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On the Existence of a Locally Stable Dynamic Process with a Statically Minimal Message Space

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1. Introduction

In his seminal paper, Hurwicz (1960) introduced a formal model of an adjustment process. The aim of the line of research he started with that paper is to provide a theoretical foundation for evaluating and comparing systems for coordinating economic activity. The model he introduced is dynamic, an adjustment process was defined by a system of difference equations which model the process of communication among economic agents, enabling them eventually to arrive at an allocation of resources. One of the important properties entering into the evaluation of such a system is the "amount" of communication required in order that the system achieve or realize a specific (gross) performance, for instance, that its outcome is guaranteed to be Pareto-optimal. This is an important property because resources devoted to communication are not available for other economic uses and because the capacities of economic agents to communicate, even when aided by equipment, may be limited.

The communication requirements of resource allocation have been studied extensively, but almost entirely in a static framework. The literature almost exclusively has studied the size of the message space needed to support the equilibria of resource allocation mechanisms capable of realizing a given performance. This amounts to asking how much information (dimension or size of message) is needed in order that each agent be able to check independently whether a given message is an equilibrium message. It is nat-

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ural to ask whether that amount of information is also sufficient to allow the agents to find equilibrium. Reiter (1979) first studied this question in the setting of an example, specifically an exchange economy with two commodities, two agents with quadratic utility functions, and a Walrasian performance function. He considered the class of adjustment processes introduced by Hurwicz, represented by difference equations in the messages of the agents. He asked whether there is such an adjustment process using a two-dimensional message space (known to be the minimum size for a static mechanism in this problem) which is stable (in the sense of Lyapunov) at an equilibrium. In this example it is seen that if the process is such that the characteristic equation is defined, then the process must be unstable near equilibrium. Reiter also gave an adjustment process, with, in effect, a four-dimensional message space in which the two-dimensional set of equilibrium messages is embedded, which is locally stable in this example.

Jordan (1982) has considered the informational requirements imposed by local stability over and above those of static realization, and has obtained more general results. He introduces a more general kind of dynamical system (in differential rather than difference equations) in which there are two kinds of message variables: *state messages*, which are the messages of the Mount-Reiter equilibrium formulation (see Mount and Reiter 1974); and *control messages*, which play the role of dynamic adjusters of the state messages. Jordan introduces this type of process in order to preserve the generality of the Mount-Reiter equilibrium formulation. In that model it is not generally possible to associate particular coordinates of the message space with an individual agent. It is therefore not clear how to define an adjustment process. (Section 2 of Jordan's paper contains an excellent discussion of the background of this problem.)

The class of dynamic processes introduced by Jordan is much larger than the class of adjustment processes heretofore studied. Jordan has pointed out that the equilibrium message correspondence defined by the stationary messages of the system need not be a coordinate correspondence, that is, not a statically privacy-preserving correspondence. This, in light of a certain feature of the dynamic system, raises the question whether Jordan processes are privacy preserving. To help answer this question, we give a definition of privacy applicable to a general dynamic system and a sufficient condition that a Jordan process be privacy preserving. (Definition 2 and Lemma 2)

The definition of privacy we give applies to dynamic systems in which at each point of the state space (the message space) the direction of motion, as in the dynamical systems introduced by Smale, is constrained only to lie in a given set, rather than to be in a specified direction. This, roughly speaking, defines a correspondence from the space of environments to the tangent space of the message space. A dynamic system preserves privacy if that correspondence is a coordinate correspondence. If the differential equations

of the system determine a vector field, as in a Jordan process, this definition can be interpreted as saying that each agent privately "consents" to the direction of motion specified by the vector field.

Jordan's main general theorem applies to the case in which the dimension of the space of state messages is minimal for the static problem. He shows that local stability requires the dimension of the space of control messages to be at least as large as the dimension of the (minimal) static message space; equality of these dimensions is a possibility. In this chapter we focus on the case of equality. An argument given at the end of Section 2 supports the idea that this is an important case related to privacy. Our assumptions and methods are somewhat different from those of Jordan, especially in the case where the message space is two-dimensional. The class of dynamic systems we study is unrestricted as to dependence on parameters but includes those studied by Jordan.

One other difference may be noted here. Jordan uses a definition of stability that requires a process to stabilize *all* its equilibria in order to qualify as stable; we use a weaker definition. Since our result is negative, in effect we say that an unstable process is one that can stabilize *no* equilibrium, whereas Jordan says a process is unstable if it fails to stabilize some equilibrium.

In the first of the two cases we study, the statically minimal message space is two-dimensional. In that case we show that there is no adjustment process that (locally) stabilizes any of its equilibria. Our analysis is qualitative and does not rely on examining roots of the characteristic equation. While the set of (regular) differential equations with linear part (i.e., with linear terms of the Taylor expansion at the equilibrium point not identically zero) is dense in the set of all (regular) differential equations, it might well be true that a dynamic system whose linear parts are all zero is stable and, even though in some sense rare and isolated in the set of dynamic processes, is precisely the one we find of interest. We present the argument for the two-dimensional case in graphic terms; although a formal algebraic analysis is available, the graphic exposition is much easier to follow than the algebraic version. Except for a special case, we have been unable to generalize the argument to higher dimensions.. Second, we present an analysis in higher dimensions for processes with linear part. There we show, by an algebraic argument related to the qualitative analysis used in the two-dimensional case, that there must be a positive characteristic root. This, of course, does not exclude the possibility of a stable process with zero linear part.

2. Dynamic Systems and Privacy

Suppose there are p agents, $1, \ldots, p$, and that the space of environments E is given by $E = E^1 \times \ldots \times E^p$. We usually shall suppose that the E^i and

E are Euclidean spaces. Let

$$G: E^1 \times \ldots \times E^p \rightarrow R$$

be a performance function and suppose that there is a static mechanism with message space M and equilibrium message correspondence $\mu: E \rightarrow M$, which realizes G. We are interested in the case in which M is an n-dimensional Euclidean space.

The kind of dynamic process introduced by Hurwicz is, in differential equation form, as follows. It is assumed that M is a product

$$M = M^1 \times \ldots \times M^p$$

where each space M^j has as coordinates the variables controlled by agent j. Then the response rules of the agents define the dynamic adjustment process

$$\text{(1)} \qquad \dot{m}^j = f^j(m,e^j) \quad j = 1, \ldots, p \qquad (1.1)$$

with initial conditions

$$\dot{m}^j(0) = m_0^j \quad j = 1, \ldots, p. \qquad (1.2)$$

It is evident that the equilibrium correspondence

$$\gamma: E \rightarrow M$$

given by

$$\gamma_f(e) = \{m \in M \mid f^j(m,e^j) = 0 \quad j = 1, \ldots, p\}$$

is a coordinate correspondence and hence privacy preserving in the static sense.

Jordan (1982) introduced a more general class of dynamic processes. He does not assume that the message space is a product with different coordinates to be adjusted independently by different agents. Jordan introduced auxiliary variables, called *control messages*, which control the adjustment of the static messages of M, called *state messages*. Let C^i denote the space of control messages of agent i with elements denoted c^i, $i = 1, 2 \ldots q$. Then a Jordan process (f,α) has the form

$$\text{(2)} \qquad c^i = f^i(m,e^i) \; i = 1, \ldots, p \qquad (2.1)$$

$$\dot{m}_k = \alpha^k(c^1, \ldots c^p, m) \; k = 1, 2, \ldots, \dim M \qquad (2.2)$$

$$m_k(0) = m_{k0} \; k = 1, \ldots, \dim M. \qquad (2.3)$$

In this type of process, each agent determines the value of his control variables in a "privacy-preserving fashion" based on the current state message m and his parameters. The state message is adjusted as a function of the current control messages and the current state message. Note that the ad-

justments $\dot{m}_k$ do not depend directly on the parameters, but only indirectly through the control messages.

Substituting from equations (2.1) into (2.2) gives us the dynamic process

$$\text{(3)} \qquad \dot{m} = F(m,e) \qquad (3.1)$$

$$m(0) = m_0. \qquad (3.2)$$

The equilibrium correspondence of this system is

$$\gamma_F(e) = \{m \in M \mid F(m,e) = 0\}.$$

We consider next a still more general kind of system. A dynamic system, such as that given by (3), determines for each $e \in E$ a vector field on M, determining at each point of M a tangent vector $\dot{m}$. More generally, the system might determine at each m a set of vectors in which $\dot{m}$ is constrained to lie.

