Econometrica, Vol. 56, No. 3 (May, 1988), 661–692

GAME FORMS WITH MINIMAL MESSAGE SPACES

BY STEFAN REICHELSTEIN AND STANLEY REITER[1]

This paper is concerned with the amount of communication that must be provided to implement a performance standard by a mechanism whose stationary messages have the Nash property. We study the question whether a given message space is large enough to implement a given performance standard. In general, an implementing mechanism with the Nash property in messages requires a larger message space than suffices for decentralized realization without regard to individual incentives. In particular, we study implementation of Walrasian allocations in exchange environments. We show that the smallest message space that implements Walrasian allocations is one of dimension, roughly, $n \cdot (l-1) + l/(n-1)$, where l is the number of commodities and n the number of agents. We exhibit an implementing mechanism whose message space has that dimension.

KEYWORDS: Decentralization, implementation, informational requirements, size of the message space, Walrasian allocations.

1. INTRODUCTION

THE LITERATURE ON DESIGN OF RESOURCE ALLOCATION MECHANISMS falls into two almost non-overlapping parts. One branch deals with the informational requirements, the other with incentives. Both assume that information about the environment is dispersed among the economic agents. The literature on informational requirements has typically studied the informational costs of decentralized attainment of a specified performance. The idea underlying most of this literature is that informational costs can be decomposed, at least conceptually, into components, such as the amount of communication required and the complexity of information processing. An objective of this line of research is to explore the constraints imposed on the design of allocation mechanisms by the technology of information processing, including the limited capacities of human beings. For the most part this literature has ignored issues of strategic behavior and individual's incentives. These are the subject of the incentive literature, which studies the constraints on the design of mechanisms imposed by the divergence of individual incentives from performance objectives. The possibility of using monitoring or enforcement techniques to counter undesired effects of individual incentives makes it clear that there are trade-offs between incentive costs and informational costs. Ultimately, it is therefore necessary to study informational and incentive constraints together. In this paper we analyze how much communication must be provided to implement a performance function by a mechanism whose equilibria have the Nash property in the space of messages, i.e., the property that no agent can improve the outcome for himself by changing his message, given the messages of others.

[1] We would like to thank Thomas Marschak, Steve Matthews, Andreu Mas-Colell, Hans Weinberger, Barath Sarath, Steve Williams, and especially Leonid Hurwicz for helpful comments and suggestions. Earlier drafts of this work were presented at U. C. Berkeley, Stanford University, Cal. Tech., and the University of Minnesota. Financial support through NSF Grants IST8313704 and IST8314504, respectively, is gratefully acknowledged.

The concept of an allocation mechanism was first formalized by Hurwicz (1960). Such a mechanism can be viewed as an abstract planning procedure; it consists of a message space (the language in which communication takes place), rules by which the agents form messages, and an outcome function, which translates messages into outcomes. Mechanisms are imagined to operate iteratively; at each step each agent responds to the current message in a way that depends on his information about the environment. Attention may be focused on mechanisms that have stationary or equilibrium messages for each possible environment. A mechanism is said to realize a given performance standard if the outcomes given by the outcome function when applied to the stationary messages agree with the performance standard. The question of how much communication must be provided to realize a given performance has been addressed in papers by Hurwicz (1972), Mount and Reiter (1974), and Hurwicz, Reiter, and Saari (1985), among others. They analyze the size of the message space needed to realize a given performance. In the case of Euclidean spaces size can be identified with dimension. One of the well-known results in this literature established the minimality of the competitive mechanism. Under certain regularity conditions, every mechanism that achieves Pareto-optimal allocations on a sufficiently rich class of exchange economies must use a message space at least as large as that of the competitive mechanism. A corresponding result exists for the Lindahl mechanism in environments with public goods.[2] These results were obtained for mechanisms in stationary form, i.e., the dynamics are suppressed and analysis focuses exclusively on the set of stationary messages. If the dynamics of allocation mechanisms are considered explicitly and stability (local and structural stability have been studied) is required then the size of the message space generally has to increase. This has been demonstrated in the papers of Reiter (1979), Mount and Reiter (1987), and Jordan (1987).

The incentives literature has studied game forms which consist of a message space and an outcome function. Game forms differ from allocation mechanisms of the Hurwicz type only in that the former do not include message forming rules for the individual agents. In the game form these are left to the choice of the individual agent and are therefore determined by individuals' incentives. Direct revelation mechanisms, in which the message space is taken to be the space of all conceivable environments, have played an important part in this literature. The revelation principle asserts that for a number of noncooperative equilibrium concepts, including dominant strategy, maxmin, Nash and Bayesian-Nash equilibrium, there is "no loss of generality" in restricting attention to revelation mechanisms; if a specified performance standard can be implemented by some abstract mechanism, then it can also be implemented by a revelation mechanism. A direct revelation mechanism may not be sufficient for implementation if the solution concept is Nash equilibrium in the space of messages. Maskin (1977) and Williams (1984) construct game forms wherein the message space amounts to the n-fold Cartesian product of the message space of a direct revelation mecha-

[2] A survey of the literature on these topics can be found in Hurwicz (1986).

nism. Such a mechanism is likely to be infeasible on information grounds. If the space of environments is infinite dimensional, as in the case of an exchange economy where agents may have any continuous and convex preferences, the direct revelation message space would have to be infinite dimensional. How much better than direct revelation (or n-fold direct revelation) can be done for a given performance standard? In economically interesting cases, Reichelstein (1984a) has shown that there are dominant strategy mechanisms that offer substantial informational savings over direct revelation schemes. In general, though, dominant strategy implementation imposes a dimensional cost compared to mechanisms in which agents follow given rules without regard to individual incentives. A converse problem has been studied by Green and Laffont (1987) who analyze the performance that is attainable with a message space of exogenously given size. For implementation in Nash equilibrium strategies, Williams (1986) and Saijo (1985) show that for certain performance standards one can do substantially better than the n-fold revelation scheme of Maskin's construction.

This paper goes a step further by attempting to solve for the minimal message space needed for mechanisms whose equilibria have the Nash property in the message space. A major part of this paper deals with the implementation of Walrasian allocations in pure exchange economies, a problem first addressed by Hurwicz (1979a) and Schmeidler (1980). The implementing mechanisms of Hurwicz and Schmeidler each use a finite dimensional Euclidean message space. The size of this space is about twice the size of the message space of the competitive mechanism, which is a Euclidean space of dimension $n \cdot (l-1)$ where n is the the number of agents and l is the number of goods. As noted above, it has been established that the message space of the competitive mechanism is minimal among (regular) mechanisms that realize Pareto-optimal allocations in exchange economies. We show that implementation of Walrasian allocations by a smooth mechanism whose stationary points have the Nash equilibrium property necessarily requires an increase in the size of the message space over that of the competitive mechanism. The dimensional increment over that of the competitive mechanism is shown to be roughly $l/(n-1)$. When there are more agents than goods, only one extra dimension is needed. The essential intuition behind this formula is that in order to implement Walrasian allocations, the mechanism must induce "price-taking" behavior on the part of all agents, i.e., agents trade at "prices" over which they have no control. Price-taking cannot be induced with only $n \cdot (l-1)$ message variables; additional message variables are needed. The dimensional increment $l/(n-1)$ arises because one additional message variable can ensure price-taking behavior for at most $(n-1)$ goods in an economy with n agents. We also show that this lower bound is attainable by constructing an implementing mechanism with that message space.

Though our basic view of allocation mechanisms is that they are and should be dynamic, our analysis in this paper is static; we solve for the message space size needed in order that stationary messages have the Nash equilibrium property. This does not mean, however, that our results should be interpreted as an analysis of the dimensional requirements of one-step mechanisms. Such an interpretation would raise questions about the appropriateness of Nash equilibrium as the solution concept. If an agent knows his own environment, but not those of other agents, he will typically not have enough information to calculate his Nash equilibrium message. We prefer to interpret our analysis as applying to an unspecified process in which agents grope their way to a stationary message and in which the Nash property is a necessary condition of stationarity. The experimental evidence reported by Smith (1979) indicates that for certain mechanisms agents' messages tend to converge to the Nash equilibrium points. Indeed, Smith (1979) reports that in experimental designs in which agents are given more information about the environments of others, convergence is more erratic.

An explicitly dynamic analysis would raise a number of additional issues. As noted above in connection with mechanisms that ignore incentive issues, convergence requirements generally impose an increase in the size of the message space required. The informational requirements of mechanisms that achieve convergence to Nash equilibrium messages has received little attention so far. For a certain class of mechanisms Jordan (1986) has provided an impossibility result. Jordan considers adjustment processes in continuous time with arbitrary (finite dimensional) message spaces. He shows that it is impossible to attain Walrasian allocations at Nash equilibrium points, if local and structural stability assumptions are imposed. However, as noted in Jordon (1986), stable processes converging to Nash equilibrium messages may well exist, if a broader class of dynamic processes is considered; for example, processes that are not required to be autonomous, i.e., temporally homogeneous.

Much of the recent work on one-step mechanisms has adopted the framework of games of incomplete information using Bayesian-Nash equilibrium as the solution concept; see, for example, Laffont and Maskin (1982), Ledyard (1986), and Postlewaite and Schmeidler (1986). A parallel analysis of the informational requirements of this type of implementation is called for.[3] The results in this paper contribute to that analysis, if the Bayesian equilibria are required to have the Nash property in the space of messages, although in general this is not the case. Examples in which Bayesian equilibria have the Nash property include the uniformly incentive compatible decision rules of Holmstrom and Myerson (1983) which, when individuals' utilities do not depend directly on the utilities of others, amount to dominant strategies and the ex-post Nash equilibrium of Crémer and McLean (1985) which in strength lies between dominant strategy and Bayesian equilibrium. In those cases the analysis of the informational requirements of the Nash property in messages applies directly.

The remainder of this paper is organized as follows. Section 2 gives basic definitions and shows that the Nash property in messages entails at least as large a message space as does decentralized realization without regard to incentives. A simple example also demonstrates that the Nash equilibrium requirement, in general, strictly increases the message space requirements. In Section 3 we study

[3] Roberts (1987) studies the Bayesian equilibria of a two-stage allocation process with exogenous constraints on the message space.

the question whether a message space of given dimension is large enough to implement a given performance. The argument is in terms of a correspondence that maps messages to environments and satisfies a set of inequalities expressing the Nash property. If differentiability is assumed, these conditions can be translated into dimensional requirements. Theorems 3.1 and 3.2 state necessary and sufficient conditions that a message space be big enough to implement a given choice rule.

In Section 4 this machinery is applied to the problem of attaining Walrasian allocations in pure exchange environments. Theorem 4.1 deals with a special class of economies in which agents preferences can be described (locally) by quadratic utility functions. We show that implementation of Walrasian allocations on this class of environments requires a message space whose dimension is approximately $n \cdot (l-1) + l/(n-1)$. Though this theorem is stated for a restricted class of mechanisms, we believe that the dimensional formula is valid for the entire class of smooth mechanisms. We analyze the general case in the Appendix; there the argument is presented for the case of three agents and three commodities because of the complexity of the notation required. Theorem 4.2 shows that an implementing mechanism exists whose message space matches the lower bound given by the formula in Theorem 4.1. This mechanism implements Walrasian allocations on a broad class of exchange economies.

2. REALIZATION VERSUS IMPLEMENTATION

Our analysis takes as given a social choice rule or performance standard which associates a set of desired alternatives with every environment. Formally, we consider a correspondence

$$F: E \twoheadrightarrow Z$$

where E represents the class of environments (sometimes called the parameter space) and Z is the space of alternatives. An environment $e \in E$ includes a complete description of every agent, i.e., preferences, private information, initial endowments, etc. Throughout this paper it will be assumed that the class of environments is decomposable, so that $E = \times_{i=1}^{n} E_i$ where $N = \{1, \ldots, n\}$ indexes the set of agents.

