (m_t^1, \ldots, m_t^n)$ denotes the message at time t. The message an agent emits at time t can depend on anything he knows at that time. This consists of what the agent knows about the environment by direct observation, by assumption, (*privacy*) his own characteristics, e^i for agent i, and what he has learned from others via the messages received from them. The agents' behaviour is represented by *response functions*, which show how the current message depends on the information at hand. Agent i's message at time t is

$$m_t^i = f^i(m_{t-1}, m_{t-2}, \ldots; e^i), \quad i = 1, \ldots, n, \quad t = 0, 1, 2, \ldots$$

If it is assumed that memory is finite, and bounded, it is possible without loss of generality to take the number of past periods remembered to be one. (If memory is unbounded, taking the number of periods remembered to be one excludes the possibility of a finite dimensional message space.) In that case the response equations become a system of first order temporally homogeneous difference equations in the messages. Thus:

$$m_t^i = f_i(m_{t-1}; e^i) \qquad i = 1, \ldots, n, \quad t = 0, \ldots,$$

which can be written more compactly as

$$(*) \qquad m_t = f(m_{t-1}; e).$$

(This formulation can accommodate the case of directed communication, in which some agents do not receive some messages; if agent i is not to receive the message of j, then f^i is independent of m^j, although m^j appears formally as an argument.) Analysis of informational properties of mechanisms is to begin with separated from that of incentives. When the focus is on communication and complexity questions, the response functions are not regarded as chosen by the agent, but rather by the designer of the mechanism.

The iterative interchange of messages modelled by the difference equation system (*) eventually comes to an end, by converging to a stationary message. (It is also possible to have some stopping rule, such as to stop after a specified number of iterations.) The stationary message, which will be referred to as an *equilibrium message*, is then translated into an outcome, by means of the *outcome function*:

$$h: M \to Z,$$

where Z is the space of outcomes, usually allocations or trades. An allocation mechanism so modelled is called an *adjustment process*; it consists of the triple (M, f, h). Since no production or consumption takes place until all communication is completed, these processes are *tâtonnement* processes.

A more compact and general formulation was given by Mount and Reiter (1974) by looking only at message equilibria when attention is restricted to static properties. A correspondence is defined, called the *equilibrium message correspondence*. It associates to each environment the set of equilibrium messages for that environment. In order to satisfy the requirement of privacy, namely that each agent's message depend on the environment only through the agent's characteristic, the equilibrium message correspondence must be the intersection of individual message correspondences, each associating a set of messages acceptable to the individual agent as equilibria in the light of his own characteristic. Thus the equilibrium message correspondence

$$\mu: E \to M,$$

is given by

$$\mu(e) = \bigcap_i \mu^i(e^i),$$

where μ^i: $E^i \to M$ is the individual message correspondence of agent i. Note that here the message space M need not be the Cartesian product of individual languages. In the case of an adjustment process, the equilibrium message correspondence is defined by the conditions

$$\mu^i(e^i) = \{m \text{ in } M \mid f^i(m; e^i) = m^i\}, \qquad i = 1, \ldots, n$$

together with the condition that μ is the intersection of the μ^i. Specification of the outcome function h: $M \to Z$ completes the model, (M, μ, h).

The performance of a mechanism of this kind can be characterized by the mapping defined by the composition of the equilibrium message correspondence μ and the outcome function h. The mapping $h\mu$; $E \to Z$, possibly a correspondence, specifies the outcomes that the mechanism (M, μ, h) generates in each environment in E. A mechanism, whether in the form of an adjustment process, or in the equilibrium form, is called *Pareto-satisfactory* (Hurwicz, 1960) if for each environment in the class under consideration, the set of outcomes generated by the mechanism coincides with the set of Pareto optimal outcomes for that environment. Allowance must be made for redistribution of initial endowment, as in the case of the second welfare theorem. (A formulation in the framework of mechanisms is given in Mount and Reiter, 1977).