DEFINITION 1. A *dynamic system* (Φ, X, M) is given by a correspondence

$$\Phi: X \times M \rightarrow \dot{M} \times M,$$

such that

$$\Phi(x,m) \subset \dot{M} \times \{m\},$$

where $\dot{M}$ is the tangent space of M at the origin, and X is a space of parameters.[1]

If Φ is single valued, we call the dynamic system *single valued*. We can identify a single-valued dynamic system whose correspondence is given by a differential equation such as (3) with that equation.

DEFINITION 2. A dynamic system (Φ, E, M), $E = E^1 \times \ldots \times E^n$, with Φ: $E^1 \times \ldots \times E^n \times M \rightarrow \dot{M} \times M$, is *privacy preserving* if there are dynamic systems (ϕ^i, E^i, M) ϕ^i: $E^i \times M \rightarrow \dot{M} \times M$ such that $\Phi(e^1, \ldots, e^n, m) = \bigcap_{i=1}^n \phi^i(e^i, m)$ for all (e,m) in $E^1 \times \ldots \times E^n \times M$.

DEFINITION 3. Let Φ: $X \times M \rightarrow \dot{M} \times M$ be a dynamic system. The *zero correspondence of Φ* is given by

$$Z_\Phi(x) = \Phi(x, M)\,(\{0\} \times M) = \{m \in M \mid (0,m) \in \Phi(x,m)\}.$$

If Φ is single valued, then $Z_\Phi(x) = \{m \in M \mid \Phi(x,m) = (0,m)\}$; that is, *$Z_\Phi$ is the equilibrium correspondence* of the system.

We show next that a privacy-preserving dynamic system has an equilibrium correspondence that is privacy preserving.

[1] In general $\dot{M} \times M$ would be replaced by the tangent bundle to M and we would require $\Phi(e^1, \ldots, e^n, m)$ to lie in the fiber over m. When M is Euclidean, we can write the tangent bundle as a product $\dot{M} \times M$.

LEMMA 1. Let $\Phi: E^1 \times \ldots \times E^n \times M \rightarrow \dot{M} \times M$ be a privacy preserving dynamic system, and let Φ be single valued. Then the equilibrium correspondence $\gamma_\Phi \equiv \gamma$ of the system is privacy preserving, that is, a coordinate correspondence into M.

PROOF. Since Φ is a privacy preserving dynamic system, by Definition 2, there exist correspondences ϕ^i: $E^i \times M \rightarrow \dot{M} \times M$ such that for all $e \in E$ and $m \in M$, $\Phi(e^1,\ldots,e^n,m) = \phi^1(e^1,m) \cap \ldots \cap \phi^n(e^n,m)$. Now, $m \in \gamma(e^1,\ldots,e^n)$ if and only if

$$\{(0,m)\} = \Phi(e^1,\ldots,e^n,m) = \phi^1(e^1,m) \cap \ldots \cap \phi^n(e^n,m)$$

since, when Φ is single valued, $\gamma(e^1,\ldots,e^n) = Z_\Phi(e^1,\ldots,e^n)$. Therefore $(0,m) \in \phi^i(e^i,m)$ for each $i = 1, \ldots, n$. Therefore $\gamma(e^1,\ldots,e^n) \subseteq Z_{\phi^i}(e^i)$ for $i = 1, \ldots, n$. Hence $\gamma(e^1,\ldots, e^n) \subseteq \cap_{i=1}^n Z_{\phi^i}(e^i)$. Now suppose that $(0,m) \in \cap_{i=1}^n Z_{\phi^i}(e^i)$. It follows that

$$(0,m) \in \phi^i(e^i,m) \cap [\{0\} \times M\,] \text{ for } i = 1, \ldots, n.$$

Therefore $(0,m) \in \cap_{i=1}^n \phi^i(e^i,m) = \Phi(e^1,\ldots,e^n,m)$. Since $(0,m) \in \{0\} \times M$, it follows that

$$(0,m) \in Z_\Phi(e^1,\ldots,e^n) = \Phi(e^1,\ldots,e^n,m) \cap [\{0\} \times M\,].$$

Therefore $m \in \gamma(e^1,\ldots e^n) = Z_\Phi(e^1,\ldots,e^n)$, since Φ is single valued. We have shown that

$$\gamma(e^1,\ldots,e^n) = \bigcap_{i=1}^{n} Z_{\phi^i}(e^i).$$

REMARK. Lemma 1 states that every single-valued, privacy preserving dynamic system has a statically privacy preserving equilibrium correspondence. It is also true that there are no other statically privacy preserving, single-valued equilibrium correspondences than those arising from privacy preserving, single-valued dynamic systems. To see this, notice that if γ: $E^1 \times \ldots \times E^n \rightarrow M$ is a privacy preserving function, then we can construct a dynamic process by defining

$$\phi^i(e^i,m) = \{\dot{m} \in \dot{M}\,|\dot{m} = m - a \text{ for } a \in \gamma^i(e^i)\}\ i = 1, \ldots, n$$

where γ^i: $E^i \rightarrow M$ are the coordinate correspondences of γ.

The Jordan process has a single-valued Φ, as seen from equations (3). But, as Jordan (1982) points out, its equilibrium correspondence is not necessarily privacy preserving. It follows from Lemma 1 and the Remark that not all Jordan processes are privacy preserving as dynamic systems. Lemma 2 states a sufficient condition that a Jordan process be a privacy preserving dynamic system.

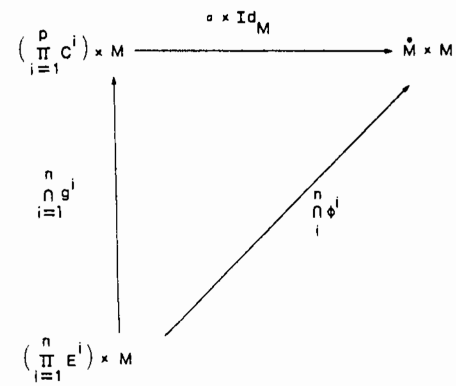

Figure 6.1. Commuting Diagram.

LEMMA 2. Let (f,α) be a Jordan process as given by (2). If for each $m \in M$

$$\alpha(\cdot,m): C \times \{m\} \rightarrow \dot{M} \times \{m\}$$

is one-to-one, then (f,α) is (corresponds to) a privacy-preserving dynamic process.

PROOF. Let $\hat{f}^i$: $E^i \times M \rightarrow \Pi_{i=1}^p C^i$ be defined by

$$\hat{f}^i(e^i,m) = C^1 \times \ldots \times C^{i-1} \times \{f^i(m,e^i)\} \times C^{i+1} \times \ldots \times C^p.$$

Define

$$g^i: E^i \times M \rightarrow \prod_{i=1}^{p} C^i \times M$$

by

$$g^i(e^i,m) = (\hat{f}(e^i,m),m).$$

Figure 6.1 shows these maps.

If $\alpha \times Id_M$ is one-to-one, then this diagram commutes, that is, the map Φ given by $\cap_i g^i$ followed by $\alpha \times Id_M$ is a coordinate correspondence.

3. Stability in Two Dimensions

Jordan has investigated the implications of local stability for the size of the space of control messages. As we have seen, Jordan's dynamic process can be written in the form of (3). He studies the local stability of (3) using a definition of local stability (Jordan 1982: 2.4 Definitions) that incorporates (i) a requirement of "Lyapunov stability" where a small displacement of equilibrium will not cause the system to move far from equilibrium; (ii) a requirement of "asymptotic stability" where the system eventually will converge to equilibrium; (iii) a requirement that *every* equilibrium corresponding to a particular environment be stabilized by the system in the senses of (i) and (ii); and (iv) a requirement that perturbations of the environment as well as of the state messages be allowed.

Under the conditions that the f^i be continuous (jointly in m and e^i), and for each e^i satisfy a Lipschitz condition in m, and that α is Lipschitzian jointly in its arguments, and under the assumption that (i) E has a subset E^0 which is homeomorphic to an open subset of R^n where $n = \dim M$; and (ii) for any $e, e' \in E^0$, $\mu(e) \cap \mu(e') = \phi$ if $e \neq e'$, Jordan has shown (1982: 2.14 Theorem) that if the adjustment process is locally stable, then $\dim C \geq n = \dim M$, where C is the space of control messages.