Initially, each agent is assumed to know only his component of the environment, $e_i \in E_i$. To achieve the desired outcomes the organization has to set up a message process through which relevant information is exchanged. Let $M = \times_{i=1}^{n} M_i$ denote the message space with M_i representing the set of messages that agent i can send. We do not study explicitly the dynamics of the message process but require that for every environment there exists a set of stationary messages $\mu(e)$. A message is stationary if it is stationary from every agent's point of view based on his information about the environment. Hence, we require the existence of correspondences μ_i: $E_i \twoheadrightarrow M$ such that $\mu(e) = \bigcap_{i=1}^{n} \mu_i(e_i)$ for all $e \in E$. Stationary messages are translated into outcomes or allocations by an outcome function. Formally, we adopt the following definition of an allocation mechanism, following Mount and Reiter (1974).

DEFINITION 2.1: The mechanism $\Lambda = \langle M, \mu, g \rangle$ realizes the choice rule $F\colon E \twoheadrightarrow Z$ if and only if

(i) $\quad \forall e \in E: \quad \mu(e) \equiv \bigcap_{i=1}^{n} \mu_i(e_i) \neq \varnothing,$

(ii) $\quad \forall \overline{m} \in \mu(e), \forall e \in E: \quad h(\overline{m}) \in F(e).$

The realization requirement amounts to verifying that for every environment there are stationary messages and that every stationary message yields an outcome that agrees with the performance standard F. In general the message space will have to be larger than the image set $F(E)$. To see this, suppose that F is single-valued and for two environments $e, \bar{e}$, $F(e) = F(\bar{e})$. If $m \in \mu(e) \cap \mu(\bar{e})$ then $m \in \mu(\tilde{e})$ for any point $\tilde{e}$ on the "cube" formed by e, and $\bar{e}$, i.e., any environment whose n-components $(\tilde{e}_1 \cdots \tilde{e}_n)$ are drawn either from e or $\bar{e}$. This follows immediately from the fact that μ is privacy preserving. For this to be compatible with the realization requirement, it must be the case that also $F(\tilde{e}) = F(e)$ for all $\tilde{e}$ on the "cube." Message space size is just one among several relevant measures of informational complexity. Recently, Mount and Reiter (1983) introduced a measure of computational complexity; in examples they exhibit a trade-off between message space size and computational complexity.

The literature on incentives has approached the mechanism design problem as one of designing an appropriate noncooperative game. Every player is given a set of messages, again represented by M_i. Let $R(e_i)$ denote the complete, binary and reflexive preordering that describes the ith agent's preferences over alternatives in Z, when his type is $e_i \in E_i$. Given an environment $e \in E$, the pair $\langle \times_{i=1}^{n} M_i, g \rangle$, with $g\colon M \to Z$ induces a game in normal form. Implementation requires that for every environment Nash equilibria exist in the induced game and, secondly, that outcomes corresponding to Nash equilibrium messages agree with F.

DEFINITION 2.2: $\langle M, g \rangle$ implements $F\colon E \twoheadrightarrow Z$ in Nash equilibrium messages, if:[4]

(i) $\quad \forall e \in E: \quad \rho(e) \equiv \bigcap_{i=1}^{n} \rho_i(e_i) \neq \varnothing,$

where $\rho_i(e_i) \equiv \{ m^* \in M \mid g(m^*) R(e_i) g(m^*_{-i}, \overline{m}_i) \forall \overline{m}_i \in M_i \}$,

(ii) $\quad \forall e \in E, \forall m^* \in \rho(e): \quad g(m^*) \in F(e).$

In a social choice framework, Maskin (1977) identified two properties of social choice rules, namely, monotonicity and no veto power, as central for implementability. Monotonicity turns out to be necessary and in conjunction with no veto power also sufficient for a choice rule to be Nash implementable. For economic settings in which agents have nonsatiable preferences, the no veto power condition is always satisfied.

[4] By $(m_{-i}, \overline{m}_i)$ we denote the vector in which the ith component of m is replaced by $\overline{m}_i$.

Definitions 2.1 and 2.2 appear to be closely related. The principal difference is that the message or strategy rule $\rho(\cdot)$ in 2.2 is not a design variable but is induced by the outcome function and the behavioral equilibrium concept. The formal relationship between the two concepts is given in Theorem 2.4 below.

DEFINITION 2.3: A mechanism $\langle M, \mu, g\rangle$ is said to have the Nash property, if:

$$\forall e \in E, \forall i \in N, \forall m \in M: \quad m \in \mu_i(e_i) \Leftrightarrow g(m) R(e_i) g(m_{-i}, \bar{m}_i) \quad \forall \bar{m}_i \in M_i.$$

THEOREM 2.4: *The following two statements are equivalent*: (i) $\langle M, g\rangle$ *implements* $F: E \twoheadrightarrow Z$. (ii) *There exists a privacy preserving correspondence* $\mu: E \twoheadrightarrow M$ *such that* $\langle M, \mu, g\rangle$ *has the Nash property and realizes* F.

PROOF: (i) $\Rightarrow$ (ii). Consider the induced Nash correspondence $\rho: E \twoheadrightarrow M$. Let $\rho_i(e_i)$ represent the ith agent's best reply correspondence. Since $\rho(e) = \bigcap_{i=1}^{n} \rho_i(e_i)$, the Nash correspondence is, in fact, privacy preserving. Consequently $\langle M, \rho, g\rangle$ is an informationally decentralized mechanism which has the Nash property and realizes F.

(ii) $\Rightarrow$ (i). Given $\langle M, \rho, g\rangle$, the Nash correspondence ρ induced by $\langle M, g\rangle$ coincides with μ, since $\langle M, \mu, g\rangle$ has the Nash property. Hence, $\langle M, g\rangle$ implements F.

Though Theorem 2.4 follows directly from the definitions, it yields an immediate lower bound on the message space needed for implementation in terms of the message space size needed for realization. Given any implementation of F, there exists an equivalent realization which, in addition, has the Nash property. To make general use of Theorem 2.4, we need to make precise the notion of size of a space and identify appropriate regularity conditions for the class of allocation mechanisms considered.

Throughout this paper we will confine ourselves to Euclidean message spaces. The natural measure of size then becomes dimension. It is well known that, in the absence of any regularity conditions, an essentially unlimited amount of information can be encoded in a one-dimensional space. For example, the inverse of the Peano space-filling curve could be used to encode a k-dimensional space on the real line. The continuous Peano function could retrieve this information, since it maps a real interval onto a k-dimensional interval.

Various restrictions on the message rule have been proposed to prevent such "smuggling" of information. Mount and Reiter (1974), for example, require the message correspondence to be locally threaded, i.e., to have locally a continuous, single-valued selection.[5] Alternatively, Hurwicz (1977) requires that the inverse of the message correspondence have, locally, a selection which is Lippschitz continuous.

[5] A correspondence $\mu: X \twoheadrightarrow Y$ is locally threaded if for every $x \in X$ there exists a neighborhood $U(x) \subset X$ and a continuous function $f: U(x) \to Y$ such that $f(x') \in \mu(x')$ for all $x' \in U(x)$.

The problem of information "smuggling" also arises in connection with implementing game forms. Suppose that $\langle M, g\rangle$ implements $F: E \twoheadrightarrow Z$ such that M_i is a convex and compact subset of $\mathscr{R}^k$ and g is continuous. Then there exists another mechanism $\langle \bar{M}, \bar{g}\rangle$ which also implements $F: E \twoheadrightarrow Z$ such that $\bar{M}_i$ is a compact interval on the real line and $\bar{g}$ is continuous. The function $\bar{g}$ is obtained as the composition of g with $(\phi_1 \cdots \phi_n)$, where ϕ_i is the Peano function which maps $\bar{M}_i$ continuously onto M_i. This argument shows that a meaningful theory of the dimensional requirements for implementation has to impose smoothness conditions beyond continuity of the outcome function.

As an application of Theorem 2.4, consider first the familiar problem of allocating resources in an economy with public goods. For simplicity, assume that there are m-public goods and one universal private good (numeraire). Individuals only know their own characteristics, i.e., preferences and endowments of the private good. It can be shown (see Sato (1981)) that every mechanism that achieves interior Pareto optima on a class of parametric utility functions, such as Cobb-Douglas or quadratic utility functions, has to use a message space at least as large as $\mathscr{R}^{n\cdot m}$, provided the message correspondence is locally threaded. Sato (1981) established a process (the Lindahl mechanism), which satisfies this regularity condition, uses $\mathscr{R}^{n\cdot m}$ as its message space and attains the desired allocations on a wide class of economies with convex and continuous preferences. It follows from Theorem 2.4 that, subject to local threadedness of the Nash correspondence, every implementing mechanism has to use a strategy space of dimension greater than or equal to $n\cdot m$. Walker (1981) showed that this lower bound is also attainable; he constructed a game form that uses a message space of exactly that size and implements Lindahl allocations on a wide class of economies.

On the other hand, we provide a simple example in which Nash implementation requires an increase in dimensionality. Consider a class of exchange economies with $n=2$ traders and $l=2$ commodities. The first good is represented by the letter X, while the second good, which serves as numeraire, is denoted by Y. Each agent is characterized by a single parameter:

$$E_i=[a,b], \quad a>1.$$

The social choice rule

$$F: E_1\times E_2 \to Z_1\times Z_2$$

assigns net trades to both agents, $Z_i\subset\mathscr{R}^2$. In particular, let

$$F_i^x(e)=\tfrac{1}{2}(e_i-e_{i+1}), \qquad 1\le i\le 2,$$

$$F_i^y(e)=-\tfrac{1}{2}(e_1+e_2-2)\cdot F_i^x(e).$$

The subscripts are understood "modulo 2." If agents' preferences can, at least locally, be represented by linear-quadratic utility functions of the form

$$U(x,y|e_i)=e_i\cdot x-\frac{x^2}{2}+y$$

and initial endowments of the X-good are fixed at the unit level, then the above performance function is exactly the Walrasian rule. Clearly, F can be realized with a two-dimensional message space, simply by employing a revelation mechanism. With the usual regularity conditions, such as local threadedness, a two-dimensional space can also be shown to be minimal. However, there does not exist a smooth Nash implementation that works with a two-dimensional message space.[6] Assume to the contrary that

$$M_i = R, \quad 1 \le i \le 2,$$

and

$$g: M_1 \times M_2 \to Z_1 \times Z_2$$

is a differentiable outcome function such that $\langle M, g \rangle$ implements F. Let $v(\cdot)$ represent a thread of the induced Nash correspondence $\rho: E \twoheadrightarrow M$. Implementation requires that

$$(g \circ v)(e) = F(e), \quad \forall e \in E.$$

Since $F(E)$ contains a two-dimensional manifold in Z, there exists a point $m^0 \in (E)$ and a neighborhood $O(m^0)$ such that $g|_{O(m^0)}$ is a diffeomorphism. If $e^0 \in v^{-1}(m^0)$, then

$$v = g^{-1} \circ F \quad \text{on some neighborhood } \tilde{O}(e^0).$$

Hence, $v(\cdot)$ itself is a diffeomorphism on $\tilde{O}(e^0)$. Since the mechanism is assumed to implement Walrasian allocations, the following two equations have to hold:

$$(1) \qquad g_i^y(v(e)) = -p(v(e)) g_i^x(v(e)) \quad \forall e \in \tilde{O}(e^0), \quad 1 \le i \le 2.$$

Here, $p(v(e)) = \frac{1}{2}(e_1 + e_2 - 2)$ is the equilibrium price for the economy $e = (e_1, e_2)$.