The competitive mechanism formalized as a static mechanism is as follows. (Hurwicz, 1960, has given a different formulation, and Sonnenschein, 1974, has given an axiomatic characterization of the competitive mechanism from a somewhat different point of view.) The message space M is the space of prices and quantities of commodities going to each agent (it has dimension $n(l-1)$ when there are n agents and l commodities, taking account of budget constraints and Walras' Law), the individual message correspondence μ^i maps agent i's characteristic e^i to the graph of his excess demand function. The equilibrium message is the intersection of the individual ones, and is therefore the price–quantity combinations that solve the system of excess demand equations. The outcome function h is the projection of the equilibrium message onto the quantity components of M. Thus $h\mu(e)$ is a competitive equilibrium allocation (or trade) when the environment is e. The classical welfare theorems state that for each e in E_c, $h[\mu(e)] = P(e)$, where E_c denotes the set of classical environments and P is the Pareto correspondence. (Allowance must be made for redistribution of initial endowment in connection with the second welfare theorem. Explicit treatment of this is omitted to avoid notational complexity. The decentralized redistribution of initial endowment is, as in the case of the second welfare theorem, not addressed.) The welfare theorems can be summarized in the Mount–Reiter diagram (Figure 2) (Reiter, 1977).

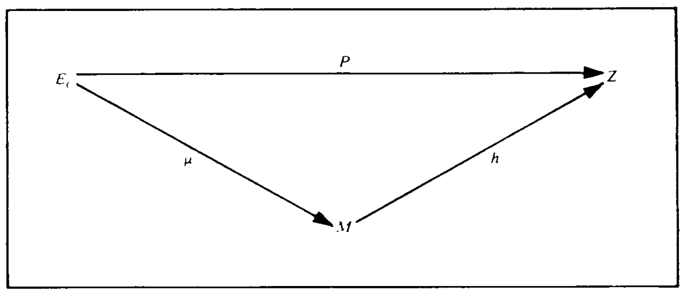

Figure 2

The welfare theorems state that this diagram *commutes* in the sense that starting from any environment e in E_c one reaches the same allocations via the mechanism, that is, via $h\mu$, as via the Pareto correspondence P.

With welfare theorems as a guide, the class of environments E_c can be replaced by some other class E, and the Pareto correspondence can be replaced by a correspondence, P, embodying another criterion of optimality, and one can ask whether there is a mechanism, (M, μ, h) that makes the diagram commute, or, in other words, *realizes P*? Without further restrictions on the mechanism, this is a triviality, because one agent can act as a central agent to whom all others communicate their environmental characteristics; the central agent then has the information required to evaluate P.

The concept of an *informationally decentralized mechanism* defined by Hurwicz (1960) makes explicit intuitive notions underlying the view that the price mechanism is decentralized.

Informationally decentralized processes are a subclass of so-called *concrete processes*, introduced by Hurwicz (1960). These are processes that use a language and response rules that allow production and distribution plans to be specified explicitly. The informationally decentralized processes are those whose response rules permit agents to transmit information only about their own actions, and which in effect require each agent to treat the rest of the economy either as one aggregate, or in a symmetrical way that, like the aggregate, gives anonymity to the other agents.

In the case of static mechanisms, the requirements for informational decentralization boil down to the condition that the message space have no more than a certain finite dimension, and in some cases only that it be of finite dimension. In the case of classical environments this can be seen to include the competitive mechanism, and to exclude the obviously centralized one mentioned above.

Without going deeply into the matter, an objective of this line of research is to analyse explicitly the consequences of constraints on economic organization that come from limitations on the capacity of economic agents to observe, communicate and process information. One important result in this field is that there is no mechanism (M, μ, h) where μ preserves privacy, that uses messages smaller (in dimension) than those of the competitive mechanism (Hurwicz, 1972b; Mount and Reiter, 1974; Walker, 1977; Osana, 1978). Similar results have been obtained for environments with public goods, showing that the Lindahl mechanism uses the minimal message space (Sato, 1981). Another objective is to analyse effects on incentives arising from private motivations in the presence of private information; that is, information held by one agent that is not observable by others, except perhaps at a cost. (There is a large literature on this subject under the rubric 'incentive compatibility', or 'strategic implementation' (Dasgupta, Hammond and Maskin, 1979; Hurwicz, 1971, 1972a). The informational requirements of achieving a specified performance taking some aspects of incentive compatibility into account have been studied by Hurwicz (1976), Reichelstein (1984a, 1984b) and by Reichelstein and Reiter (1985).