We focus on the case where $\dim C = \dim M$, especially where $\dim C = \dim M = 2$. This case is of special interest for at least the following reason. Jordan has shown that for any process in his class, $\dim C \geq \dim \dot{M}$. Since $\dim \dot{M} = \dim M$, his theorem tells us that $\dim C \geq \dim \dot{M}$. We know from Jordan's Assumption 1 that $\alpha(\cdot,m)$: $C \times \{m\} \to \dot{M} \times \{m\}$ is continuous and from Lemma 2 that injectiveness of α is a sufficient condition that the Jordan process (f,α) be privacy preserving. It can be shown that if α is continuous and injective, then $\dim C \leq \dim \dot{M}$. This follows from the propositions (i) that a continuous one-to-one function from a compact space to a Hausdorff space is a homeomorphism, and (ii) that an open subset of a Euclidean space has the same dimension as the space.

Applying this result to the mapping $\alpha(\cdot,m)$: $C \times \{m\} \to \dot{M} \times \{m\}$, we see that in the class of cases when the sufficient condition of Lemma 2 is satisfied, the only case to examine is that in which $\dim C = \dim M$.

We consider a broader class of processes than those introduced by Jordan, in particular, those that correspond to single-valued dynamic systems in the sense of Definition 1. In that case, the equations of (3) are not required to have the structure defined by (2), nor is the system required to preserve privacy. We also use a weaker concept of local stability, which, since our result is a negative one, gives a stronger theorem.

Let the differential equations $\dot{m}_i = h^i(m,e)$ $i = 1, \ldots, n$ be given. We may write this system in the form $\dot{m} = h(m,e)$. In order to guarantee existence and uniqueness of solutions, we make the following Assumption. (See Coddington and Levinson 1955: Theorem 7.1, p. 22.)

ASSUMPTION 1. For each $i = 1, \ldots, n$, h^i is continuous on $M \times E$, and for each $e \in E$, the function $h^i(\cdot,e)$ is locally Lipschitzian. That is, for any $m^0 \in M$ there is a *neighborhood* M^1 of m^0 and a constant $K > 0$ such that for any $m, m' \in M^1$,

$$\|h^i(m,e) - h^i(m',e)\| \leq K\|m - m'\|.$$

Next, we define local stability.

DEFINITION 4. For each $(m^0,e^0) \in M \times E$ consider the differential equation (with initial condition)

$$\dot{m} = h(m,e^0)$$
$$m(0) = m^0. \tag{4}$$

Let $D_{e^0} = \{m^0 \in M \mid (4) \text{ has a solution on the entire time domain } [0,\infty]\}$. Define $m^*_{e^0}$: $D_{e^0} \times [0,\infty) \to M$, by setting $m^*_{e^0}(m,t)$ equal to the solution of (4) at time t.

Let $m \in \mu(e^0)$. The process (4) is locally *stable* at an equilibrium m of the environment e^0 if for each neighborhood V of m there exists a neighborhood U of m with $U \subset V$ and $U \subset D_{e^0}$ and a function μ^*: $U \to M$ satisfying for each $m^0 \in U$ the conditions:

(a) $m^*_{e^0}(m^0,t) \in V$, for all t (Lyapunov stability)

(b) (a) holds for every environment in $E_m \subset E$, where E_m is the class of environments $e \in E$ such that m is an equilibrium of e.

The distinction between Jordan's definition of local stability and ours is as follows. First, Jordan studies the variation of equilibria under perturbation of the environment. We do not. We fix an equilibrium and require that it be Lyapunov stable for all environments for which it is an equilibrium. Second, Jordan requires the process to stabilize all equilibria, in the sense of both Lyapunov and asymptotic stability; our definition applies to Lyapunov stability of a particular equilibrium. Thus it is conceivable that an adjustment process might stabilize some equilibria of a particular environment but not others. The case of an environment with multiple competitive equilibria, some of which are locally stable (with Walrasian price adjustment) and others of which are unstable, illustrates the difference. Jordan's definition requires that this process be called unstable, whereas ours allows it to be called stable at the right equilibrium.

We now turn to the case $n = 2$. In that case Jordan's theorem tells us that $\dim C \geq 2$. We study the case of equality, that is, $\dim C = 2$. In that case the system (3) can be written

$$\dot{m}_1 = h^1(m,e)$$
$$\dot{m}_2 = h^2(m,e) \tag{5}$$

where the functions h^i, $i = 1,2$ may each have all components of e as arguments. Define the correspondences by the equations:

$$\gamma^1(e) = \{m \in M \mid h^1(m,e) = 0\} \text{ and}$$
$$\gamma^2(e) = \{m \in M \mid h^2(m,e) = 0\}. \qquad (6)$$

Then

$$\gamma(e) = \gamma^1(e) \cap \gamma^2(e).$$

Since $\gamma^i(e)$ is the set of zeroes of $h^i(\cdot,e)$, we sometimes refer to it as the zero-locus of $h^i(\cdot,e)$.

If (μ,M,g) is a static mechanism (realizing a performance function G), we say that the dynamic process (5) realizes μ (and G) if $\gamma = \mu$, that is, if the stationary points of (5) are those given by μ.

We assume that for each environment $e \in E$, $\mu(e)$ is not empty. Therefore if the dynamic process given by (5) is to realize μ, the equations (6) must have at least one solution for each environment. Geometrically, for each environment, the sets given by (6) intersect in M at least once.

We take such a point of intersection. Without loss of generality we may, by a translation of coordinates, take that point to be the origin.

ASSUMPTION 2. In a neighborhood of the origin, the set of zeros of $h^i(\cdot,e)$ is a submanifold. (Golubitsky and Guillemin 1973: Definition 2.7, p. 9)

Note that the function $h^i(\cdot,e)$ need not be one of the coordinate functions that define the submanifold. In particular Assumption 2 does not imply that $h^i(\cdot,e)$ is differentiable.

The zero-locus of $h^i(\cdot,e)$ might be empty, consist of a finite number of points, be a curve in R^2, or be two-dimensional. The only interesting case is that in which it is a one-dimensional curve in R^2. We confine attention to that case. Given an equilibrium point, m, there is a subset E_m of environments which have the same point as an equilibrium. We may, without loss of generality, take $m = 0$. Thus, let

$$E_0 = \{e \in E \mid h^i(0,e) = 0\} \; i = 1, 2.$$

Since $E_0 \subset E$, if there is no dynamic process stabilizing 0 for all environments in E_0, then there is certainly no such process for all environments in E. We can confine attention to E_0.

It follows from the assumption that $\gamma^i(e)$, $i = 1, 2$ is a submanifold, that for each $e \in E_0$, the curve $\gamma^i(e)$ has a unique tangent at the origin. In a neighborhood of the origin on which $\gamma^i(e)$ is a submanifold, the curve $\gamma(e)$ divides the space into two regions, which we orient as follows.

Let $L(\gamma^i(e); m_1,m_2)$ be a real-valued continuous function, taking the value 0 at each point of $\gamma^i(e)$ and changing sign across $\gamma^i(e)$. This function exists because $\gamma^i(e)$ is a submanifold. Thus $L(\gamma^i(e),m_1,m_2)$ is positive for all (m_1,m_2) on one side of $\gamma^i(e)$ and negative on the other.

Let

$$H^i_+(e) = \{(m_1,m_2) \in M \mid L(\gamma^i(e),m_1,m_2) > 0\}$$
$$H^i_-(e) = \{(m_1,m_2) \in M \mid L(\gamma^i(e),m_1,m_2) < 0\}.$$

We call $H^i_+(e)$ the *upper* (or positive) *half-space* and $H^i_-(e)$ the *lower* (or negative) *half-space* of $\gamma^i(e)$. Of course, these regions are literally half-spaces only when $\gamma^i(e)$ is a line; generally they will be regions with a curved boundary.

For $e \in E_0$, let $\sigma^i(e)$ denote the unit vector normal to $\gamma^i(e)$ at the origin pointing into $H^i_+(e)$. Thus σ^i: $E_0 \to S^1$ $i = 1, 2$, where S^1 is the unit circle in R^2.

