

CHAPTER XI

ON THE CHOICE OF A CROP ROTATION PLAN

BY CLIFFORD HILDRETH AND STANLEY REITER

This chapter is concerned with the application of a linear production model to the problem of the selection of a crop rotation plan by an individual farmer. The analysis presented here is static and is relevant to the long-run decision as to which basic rotation and cultivation plan to adopt as a fairly permanent practice. It does not bear on problems of possible year-to-year deviations in plans due to weather or economic conditions experienced in a particular season. For purposes of exposition a number of simplifying assumptions are made. After a simple model has been developed, the effects of relaxing some of the assumptions can be easily indicated.

The farmer is visualized as dealing exclusively in competitive markets. This means that the prices he pays for inputs and receives for outputs are market determined and are independent of his production decisions.

The crops used for illustrations in this chapter are several that are common on Corn Belt farms. A rotation plan is a specification of a sequence of crops to be grown in successive years on a selected parcel of land. A rotation consisting of corn, oats, hay, abbreviated COH, for example, would mean that the parcel was to be planted in corn the first year, oats in the second, hay in the third, corn in the fourth, oats in the fifth, and so forth. This would be called a three-year rotation. A farmer who adopted this rotation would probably divide his cropland into subparcels and start some of his land at each stage of the rotation. This would spread his work more evenly through the year and provide to some degree a hedge against failure of a particular crop in one year. Thus, in considering long-run effects, we regard COH and OHC as the same rotation, but CHO would have to be considered a different rotation (i.e., hay after corn instead of after oats might have a different effect on the soil and might result in different average yields of the three crops).

For simplicity it will be assumed throughout most of this chapter that with each rotation is associated a particular cultivation plan (i.e., a particular sequence of soil treatments). The effect of recognizing that a particular rotation can be carried out with various cultural practices

will be briefly considered in Section 3. It will also be assumed initially that the available land is homogeneous, and effects of relaxing this assumption will also be considered in Section 3. Our example will be developed in somewhat more detail than would be necessary just to present the practical problem considered. This will be done in order to illustrate some of the elementary properties of linear production models. These have already been developed by Koopmans [III] and are included here because the crop rotation application seems to be a convenient expository device.

1. The significant consequences of using a particular rotation are the crop yields the rotation will furnish and the input requirements (acres of land, hours of labor, gallons of fuel, etc.) necessary to carry out the rotation. A rotation may be identified with a vector specifying these quantities. For convenience we shall think of the quantities that represent a particular rotation as the average annual yields of crops and average annual inputs used for each acre devoted to this rotation (i.e., the rotation vector is normalized on land input).

Each rotation that is considered represents an activity in a linear model. Each crop produced and each input used is treated as a commodity. If we first consider only two rotations, say corn every year (CCC) and hay every year (HHH), and assume that land is the only input required, then the model appears as in Table I.¹

TABLE I

Commodities	Activities	
	Rotation 1 CCC x_1	Rotation 2 HHH x_2
y_1 = corn output	a_{11}	0
y_2 = hay output	0	a_{22}
y_3 = land input	-1	-1

¹ As in Chapter III, a negative coefficient in an activity vector indicates that the associated commodity is used up in the activity. In this chapter the term "input" is applied to a commodity that is typically used up in the kind of activities being considered, and refers to the negative number whose absolute value measures the extent of this using up. This differs somewhat from Chapter III, where the term "input" refers to that absolute value rather than to its negative.

We assume first that the input of land is fixed, say $y_3 = -k$ and, therefore, $x_1 + x_2 = k$. Figure 1 then shows the alternative combinations of corn output and hay output that can be obtained by varying the levels x_1 and x_2 of the activities.

If all the land available is devoted to rotation 1, we get the point Q_1 (coordinates ka_{11} and 0); if all the land is devoted to rotation 2, we get the point Q_2 (0, ka_{22}). All other points on the line $\overline{Q_1Q_2}$ are obtained by apportioning the land between the two rotations in amounts αk and $(1 - \alpha)k$, where $0 \leq \alpha \leq 1$.

If the products cannot be destroyed or thrown away, the line $\overline{Q_1Q_2}$ is the set of all possible combinations of corn and hay from a given land input. It is also the set of efficient points for the given land input since, for every point, one coordinate can be increased only by decreasing the other.

If products can be disposed of, the set of possible points is the triangle $\overline{Q_1O'Q_2}$ since all points on or inside this triangle can be reached, for instance, by producing some combination on the line $\overline{Q_1Q_2}$ and throwing away appropriate quantities of product. However, the efficient point set remains the line $\overline{Q_1Q_2}$.