If $v(e)$ is a Nash equilibrium, the following first order conditions have to hold:

$$(2) \qquad (e_i - g_i^x(v(e))) \frac{\partial}{\partial m_i} g_i^x(v(e)) + \frac{\partial}{\partial m_i} g_i^y(v(e)) = 0, \quad 1 \le i \le 2.$$

In equilibrium, marginal utility has to equal the price, i.e.,

$$e_i - g_i^x(v(e)) = p(v(e)).$$

Total differentiation of (1) yields:

$$(3) \qquad \left[\nabla g_i^y(v(e)) + p(v(e)) \nabla g_i^x(v(e)) + \nabla p(v(e)) g_i^x(v(e))\right] \circ Dv(e) = 0.$$

Recalling that $g_1^y(v(e)) = -g_2^y(v(e))$ and $g_1^x(v(e)) = -g_2^x(v(e))$, we may substitute (2) into (3) and obtain:

$$g_i^x(v(e)) \cdot \nabla p(v(e)) \circ Dv(e) = 0.$$

This contradicts the requirements that $p(v(e)) = \frac{1}{2}(e_1 + e_2 - 2)$ and $v(\cdot)$ be a diffeomorphism.

[6] Note that we allow for unbalanced net trades out of equilibrium. If one insisted on balanced outcomes, i.e., the image of the outcome function is contained in the set $\hat{Z} = \{(z_1, z_2) | z_1 + z_2 = 0\}$ then, independently of the allowed message space size, there will not exist a *smooth* outcome function implementing the Walrasian choice rule; see Reichelstein (1984b). Hurwicz (1979c) established that there are *discontinuous* outcome functions that select balanced net trades and implement Walrasian allocations at Nash equilibrium points.

This example shows that implementation generally requires a larger message space than realization. The example also illustrates the well-known fact that the revelation principle does not apply to implementation in Nash equilibrium strategies. This follows from the observation that a revelation mechanism would use a two-dimensional message space in this case. In the above example, implementation becomes possible, however, if one additional dimension is added to the message space, as can be seen from the following mechanism:

$$M_1 = \mathscr{R} \quad \text{and} \quad M_2 = \mathscr{R}^2.$$

Denoting the second agent's messages by (m_2, m_3), we define the following outcome function:

$$g_1^x(m) = m_1 - m_2,$$
$$g_2^x(m) = m_2 - m_1,$$
$$g_1^y(m) = -m_3(m_1 - m_2),$$
$$g_2^y(m) = -m_1(m_2 - m_1) - (m_1 - m_3)^2.$$

It is easily verified that this mechanism implements the Walrasian choice rule[7] on a wide class of economies.[8] Any Nash equilibrium is such that $m_1 = m_3$, since the second player has, independently of his actual characteristics, a global incentive to match his second variable with m_1. Hence, the set of Nash equilibria is contained in a hyperplane of the message space. Messages that are not in this hyperplane are not needed for the actual implementation task, yet they are necessary to ensure that the messages on the hyperplane, and only those, have the Nash property. The interesting feature here is that the set of Nash equilibria is contained in a space of no larger dimension than is needed for ordinary realization (in our example, a space of dimension two), yet this set needs to be embedded in a message space of higher dimension.

3. GENERAL THEORY

This section identifies necessary and sufficient conditions for a given message space to admit a Nash implementation of the social choice rule $F: E \twoheadrightarrow Z$. These conditions are stated in terms of a mapping that takes messages to environments, while satisfying a set of inequalities. If the domain of the map, i.e., the message space, is too small, it is impossible to satisfy the inequalities. Before stating the results two structural assumptions are introduced. First, we suppose that the ith agent's preferences are defined on some space Z_i and that $Z \subset \times_{i=1}^{n} Z_i$. For example, Z_i could be the commodity space, the dimension of which equals the number of commodities in the economy, while Z is the set of feasible net trades. Secondly, the choice rule is assumed to be a function. To analyze the dimensional requirements for realization or implementation of social choice rules it is often-

[7] We note that this mechanism does not necessarily produce individually feasible outcomes. For nonequilibrium messages agents could be assigned net trades that lead to negative consumption. Throughout this paper the issue of individual feasibility (see Hurwicz, Maskin and Postlewaite (1984) and Reichelstein (1987)) is ignored.

[8] The exact description of this class of economies is given in Section 4 preceding Theorem 4.2 for an arbitrary number of agents and commodities.

times helpful to derive a lower bound on the size of the message space by finding the dimensional requirements for a special class of environments only. In Section 4, for example, we focus first on special economies in which traders' preferences can be represented by quadratic utility functions. On this class of economies the Walrasian choice rule is single-valued.

THEOREM 3.1: *Let* $\langle M, g\rangle$ *implement* $F=(F_1 \cdots F_n)\colon E \to Z$ *and denote by* D *the set of Nash equilibria in* M, *i.e.* $D \equiv \rho(E)$. *If* $F_i(E)=Z_i$, *there exist correspondences*

$$\gamma_i\colon M \twoheadrightarrow E, \quad 1 \le i \le n,$$

satisfying the following conditions:

(i) γ_i *is* F_i*-compatible, i.e., for every* $m \in M$: $e, \bar{e} \in \gamma_i(m)$ *implies* $F_i(e)=F_i(\bar{e})$; *there exists an onto correspondence* $\gamma\colon D \twoheadrightarrow E$ *such that for each* $i \in N$ *the restriction of* γ_i *to* D *agrees with* γ.

(ii) $\forall m \in D$, $\forall e \in \gamma(m)$, $\forall i \in N$:

$$F_i(\gamma_i(T_i(m))) \subset L(e_i, F_i(e))$$

where

$$T_i(m) \equiv \{\bar{m} \in M | \bar{m}=(m_{-i}, \bar{m}_i), \bar{m}_i \in M_i\}$$

and

$$L(e_i, z_i) \equiv \{\bar{z}_i \in Z_i | z_i R(e_i) \bar{z}_i\}.$$

(iii) $\forall m \in M$, $\forall e \in E$:

$$\text{If } \forall i \in N\colon \quad F_i(\gamma_i(T_i(m))) \subset L(e_i, F_i(\gamma_i(m))),$$

$$\text{then } F_i(\gamma_i(m)) = F_i(e) \quad \forall i \in N.$$

PROOF: See Section 5.

The basic idea of Theorem 3.1 is that, if M can serve as a message space for implementation, then there will exist mappings which take messages "back" to environments. When restricted to the set D of Nash equilibria, these mappings are identical to the inverse of the Nash correspondence.

Let $m \in M$ be a Nash equilibrium for $e \in E$. Consider the image of the mapping γ_i under the intersection of messages which the ith agent can effect by unilateral deviation from the set D. The choice rule must map this image set in the class of environments into the ith agent's lower contour set relative to the allocation prescribed by the choice rule at $e \in E$. Condition (ii) can be given the following calculus representation. For $m \in D$, assume that γ has (locally) a differentiable thread $t(\cdot)$. Assume furthermore that the γ_i's have (locally) differentiable threads $t_i(\cdot)$ such that the restrictions of $t_i(\cdot)$ to D agree with $t(\cdot)$. Assume, furthermore, that agents' preferences are representable by differentiable utility functions $U(z_i|e_i)$. Condition (ii) then implies that for every $m \in D$:

$$(4) \qquad \nabla U(F_i(t(m))|(t(m))_i) \circ DF_i(t(m)) \circ \begin{pmatrix} 0,\dots,0 & \frac{\partial t_i^1}{\partial m_i^1} & \cdots & \frac{\partial t_i^1}{\partial m_i^{k_i}} & 0,\dots,0 \\ & \vdots & & \vdots & \\ 0,\dots,0 & \frac{\partial t_i^q}{\partial m_i^1} & \cdots & \frac{\partial t_i^q}{\partial m_i^{k_i}} & 0,\dots,0 \end{pmatrix} = [0,\dots,0].$$

Here, the function $t_i(\cdot)$ maps from the message space $\mathscr{R}^{k_1} \times \cdots \times \mathscr{R}^{k_n}$ to $\mathscr{R}^q$, where a point in $\mathscr{R}^q$ represents an environment. Equation (4) says that for every $e = t(m)$ and $i \in N$ the gradient of the indirect utility function $V(e|e_i) \equiv U(F_i(e)|e_i)$ must be orthogonal to those columns of $Dt_i(m)$ that correspond to the ith agent's strategies. This constrains the rank of $Dt_i(m)$, potentially leading to a conflict with the "onto" requirement in (i).

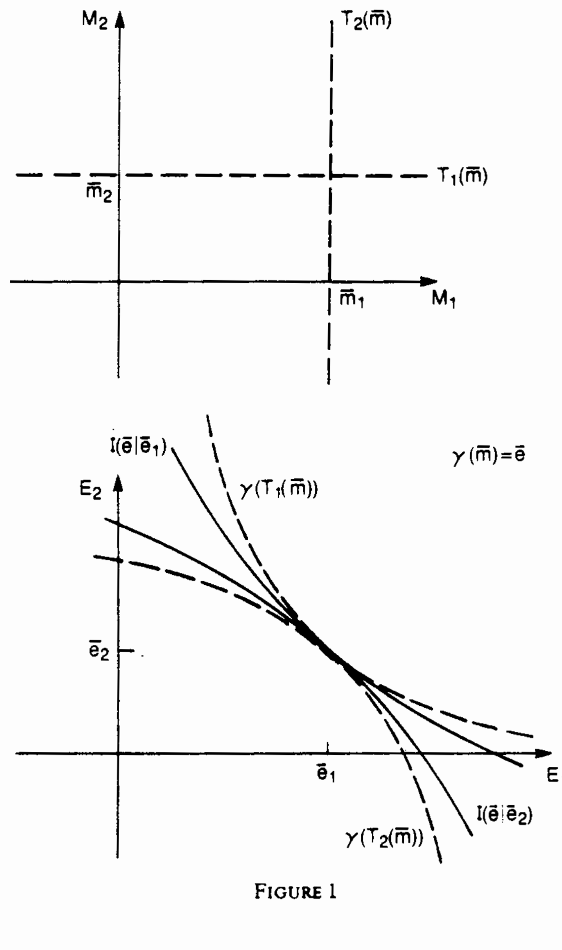

FIGURE 1

To illustrate this result we return to the example discussed in the preceding section. Consider again the Walrasian choice rule on the class of environments in which traders' preferences are described by linear-quadratic utility functions.

In Figure 1, the curve $I(\bar{e}|\bar{e}_i)$, $1 \le i \le 2$, represents the indifference curve of the indirect utility function $V(e_1, e_2|\bar{e}_i) \equiv U(F_i(e_1, e_2)|\bar{e}_i)$ through the point $(e_1, e_2) = (\bar{e}_1, \bar{e}_2)$. Pareto optimality of Walrasian allocations implies that the two indifference curves have to be tangent at $(\bar{e}_1, \bar{e}_2)$.

Next, suppose that $M = \mathscr{R}^2$. In that case the set of Nash equilibria, D, must contain an open set in M. Hence, $\gamma_1 = \gamma_2 = \gamma$ on this open set. Also, the correspondence γ is a function, since the γ_i are F_i compatible as required by condition (i) in Theorem 3.1. Finally, γ is differentiable provided the mechanism's outcome function is smooth. Let $\gamma(\bar{m}) = \bar{e}$; condition (ii) in Theorem 3.1 says that $\gamma(T_1(\bar{m}))$ is contained in the set $L(F_1(\bar{e}), \bar{e}_1)$ and, consequently, the line tangent to $I(\bar{e}|\bar{e}_1)$ at $\bar{e}$, is also tangent to $\gamma(T_1(\bar{m}))$ at $\gamma(\bar{m})$. Repeating the same argument for the second agent shows that the tangent lines of $\gamma(T_1(\bar{m}))$ and $\gamma(T_2(\bar{m}))$ are in fact identical at $\gamma(\bar{m})$. However, this contradicts the requirement that $\gamma(\cdot)$ be onto, which would imply that the tangent spaces of the images of the two perpendicular manifolds $T_1(\bar{m})$ and $T_2(\bar{m})$ span the entire two dimensional space of environments.