Let $\mathcal{H}$ denote the class of all functions h: $M \times E_0 \to R$ which satisfy Assumption 1 and write $\mathcal{H}_e$ for the collection of all functions $h \in \mathcal{H}$ with e a fixed element of E_0, that is, $h(\cdot,e)$: $M \times \{e\} \to R$. The collection includes the right-hand side functions, which we shall refer to as *adjustment functions*, of dynamic processes given by (5) and, hence, covers the Jordan and, therefore, also the Hurwicz processes. We now introduce an assumption about the way the zero-locus of the function h^i depends on the environment.

ASSUMPTION 3. The mapping

$$\sigma = (\sigma^1,\sigma^2): E_0 \to S^1 \times S^1$$

satisfies the following conditions:

(i) There exists a point $w \in S^1$ and a neighborhood of (w,w), $V(w,w) \in S^1 \times S^1$ such that $\sigma(E_0)$ contains $V(w,w)$

(ii) the correspondence σ^{-1} is locally threaded at the point $(w,w) \in S^1 \times S^1$. That is, there exists a neighborhood $W(w,w) \subset V(w,w) \subset S^1 \times S^1$ and a continuous function ξ: $W(w,w) \to E_0$ such that $(x,y) \in W(w,w)$ implies $\xi(x,y) \in \sigma^{-1}(x,y)$.

We shall now show there is no dynamic process satisfying our assumptions that is locally stable at the origin for all environments in E_0.

THEOREM. Let m be a point of R^2. There does not exist a dynamic process, as given by equations (5) with $(h^1,h^2) \in \mathcal{H}$, satisfying Assumptions 1, 2 and 3, for which m is a locally stable equilibrium.

PROOF. Let $(h^1,h^2) \in \mathcal{H}$ be the adjustment functions of an arbitrary system given by (5). It follows from the continuity of h^i that for $i = 1, 2$, and each $e \in E_0$, $h^i(\cdot,e)$ can change sign only at points of $\gamma^i(e)$. That is, $h^i(\cdot,e)$ must have one sign on each half-space $H^i_+(e)$ and $H^i_-(e)$. Let $w^1 \in S^1$ be a point whose existence is guaranteed by Assumption 3(i). We may, without loss of generality (by permuting coordinates and possibly changing signs), assume that w^1 is in the interior of the second quadrant, for example, $w^1 = (-1/\sqrt{2},1/\sqrt{2})$.

By Assumption 3(i) there exists $e_0 \in E_0$ such that $\sigma(e_0) = (w^1, w^1)$. Since h^i is continuous, taking a point m in the interior of $H^i_+(e_0)$ at which $h^i(m,e_0)$ is, say, positive, and letting $\varepsilon = h^i(m,e_0)/2$, there is a neighborhood $N^i(e_0)$ such that $e \in N^i(e_0)$ implies $h^i(m,e)$ has the same sign as $h^i(m,e_0)$ and therefore has the same sign as $h^i(\cdot,e_0)$ on $H^i_+(e)$(resp. $H^i_-(e)$) as $h^i(\cdot,e_0)$ has on $H^i_+(e_0)$(resp. $H^i_-(e_0)$), for $i = 1, 2$.

Since ξ: $S^2 \times S^2 \rightarrow E$ is continuous on $W(w^1,w^1) \subset S^1 \times S^1$, for every neighborhood $U(e_0) \subset E_0$, there is a neighborhood $V(w^1,w^1) \subset W(w^1,w^1) \subset S^1 \times S^1$ such that $(w^1,w^2) \in V(w^1,w^1)$ implies $\xi(w^1,w^2) \in U(e_0)$.

For $\lambda > 0$, and given w^1, let

$$w^2(\lambda) = \frac{(1-\lambda)(-1,0) + \lambda w^1}{\|(1-\lambda)(-1,0) + \lambda w^1\|}.$$

Then for every neighborhood $V(w^1,w^1) \subset S^1 \times S^1$ there exists $\delta > 0$ such that $|\lambda - 1| < \delta$ implies $(w^1,w^2(\lambda)) \in V(w^1,w^1)$.

Let $U(e_0) = N^1(e_0) \cap N^2(e_0)$. Then for this neighborhood $U(e_0)$, there is $V(w^1,w^1) \subset S^1 \times S^1$ and $\delta > 0$ so that $1 - \delta < \lambda < 1$ implies

α) $(w^1,w^2(\lambda)) \in V(w^1,w^1)$;
β) $e(\lambda) = \xi(w^1,w^2(\lambda)) \in U(e_0) = N^1(e_0) \cap N^2(e_0)$; and
γ) the slope of $\gamma^2(e(\lambda))$ at the origin is positive and greater than the slope of $\gamma^1(e(\lambda))$ at the origin, which is also positive.

It follows from Assumption 2 that there is a neighborhood of the origin in M such that $\gamma^1(e(\lambda))$ and $\gamma^2(e(\lambda))$ intersect in that neighborhood only at the origin. From now on we shall confine attention to this neighborhood, and in our diagrams we shall picture this neighborhood as if it were the whole of R^2. In this way we can avoid inessential notational complexities.

Since $h^i(\cdot,e(\lambda))$ must have one sign on each half-space $H^i_+(e(\lambda))$ and $H^i_-(e(\lambda))$, we can partition $\mathscr{H}_{e(\lambda)}$ into equivalence classes based on the sign pattern of the h^i functions.

Let

$$A^i(e(\lambda)) = \{h^i(\cdot,e(\lambda)) \in \mathscr{H}_{e(\lambda)} | h^i(m,e(\lambda)) > 0 \text{ for } m \in H^i_+(e(\lambda)) \cup H^i_-(e(\lambda))\}$$

$$B^i(e(\lambda)) = \{h^i(\cdot,e(\lambda)) \in \mathscr{H}_{e(\lambda)} | h^i(m,e(\lambda)) > 0 \text{ for } m \in H^i_+(e(\lambda)),\ h^i(m,e(\lambda)) < 0 \text{ for } m \in H^i_-(e(\lambda))\}$$

$$C^i(e(\lambda)) = \{h^i(\cdot,e(\lambda)) \in \mathscr{H}_{e(\lambda)} | h^i(m,e(\lambda)) < 0 \text{ for } m \in H^i_+(e(\lambda)),\ h^i(m,e^i(\lambda)) > 0 \text{ for } m \in H^i_-(e(\lambda))\}$$

$$D^i(e(\lambda)) = \{h^i \in \mathscr{H}_{e(\lambda)} | h^i(m,e(\lambda)) < 0 \text{ for } m \in H^i_+(e(\lambda)) \cup H^i_-(e(\lambda))\}.$$

It follows that the arbitrary adjustment process we began with must have its response functions in one of 16 subsets of functions in $\mathscr{H}_{e(\lambda)}$, namely,

$$A^1(e(\lambda)) \times A^2(e(\lambda))$$
$$A^1(e(\lambda)) \times B^2(e(\lambda))$$
$$\cdots$$
$$D^1(e(\lambda)) \times D(e(\lambda)),$$

generated by the four ways in which signs can be assigned independently to each of the two adjustment functions.

The two half-spaces $H^i_+(e(\lambda))$ and $H^i_-(e(\lambda))$ for each agent determine four regions of R^2 which we may label

$$\text{I}(e(\lambda)) = H^i_+(e(\lambda)) \cap H^2_-(e(\lambda))$$
$$\text{II}(e(\lambda)) = H^1_+(e(\lambda)) \cap H^2_+(e(\lambda))$$
$$\text{III}(e(\lambda)) = H^1_-(e(\lambda)) \cap H^2_+(e(\lambda))$$
$$\text{IV}(e(\lambda)) = H^1_-(e(\lambda)) \cap H^2_-(e(\lambda)).$$

(If the curves $\gamma^1(e(\lambda))$ and $\gamma^2(e(\lambda))$ intersected in some other way at the origin, for instance, if they were tangent there, some of these regions might be empty.)