Let us now permit land input to vary. Then the triangle of Figure 1 is replaced by the cone $\overline{OO'Q_1Q_2}$ in the three-dimensional commodity space shown in Figure 2. Figure 1 may then be regarded as the intersection of this cone with the plane $y_3 = -k$. Alternatively, the cone may be regarded as obtained by multiplication out of the origin of the triangle in Figure 1 by a variable nonnegative factor.

If disposal of commodities is ruled out, the efficient point set and the possible point set coincide and consist of the "front" facet of the cone (i.e., the two halflines from 0 through Q_1 and from 0 through Q_2 and the points of the plane angle spanned by these halflines).

If disposal of products is permitted, any combination of corn and hay represented by a point of the cone is possible. However, all such points cannot be efficient for, starting from an interior point, one can obtain more hay (corn) with the same amount of land and without giving up any corn (hay). Alternatively, for any interior point, it is possible to produce that combination of corn and hay with less land. Only those points lying on the "front" facet of the cone are efficient.

The equation of the plane through 0, Q_1 , Q_2 determines the rates of substitution or transformation between commodities in efficient produc-

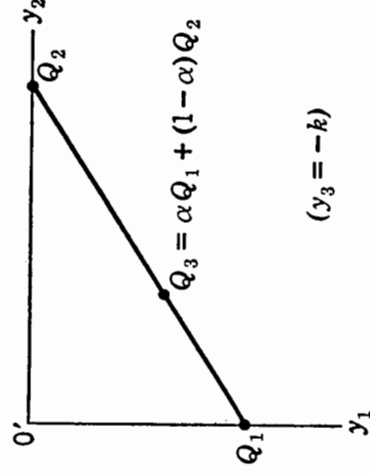


FIGURE 1

tion. In the notation of Table I, the equation of this plane is

$$(1) \quad y_3 = -\frac{1}{a_{11}}y_1 - \frac{1}{a_{22}}y_2.$$

Thus the marginal rate of substitution of hay for corn is a_{22}/a_{11} , and the marginal rates of transformation are a_{11} between corn and land and a_{22} between hay and land.

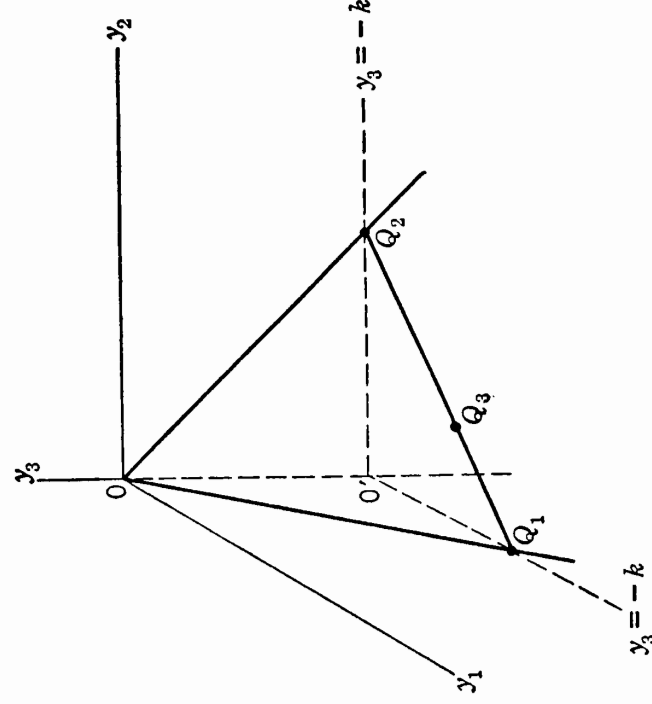


FIGURE 2

It is clear in this simple case that a farmer with a fixed amount of land and these two production alternatives, seeking to maximize the return to his land and entrepreneurship, would choose between corn and hay on the basis of their relative prices. If the ratio of the price² of corn (p_1) to the price of hay (p_2) exceeds the equivalence ratio for hay in terms of corn ($p_1/p_2 > a_{22}/a_{11}$), then corn will be chosen, and conversely for hay. The case where $p_1/p_2 = a_{22}/a_{11}$ is one of indifference in which corn, hay, and any combination of αk acres of corn and $(1 - \alpha)k$ acres of hay ($0 < \alpha < 1$) would be equally profitable.