The same conclusion is obtained through the use of equation (4). Straightforward calculations yield:

$$V(e|\bar{e}_1) \equiv U(F_1(e)|\bar{e}_1)$$
$$= \bar{e}_1 \cdot (F_1^x(e) + 1) - \tfrac{1}{2}(F_1^x(e) + 1)^2 - p(e)F_1^x(e)$$

where $p(e) = \frac{1}{2}(e_1 + e_2 - 2)$,

$$\nabla U(F_1(e)|e_1) \circ DF_1(e)$$
$$= (e_1 - (F_1^x(e) + 1), 1)$$
$$\times \circ \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2}[e_1 - 1] & -\frac{1}{2}[e_2 - 1] \end{pmatrix} = -\tfrac{1}{2}(F_1^x(e), F_1^x(e)).$$

A similar computation for the second agent shows that

$$\nabla U(F_2(e)|e_2) \circ DF_2(e) = -\tfrac{1}{2}(F_2^x(e), F_2^x(e)).$$

Theorem 3.1 says that there exists an onto function γ: $M \to \mathscr{R}^2$ such that[9]

$$\nabla U\big(F_1(\gamma(m))|\gamma^1(m)\big) \circ DF_1(\gamma(m)) \circ \begin{pmatrix} \dfrac{\partial \gamma^1}{\partial m_1} & 0 \\ \dfrac{\partial \gamma^2}{\partial m_1} & 0 \end{pmatrix} = [0, 0]$$

[9] Here, γ^i stands for the ith component of the function γ.

and

$$\nabla U\big(F_2(\gamma(m))|\gamma^2(m)\big)\circ DF_2(\gamma(m))\circ\begin{pmatrix}0 & \dfrac{\partial\gamma^1}{\partial m_2}\\ 0 & \dfrac{\partial\gamma^2}{\partial m_2}\end{pmatrix}=[0,0].$$

Since the vectors $\nabla U(F_i(\gamma(m))|\gamma^i(m))\circ DF_i(\gamma(m)), 1\le i\le 2$, are collinear, it follows that $D\gamma(m)$ has rank one, which contradicts the requirement that γ be onto.

In contrast, for the mechanism given at the end of Section 2, where the strategy space has one additional dimension, the following equations emerge:[10]

$$\gamma_1(m)=(1+m_3+m_1-m_2, 1+m_3-m_1+m_2),$$

$$\gamma_2(m)=\left(1+2m_1-m_2+\frac{(m_1-m_3)^2}{(m_2-m_1)}, 1+\frac{(m_1-m_3)^2}{(m_2-m_1)}+m_2\right)\quad \text{if}\quad m_2\neq m_1,$$

$$D=\{m\in M_1\times M_2|m_1=m_3\}.$$

The restrictions of γ_i to the set D agree with each other, i.e.,

$$\gamma_1|_D=\gamma_2|_D=\gamma\quad\text{and}$$

$$\gamma(m)=(1+2m_1-m_2, 1+m_2).$$

For every $m\in M$, the set $\gamma_2(T_2(m))$ is a half-space. By setting $m_3=m_1$, the second agent moves to the boundary of this half-space.

We conclude this section by providing a partial converse of Theorem 3.1, partial because we now assume that the space Z is the Cartesian product of Z_i's, while Theorem 3.1 required only that $Z\subset \times_{i=1}^n Z_i$.

THEOREM 3.2: *If* $Z=\times_{i=1}^n Z_i$, *the existence of* γ_i: $M\twoheadrightarrow E$, *satisfying conditions* (i)–(iii) *in Theorem* 3.1, *is sufficient for the existence of a mechanism* $\langle M, g\rangle$ *which implements* F: $E\to Z$.

PROOF: See Section 5.

4. IMPLEMENTING WALRASIAN ALLOCATIONS

This section examines the dimensional requirements for the implementation of Walrasian allocations. Our framework is one of pure exchange; there are n-traders and l-commodities in the economy. Agents are described by their characteristics

$$e_i=(X_i, R_i, w_i)$$

where $X_i\subset\mathscr{R}^l$ represents the ith agent's consumption set, R_i denotes his

[10] We note that γ_2 is differentiable at any $m\in M$ provided that m is a regular point of the function g.

complete binary and reflexive preference relation on X_i and $w_i \in \mathscr{R}^l_+$ represents his initial endowment. By Z we denote the space of net-trades, i.e.,[11]

$$Z = \{ z = (z_1, \ldots, z_n) | z_i \in \mathscr{R}^l \}.$$

For a given class of economies $E = \times_{i=1}^n E_i$, the Walrasian choice rule

$$W: E \twoheadrightarrow Z$$

is defined as follows: $z \in W(e)$ iff $\sum_{i=1}^n z_i = 0$ and $\exists p \in \mathscr{R}^l_+$ such that $\forall i \in N$:

(i) $z_i + w_i \in X_i, \qquad p \cdot z_i \leq 0,$

(ii) $(z_i + w_i) R_i (\bar{z}_i + w_i) \quad \forall \bar{z}_i \in \mathscr{R}^l$ such that $p \cdot \bar{z}_i \leq 0$.

To find lower bounds on the message space needed for the implementation of $W(\cdot)$, it will be convenient to analyze the dimensional requirements of a subclass $\bar{E} \subset E$. Environments in this class have the following special features:

$$X_i = \mathscr{R}^l_+, \quad w_i = (1, \ldots, 1, \bar{w}^l).$$

Writing a consumption vector in $\mathscr{R}^l$ as $(x^1, \ldots, x^{l-1}, y)$, each agent's preferences can be locally represented by a linear-quadratic utility function of the form:[12]

$$U(x, y | e_i) = e_i \cdot x - \tfrac{1}{2} x^t \cdot x + y \qquad \text{where} \qquad e_i \in \mathbb{R}^{l-1}_+, \quad e_i^j > 1.$$

If agents' endowments of the numeraire good are fixed at some level, say $\bar{w}^l$,[13] an environment in $\bar{E}$ can be identified with the point $(e_1, \ldots, e_n) \in \mathscr{R}^{n(l-1)}$.

For each economy in the class $\bar{E}$ there exists a unique price equilibrium; hence $W(\cdot)$ is single-valued on $\bar{E}$. The informational requirements literature has shown that, subject to the regularity conditions discussed above, $\mathscr{R}^{n \cdot (l-1)}$ is the minimal message space needed to realize W on $\bar{E}$. This bound is attained by the competitive mechanism, as represented in Hurwicz (1977) and Mount and Reiter (1974), which uses $\mathscr{R}^{n \cdot (l-1)}$ as its message space and attains Walrasian equilibria on a broad class of economies with convex and continuous preferences.

From the discussion in Section 2, which dealt with the case $n = 2$ and $l = 2$, we expect a dimensional increase for Nash implementation. The exact magnitude of that increase is given in Theorem 4.1 below. We call an environment $\bar{e} \in \bar{E}$ regular if for all $i \in N$ the Jacobian $DW_i(\bar{e})$ has rank l.

THEOREM 4.1: *Let $\langle M, g \rangle$ implement $W: \bar{E} \to Z$, with $M_i = \mathscr{R}^{k_i}$ and $g: M \to Z$ continuously differentiable. Suppose that for a regular environment $\bar{e} \in \bar{E}$ and a neighborhood $O(\bar{e}) \subset \bar{E}$ the restriction of the Nash correspondence to $O(\bar{e})$, i.e.,*

[11] Note that we do not insist on balanced net trades, i.e., $\sum_{i=1}^n z_i = 0$.

[12] Jordan (1986) shows that it is possible to extend preferences that can on some neighborhood be described by a quadratic utility function to the entire consumption set such that the extended preference relation satisfies the usual regularity conditions, i.e., continuity, convexity, and monotonicity. Furthermore, the extension can be chosen such that the Walrasian allocation is unique and lies in the neighborhood in which preferences are described by the quadratic utility function.

[13] Since preferences are linear in the numeraire good, Walrasian allocations will not depend on the numeraire good endowment $\bar{w}^l$, provided this endowment is sufficiently large.

$\rho|_{O(\bar{e})}$, *has a linear thread. Then*

$$\sum_{i=1}^{n} k_i \geq n \cdot (l-1) + \psi(n,l)$$

where $\psi(n,l)$ *is defined as*:

$$\psi(n,l) = \min_{\bar{k} \in \mathcal{N}} \{\bar{k} | (n-1) \cdot \bar{k} \geq (l-1)\}, \quad \mathcal{N} \equiv \{1,2,3,\dots\}.$$

PROOF: See Section 5.

According to Theorem 4.1 the dimensional increment depends on the relationship between n and l. In particular, if $n \geq l$, one extra dimension is needed: If $n = 2$, then the dimension of the message space has to increase by $(l-1)$ over that required for realization. For general n and l, $\psi(n,l)$ is the unique positive integer satisfying the equation:

$$(l-1) = [\psi(n,l) - 1](n-1) + \bar{h}, \quad 0 < \bar{h} \leq n-1.$$

There is always a unique pair $(\psi(n,l), \bar{h})$ satisfying this equation. The condition of linear threadedness in the Nash correspondence will greatly simplify the proof of our dimensional formula. Instead of having to work with a system of partial differential equations, the argument reduces to an analysis of the rank of a system of linear equations.[14] We believe, however, that our formula remains valid even when the thread of the Nash correspondence is not required to be linear. For the case $n \geq l$, we have found a class of solutions for the corresponding system of partial differential equations. For this class of solutions the argument in the Appendix demonstrates the validity of the formula. However, it remains a conjecture that the class of solutions analyzed in the Appendix includes all solutions.

We next construct a mechanism that implements Walrasian allocations with a message space of the indicated size. Consider a class of economies E characterized by the following conditions:

(i) $w_i \in X_i$, $\{w_i\} + \mathscr{R}^l_+ \subset X_i$.

(ii) Preferences are monotone increasing in the numeraire good.

Since nonequilibrium outcomes may be individually infeasible, it is necessary to extend the preference relation to all of $\mathscr{R}^l$. This extended preference relation is denoted by $\bar{R}_i$ and is chosen such that it preserves the preferences R_i on X_i and every point outside the consumption set is strictly inferior to any point in X_i. Formally,

if $x, x' \in X_i$, then $x\bar{R}_i x'$ if and only if $xR_i x'$;

if $x \in X_i$, $x' \notin X_i$, then $x\bar{R}_i x'$ but not $x'\bar{R}_i x$.

[14] It should be noted that the familiar mechanisms of Groves and Ledyard (1977), Hurwicz (1979a), Schmeidler (1980), and Walker (1981) all meet the linear threadedness condition on $\bar{E}$.

THEOREM 4.2: *The (smooth) mechanism* $\langle \times_{i=1}^{n} M_i, g\rangle$, $M_i = \mathscr{R}^{k_i}$, *as defined in equations* (5)–(8), *implements* $W\colon E \twoheadrightarrow Z$ *such that*

$$\sum_{i=1}^{n} k_i = n\cdot(l-1) + \psi(n,l).$$

PROOF: Let $M_i = \mathscr{R}^{l-1}$, $1 \le i \le n-1$, and $M_n = \mathscr{R}^{l-1} \times \mathscr{R}^{\psi(n,l)}$. An element in M_n will be represented as (m_n, u) and $n \equiv (m_1, \dots, m_n, u)$. We define:

(5) $$g_i^j(m) = m_i^j - m_{i+1}^j, \quad 1 \le i \le n, \quad 1 \le j \le l-1,$$

where the subscript $(n+1)$ is understood "modulo n,"

(6) $$g_i^l(m) = -\sum_{j=1}^{l-1} p_i^j(m) g_i^j(m), \quad 1 \le i \le n-1,$$

(7) $$g_n^l(m) = -\sum_{j=1}^{l-1} p_n^j(m) g_n^j(m) - \sum_{s=1}^{\psi(n,l)} \left(u_s - \sum_{r=1}^{\beta(s)} m_r^{\alpha(r,s)} \right)^2,$$

where $\alpha(r,s) \equiv (s-1)(n-1) + r$,

$$\beta(s) = \begin{cases} n-1 & \text{if } s < \psi(n,l), \\ \bar{h} & \text{if } s = \psi(n,l). \end{cases}$$

We recall that $\bar{h}$ is given by the equation

$$(l-1) = (\psi(n,l) - 1)(n-1) + \bar{h}.$$

Define $h(j)$, $k(j)$, by the equation

$$j = (k(j)-1)(n-1) + h(j),$$
$$1 \le k(j) \le \psi(n,l), \quad 0 < h(j) \le n-1.$$

