

**Economic Environments for which there are Pareto Satisfactory Mechanisms**



Kenneth Mount; Stanley Reiter

*Econometrica*, Vol. 45, No. 4 (May, 1977), 821-842.

Stable URL:

<http://links.jstor.org/sici?sici=0012-9682%28197705%2945%3A4%3C821%3AEEFWTA%3E2.0.CO%3B2-0>

*Econometrica* is currently published by The Econometric Society.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/econosoc.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

## ECONOMIC ENVIRONMENTS FOR WHICH THERE ARE PARETO SATISFACTORY MECHANISMS

BY KENNETH MOUNT AND STANLEY REITER<sup>1</sup>

In recent studies of economic organization [9 and 15] it has been found necessary to impose certain restrictions (regularity conditions) on the communication process. Such restrictions limit the class of environments for which the organization is Pareto satisfactory. We show that a class of environments admits a Pareto satisfactory (and regular) resource allocation mechanism if and only if the graph of the Pareto correspondence is a union of continuous functions. We also study the shape of the Pareto set for a given environment and the way that set varies as the environment varies. We present examples showing that near any environment whose Pareto utility frontier is homeomorphic to a simplex there is an environment whose Pareto frontier is badly chopped up. We also give an example of a class of environments for which the Pareto correspondence has no continuous selection through a given point.

### 1. INTRODUCTION

ECONOMICS HAS LONG BEEN CONCERNED with the possibility of decentralized economic organization. Interest attaches on the one hand to the theory of decentralized processes, and on the other to identifying those properties of economic environments which distinguish between those that can be successfully coordinated by a decentralized process and those which cannot be so organized. The broad question addressed in this paper is: "What class of economic environments can be satisfactorily (with respect to the Pareto criterion) organized by means of an economic mechanism which is informationally decentralized?" We study this question on the basis of formulations and results in [7 and 9], and especially in [16], where background material and fuller discussion of basic ideas may be found.

Resource allocation processes (which are abstract models of economic organization) typically involve an explicit model of communication using formal messages [16 and 17]. Informational decentralization of processes has been defined in terms of two properties. The first, involving the concept of "privacy", amounts to the requirements that all information, except what is internal to an agent (e.g., his preferences), must come to him via messages; the second restricts the messages used by the process to be vectors whose dimension is bounded by a number related to the dimension of the commodity space [7]. In this way the information-carrying capacity of messages is constrained, and therefore the information that can be communicated is constrained. Thus, the dimension or "size" of messages is a basic component of the concept of informational decentralization. However, without further conditions on communication restricting the dimension of messages does not impose a meaningful restriction on the amount of information communicated via those messages. It seems intuitive that two-dimensional messages are "bigger" than one-dimensional ones in the sense that "more information" should be

<sup>1</sup> We are indebted to Leo Hurwicz, Hugo Sonnenschein, and Mark Walker for helpful discussions and comments.

conveyed via a two-dimensional message than a one-dimensional. However, a communication process using two-dimensional messages can be replaced by one which uses one-dimensional messages without sacrifice of performance. This paradoxical possibility exists because there are (continuous) functions which map a lower dimensional space onto one of higher dimension. One such map is the Peano function, which maps the unit interval continuously onto the unit square. The inverse of the Peano function can be used to encode two-dimensional messages into one dimension; the Peano function can be used to recover the original two-dimensional information. It is therefore necessary to impose a condition on the communication process which excludes such smuggling of information.

One such condition, given in [16], is that the message correspondence, which models communication, is locally threaded.<sup>2</sup> (Another such condition given by Hurwicz is that a certain function, which plays a role analogous to the inverse of our message correspondence, should satisfy a Lipschitz condition.) These conditions, which relate to the smoothness of the communication process, have implications for the smoothness of the relation between the outcome of the process and the environment. Specifically, if the message correspondence is locally threaded, then the performance of the process is given by a continuous function from environments to outcomes (Lemma 2.1).

We study organizations whose performance is satisfactory with respect to a criterion (usually the Pareto criterion) which can be expressed formally by means of a correspondence  $\mathcal{P}$  from environments to outcomes. We consider a collection of processes using the same message space and outcome function, which we call a *mechanism* (Definition 2.2), and show that a mechanism whose message correspondences are locally threaded can be, e.g., Pareto satisfactory on a class of environments if and only if the Pareto correspondence is a union of continuous functions (a completely threaded correspondence) on that class of environments (Theorems 2.1 and 2.2).