The market price ratio p_1/p_2 determines a family of parallel lines in the (y_1, y_2) -space such that all points on a given line represent combinations of y_1 and y_2 that have equal market value. Combinations of equal market value are indicated by dotted lines in Figure 3. The interior angle formed by the intersection of such a line with the positive y_1 -axis is $\theta_1 = \arctan(p_1/p_2)$. The line through the origin perpendic-

² For a comparison of the notation for prices in this chapter with that of Chapter III, see footnote 3 of the Introduction (p. 9).

ular to the constant-value lines, $\overline{O'V}$ in Figure 3, has the property that the market value of any point (y_1, y_2) can be measured by the projection of the point on this line. Thus it may be called the value axis,³ and it intersects the positive y_2 -axis at an angle equal to θ_1 .

So long as y_1 and y_2 have positive prices the value axis will lie in the positive quadrant of the (y_1, y_2) -plane. $\overline{O'V'}$, perpendicular to $\overline{Q_1Q_2}$, has the interesting property that all price combinations whose value

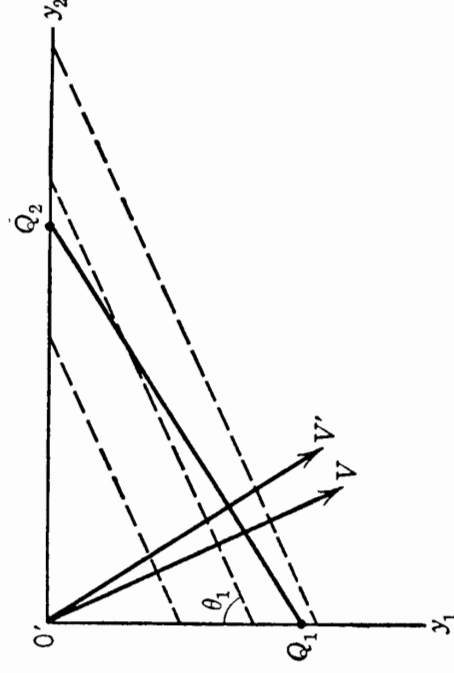


FIGURE 3

axes lie between $\overline{O'V'}$ and the y_1 -axis make y_1 the more profitable crop. This is just another way of phrasing the statement above that corn is more profitable if $p_1/p_2 > a_{22}/a_{11}$.

2. The ideas developed so far carry over into somewhat more complex cases quite readily. Consider the four rotations described in Table II.

TABLE II

Commodities	Activities			
	Rotation 1 CCC x_1	Rotation 2 HHH x_2	Rotation 3 CCH x_3	Rotation 4 CHH x_4
y_1 = corn output	a_{11}	0	a_{13}	a_{14}
y_2 = hay output	0	a_{22}	a_{23}	a_{24}
y_3 = land input	-1	-1	-1	-1

³ If p_1 and p_2 are measured in dollars, any point (y_1, y_2) that projects to the point $[p_1/(p_1^2 + p_2^2), p_2/(p_1^2 + p_2^2)]$ on the value axis will have a market value of one dollar. This may be regarded as the unit point on the value axis.

Again, assume land input fixed at k acres ($x_1 + x_2 + x_3 + x_4 = k$). The results of raising corn only or hay only are again indicated by points Q_1 and Q_2 in Figure 4. The results of growing two corn crops followed by a hay crop are indicated by Q_3 (coordinates ka_{13}, ka_{23}). Had Q_3 been an interior point of the old set of possible points, say at Q'_3 , this would have indicated that land reacted unfavorably to alternation of crops. Q'_3 would not add to the set of possible points and would not be an efficient point since there are combinations of rotations 1 and 2 that produce more of both crops. Q_3 does add possible points and is

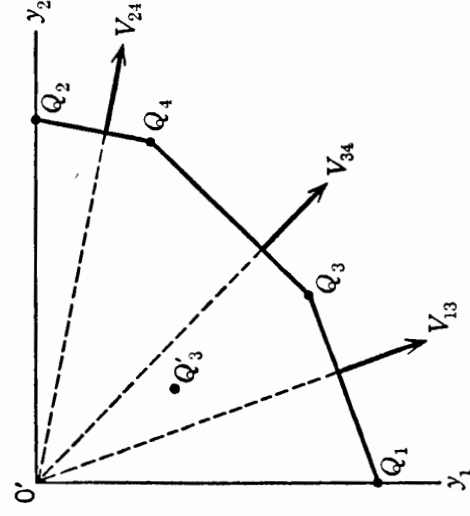


FIGURE 4

itself an efficient point. Q_4 represents the results of growing one corn crop and two hay crops on each parcel of land in each three-year period. The new set of possible points is the interior and boundary of the polygon formed by the axes and $\overline{Q_1Q_3Q_4Q_2}$. The broken line $\overline{Q_1Q_3Q_4Q_2}$ is the corresponding set of efficient points and is comparable to a product substitution curve of the usual economic theory. There are now three marginal rates of substitution of hay for