Finally,

(8) $$p_i^j(m) = \begin{cases} m_{h(j)}^j & \text{if } h(j) \ne i, \\ u_{k(j)} - \sum\limits_{\substack{r=1 \\ r \ne i}}^{\beta(k(j))} m_r^{\alpha(r,k(j))} & \text{if } h(j) = i. \end{cases}$$

We note in passing that this outcome function is balanced for all but the numeraire good. Interpreting the $p_i^j(\cdot)$ as prices at which the ith agent trades the jth good we note that p_i^j does not depend on the ith agent's message choice. In other words, every agent acts like a price taker. Now, suppose that $m^* \in \rho(e)$. Since the nth agent's preferences are monotone increasing for the numeraire good, m^* is a Nash equilibrium only if

$$u_s^* - \sum_{r=1}^{\beta(s)} m_r^{*\alpha(r,s)} = 0, \quad 1 \le s \le \psi(n,l).$$

Therefore, at a Nash equilibrium,

$$p_i^j(m^*) = p_{i'}^j(m^*), \quad 1 \le i, i' \le n, \quad 1 \le j \le l-1,$$

and

$$\sum_{i=1}^{n} g_i^j(m^*) = 0, \quad 1 \le j \le l.$$

Furthermore,

$$w_i + g_i(m^*) R_i(e_i) w_i + z_i$$

for all

$$z_i \in \{ \bar{z}_i | \bar{z}_i = g_i(m^*_{-i}, \overline{m}_i), \overline{m}_i \in M_i \} = \left\{ \bar{z}_i \middle| \sum_{j=1}^{l-1} p_i^j(m^*) \bar{z}_i^j + \bar{z}_i^l = 0 \right\}.$$

In particular, $\bar{z}_i = 0$ is one attainable alternative, so that, if m^* is a Nash equilibrium, it must be the case that $w_i + g_i(m^*) \in X_i$. It follows that $(g_1(m^*) \cdots g_n(m^*))$ is a Walrasian allocation with equilibrium price vector $(p^1(m^*) \cdots p^{l-1}(m^*), 1)$.

Conversely, for $e \in E$, let $(\bar{x}_1, \dots, \bar{x}_n) \in \mathscr{R}^{n \cdot l}$ be a Walrasian allocation[15] with equilibrium prices $(\bar{p}^1, \dots, \bar{p}^{l-1}, 1)$. We construct a message-tuple that is a Nash equilibrium at $e \in E$ and induces the Walrasian allocation. Set $m^j_{h(j)} = \bar{p}^j$ for $1 \le j \le l-1$ where, again, $h(j)$ is the unique positive integer satisfying

$$j = (k(j) - 1)(n - 1) + h(j),$$
$$1 \le k(j) \le \psi(n, l), \quad 0 < h(j) \le n - 1.$$

Next, consider the system of linear equations

$$\bar{x}_i^j = m_i^j - m_{i+1}^j, \quad 1 \le j \le l - 1, \quad 1 \le i \le n.$$

For every good j there are $(n-1)$ independent linear equations in $(n-1)$ variables (the value of $m^j_{h(j)}$ is already fixed). Therefore, there exists a unique solution such that:

$$(g_1(m), \dots, g_n(m)) = (\bar{x}_1, \dots, \bar{x}_n) \quad \text{and} \quad m^j_{h(j)} = \bar{p}^j.$$

Finally, the "auxiliary" variables $\{u_s\}$ are determined by the equations:

$$u_s = \sum_{r=1}^{\beta(s)} m_r^{\alpha(r,s)}$$

where again $\alpha(r, s) = (s-1)(n-1) + r$, $1 \le s \le \psi(n, l)$, and

$$\beta(s) = \begin{cases} n - 1 & \text{if } s < \psi(n, l), \\ \bar{h} & \text{if } s = \psi(n, l). \end{cases}$$

As shown above, the set of allocations attainable for the ith agent by unilateral deviation from $m \in M$ is the budget plane given by the price system $(\bar{p}^1 \cdots \bar{p}^{l-1})$ and endowments $w_i \in X_i$.

Since $(g_1(m), \dots, g_n(m)), (\bar{p}^1 \cdots \bar{p}^{l-1}, 1)$ is a competitive equilibrium by hypothesis, m is a Nash equilibrium. This completes the proof of Theorem 4.2.

[15] It should be noted that Walrasian allocations do not necessarily exist for all $e \in E$. If $W(e) = \varnothing$, our claim is vacuous.

The mechanism presented, like the one in Section 2, has the feature that the set of Nash equilibria forms an $n \cdot (l-1)$ dimensional linear manifold within the message space; it is the intersection of $\psi(n, l)$ different hyperplanes. We note that the linear threadedness condition of Theorem 4.1 is satisfied. The basic intuition underlying our mechanism is that monopolistic behavior has to be ruled out, i.e., no agent may control the prices at which he trades commodities. To accomplish this, we first give every agent one message variable with which he influences the allocation of the jth good. Because of the balancing requirement, there are only $(n-1)$ independent allocations to be made for any particular good. Therefore, one message variable is left to determine the price. In our mechanism, the nth agent also chooses a number of auxiliary variables. Independently of his characteristics, the nth agent has an incentive to choose these auxiliary variables such that each one of them lies in a given plane with, at maximum, $(n-1)$ different price-setting variables, one provided by each of the remaining agents. Thus, one auxiliary variable can ensure price taking behavior for at most $(n-1)$ commodities in an economy with n agents. Accordingly, the number of auxiliary variables needed depends on the ratio between the number of commodities and the number of traders.

Though the mechanism presented in Theorem 4.2 is efficient with respect to the size of the message space, it has the unappealing feature that net trades may not be balanced if agents fail to reach an equilibrium. In case $n > l$, this problem can be avoided by a slight modification of our mechanism. The basic idea is to allocate the residual quantity of the numeraire good (recall that our mechanism is balanced for all other goods) among those agents that are not price-setters. Provided that $n > l$ there is at least one such agent. It turns out that the residual quantity is independent of those agents' messages. Therefore, balancedness can be achieved through a system of transfer payments that do not affect incentives. In equilibrium the transfers are zero, since the original mechanism is balanced in equilibrium.

However, this construction fails if $n \leq l$. Some preliminary analysis indicates that, in case $n = 3 = l$, it is impossible to implement Walrasian allocations with a seven-dimensional strategy space while maintaining balancedness throughout. A possibility result obtains when the size of the message space is raised to eight dimensions. This observation suggests that the minimal message space needed for implementation may change as the set of permissible outcomes is enlarged from Z to some set $\tilde{Z}$, where $F(E) \subset Z \subset \tilde{Z}$. For Walrasian allocations $(n = 3 = l)$, the minimal message space decreases by one dimension as the permissible choice set is expanded from the six dimensional set of balanced net trades to the nine dimensional set of possible net trades. While this problem remains to be analyzed in detail, a first insight is gained by observing that the outcome functions g_i are functionally dependent if net trades have to be balanced. As demonstrated in Sections 3 and 4, the dimensional requirements for implementation depend on the existence of correspondences γ_i: $M \twoheadrightarrow E$ satisfying certain conditions. Since $\gamma_i(m) \subset (F_i^{-1} \circ g_i)(m)$, the γ_i's will be constrained further if the g_i's are functionally dependent.

5. PROOFS

PROOF OF THEOREM 3.1: Define:

$$\gamma_i(m) = \begin{cases} \rho^{-1}(m) & \text{if } m \in D, \\ (F_i^{-1} \circ g_i)(m) & \text{if } m \notin D. \end{cases}$$

(i) Clearly, $\forall m \in M$ $\gamma_i(m) \subset (F_i^{-1} \circ g_i)(m)$. Hence, $e, \bar{e} \in \gamma_i(m)$ implies $F_i(e) = F_i(\bar{e})$. By construction, $\gamma_1 = \cdots = \gamma_n = \gamma$ for $m \in D$. Since $\langle M, g \rangle$ implements F, γ is onto.

(ii) Let $i \in N$, $m \in D$, and $e \in \gamma(m)$. Then $m \in \rho(e)$ and $F_i(e) R(e_i) g_i(m_{-i}, \bar{m}_i)$, $\forall \bar{m}_i \in M_i$. Now, $\bar{e} \in \gamma_i(m_{-i}, \bar{m}_i)$ implies $F_i(\bar{e}) = g_i(\bar{m}_i, m_{-i})$, and consequently, $F_i(\bar{e}) \in L(e_i, F_i(e))$.

(iii) Let $m \in M$, $e \in E$, and $\forall i \in N$: $F_i(\gamma_i(T_i(m))) \subset L(e_i, F_i(\gamma_i(m)))$. By construction, $L(e_i, F_i(\gamma_i(m))) = L(e_i, g_i(m))$ and $F_i(\gamma_i(T_i(m))) = g_i(T_i(m))$. Hence, m is a Nash-equilibrium at $e \in E$ which implies $g(m) = F(e)$, since $\langle M, g \rangle$ implements F.

PROOF OF THEOREM 3.2: Define g_i: $M \to Z_i$ by $g_i = F_i \circ \gamma_i$. It must be shown that (a) for every environment Nash equilibria exist and (b) all Nash equilibria yield desired outcomes.

(a) Given $e \in E$, there exists an $m \in D$ such that $e \in \gamma(m)$. Condition (ii) implies:

$$\forall \bar{m}_i \in M_i$$
$$g_i(m) = (F_i \circ \gamma_i)(m) = F_i(e) R_i(e_i) F_i(\gamma_i(m_{-i}, \bar{m}_i)) = g_i(m_{-i}, \bar{m}_i).$$

(b) Suppose $m \in M$ is a Nash equilibrium for $\bar{e} \in E$, i.e.,

$$g_i(m) R_i(\bar{e}_i) g_i(m_{-i}, \bar{m}_i) \quad \forall (\bar{m}_i, m_{-i}) \in T_i(m).$$

This implies that

$$F_i(\gamma_i(T_i(m))) \subset L(\bar{e}_i, F_i(\gamma_i(m))).$$

Hence, Condition (iii) says that $g_i(m) = F_i(\bar{e})$.

PROOF OF THEOREM 4.1: *Step 1:* If $\langle M, g \rangle$ implements W on $\bar{E}$, then for any two points $e, \bar{e} \in \bar{E}$:

$$\rho(e) \cap \rho(\bar{e}) = \varnothing.$$

Suppose to the contrary that two environments have a Nash equilibrium in common, i.e., $m^* \in \rho(e) \cap \rho(\bar{e})$. Implementation requires that $W(e) = W(\bar{e})$. Since the Nash correspondence $\rho(\cdot)$ is a coordinate correspondence, it follows that: $m^* \in \rho(\tilde{e})$ $\forall \tilde{e} \in \bar{E}$ with $\tilde{e}_i \in \{e_i, \bar{e}_i\}$. All environments on the "cube" formed by the e and $\bar{e}$ have to have $m^* \in M$ as a Nash equilibrium. Hence, $W(e) = W(\tilde{e})$ for all $\tilde{e}$ on the cube. However, a straightforward calculation shows that

$$W_i^j(e) = \frac{1}{n}\left[(n-1)e_i^j - \sum_{i \neq i'} e_{i'}^j\right] \quad \text{for} \quad e \in \bar{E}.$$

These linear equations imply that $W_i^j(e) = W_i^j(\tilde{e})$ for all $\tilde{e}$ on the cube formed by e and $\bar{e}$, only if, in fact, $e = \bar{e}$.