Some important results for non-neoclassical environments can be mentioned. Hurwicz (1960, 1966, 1972a) has shown that there can be no informationally decentralized mechanism that realizes Pareto optimal performance on a class of environments that includes those with externalities. Calsamiglia (1977, 1982) has shown in a model of production that if the set of environments includes a sufficiently rich class of those with increasing returns to scale in production, then the dimension of the message space of any mechanism that realizes efficient production cannot be bounded.

EFFICIENT ALLOCATION WITH INFINITELY MANY COMMODITIES

An infinite dimensional commodity space is needed when it is necessary to make infinitely many distinctions among goods and services. This is the case when commodities are distinguished according to time of availability and the time horizon in the model is not bounded or when time is continuous, or according to location when there is more than a finite number of possible locations; differentiated commodities provide other examples, and so does the case of uncertainty with infinitely many states. The bulk of the literature deals with the infinite horizon model of allocation over time, though recently more attention is given to models of product differentiation. Ramsey (1928) studied the problem of saving in a continuous time infinite horizon model with one consumption good and an infinitely lived consumer. He used as the criterion of optimality the infinite sum (integral) of undiscounted utility. Ramsey's contribution was largely ignored, and rediscovered when attention returned to problems of economic growth. A model of maximal sustainable growth based on a linear technology with no unproduced inputs was formulated by von Neumann (1937 in German; English translation, 1945–6). This contribution was unknown among English-speaking economists until after World War II. Study of intertemporal allocation by Anglo-American economists effectively began with the contributions of Harrod (1939) and Domar (1946). These models were concerned with stationary growth at a constant sustainable rate (stationary growth paths) rather than full intertemporal efficiency. Malinvaud (1953) first addressed this problem in a pioneering model of intertemporal allocation with an infinite horizon.

Efficient allocation over (discrete) time would be covered by the finite dimensional models described above if the time horizon were finite. It might be thought that a model with a sufficiently large but still finite horizon would for all practical purposes be equivalent to one with an infinite horizon, while avoiding the difficulties of infinity, but this is not the case, because of the dependence of efficient or optimal allocations on the value given to final stocks, a value that must depend on their uses beyond the horizon.

Malinvaud (1953) formulated an important infinite horizon model, which is the infinite dimensional counterpart of the linear activity analysis model of Koopmans. In Malinvaud's model time is discrete. The time horizon consists of an infinite sequence of time periods. At each date there are finitely many commodities. All commodities are desired in each time period, and no distinction is made between desired, intermediate and primary commodities. As in the activity analysis model, there is no explicit reference to preferences of consumers. Productive efficiency over time is analysed in terms of the output available for consumption, rather than the resulting utility levels.

Technology is represented by a production set X^t for each time period $t = 1, 2, \ldots$, an element of X^t being an ordered pair (a^t, b^{t+1}) of commodity bundles where a^t represents inputs to a production process in period t, and b^{t+1} represents the outputs of that process available at the beginning of period $t + 1$. Here both a^t and b^{t+1} are non-negative. The set X^t is the aggregate production set for the economy during period t. The net outputs available for consumption are given by

$$y^t = b^t - a^t, \qquad \text{for } t \geqslant 1,$$

where b^1 is the initial endowment of resources available at the beginning of period 1. A *programme* is an infinite sequence $\langle(a^t, b^{t+1})\rangle$; it is a *feasible programme* if (a^t, b^{t+1}) is in X^t, and $b^t - a^t \geqslant 0$ for each $t \geqslant 1$, given b^1. The sequence $y = \langle y^t \rangle$ is called the *net output programme* associated with the given programme; it is a *feasible net output programme* if it is the net output programme of a feasible programme. A programme is *efficient* if it is (1) feasible and (2) there is no other programme that is feasible, from the same initial resources b^1, and provides at least as much net output in every period and a larger net output in some period. This is the concept of efficient production, already seen in the linear activity analysis model, now extended to an infinite horizon model. The main aim of this research is to extend to the infinite horizon model the characterization of efficient production by prices seen in the finite model. This goal is not quite reached, as is seen in what follows.