corn, one corresponding to each segment of the efficient point set. Along the segment $\overline{Q_1Q_3}$ the rate is $a_{23}/(a_{11} - a_{13})$, along $\overline{Q_3Q_4}$ it is $(a_{24} - a_{23})/(a_{13} - a_{14})$, and along $\overline{Q_4Q_2}$ it is $(a_{22} - a_{24})/a_{14}$. As before, any pair of positive prices for corn and hay give rise to a value axis passing through the origin. $\overline{O'V_{13}}$ is the value axis corresponding to pairs of prices such that $p_1/p_2 = a_{23}/(a_{11} - a_{13})$. At these prices rotations 1 and 3 and all combinations of them are equally profitable.⁴ For price combinations such that $p_1/p_2 > a_{23}/(a_{11} - a_{13})$, rotation 1 would be most profitable and the value axis would lie between $\overline{O'V_{13}}$ and the y_1 -axis. Similarly, if $a_{23}/(a_{11} - a_{13}) < p_1/p_2 < (a_{24} - a_{23})/(a_{13} - a_{14})$, the value axis lies between $\overline{O'V_{13}}$ and $\overline{O'V_{34}}$ and rotation 3 is the most profitable. Corresponding statements can be made about price ratios in the other two ranges. Thus the three lines perpendicular to segments of the efficient point set classify possible combinations of market prices into four groups, each group containing those price combinations at which a particular rotation plan is most profitable.

⁴ Profitability in this context is measured by the total value of the two crops raised. This would include both economic profit and rent.

As in Section 1, if we consider land a variable, the set of possible points becomes a cone and the set of efficient points becomes part of its boundary. This is shown in Figure 5. The efficient point set now has three facets corresponding to the three efficient line segments of Figure 4.

The set of efficient points is conceptually the same as the transformation surface usually employed in the theory of the firm. Marginal rates of substitution and transformation are usually visualized as varying continuously on the transformation surface, whereas, under the assumptions

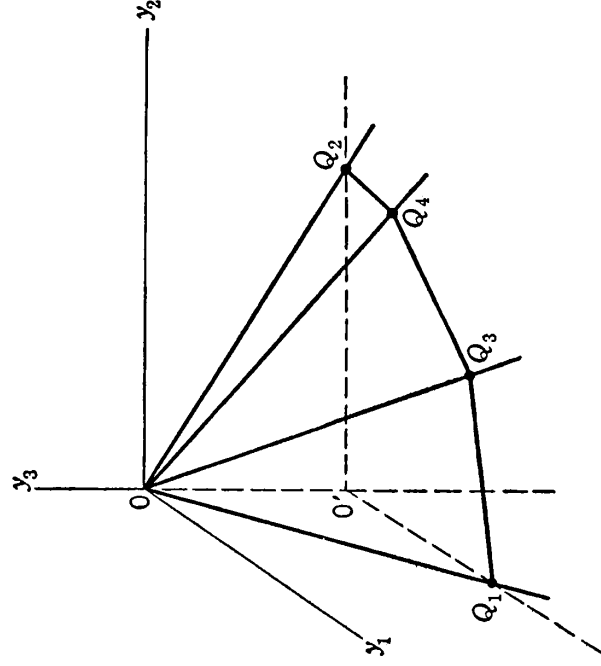


FIGURE 5

employed here, the marginal rates change discontinuously at the edges of facets of the efficient point set and are constant at all points in the interior of a facet. Each of the three "front" facets of the cone in Figure 5 determines a set of marginal rates of substitution and transformation. For example, the equation of the plane through 0 , Q_1 , Q_3 is

$$(2) \quad y_3 = -\frac{1}{a_{11}}y_1 - \left(\frac{1}{a_{23}} - \frac{a_{13}}{a_{11}a_{23}} \right) y_2.$$

This determines the marginal rate of substitution of hay for corn as $a_{23}/(a_{11} - a_{13})$ and the marginal rate of transformation between hay and land as $a_{11}a_{23}/(a_{11} - a_{13})$.