Step 2: Suppose that $M_i = \mathscr{R}^{k_i}$ and $k \equiv \sum_{i=1}^{n} k_i = n \cdot (l-1) + \bar{k}$. We have to show that $\bar{k} \geq \psi(n, l)$. By assumption, at the regular environment $\bar{e} \in \bar{E}$, referred to in the statement of the theorem, the Nash correspondence has a linear thread on some neighborhood $O(\bar{e})$. Denote this linear thread by t^{-1}. It follows from Step 1 that t^{-1} is one-to-one and consequently $t^{-1}(O(\bar{e}))$ forms a linear $n \cdot (l-1)$ dimensional manifold in the message space M. Recall that $n \cdot (l-1)$ is the dimension of $\bar{E}$. Next, the inverse of the Nash-correspondence $\gamma: \rho(\bar{E}) \to \bar{E}$ has to be a function, because $\rho(\cdot)$ is injective. Denoting the inverse of $t^{-1}(\cdot)$ by $t: V(\bar{m}) \to \bar{E}$, where $\bar{m} = t^{-1}(\bar{e})$ and $V(\bar{m}) \equiv t^{-1}(O(\bar{e}))$, it follows that $\gamma|_{V(\bar{m})} = t$.

For $1 \leq i \leq n$, let $t_i: V(\bar{m}) \to \bar{E}$ denote differentiable threads from the correspondences $(\gamma_1 \cdots \gamma_n)$ as defined in Theorem 3.1. To demonstrate that differentiable threads exist, it suffices to show that on some neighborhood $O(\bar{m}) \subset M$ the correspondence

$$\hat{\gamma}_i(m) = \begin{cases} t(m) & \text{if } m \in V(\bar{m}), \\ (F_i^{-1} \circ g_i)(m) & \text{if } m \notin V(\bar{m}), \end{cases}$$

has a differentiable thread. Let $f_i: O(\bar{m}) \to V(\bar{m})$ be any differentiable function such that $f_i(m) = m$ if $m \in V(\bar{m})$ and $g_i(f_i(m)) = g_i(m)$ for all $m \in O(\bar{m})$. Such a function f_i will always exist, if the Jacobian $Dg_i(m)$ has rank l on the neighborhood $O(\bar{m})$. Since $g_i(t^{-1}(e)) = F_i(e)$ for all $e \in O(\bar{e})$, it follows that $Dg_i(\bar{m})$ has rank l, if $\bar{e}$ is a regular environment. Finally, it is easy to check that

$$t_i \equiv (t \circ f_i): O(\bar{m}) \to \bar{E}$$

is a differentiable thread of $\hat{\gamma}_i$ and

$$t_i|_{V(\bar{m})} = t \quad \text{for all} \quad 1 \leq i \leq n.$$

This implies the existence of a $k \times k$ matrix A which has rank $n \cdot (l-1)$ such that for each $m \in V(\bar{m})$:

$$Dt_1(m) \circ A = \cdots Dt_n(m) \circ A = Dt(m).$$

We write the derivative of the linear function $t(\cdot)$ as:

$$Dt(m) = [\beta_1, \ldots, \beta_k], \quad \beta_j \in \mathscr{R}^{n \cdot (l-1)}.$$

Then the derivatives of the functions $t_i(\cdot)$ can be expressed uniquely in the form:

$$Dt_i(m) = \left[\beta_1 + \sum_{j=1}^{\bar{k}} s_1^j \lambda^j(m), \ldots, \beta_k + \sum_{j=1}^{\bar{k}} s_k^j \lambda^j(m)\right]$$

for all $m \in V(\bar{m})$. The $\lambda^j(m)$ are arbitrary vectors in $\mathscr{R}^{n(l-1)}$ and the vectors $\{s^1 \cdots s^{\bar{k}}\}$ span the nullspace of A (recall that by definition $\bar{k} = k - n \cdot (l-1)$).

To invoke the first order conditions for Nash equilibrium as stated in (4), Section 3, we first calculate the gradient of the indirect utility function

$V(m|(t^{-1}(m))_i) \equiv U(W_i(t^{-1}(m))|(t^{-1}(m))_i)$ which becomes:[16]

$$\nabla U\big(W_i(t^{-1}(m))|(t^{-1}(m))_i\big) \circ DW_i(t^{-1}(m)) = -\sum_{j=1}^{l-1} W_i^j(t^{-1}(m)) \cdot \eta^j$$

where $\eta^j \in \mathscr{R}^{n(l-1)}$ is the following vector:

$$\eta^j = (\overbrace{0 \cdots \underset{\substack{\uparrow \\ j\text{th}}}{1} \cdots 0}^{l-1}, \overbrace{0 \cdots \underset{\substack{\uparrow \\ j\text{th}}}{1} \cdots 0}^{l-1}, \ldots, \overbrace{0 \cdots \underset{\substack{\uparrow \\ j\text{th}}}{1} \cdots 0}^{l-1}).$$

η^j has n 1's and 0's otherwise. $W_i^j(t^{-1}(m))$ is just the Walrasian allocation of the jth good to the ith agent at the environment $t^{-1}(m)$.

It may be noted at this point that the gradients of the indirect utility functions, which are elements of $\mathscr{R}^{n(l-1)}$, all lie in some $(l-1)$ dimensional subspace spanned by the vectors $(\eta^1 \cdots \eta^{l-1})$. This feature must be considered special to the Walrasian choice rule and ultimately explains the increase in dimensional requirements.

First-order conditions for the ith agent then read as follows:

$$(9) \qquad -\sum_{j=1}^{l-1} W_i^j(t^{-1}(m)) \cdot \eta^j \circ \left[0,\ldots,0,\beta_{k_{i-1}+1} + \sum_{j=1}^{\bar{k}} s_{k_{i-1}+1}^j \lambda^j(m),\ldots, \beta_{k_i} + \sum_{j=1}^{\bar{k}} s_{k_i}^j \lambda^j(m), 0 \cdots 0\right] = [0,\ldots,0].$$

The columns in $Dt_i(t^{-1}(m))$ that correspond to the ith agent's messages $(m_{k_{i-1}+1} \cdots m_{k_i})$ have to be orthogonal to the gradient of the ith agent's indirect utility function.

If the terms involving $\lambda^j(m)$ were zero in equation (9), that equation would imply that the β's belong to the orthogonal complement of the space spanned by $\{\eta^1 \cdots \eta^{l-1}\}$. It is shown in Step 3 that if there are $\lambda^j(m)$ satisfying (9), then there must be fixed vectors $\lambda_i^1 \cdots \lambda_i^{\bar{k}}$ such that the k_i vectors:

$$(10) \qquad \left\{\beta_{k_{i-1}+1} + \sum_{j=1}^{\bar{k}} s_{k_{i-1}+1}^j \lambda_i^j, \ldots, \beta_{k_i} + \sum_{j=1}^{\bar{k}} s_{k_i}^j \lambda_i^j\right\}$$

lie in the orthogonal complement of the space spanned by $\{\eta^1 \cdots \eta^{l-1}\}$ provided $k_i \geq \bar{k}$. We denote this $(n-1)\cdot(l-1)$ dimensional subspace by $L(\eta^1 \cdots \eta^{l-1})^{\perp}$.

[16] This calculation is shown explicitly for the case $n = 2 = l$ in Section 2 and for the case $n = 3 = l$ in the Appendix.

Step 3: To show that (9) implies the existence of vectors $\lambda_i^1 \cdots \lambda_i^{\bar{k}}$ such that the k_i vectors in (10) lie in $L(\eta^1 \cdots \eta^{l-1})^{\perp}$, we omit the subscript i, since the argument can be made for each agent separately.

Define $\eta(m) \equiv \sum_{j=1}^{l-1} W_i^j(t^{-1}(m))\eta^j$ and write

$$\lambda^j(m) = c^j(m)\eta(m) + \sum_{v=1}^{(n-1)(l-1)} \bar{c}_v^j(m)\bar{\eta}^v + \sum_{v=1}^{l-2} c_v^j(m)\tilde{\eta}^v(m)$$

where $\{\bar{\eta}^1 \cdots \bar{\eta}^{(n-1)(l-1)}\}$ are orthogonal to $L(\eta^1 \cdots \eta^{l-1})$ and the vectors $\{\tilde{\eta}^1(m) \cdots \tilde{\eta}^{l-2}(m)\}$ are chosen such that $\{\bar{\eta}^1 \cdots \bar{\eta}^{(n-1)(l-1)}, \eta(m), \tilde{\eta}^1(m) \cdots \tilde{\eta}^{(l-2)}(m)\}$ is an orthogonal system. The conditions in (9) require that[17]

$$\begin{aligned} &\langle \beta_1, \eta(m)\rangle + \sum_{j=1}^{\bar{k}} s_1^j c^j(m)\langle \eta(m), \eta(m)\rangle = 0, \\ &\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ &\langle \beta_k, \eta(m)\rangle + \sum_{j=1}^{\bar{k}} s_k^j c^j(m)\langle \eta(m), \eta(m)\rangle = 0. \end{aligned} \tag{11}$$

Consider first the case $\bar{k} = 1$. The first two equations in (11) then read as follows:

$$\langle \beta_1, \eta(m)\rangle + s_1^1 c^1(m)\langle \eta(m), \eta(m)\rangle = 0,$$
$$\langle \beta_2, \eta(m)\rangle + s_2^1 c^1(m)\langle \eta(m), \eta(m)\rangle = 0.$$

Solving the first equation for $c^1(m)$ yields:

$$c^1(m) = -\frac{1}{s_1^1}\left[\frac{\langle \beta_1, \eta(m)\rangle}{\langle \eta(m), \eta(m)\rangle}\right].$$

(If $s_1^1 = 0$ the claim follows immediately.)

Let λ be an arbitrary vector such that

$$\beta_1 - s_1^1\lambda \in L(\eta^1 \cdots \eta^{l-1})^{\perp}.$$

Substitution into the second equation yields:

$$\langle \beta_2, \eta(m)\rangle + \frac{s_2^1}{s_2^2}\langle \theta + s_1^1\lambda, \eta(m)\rangle = 0$$

with $\theta \in L(\eta^1 \cdots \eta^{l-1})^{\perp}$.

Since θ is orthogonal to $\eta(m)$ for all $m \in V(\overline{m})$, it follows that

$$\langle \beta_2 - s_2^1\lambda, \eta(m)\rangle = 0.$$

Since $\eta(m)$ covers the entire space spanned by $\{\eta^1 \cdots \eta^{l-1}\}$ as m varies, it follows that:

$$\beta_2 - s_2^1\lambda \in L(\eta^1 \cdots \eta^{l-1})^{\perp}.$$

The same argument can be made for the vectors $\beta_3 \cdots \beta_k$.

[17] Subsequently, $\langle \cdot, \cdot \rangle$ will denote the inner product.

For general $\bar{k}$, the first $\bar{k}$ equations in (11) can be expressed in the form:

$$(12)\qquad -\begin{pmatrix} \tilde{\beta}_1(m) \\ \vdots \\ \tilde{\beta}_{\bar{k}}(m) \end{pmatrix} = \begin{pmatrix} s_1^1 & \cdots & s_1^{\bar{k}} \\ \vdots & & \vdots \\ s_{\bar{k}}^1 & \cdots & s_{\bar{k}}^{\bar{k}} \end{pmatrix} \circ \begin{pmatrix} c^1(m) \\ \vdots \\ c^{\bar{k}}(m) \end{pmatrix}$$

with $\tilde{\beta}_i(m) \equiv \langle \beta_i, \eta(m)\rangle / \langle \eta(m), \eta(m)\rangle$. We can write (12) in more compact form as:

$$-\tilde{\beta}(m) = S \circ c(m).$$

Provided that S has full rank (an argument along the same lines can be made, if this is not the case), we obtain

$$-S^{-1} \circ \tilde{\beta}(m) = c(m).$$

We denote the jth row of S^{-1} by α^j. Let $\{\lambda^1 \cdots \lambda^{\bar{k}}\}$ be arbitrary vectors such that

$$(13)\qquad \beta_i - \sum_{j=1}^{\bar{k}} s_i^j \lambda^j \in L(\eta^1 \cdots \eta^{l-1})^{\perp}, \quad 1 \le i \le \bar{k}.$$

Equation $(\bar{k}+1)$ in (11) requires:

$$\langle \beta_{\bar{k}+1}, \eta(m)\rangle + \sum_{j=1}^{\bar{k}} s_{\bar{k}+1}^j c^j(m) \langle \eta(m), \eta(m)\rangle = 0$$

or equivalently,

$$\langle \beta_{\bar{k}+1}, \eta(m)\rangle + \sum_{j=1}^{\bar{k}} s_{\bar{k}+1}^j \langle -\alpha^j, \tilde{\beta}(m)\rangle \cdot \langle \eta(m), \eta(m)\rangle = 0,$$

$$\langle \beta_{\bar{k}+1}, \eta(m)\rangle - \sum_{j=1}^{\bar{k}} s_{\bar{k}+1}^j \langle \sum_{i=1}^{\bar{k}} \alpha_i^j \beta_i, \eta(m)\rangle = 0.$$

Substituting (13) into the last equation implies

$$\langle \beta_{\bar{k}+1}, \eta(m)\rangle - \sum_{j=1}^{\bar{k}} s_{\bar{k}+1}^j \langle \sum_{i=1}^{\bar{k}} \alpha_i^j \left[\sum_{v=1}^{\bar{k}} s_i^v \lambda^v + \theta_i \right], \eta(m)\rangle = 0,$$

where again θ_i is an arbitrary element of $L(\eta^1 \cdots \eta^{l-1})^{\perp}$.