The main difficulties presented by the infinite horizon are already present in a special case of the Malinvaud model with one good and no consumers. Let Y be the set of all non-negative sequenes $y = (y_t)$ that satisfy $0 \leqslant y_t = f(a_{t-1}) - a_t$ for $t \geqslant 1$, and $0 \leqslant y^0 = b^1 - a^0, b^1 > 0$, where f is a real-valued continuous concave function on the non-negative real numbers (the production function), $f(0) = 0$, and b^1 is the given initial stock. The set Y is the set of all feasible porammes. A programme $y' - y > 0$. A price system is an infinite sequence $p = (p^t)$ of non-negative numbers. Denote by P the set of all price systems.

Malinvalud recognized the possibility that an efficient net output programme (y^t) need not have an associated system of non-zero prices (p^t) relative to which the production

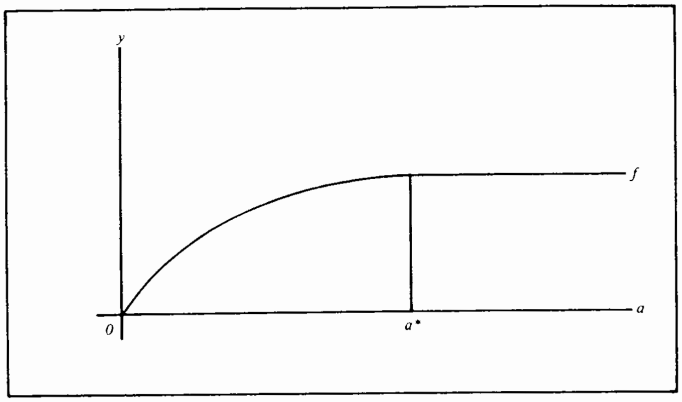

Figure 3

programme generating y satisfies the condition of intertemporal profit maximization, namely that

$$p^{t+1}f(a^t) - p^t a^t \geqslant p^{t+1}f(a) - p^t a$$

for all t and every $a \geqslant 0$. (Here (a^t) is the sequence of inputs producing y.) A condition introduced by Malinvaud, called *nontightness*, is sufficient for the existence of such nonzero prices. Alternative proofs of Malinvaud's existence theorem were given by Radner (1967) and Peleg and Yaari (1970). (An example showing the possibility of non-existence given by Peleg and Yaari (1970) is as follows. Suppose f is as shown in figure 3.

At an interior efficient, and therefore value maximizing, programme the first-order necessary conditions for a maximum imply $p^{t+1}f'(a^t) = p^t$. If there is a time at which $a^t = a^*$, in an efficient programme, then, since $f'(a^*) = 0$, it follows that prices at all prior and future times are 0. Nontightness rules out such examples.)

On the side of sufficiency, Malinvaud showed that intertemporal profit maximization relative to a strictly positive price system p is not enough to ensure that a feasible programme is efficient. An additional (transversality) condition is needed. In the present model the following is such a condition;

$$\lim_{t \to \infty} p^t y^t = 0.$$

Cass (1972) has given a criterion that completely characterizes the set of efficient programmes in a one good model with strictly concave and smooth production technology that satisfies end-point conditions $0 \leqslant f'(\infty) < 1 < f'(x) < \infty$ for some $x > 0$. Cass's criterion, states that a programme is *inefficient* if and only if the associated competitive prices – that is, satisfying $p^{t+1}f'(a^t) = p^t$ – also satisfy $\Sigma_{t=1}^{\infty}(1/p^t) < \infty$. This criterion may be interpreted as requiring the terms of trade between present and future to deteriorate sufficiently fast. Other similar conditions have been presented (Benveniste and Gale, 1975; Benveniste, 1976; Majumdar, 1974; Mitra, 1979). It is hard to see how any transversality condition can be interpreted in terms of decentralized resource allocation.