3. It may now be useful to consider the following practical situation. A farmer has the use of a certain parcel of land, say k acres, assumed to be homogeneous, and wishes to choose among several rotations. Suppose that data are available from technical experiments to show the average yield of various crops to be expected under each of the alter-

native rotations.⁵ Suppose further that the farmer can estimate the various resources that would be required to carry out each rotation. This information could be summarized in a form such as Table III,

TABLE III

Commodities	Activities			
	Rotation 1 CCC x_1	Rotation 2 HHH x_2	Rotation 3 CCH x_3	Rotation 4 CHH x_4
y_1 = corn output	a_{11}	0	a_{13}	a_{14}
y_2 = hay output	0	a_{22}	a_{23}	a_{24}
y_3 = land input	-1	-1	-1	-1
y_4 = labor input	a_{41}	a_{42}	a_{43}	a_{44}
y_5 = equipment input	a_{51}	a_{52}	a_{53}	a_{54}
y_6 = fuel input	a_{61}	a_{62}	a_{63}	a_{64}

which differs from Table II in that it contains rows for inputs other than land. The coefficients in the last four rows represent inputs used per acre of land cultivated and are therefore negative.

The farmer's profit, π , can be written

$$(3) \quad \pi = \sum_{i=1}^6 p_i y_i,$$

where p_i is the price of the i th commodity. His profit if he chooses the j th rotation will be

$$(4) \quad \pi_j = k \sum_{i=1}^6 a_{ij} p_i.$$

The difference between profit under the j th rotation and under the l th rotation is

$$(5) \quad \pi_j - \pi_l = k \sum_{i=1}^6 (a_{ij} - a_{il}) p_i.$$

For given values of the a 's, the equation

$$(6) \quad \pi_j - \pi_l = 0$$

⁵ Such data are available for some rotations on certain types of land; see e.g., Browning, *et al.* [1948].

is a hyperplane in the 6-dimensional space of all possible commodity prices and divides the space into two sectors, one including those price combinations for which the j th rotation is more profitable ($\pi_j - \pi_l > 0$) and the other including those prices for which the l th rotation is more profitable ($\pi_j - \pi_l < 0$). Such a plane exists for each pair of rotations, and together they would divide the price space into four subsets which we shall refer to as sectors, each sector consisting of those price combinations for which a given rotation is most profitable. On the boundaries of these sectors two or more rotations are equally profitable.

Since, in our example, all planes are parallel to the land-price axis, no information would be lost by considering only a 5-dimensional price space, omitting p_3 . The sectors corresponding to the alternative rotations are convex since $\pi_j - \pi_l > 0$ for p , and $\pi_j - \pi_l > 0$ for p^* implies $\pi_j - \pi_l > 0$ for $\alpha p + (1 - \alpha)p^*$, ($0 \leq \alpha \leq 1$). With some approximations, the information in these sectors could be represented in a 2-dimensional figure. Generally speaking, prices paid by farmers for factors of production are more stable than prices of crops raised. An approximate representation might be obtained by inserting average prices for a recent period for p_4, p_5, p_6 and regarding these as constants, thus reducing equations like (6) to lines in the (p_1, p_2) -space.

The equation $\pi_j - \pi_l = 0$ could then be written

$$(7) \quad (a_{1j} - a_{1l})p_1 + (a_{2j} - a_{2l})p_2 + (c_j - c_l) = 0,$$

where $c_j = \sum_{i=4}^6 a_{ij}p_i$ ($j = 1, \dots, 4$) and is regarded as a constant. The six equations like (7) determine boundaries of rotation sectors in the (p_1, p_2) -space as illustrated in Figure 6. The six lines of equal

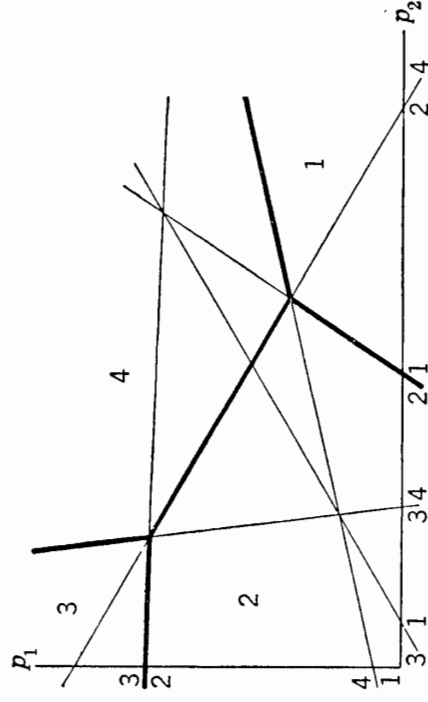


FIGURE 6

profits are shown with the numbers appearing along the coordinate axes showing which two rotations are compared by a particular line. The rotation numbers also show on which side of a line a particular rotation