Now, each pair of functions $(h^1(\cdot,e(\lambda)), h^2(\cdot,e(\lambda)))$ determines a pair of signs (s_1,s_2), where $s_i \in \{+,-\}$, $i = 1, 2$, on each of the sets I$(e(\lambda))$, II$(e(\lambda))$, III$(e(\lambda))$, and IV$(e(\lambda))$. Since these are the signs of $(\dot{m}_1,\dot{m}_2)$, to each pair of functions $(h^1(\cdot,e(\lambda)), h^2(\cdot,e(\lambda)))$ in an equivalence class of $\mathscr{H}_{(e(\lambda))}$, in each of the regions I$(e(\lambda))$,...,IV$(e(\lambda))$, there is a unique quadrant of directions in which the vector $(\dot{m}_1,\dot{m}_2)$ must lie at every point of the region. Since $\dot{m}_i = 0$ if $m \in \gamma^i(e(\lambda))$, we also know the unique direction of $(\dot{m}_1,\dot{m}_2)$ along the boundaries of these regions.

Figure 6.2 shows the two curves $\gamma^1(e(\lambda))$ and $\gamma^2(e(\lambda))$ and the four regions into which they partition the plane.

The dynamic process with adjustment functions (h^1,h^2) falls into one of the 16 equivalence classes $A^1(e(\lambda)) \times A^2(e(\lambda))$, ..., $D^1(e(\lambda)) \times D^2(e(\lambda))$. We study the solution path determined by the differential equation (5) from each initial point of the message space. There are, of course, 16 cases, which we examine in turn.

Case 1. Suppose $(h^1,h^2) \in B^1(e(\lambda)) \times C^2(e(\lambda))$; that is, $h^1(\cdot,e(\lambda)) > 0$ on $H^1_+(e(\lambda))$ and $h^1(\cdot,e(\lambda)) < 0$ on $H^1_-(e(\lambda))$ and $h^2(\cdot,e(\lambda)) < 0$ on $H^2_+(e(\lambda))$ and $h^2(\cdot,e(\lambda)) > 0$ on $H^2_-(e(\lambda))$. We can represent the sign of $\dot{m}_1$ by an arrow "→" or "←" showing direction but not magnitude, and similarly $\dot{m}_2$ by "↑" or "↓". Thus, in Figure 6.3, the arrows in region I$(e(\lambda))$, for example, indicate that (i) at any point interior to that region the motion must be up

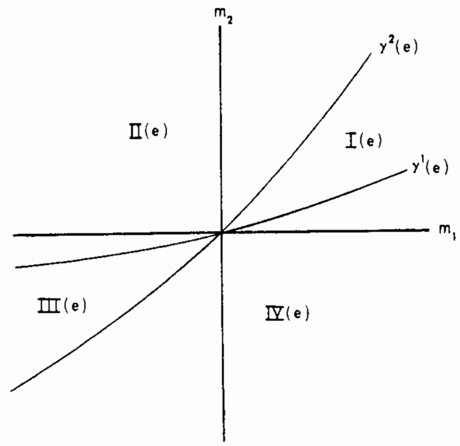

Figure 6.2. Partition.

and to the right, and (ii) at any point on the boundary given by $\gamma^1(e(\lambda))$, except the origin, the motion must be up ($\dot{m}_1 = 0$ on $\gamma^1(e(\lambda))$), and (iii) at any point on $\gamma^2(e(\lambda))$, except the origin, the motion is to the right. Therefore, if the initial point $m_0 \neq 0$ belongs to region $\mathrm{I}(e(\lambda))$, the solution path moves away from the origin and does not leave region $\mathrm{I}(e(\lambda))$. Therefore, if (h^1,h^2) is in $B^1(e(\lambda)) \times C^2(e(\lambda))$, the equilibrium corresponding to the origin must be unstable at the environment $e(\lambda)$. Figure 6.3 shows that the region $\mathrm{III}(e(\lambda))$ is also a region of instability for (h^1,h^2) in this case.

Table 6.1 summarizes the results for each of the 16 possibilities for (h^1,h^2). The table lists the case numbers, the sign patterns of the two response functions, and all of the regions of instability. A solution path emanating from a point of such a region of instability must go off to infinity (leave the neighborhood), thus violating condition (a) (Lyapunov stability) of Definition 4. The table also uses the notation Q_i, $i = 1, 2, 3, 4$ for the four quadrants, that is, $Q_1 = \{(m_1,m_2)|m_1 \geq 0,\ m_2 \geq 0\}$, $Q_2\{(m_1,m_2)|m_1 \leq 0,\ m_2 \geq 0\}$, and so on.

Figures 6.3 through 6.18 show the arrow diagrams for each of these 16 cases. For example, Figure 16.4 shows that at a point m_0, which belongs to region $\mathrm{II}(e(\lambda))$ and is to the right of the m_2-axis, the motion is down and to the right. A solution path through such a point must either go off to infinity or hit $\gamma^2(e(\lambda))$. If a path hits $\gamma^2(e(\lambda))$, it crosses into region $\mathrm{I}(e(\lambda))$, which is a region of instability. Table 6.1 reveals that in 14 of the 16 cases there is at least one region of instability. In the remaining two cases, Cases 4 and 13, spiral convergence to the origin (or to a limit cycle) cannot be excluded.

Table 6.1.

Case Number	Sign Patterns of the Response Functions: $h^1(\cdot,e(\lambda))$ $H^1_+(e(\lambda))$	$h^1(\cdot,e(\lambda))$ $H^1_-(e(\lambda))$	$h^2(\cdot,e(\lambda))$ $H^2_+(e(\lambda))$	$h^2(\cdot,e(\lambda))$ $H^2_-(e(\lambda))$	Region of Instability
1	→	←	↓	↑	$Q_1 \cup Q_3$
2	→	←	↓	↓	$(Q_1 \cap H^1_+(e(\lambda))) \cup Q_3 \cup Q_4$
3	→	←	↑	↑	$Q_1 \cup Q_2$
4	→	←	↑	↓	stability not excluded; see Case 4a
5	→	→	↓	↑	$Q_1 \cup Q_4$
6	→	→	↓	↓	$Q_1 \cup Q_3 \cup Q_4$
7	→	→	↑	↑	$Q_1 \cup Q_2 \cup Q_4$
8	→	→	↑	↓	$Q_1 \cup Q_2 \cup Q_4 \cup (Q_3 \cap H^1_+(e(\lambda)))$
9	←	←	↓	↑	$Q_2 \cup Q_3$
10	←	←	↓	↓	$Q_2 \cup Q_3 \cup Q_4$
11	←	←	↑	↑	$Q_1 \cup Q_2 \cup Q_3$
12	←	←	↑	↓	$Q_2 \cup Q_3 \cup Q_4$
13	←	→	↓	↑	stability not excluded; see Case 13a
14	←	→	↓	↓	$Q_2 \cup Q_3 \cup Q_4$
15	←	→	↑	↑	$Q_1 \cup Q_2 \cup (Q_3 \cap H^1_+(e(\lambda))) \cup Q_4$
16	←	→	↑	↓	$Q_2 \cup Q_3 \cup Q_4$

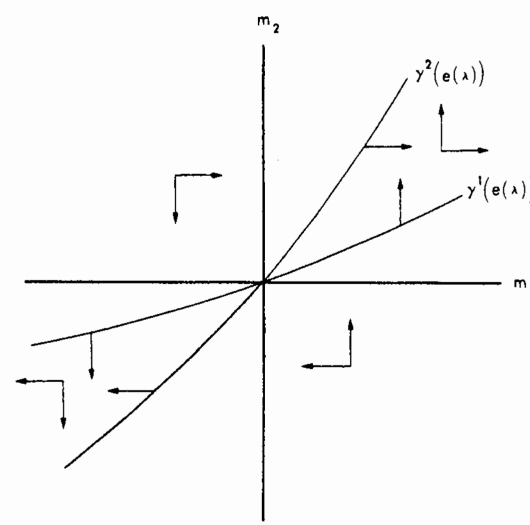

Figure 6.3. Case 1.