Since the α_i^j are the coefficients of the inverse of S it follows that

$$\sum_{v=1}^{\bar{k}} \sum_{i=1}^{\bar{k}} (\alpha_i^j s_i^v) \lambda^v = \lambda^j$$

and

$$\langle \sum_{i=1}^{\bar{k}} \alpha_i^j \left[\sum_{v=1}^{\bar{k}} s_i^v \lambda^v + \theta_i \right], \eta(m)\rangle = \langle \lambda^j, \eta(m)\rangle.$$

Therefore,

$$\langle \beta_{k+1} - \sum_{j=1}^{\bar{k}} s_{k+1}^j \lambda^j, \eta(m) \rangle = 0.$$

This implies our claim, i.e. that $\beta_{\bar{k}+1} - \sum_{j=1}^{\bar{k}} s_{\bar{k}+1}^j \lambda^j \in L(\eta^1 \cdots \eta^{l-1})^{\perp}$, since $\eta(m)$ covers the entire space spanned by $\{\eta^1 \cdots \eta^{l-1}\}$. Again, this argument can be repeated for $\beta_{\bar{k}+2} \cdots \beta_k$.

Step 4: We consider next the $k \times k$ matrix

$$B = \begin{pmatrix} \beta_1 & \cdots & \beta_k \\ \hline s_1^1 & \cdots & s_k^1 \\ \vdots & \ddots & \vdots \\ s_1^{\bar{k}} & \cdots & s_k^{\bar{k}} \end{pmatrix}$$

consisting of the $n \cdot (l-1) \times k$ matrix $[\beta_1, \ldots, \beta_k]$ with the $\bar{k}$ vectors $\{s^1 \cdots s^{\bar{k}}\}$ appended as rows. The matrix B must have full rank k, since $[\beta_1, \ldots, \beta_k] \circ A$ has rank $n \cdot (l-1)$.

From Step 3 we know that every vector β_v can be written as:

$$\beta_v = \theta_v - \sum_{j=1}^{\bar{k}} s_v^j \lambda^j \tag{14}$$

where $\theta_v \in L(\eta^1 \cdots \eta^{l-1})^{\perp}$ and the vth message belongs to agent i.

Substituting (14) into the matrix B, we note that each coefficient m_v^j appears in only one column of B. It is then possible to perform elementary matrix operations which generate 0's in the last $\bar{k}$ rows for $k_i - \bar{k}$ columns. This can be done for each agent separately. The matrix operations leave the rank of B unchanged. In the resulting matrix B' there are, corresponding to each agent i, $k_i - \bar{k}$ columns of the form:

$$\begin{pmatrix} \theta_v \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{with } \theta_v \in L(\eta^1 \cdots \eta^{l-1})^{\perp}.$$

In order for B and B' to have full rank, there cannot be more than $(n-1)(l-1)$ (the dimension of $L(\eta^1 \cdots \eta^{l-1})^{\perp}$) such vectors. Consequently:

$$\sum_{i=1}^{n} (k_i - \bar{k}) \leq (n-1)(l-1)$$

or

$$k - n\bar{k} \leq (n-1)(l-1)$$

or

$$n(l-1) + \bar{k} - n\bar{k} \leq (n-1)(l-1)$$

or

$$(l-1) \leq (n-1)\bar{k}.$$

Since $\psi(n, l)$ was defined as the smallest positive integer u satisfying $(n-1)u \geq (l-1)$, it follows that $\bar{k} \geq \psi(n, l)$. This completes the proof of Theorem 4.1.

6. CONCLUSION

This paper represents a step toward an integrated theory of incentives and communication requirements for decentralized allocation mechanisms. In terms of the size of the message space, Nash implementation is always at least as costly as decentralized realization. Our general results provide a way of testing whether a given message space is big enough to implement a social choice rule in Nash equilibria.

For the case of the Walrasian choice rule we find that it is possible to construct implementing mechanisms whose set of Nash equilibria forms a manifold of exactly the dimension needed for realization only. However, this manifold must be embedded in a message space of higher dimension in order to give the messages in the manifold, and only those, the Nash property. We do not know whether this feature is shared by other social choice rules. Lindahl allocations in public goods economies provide an example of a social choice rule for which there is no increase in dimensionality. On the other hand, Hurwicz (1979b) has shown that any Nash implementable performance standard which selects individually rational and Pareto optimal allocations has to contain all Walrasian allocations. Thus, our dimensional lower bound applies to the implementation of any such performance standard, if one insists on full implementation, i.e., every desired outcome can be supported by some Nash equilibrium.

Our analysis has paid little or no attention to the important issues of balancedness and individual feasibility of nonequilibrium allocations. It remains to integrate these issues with the theory of dimensionally efficient mechanisms.

Graduate School of Business, Stanford University, Stanford, CA 94305, U.S.A.
and
Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL 60201, U.S.A.

Manuscript received October, 1985; final revision received June, 1987.

APPENDIX

In this Appendix we reexamine our dimensional formula in Theorem 4.1 without the line threadedness assumption. We focus on the case $n \geq l$; our conjecture is that a message space dimension $n \cdot (l-1)$ is not large enough. The mechanism introduced in Theorem 4.2 shows th adding one additional dimension will be sufficient. To keep the notation tractable we confine ourselves to the case $n = 3$ and $l = 3$, though the reader will see that the argument can be generalized without substantial change for general n and l, $n \geq l$.

Assume, contrary to our claim, that there exists a smooth mechanism which implements $W(\cdot)$ on $\bar{E}$ and whose message space is six-dimensional, i.e., $M_i = \mathscr{R}^2$, $1 \leq i \leq 3$.

Let $v: \bar{E} \to M$ be a thread of the Nash correspondence. We first show that on a properly chosen neighborhood v^{-1} is a differentiable function. By definition:

$$(g \circ v)(e) = W(e) \quad \text{for all } e \in \bar{E}.$$

$W(\bar{E})$ forms a six-dimensional linear manifold in Z. The differentiable function $g: M \to Z$ maps $v(\bar{E})$ onto $W(\bar{E})$. Therefore, the set $v(\bar{E})$ contains an open set O on which g is $1-1$. Let $z^* \in g(O)$ be a regular value of g such that $g(m^*) = z^*$ and $m^* \in O$. By $O(m^*) \subset O$ we denote a neighborhood on which $g|_{O(m^*)}$ is a diffeomorphism. Let $m^0 \in O(m^*)$ denote a point for which $g(m^0)$ is a regular value of $W(\cdot)$. If $W(e^0) = g(m^0)$, then $(W|_{\tilde{O}(e^0)})^{-1} \circ g|_{\tilde{O}(m^0)} = (v^{-1})|_{\tilde{O}(m^0)}$, on properly chosen neighborhoods $\tilde{O}(e^0)$ and $\tilde{O}(m^0)$, where $\tilde{O}(m^0) \subset O(m^*)$. Thus, $(v^{-1})|_{\tilde{O}(m^0)}$ is itself a diffeomorphism.

Note that $(v^{-1})|_{\tilde{O}(m^0)} \equiv t$ is a local thread of the correspondence $\gamma: M \twoheadrightarrow \bar{E}$ as defined in Theorem 3.1. Let

$$t_1(m_1^1, m_1^2, m_2^1, m_2^2, m_3^1, m_3^2) = e_1^1,$$

$$t_2(\cdot) = e_1^2 \qquad \cdots \qquad t_5(\cdot) = e_3^1, \qquad t_6(\cdot) = e_3^2.$$

To prove our claim we recall the first-order representation of Theorem 3.1, as given in (4). It is easily verified that for all $e \in \bar{E}$:

$$\nabla U(F_i(e)|e_i) \circ DF_i(e) = -\tfrac{1}{3}\big(F_i^1(e), F_i^2(e), F_i^1(e), F_i^2(e), F_i^1(e), F_i^2(e)\big),$$

$$F_i^1(e) = \frac{1}{3}\left[2e_i^1 - \sum_{\substack{k=1\\k\neq i}}^{3} e_k^1\right], \qquad F_i^2(e) = \frac{1}{3}\left[2e_i^2 - \sum_{\substack{k=1\\k\neq i}}^{3} e_k^2\right].$$

Multiplication of $\nabla U(F_i(e)|e_i) \circ DF_i(e)$ with the corresponding two columns in the Jacobian of $t(\cdot)$, as required in (4), leads to the following six equations:

(i) $$(2t_1 - t_3 - t_5)\left(\frac{\partial t_1}{\partial m_i} + \frac{\partial t_3}{\partial m_i} + \frac{\partial t_5}{\partial m_i}\right) + (2t_2 - t_4 - t_6)\left(\frac{\partial t_2}{\partial m_i} + \frac{\partial t_4}{\partial m_i} + \frac{\partial t_6}{\partial m_i}\right) \equiv 0, \quad 1 \leq i \leq 2,$$

(ii) $$(2t_3 - t_1 - t_5)\left(\frac{\partial t_1}{\partial m_i} + \frac{\partial t_3}{\partial m_i} + \frac{\partial t_5}{\partial m_i}\right) + (2t_4 - t_2 - t_6)\left(\frac{\partial t_2}{\partial m_i} + \frac{\partial t_4}{\partial m_i} + \frac{\partial t_6}{\partial m_i}\right) \equiv 0, \quad 3 \leq i \leq 4,$$

(iii) $$(2t_5 - t_1 - t_3)\left(\frac{\partial t_1}{\partial m_i} + \frac{\partial t_3}{\partial m_i} + \frac{\partial t_5}{\partial m_i}\right) + (2t_6 - t_2 - t_4)\left(\frac{\partial t_2}{\partial m_i} + \frac{\partial t_4}{\partial m_i} + \frac{\partial t_6}{\partial m_i}\right) \equiv 0, \quad 5 \leq i \leq 6.$$

This system of partial differential equations has many solutions, in particular linear solutions (leading us back to the case considered in Theorem 4.1). We shall show that, subject to a conjecture about the class of solutions to this system of equations, no solution can be onto, i.e., the rank of its Jacobian is always less than six. It will be convenient to rewrite the system in the following way:

$$\beta_1(m) = t_1(m) + t_3(m) + t_5(m),$$

$$\beta_2 = t_2 + t_4 + t_6, \qquad \beta_3 = 2t_1 - t_3 - t_5,$$

$$\beta_4 = 2t_2 - t_4 - t_6, \qquad \beta_5 = 2t_3 - t_1 - t_5,$$

$$\beta_6 = 2t_4 - t_6 - t_2.$$

Equations (i)–(iii) can then be represented as:

$$\beta_3\frac{\partial\beta_1}{\partial m_i}+\beta_4\frac{\partial\beta_2}{\partial m_i}\equiv 0,\quad 1\le i\le 2,$$

$$\beta_5\frac{\partial\beta_1}{\partial m_i}+\beta_6\frac{\partial\beta_2}{\beta m_i}\equiv 0,\quad 3\le i\le 4,$$

$$(\beta_3+\beta_5)\frac{\partial\beta_1}{\partial m_i}+(\beta_4+\beta_6)\frac{\partial\beta_2}{\partial m_i}\equiv 0,\quad 5\le i\le 6.$$

Note that the two systems are equivalent, since the linear transformation between $t(\cdot)$ and $\beta(\cdot)$ is one-to-one. We consider, what will be called the reduced system,

$$\alpha(m)\frac{\partial\beta_1}{\partial m_i}+\frac{\partial\beta_2}{\partial m_i}\equiv 0,\quad 1\le i\le 2,$$

$$\zeta(m)\frac{\partial\beta_1}{\partial m_i}+\frac{\partial\beta_2}{\partial m_i}\equiv 0,\quad 3\le i\le 4,$$

$$\varepsilon(m)\frac{\partial\beta_1}{\partial m_i}+\frac{\partial\beta_2}{\partial m_i}\equiv 0,\quad 5\le i\le 6,$$

where

$$\alpha(m)\equiv\beta_3(m)/\beta_4(m),$$

$$\zeta(m)\equiv\beta_5(m)/\beta_6(m),$$

$$\varepsilon(m)\equiv(\beta_3(m)+\beta_5(m))/(\beta_4(m)+\beta_6(m)).$$

If the Jacobian of $(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6)$ has full rank, then the Jacobian of $(\beta_1,\beta_2,\alpha,\zeta,\varepsilon)$ has full rank as well, i.e., rank five. Suppose that $(\beta_1,\beta_2,\alpha,\zeta,\varepsilon)$ is a solution to the reduced system.