An alternate approach to characterizing efficient programmes was taken by Radner (1967), based on value functions as introduced in connection with valuation equilibrium by Debreu (1954). (Valuation equilibrium was discussed in connection with Arrow's exceptional case, above.) The value function approach was followed up by Majumdar (1970, 1972) and by Peleg and Yaari (1970). A price system defines a continuous linear functional, (a real-valued linear function) on the commodity space. This function assigns to a programme its present value. The present value may not be well-defined, because the infinite sequence that gives it diverges. This creates certain technical problems passed over here. A more important difficulty is that linear functionals exist that are not defined by price systems. Radner's approach was to characterize efficient programmes in terms of maximization of present value relative to a linear functional on the commodity space. Radner showed, technical matters aside, that:

1. If a feasible programme maximizes the value of net output (consumption) relative to a strictly positive continuous linear functional, then it is efficient.
2. If a given programme is efficient, then there is a nonzero non-negative continuous linear functional such that the given programme maximizes the value of net output relative to that functional on the set of feasible programmes.

These propositions seem to be the precise counterparts of the ones characterizing efficiency in the finite horizon model. Unfortunately, a linear functional may not have a representation in the form of the inner product of a price sequence with a net output sequence. (The production function $f(a) = a^{\beta}$, with $0 < \beta < 1$ provides an example. It is known that the programme with constant input sequence $x_t = (1/\beta)^{\beta/\beta-1}$ and output sequence $y_t = (1/\beta)^{\beta/\beta-1} - (1/\beta)^{1/\beta-1} t = 1, 2, \ldots,$ is efficient, and therefore there is a continuous linear functional relative to which it is value maximizing. But there is no price sequence (p^t) that represents that linear functional.) This presents a serious problem, because in the absence of such a representation it is unclear whether this characterization has an interpretation in terms of decentralized allocation processes; profit in any one period can depend on 'prices at infinity'.

This approach has the advantage that it is applicable not only to infinite horizon models, but to a broader class in which the commodity space is infinite dimensional. Bewley (1972), Mas-Colell (1977) and Jones (1984) among others discuss Pareto optimality and competitive equilibrium in economies with infinitely many commodities. Hurwicz (1958) and others analysed optimal allocation in terms of nonlinear programming in infinite dimensional spaces. Theorems of programming in infinite dimensional spaces are also used in some of the models mentioned in this discussion.

The basic difficulties encountered in the one-good model, apart from the numerous technical problems that tend to make the literature large and diverse as different technical structures are investigated, are on the one hand the fact that transversality conditions are indispensable, and on the other the possibility that linear functionals, even when they exist, may not be representable in terms of price sequences. These problems raise strong doubt about the possibility of achieving efficient intertemporal resource allocation by decentralized means, though they leave open the possibility that some other decentralized mechanism, not using prices, might work. Analysis of this possibility has just begun, and is discussed below.