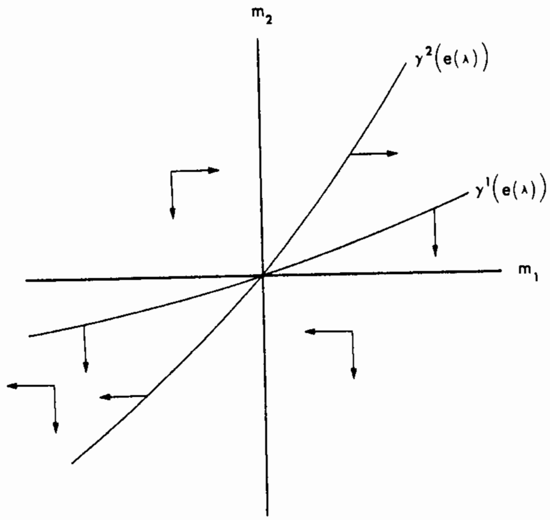

Figure 6.4. Case 2.

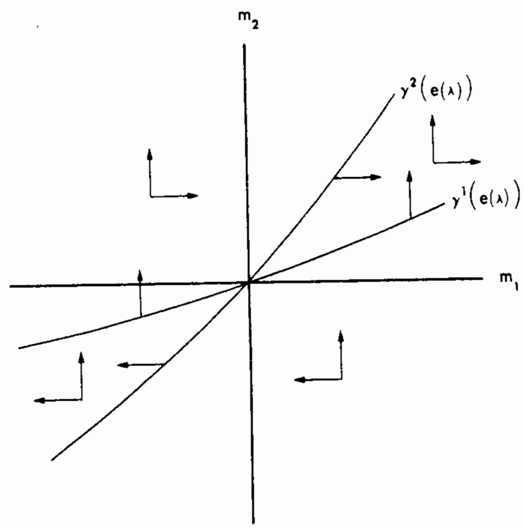

Figure 6.5. Case 3.

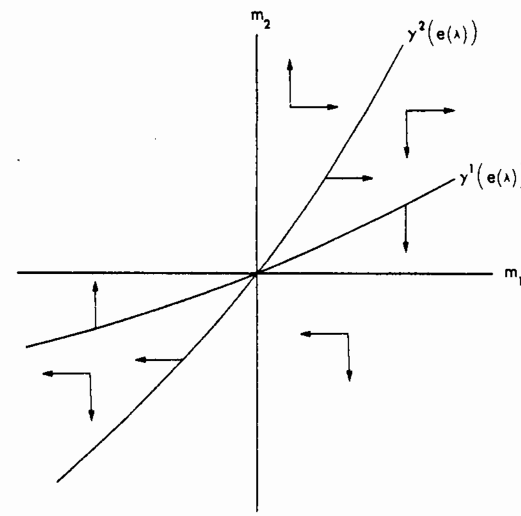

Figure 6.6. Case 4.

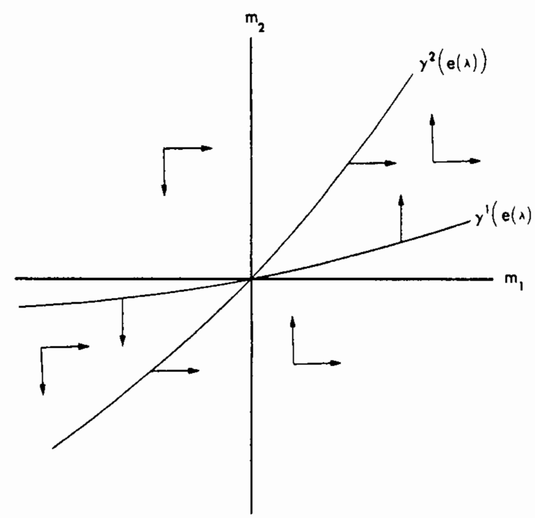

Figure 6.7. Case 5.

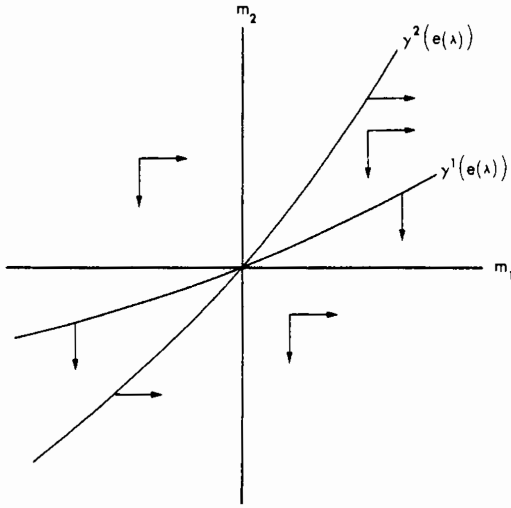

Figure 6.8. Case 6.

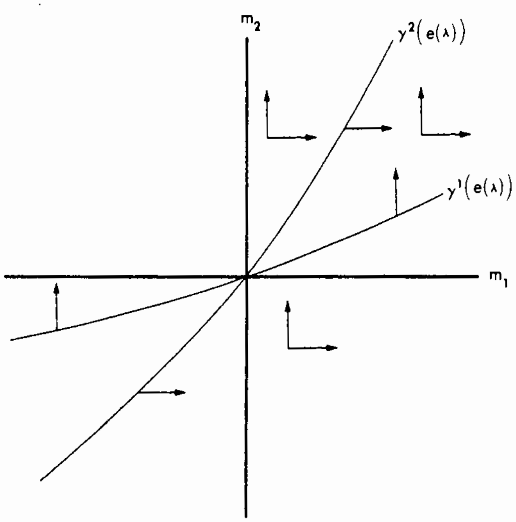

Figure 6.9. Case 7.

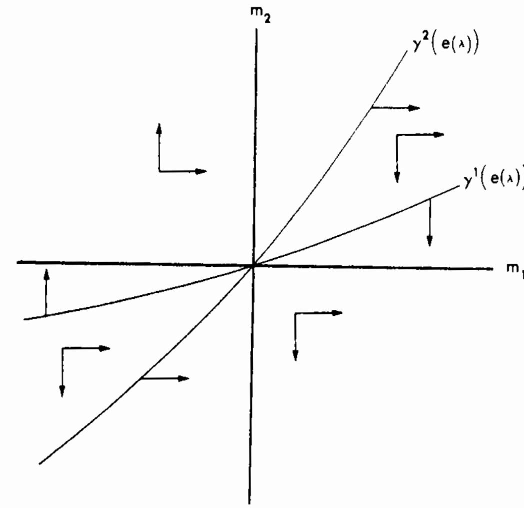

Figure 6.10. Case 8.

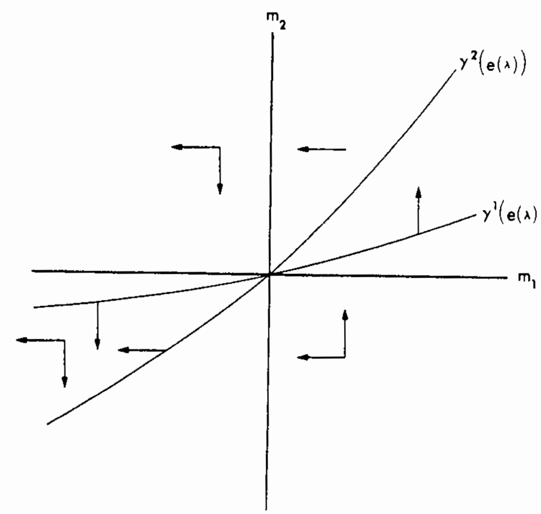

Figure 6.11. Case 9.

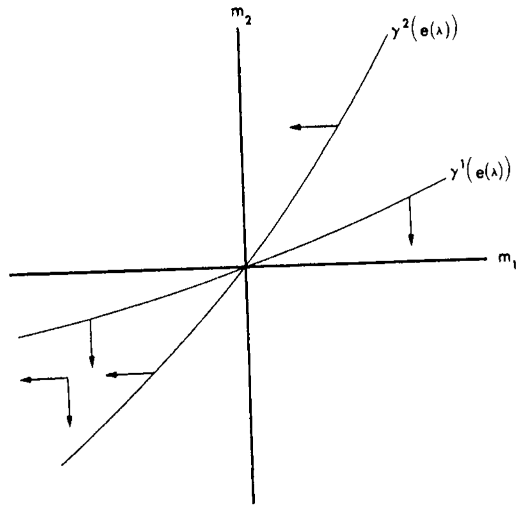

Figure 6.12. Case 10.