We observe that

$$\frac{\partial\beta_2}{\partial m_1}\Big/\frac{\partial\beta_1}{\partial m_i}=\frac{\partial\beta_2}{\partial m_2}\Big/\frac{\partial\beta_1}{\partial m_2}=-\alpha(m)$$

implies that the functions β_1 and β_2 have the same isoquants in (m_1,m_2) space, given $(m_3\cdots m_6)$. Similarly, β_1 and β_2 have the same isoquants in (m_3,m_4) space as well as in (m_5,m_6) space. This suggests that we consider functions β_1 and β_2 of the form:

$$\beta_i=\Gamma_i(a(m),b(m),c(m))$$

where $\Gamma_i\colon\mathscr{R}^3\to\mathscr{R}$, $1\le i\le 2$, and $a(m)$, $b(m)$, $c(m)$ are arbitrary functions. (We have no proof that every pair of functions β_1 and β_2 can be written in this form whenever they have the same isoquants in the three two-dimensional subspaces.)

Differentiating β_i with respect to m_1 and m_2 and using the requirement

$$\frac{\partial\beta_1}{\beta m_1}\Big/\frac{\partial\beta_1}{\partial m_2}=\frac{\partial\beta_2}{\partial m_1}\Big/\frac{\partial\beta_2}{\partial m_2}$$

leads to the equation:

$$(15)\qquad \left(\Gamma_1^a a_1+\Gamma_1^b b_1+\Gamma_1^c c_1\right)\cdot\left(\Gamma_2^a a_2+\Gamma_2^b b_2+\Gamma_2^c c_2\right)$$
$$=\left(\Gamma_1^a a_2+\Gamma_1^b b_2+\Gamma_1^c c_2\right)\cdot\left(\Gamma_2^a a_1+\Gamma_2^b b_1+\Gamma_2^c c_1\right)$$

where $\Gamma_1^a\equiv\partial\Gamma_1/\partial a$ and $a_1\equiv\partial a/\partial m_1$.

Multiplying out in (15) yields:

$$\left(\Gamma_1^a\Gamma_2^b-\Gamma_1^b\Gamma_2^a\right)a_1b_2+\left(\Gamma_1^a\Gamma_2^c-\Gamma_1^c\Gamma_2^a\right)a_1c_2$$
$$+\left(\Gamma_1^b\Gamma_2^a-\Gamma_1^a\Gamma_2^b\right)a_2b_1+\left(\Gamma_1^b\Gamma_2^c-\Gamma_1^c\Gamma_2^b\right)b_1c_2$$
$$+\left(\Gamma_1^c\Gamma_2^a-\Gamma_1^a\Gamma_2^c\right)a_2c_1+\left(\Gamma_1^c\Gamma_2^b-\Gamma_1^b\Gamma_2^c\right)b_2c_1=0.$$

Collecting terms, (15) takes the form:

$$\left(\Gamma_1^a\Gamma_2^b-\Gamma_1^b\Gamma_2^a\right)(a_1b_2-b_1a_2)+\left(\Gamma_1^a\Gamma_2^c-\Gamma_1^c\Gamma_2^a\right)(a_1c_2-a_2c_1)$$
$$+\left(\Gamma_1^b\Gamma_2^c-\Gamma_1^c\Gamma_2^b\right)(b_1c_2-b_2c_1)=0.$$

We define new variables:

$$X=\Gamma_1^a\Gamma_2^b-\Gamma_1^b\Gamma_2^a,$$
$$Y=\Gamma_1^a\Gamma_2^c-\Gamma_2^a\Gamma_1^c,$$
$$Z=\Gamma_1^b\Gamma_2^c-\Gamma_2^b\Gamma_1^c,$$
$$s_{12}^{ab}=a_1b_2-b_1a_2,\quad s_{12}^{ac}=a_1c_2-c_2a_1,\quad s_{12}^{bc}=b_1c_2-c_1b_2.$$

The next step is to perform the same calculations for (m_3, m_4) and (m_5, m_6) respectively, keeping in mind that

$$\frac{\partial\beta_1}{\partial m_3}\bigg/\frac{\partial\beta_1}{\partial m_4}=\frac{\partial\beta_2}{\partial m_3}\bigg/\frac{\partial\beta_2}{\partial m_4}$$

and

$$\frac{\partial\beta_1}{\partial m_5}\bigg/\frac{\partial\beta_1}{\partial m_6}=\frac{\partial\beta_2}{\partial m_5}\bigg/\frac{\partial\beta_2}{\partial m_6}.$$

This leads to the system of equations:

$$\text{(16)}\qquad \begin{pmatrix} s_{12}^{ab} & s_{12}^{ac} & s_{12}^{bc}\\ s_{34}^{ab} & s_{34}^{ac} & s_{34}^{bc}\\ s_{56}^{ab} & s_{56}^{ac} & s_{56}^{bc}\end{pmatrix}\circ\begin{pmatrix}X\\Y\\Z\end{pmatrix}=0.$$

We classify the constraints on $(\Gamma_1, \Gamma_2, a, b, c)$, according to the rank of the matrix in (16), which we denote S for brevity.

Case I: Rank $(S)=0$. Then all entries in S are zero. In this case:

$$\beta_i=\Gamma_i\big(a(m_1,m_2),b(m_3,m_4),c(m_5,m_6)\big),\quad 1\le i\le 2,$$

is one possible solution to our differential equations system, but there are others.[18] It will be shown that for any solution the rank of the Jacobian of the reduced system is less than five.

First consider the Jacobian of (a, b, c):

$$\text{(17)}\qquad J_{abc}=\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6\end{pmatrix}.$$

[18] Leo Hurwicz found this solution, and one in which β_1 and β_2 are dependent, by a direct argument based on the observation that β_1 and β_2 have the same isoquants in the three subspaces noted above.

If all entries in S are zero, then there are multipliers (depending on m) $(\lambda_1, \lambda_2, \lambda_3, \eta_1, \eta_2, \eta_3)$ such that (17) can be expressed as:

$$(17') \qquad J_{abc} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \lambda_1 a_1 & \lambda_1 a_2 & \lambda_2 a_3 & \lambda_2 a_4 & \lambda_3 a_5 & \lambda_3 a_6 \\ \eta_1 a_1 & \eta_1 a_2 & \eta_2 a_3 & \eta_2 a_4 & \eta_3 a_5 & \eta_3 a_6 \end{pmatrix}.$$

We now consider the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$. It is readily verified that

$$\frac{\partial \beta_1}{\partial m_1} \Big/ \frac{\partial \beta_1}{\partial m_2} = \frac{a_1}{a_2} = \frac{\partial \beta_2}{\partial m_1} \Big/ \frac{\partial \beta_2}{\partial m_2}.$$

A straightforward, but laborious, calculation shows that in this case also:

$$\frac{\partial \alpha}{\partial m_1} \Big/ \frac{\partial \alpha}{\partial m_2} = \frac{a_1}{a_2} = \frac{\partial \zeta}{\partial m_1} \Big/ \frac{\partial \zeta}{\partial m_2} = \frac{\partial \varepsilon}{\partial m_1} \Big/ \frac{\partial \varepsilon}{\partial m_2}.$$

Hence, the first two columns in the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ are proportional, so are the third and fourth, and fifth and sixth. This shows that the rank can be at most three.

Case II: Rank $(S) = 3$. The only solution to (16) is $(X, Y, Z) = (0, 0, 0)$. This implies that Γ_1 and Γ_2 are functionally dependent since the rank of the matrix

$$\begin{pmatrix} \Gamma_1^a & \Gamma_1^b & \Gamma_1^c \\ \Gamma_2^a & \Gamma_2^b & \Gamma_2^c \end{pmatrix}$$

is at most one. Thus, $\Gamma_2 = f(\Gamma_1)$ and

$$\beta_2(m) = f(\Gamma_1(a(m), b(m), c(m))) = f(\beta_1(m)).$$

Hence, β_1 and β_2 are functionally dependent, implying that the first two rows in the Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ are proportional and hence the Jacobian cannot have rank five.

Case III: Rank $(S) = 2$. Assume, for simplicity, that the nullspace of S is spanned by vectors $[1, 0, 0]$ and $[0, 1, 0]$. It follows that

$$s_{12}^{ab} = s_{34}^{ab} = s_{56}^{ab} = 0 \quad \text{and}$$

$$s_{12}^{ac} = s_{34}^{ac} = s_{56}^{ac} = 0.$$

Therefore J_{abc} in (17) can again be expressed in the form given in (17') leading to the same conclusion. A longer argument is needed when the nullspace is an arbitrary two dimensional subspace of $\mathcal{R}^3$.

Case IV: Rank $(S) = 1$. Suppose the nullspace of S is spanned by the vector $[1, 0, 0]$. Since $[X, Y, Z]$ is in the nullspace, it follows that $Y = Z = 0$. If $X = 0$, the same reasoning as in Case II applies. If $X \neq 0$, we get the following determinant conditions:

$$\det\begin{pmatrix} \Gamma_1^a & \Gamma_1^b \\ \Gamma_2^a & \Gamma_2^b \end{pmatrix} \neq 0, \qquad \det\begin{pmatrix} \Gamma_1^a & \Gamma_1^c \\ \Gamma_2^a & \Gamma_2^c \end{pmatrix} = 0, \qquad \det\begin{pmatrix} \Gamma_1^b & \Gamma_1^c \\ \Gamma_2^b & \Gamma_2^c \end{pmatrix} = 0.$$

These conditions can be satisfied only if $\Gamma_1^c = \Gamma_2^c = 0$. Since $[1, 0, 0]$ is in the nullspace, it follows also that

$$s_{12}^{ab} = s_{34}^{ab} = s_{56}^{ab} = 0.$$

Therefore (17) takes the form

$$J_{abc} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \lambda_1 a_1 & \lambda_1 a_2 & \lambda_2 a_3 & \lambda_2 a_4 & \lambda_3 a_5 & \lambda_3 a_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{pmatrix}.$$

This leads us back to the analysis in Case I. Though we have no information about the last row in J_{abc} in the present case, the conclusion remains the same since $\Gamma_1^c = \Gamma_2^c = 0$. We find again that the

Jacobian of $(\beta_1, \beta_2, \alpha, \zeta, \varepsilon)$ has at most rank three since the first and second, third and fourth, and fifth and sixth columns are respectively proportional.

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