The difficulties seen in the one-good production model persist in more elaborate ones, including multisectoral models with efficiency as the criterion, and models with consumers in which Pareto optimality is the criterion. McFadden, Mitra and Majumdar (1980) studied a model in which there are firms, and overlapping generations of consumers, as in the model first investigated by Samuelson (1958). Each consumer lives for a finite time and has a consumption set and preferences like the consumers in a finite horizon model. A model with overlapping generations of consumers presents the fundamental difficulty that consumers cannot trade with future consumers as yet unborn. This difficulty can appear even in a finite horizon model if there are too few markets. The economy is closed in the sense that there are no nonproduced resources; the von Neumann growth model is an example of such a model. Building on the results of an earlier investigation (Majumdar, Mitra and McFadden, 1976), these authors introduced several notions of price systems, of competitive equilibrium, efficiency and optimality, and sought to establish counterparts of the classical welfare theorems. To summarize, in the 1976 paper they strengthen an earlier result of Bose (1974) to the effect that the problem of proper distribution of goods in essentially a short-run problem, and that the only long-run problem, one created by the infinite horizon, is that of inefficiency through overaccumulation of capital. In the 1980 paper the focus is on the relationships among various notions of equilibrium and Pareto optimality. The force of their results is, as might be expected, that the difficulties already seen in one-good model without consumers persist in this model. A transversality condition is made part of the definition of competitive equilibrium in order to obtain the result that an equilibrium is optimal. A partial converse requires some additional assumptions on the technology (reachability) and on the way the economy fits together (nondecomposability). These results certainly illuminate the infinite horizon model with overlapping generations of consumers and producers, but the possibility of efficient or optimal resource allocation by decentralized means is not different from that in the one-good Malinvaud model.

Recently, Hurwicz and Majumdar in an unpublished manuscript dated 1983, and later Hurwicz and Weinberger (1984), have addressed this issue directly, building on the approach of mechanism theory.

Hurwicz and Majumdar have studied the problem of efficiency in a model with an infinite number of periods. In each period there are finitely many commodities, one producer who is alive for just one period, and no consumers' choices. The criterion is the maximization of the discounted value of the programme (well-defined in this model). The producer alive in any period knows only the technology in that period. The question is whether there is a (static) privacy preserving mechanism using a finite dimensional message space whose equilibria coincide with the set of efficient programmes. The question can be put as follows. In each period a message is posted. The producer alive in that period responds 'Yes' or 'No'. If every producer over the entire infinite horizon answers 'Yes', the programme is an outcome corresponding to the equilibrium consisting of the infinite succession of posted messages. Since each producer knows only the technology prevailing in the period when he is alive, the process preserves privacy. If in addition the message posted in each period is finite dimensional, the process is informationally decentralized. Period-by-period profit maximization using period-by-period prices is a mechanism of this type; the message posted in each period consists of the vector of prices for that period, and the production plan for that period, both finite dimensional. The object is to characterize all efficient programmes as equilibria of such a mechanism. This would be an analogue of the classical welfare theorems, but without the restriction to mechanisms that use prices in their messages.

The main result is in the nature of an impossibility theorem. If the technology is constant over time, and that fact is common knowledge at the beginning, the problem is trivial since knowledge of the technology in the first period automatically means knowledge of it in every period. On the other hand, if there is some period whose technology is not known in the first period, then there is no finite dimensional message that can characterize efficient programmes, and in that sense, production cannot be satisfactorily decentralized over time.

Hurwicz and Weinberger (1984) have studied a model with both producers and consumers. As with producers, there is a consumer in each period, who lives for one period. The consumer in each period has a one-period utility function, which is not known by the producer; similarly the consumer does not know the production function. The criterion of optimality is the maximization of the sum of discounted utilities over the infinite horizon. Hurwicz and Weinberger show that there is no privacy preserving mechanism of the type just described whose equilibria correspond to the set of optimal programmes. It should be noted that their mechanism requires that the first-period actions (production, consumption and investment decisions) be made in the first period, and not be subject to revision after the infinite process of verification is completed. (On the other hand, under tâtonnement assumptions it may be possible to decentralize. In this model tâtonnement entails reconsideration 'at infinity'.)

If attention is widened to efficient programmes, and if technology is constant over time, there is an efficient programme with a fixed ratio of consumption to investment. This programme can be obtained as the equilibrium outcome of a mechanism of the specified type. However, this corresponds to only one side of the classical welfare theorems. It says that the outcome of such a mechanism is efficient; but it does not ensure that every efficient programme can be realized as the outcome of such a mechanism. The latter property fails in this model.

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See also INCENTIVE COMPATIBILITY; LINEAR PROGRAMMING; ORGANIZATION THEORY; WELFARE ECONOMICS.

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