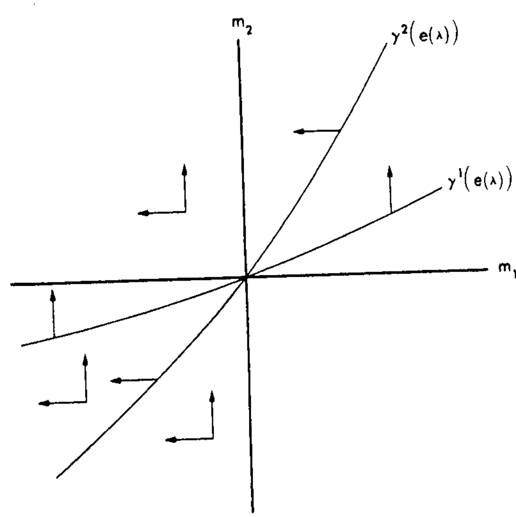

Figure 6.13. Case 11.

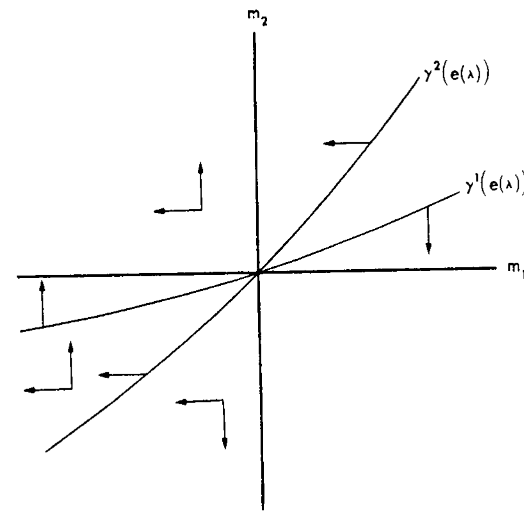

Figure 6.14. Case 12.

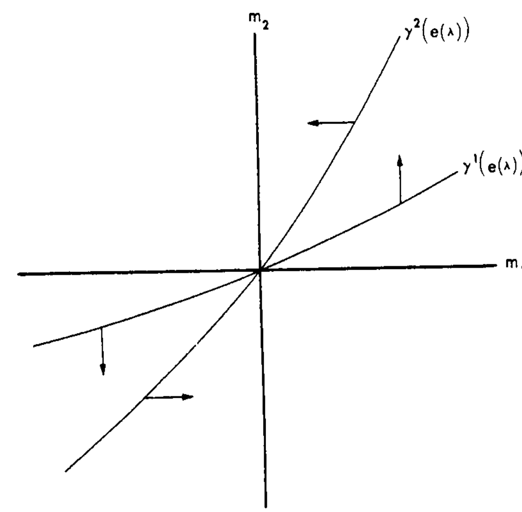

Figure 6.15. Case 13.

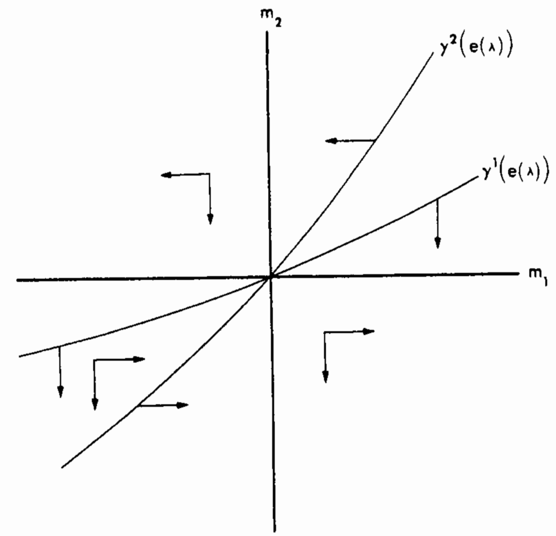

Figure 6.16. Case 14.

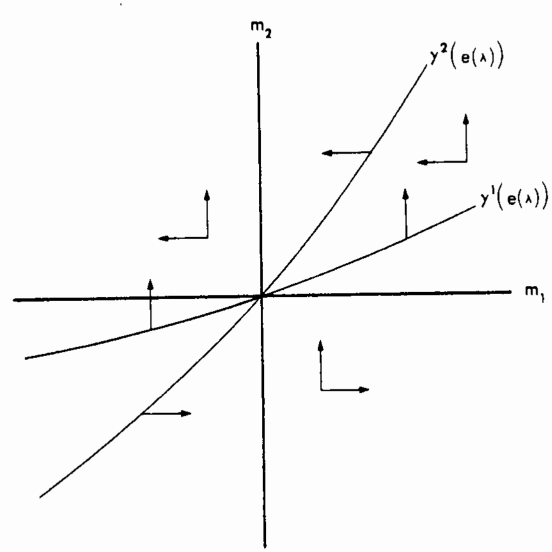

Figure 6.17. Case 15.

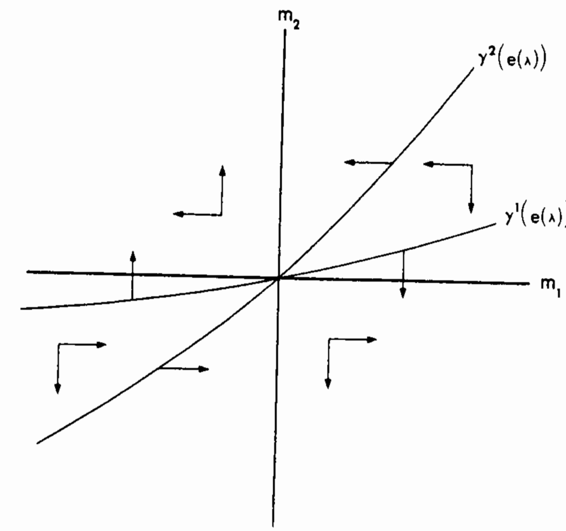

Figure 6.18. Case 16.

That is, if (h^1,h^2) is a member of the equivalence class corresponding to Case 4 (or 13), then the equilibrium 0 might be stable at the environment $e(\lambda)$. We show next that in each such case there must be another environment in E_0 for which the origin is unstable, thus violating (b) of Definition 4 of local stability.

We first consider Case 4. In that case

$$h^1(m,e(\lambda)) > 0 \text{ for all } m \text{ in } H^1_+(e(\lambda)),$$

$$h^1(\mathrm{m},e(\lambda)) < 0 \text{ for all } m \text{ in } H^1_-(e(\lambda)),$$

while

$$h^2(m,e(\lambda)) > 0 \text{ for all } m \text{ in } H^2_+(e(\lambda))$$

and

$$h^2(m,e(\lambda)) < 0 \text{ for all } m \text{ in } H^2_-(e(\lambda)).$$

Let $1 < \bar{\lambda} < 1 + \delta$; then $\xi(w^1,w^2(\bar{\lambda})) = e(\bar{\lambda}) \in N^1(e_0) \cap N^2(e_0)$. It follows that

δ) the slope of $\gamma^2(e(\bar{\lambda}))$ at the origin is positive and less than that of $\gamma^1(e(\bar{\lambda}))$, and

ϵ) the sign pattern of $h^1(\cdot,e(\bar{\lambda}))$[resp. $h^2(\cdot,e(\bar{\lambda}))$] is the same as that of $h^1(\cdot,e(\lambda))$[resp. $h^2(\cdot,e(\lambda))$]. Thus, we have the situation shown in Figure

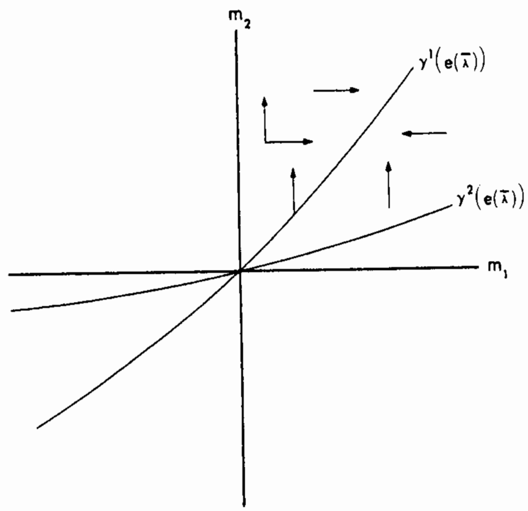

Figure 6.19. Variant of Case 4.