is more profitable. The four sectors bounded by the heavy lines inside the axes are numbered to show which rotation is most profitable for price combinations in each sector. The existence of four points at which three lines of equal profits intersect may be regarded as typical. Whenever we have a point such that $\pi_j - \pi_l = 0$ and $\pi_j - \pi_m = 0$, then it is also true that $\pi_l - \pi_m = 0$ at that point.⁶

The situation represented by Table III is a highly simplified example, but the way in which a number of complexities could be incorporated is clear. Additional rotations would add columns to the table, additional crops or resources used would add rows. If the farmer wished to consider alternative cultivation practices for some rotations, each combination of a specific rotation and a specific cultivation practice would represent a distinct activity and would add a column to the table. However, the procedure for translating the relevant technical information into sets of prices for which particular activities are more profitable would remain unchanged.

If the farmer had more than one type of land, there would be a set of activities for each type of land. If the quantities of each type of land were regarded as fixed, a separate table could be used for each type and the problems of the best rotation and cultivation plan for each type of land could be considered separately.

4. To shorten the discussion of a few points related to the previous sections, let the model be expressed in matrix form

$$(8) \quad y = Ax,$$

where y is the commodity vector whose elements are products produced and resources used as in the previous example. A is a matrix of coefficients of the sort contained in Table III, and x is a vector of levels of activities stating the extent to which each is used. Let p be the price vector whose elements are the prices of the commodities, y . Then profit, π , is given by

$$(9) \quad \pi = p'y.$$

It is plain that, if the entrepreneur is unrestricted in his selection of x , if prices are independent of his decision, and if one of the activities yields a positive profit, then the entrepreneur can make any desired profit by choosing the appropriate value for x_i , the element of x corresponding to the activity which yields a positive profit. This situation, of course, is

⁶ The four points could be identical, or it could happen that one or more of the rotations are worse than others for all prices, in which case fewer lines and sectors would appear in Figure 4.

not realized in practice. Any entrepreneur who expanded one or more activities far enough would encounter some violation of these conditions for indefinitely large profit. He would find himself bidding up prices of resources used, forcing down prices of products sold, exhausting available resources at least in their more efficient forms, or exhausting his financial resources. In any given case it is probable that several of these exist as potential limits to the expansion of an activity or group of activities. However, some of the restrictions may be ineffective in that the most profitable selection of activity levels is the same in a model which includes them and in a model which excludes them.

In Section 3 it was assumed that the only effective restriction was that the amount of land available to the farmer was fixed. In models where a limitation on one resource is the only effective restriction, the problem of selecting the optimal collection of activities reduces to the problem of selecting one activity that yields highest returns per unit of the fixed resource. Thus the discussion has been limited to a highly special situation, though it is a situation of some practical interest.

In classical competitive equilibrium theory, profit is eliminated by the bidding of entrepreneurs for land. This can be expressed in a linear model by letting the vector y and the matrix A include only commodities other than land. $\pi = p'y$ then represents the sum of profits and rent. If other resources are plentiful and land is scarce, profits will be pushed to zero and the whole quantity π will be rent. Furthermore, competition will force each entrepreneur to use the best activity to avoid losses. The problem of selection of a best activity under these conditions has been analyzed in Section 3. Let a_j^* be the column of A corresponding to a best activity (i.e., $\pi_j = kp'a_j^* \geq \pi_l$ for all l). The rent per acre is then equal to $p'a_j^*$. Similarly, if several types of land exist in an economy, we could write

$$(10) \quad y = [A \ B \ C \ \dots]x,$$

where $[A \ B \ C \ \dots]$ is a matrix of activities arranged so that activities using land of, say, type A appear at the left, activities using land of type B appear next, etc. Rents for the respective types of land would then be $p'a_j^*$, $p'b_l^*$, $p'c_m^*$, etc., where b_l^* and c_m^* are columns corresponding to best activities on land of types B and C , respectively. The difference in rent between, say, type A and type B would be equal to $p'(a_j^* - b_l^*)$. This is the difference in net value productivity and corresponds to the classical notion of differences in rents being determined by differences in productivity.

Another interesting question can be explored by considering the hyperplanes in the price space of the form $p'a_i^* = 0$. Each of these represents