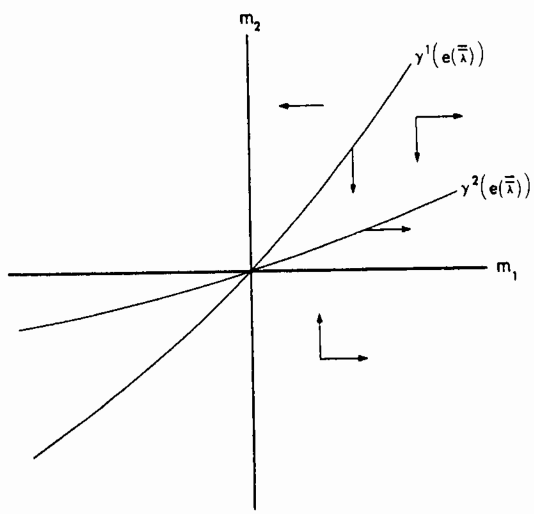

Figure 6.20. Variant of Case 13.

16.19, where it can be seen that $Q_1 \cap H^1_+(e(\bar{\lambda}))$ is a region of Lyapunov instability.[2]

The argument in Case 13 (see Figure 6.20) is completely analogous to that of Case 4.

The following examples show that Assumptions 1, 2, and 3 can be satisfied, and a modification of these examples makes it clear that some condition like Assumption 3 is indispensable.

We consider a case with two agents ($p = 2$) where the environment $e = (e^1,e^2)$ is given by $e^1 = (x,z)$, $e^2 = (x^1,z^1)$, where x, z, x^1, z^1 are real numbers, and the message space $M = R \times R$ is two dimensional. It is known that there are performance functions for which this message space is statically minimal. Consider the dynamic process defined by

$$\dot{m}_1 = z - xm_2 - m_1$$
$$\dot{m}_2 = z^1 - x^1m_1 - m_2. \tag{7}$$

The zero-loci are

$$\gamma^1(e^1) = \{(m_1,m_2) \in M \mid z - xm_2 - m_1 = 0\}$$
$$\gamma^2(e^1) = \{(m_1,m_2) \in M \mid z^1 - x^1m_1 - m_2 = 0\} \tag{8}$$

which determine the equilibria

$$\bar{m}_1 = \frac{z - z^1x}{1 - xx^1}, \bar{m}_2 = \frac{z^1 - zx^1}{1 - xx^1} \tag{9}$$

provided $xx^1 \neq 1$.

Since $1 - xx^1 \neq 0$, the solution $\bar{m}_1 = 0 = \bar{m}_2$ holds for the class of environments with $z = 0 = z^1$. In addition, when $xx^1 = 1$, and $z = z^1 = 0$, the two equations in (8) coincide. Because equations (8) are satisfied at the origin even when they coincide, the class E_0 is given by

$$E_0 = \{((x,z),(x^1,z^1)) \in R^2 \times R^2 \mid z = z^1 = 0\},$$

which, since x and x^1 are unconstrained, we may identify with $R \times R$. The mapping

$$\sigma\colon R \times R \to S^1 \times S^1$$

is given by

[2]Here we take the neighborhood of the origin in M to be sufficiently small to ensure that $\gamma^2(e(\lambda))$ intersects $\gamma^1(e(\lambda))$ only at the origin.

$$\sigma^1(x,x^1) = \theta_1(1,x)$$

$$\sigma^2(x,x^1) = \theta_2(x^1,1),$$

where θ_1 and θ_2 are scalars chosen to make the norms equal to 1, and to orient the vectors according to the prescribed convention. It is straightforward to verify that Assumptions 1, 2, and 3 are satisfied.

Now, to see that the dynamic process given by (7) might locally stabilize the origin if Assumption 3 is not satisfied, let

$$E_0^1 = \{(x,x^1) \in R \times R \mid 0 < x \cdot x^1 < 1\} \subset E_0.$$

A point on S^1 may be represented in terms of the central angle θ, so that it has coordinates (cos θ, sin θ). Suppose $\theta \neq \pi/2$. If $\sigma(x,x^1) = (\theta,\theta)$, it follows that $x = \tan\theta$ and $x^1 = \operatorname{ctn}\theta$. Hence $xx^1 = 1$. If $\theta = \pi/2$, then $x \cdot x^1 = 0$, since in that case $\cos\theta = 0$ so $x^1 = 0$.

Thus Assumption 3 cannot be satisfied.

Furthermore, the characteristic roots of (7) are

$$\lambda_1 = -1 + \sqrt{xx^1}$$

$$\lambda_2 = -1 - \sqrt{xx^1},$$

which for $(x,x^1) \in E_0^1$ are always negative. Hence the process given by (7) locally stabilizes the origin on E_0^1.

It may be noted that if we restrict attention to the environments E_0, and consider, for example, the performance function

$$G: E_0 \to R$$

given by

$$G(x,x^1) = x \cdot x^1,$$

the process

$$\dot{m}_1 = -(m_1 - x)$$

$$\dot{m}_2 = -(m_2 - m_1 \cdot x^1) \tag{10}$$

is well defined and is locally stable on E_0, but it does not satisfy Assumption 3, since for any equilibrium $\bar{m} = (\bar{m}_1,\bar{m}_2)$ of the process, the set $E_{0\bar{m}}$ of environments in E_0 that have $\bar{m}$ as an equilibrium is

$$E_{0\bar{m}} = \{(x,x^1) \in R \times R \mid x = \bar{m}_1;\ x^1 = \frac{\bar{m}_2}{\bar{m}_1}\},$$

so that

$$\sigma^1(E_{0\bar{m}}) = \{(1,0)\} \text{ or } \{(-1,0)\},$$

depending on the orientation of the line $\bar{m}_1 - x = 0$.

4. Stability in Higher Dimensions

In the proof of the theorem, Assumption 3 was used in Cases 4 and 13 to produce an unstable configuration. This instability was produced by interchanging the normals of the curves $\gamma^i(e)$ without changing the signs of the h^i. Instability might also result from changing the sign of one of the h^i. In the case in which the h^i are linear, or in which the Jacobian of (h^1,h^2) is nonsingular, either of these alternatives changes the sign of the determinant of the system.

The "folklore" of the subject includes the result that in the n-dimension nonsingular case, if n is even, then the system is unstable if the determinant is negative, and if n is odd and the determinant is positive, the system is unstable. Because the argument is short, we have included it below.

Denote by M the matrix of the system

$$\dot{m}_j = \sum_{j=1}^{n} a(ij)m_j. \tag{11}$$

The characteristic function of the system (11), $\det(M - tId)$, is a polynomial of degree n with real coefficients which we may write as

$$(-1)^n t^n + \ldots + d = (-1)^n[t^n + \ldots + (-1)^n d].$$

The constant term is of course $\det(M)$.

The Fundamental Theorem of algebra says that we may write

$$t^n + \ldots + (-1)^n d(x^1,\ldots,x^n) = (t - r(1))\ldots(t - r(u))Q_1(t)\ldots Q_v(t)$$

where each $r(i)$ is real and each $Q_i(t)$ is a quadratic with two complex conjugate roots. Thus

$$(-1)^n d(x^1,\ldots,x^n) = (-1)^u r(1)\ldots r(u) \cdot q(1)\overline{q(1)}\ldots q(v)\overline{q(v)}$$

where $q(s)$ and $\overline{q(s)}$ are the complex conjugate roots of $Q_s(t)$. Thus

$$(-1)^n d(x^1,\ldots x^n) = (-1)^n r(1)\ldots r(u) \cdot q$$

where q is positive. It follows that

$$(-1)^n \det M = (-1)^u r(1)\ldots r(u)q.$$

Now suppose that n is even. The number of nonreal roots is even (since they arise as conjugate pairs). Thus u is even and

$$\det M = r(1)\ldots r(u)q.$$

If $\det M$ is negative, then one of the real numbers $r(i)$ must be positive.

Suppose that n is odd. Then u is also odd and we have

$$-\det M = -r(1)\ldots r(u)q.$$

If $\det M$ is positive, then not all the real roots $r(1)\ldots r(u)$ can be negative.

We know that if the matrix of the system (11) has a positive real eigenvalue, then the system is unstable.

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IV. Market Mechanism