Since Theorems 2.1 and 2.2 identify a property (complete threading) of the Pareto correspondence which is necessary and sufficient for the existence of a (decentralized) mechanism whose performance is Pareto satisfactory, the question arises, on what class of environments does the Pareto correspondence have that property? That is, on what class of environments is the Pareto correspondence completely threaded? Our results show that, in the presence of certain rather standard conditions, if preferences are strictly monotone, then the Pareto utility frontier correspondence is completely threaded (Theorem 5.2). Examples show that strict monotonicity is indispensable. If, further, the set of points (allocations or trades) which are Pareto equivalent to a given Pareto optimal point is a singleton (pointedness assumption), then the "contract curve" correspondence is also completely threaded (Theorem 6.1).

The classical welfare theorems establish the Pareto satisfactoriness of the competitive mechanism on the class of convex environments. However, it is not

<sup>2</sup> See the footnote to Definition 2.1 for the definition of "locally threaded".

known whether the competitive mechanism has a locally threaded message correspondence on the full class of environments on which the welfare theorems hold. It was established in [16] that the competitive mechanism does satisfy that regularity condition on the class of pure trade environments with Cobb-Douglas utilities. Furthermore, it is clear that when the competitive equilibrium is unique (and the Walras correspondence is upper hemi-continuous), the regularity condition (local threading) is also met. The case of multiple equilibria for environments which do not satisfy the assumptions of our theorems remains open. It should be pointed out that the competitive mechanism, as it is ordinarily specified, does not meet our requirements for a mechanism in the case of multiple equilibria. We require that a particular equilibrium be selected in a continuous fashion as the environment varies. Indeed, one interpretation of our result is that the conjunction of (i) the requirement that the economic mechanism make a selection of equilibria for cases in which there are multiple equilibria, (ii) the regularity condition on communication, and (iii) Pareto satisfactoriness of performance, restricts the allowable environments to very classical ones.

## 2. $\mathcal{P}$ SATISFACTORY MECHANISMS

In this section we give the definitions needed to determine the structure of mechanisms which have satisfactory performance relative to an abstract criterion represented by a correspondence  $\mathcal{P}$  from environments to actions. We call these  $\mathcal{P}$  satisfactory mechanisms. In subsequent sections we take  $\mathcal{P}$  to be the Pareto correspondence. We begin by reproducing the definition of a resource allocation process given in [16].

**DEFINITION 2.1:** Let  $E$ ,  $M$ , and  $Z$  be topological spaces; let  $\gamma: E \rightarrow M$  be a correspondence and  $g: M \rightarrow Z$  a function. The pair  $(\gamma, g)$  is a *resource allocation process on  $E$*  (with message space  $M$ ) if (i)  $\gamma$  is locally threaded;<sup>3</sup> (ii)  $g$  is continuous; and (iii)  $g$  is compatible with  $\gamma$ , i.e., for  $e \in E$  if  $m$  and  $m'$  are in  $\gamma(e)$ , then  $g(m) = g(m')$ .

The *performance* of a resource allocation process  $(\gamma, g)$  is characterized by the function  $f = g \cdot \gamma: E \rightarrow Z$ .

If  $f: E \rightarrow Z$  is a given continuous function, we say that  $(\gamma, g)$  realizes  $f$  and that  $M$  is *sufficient for  $f$*  if  $f = g \cdot \gamma$ . See [16, p. 169, Definition 1].

The structure is as yet not fully adequate for the study of Pareto satisfactory mechanisms, since the outcome of a resource allocation process is a point, while the outcomes corresponding to equilibria of a Pareto satisfactory process must cover the Pareto set.

Let  $\Gamma: E \rightarrow M$  be a correspondence. We may, e.g., interpret  $\Gamma(e)$  to be the set of message complexes which are equilibria of a communication process in the

<sup>3</sup> For convenience we give Definition 6 of [16, p. 173]. The correspondence  $\gamma: E \rightarrow M$  is locally threaded if for each  $e \in E$  there exists an open neighborhood of  $e$ ,  $\mathcal{O}(e) \subset E$  and a continuous function  $s_e: \mathcal{O}(e) \rightarrow M$  such that  $s_e(e') \in \gamma(e')$  for all  $e' \in \mathcal{O}(e)$ .

environment  $e$ . In general,  $\Gamma$  is not compatible with the outcome function  $g: M \rightarrow Z$ . Where there are multiple equilibria, different equilibria could lead to different allocations, as in the case of multiple competitive equilibria. We may consider the family of correspondences  $\gamma$  from  $E$  to  $M$  which lie in  $\Gamma$  and are compatible with  $g$ ; i.e.,  $\Gamma = \{\gamma \mid \gamma(e) \subset \Gamma(e) \text{ for all } e \in E \text{ and } m, m' \in \gamma(e) \text{ implies } g(m) = g(m')\}$ . Each such selection from  $\Gamma$ , together with the function  $g$ , determines a resource allocation process provided that condition (i) of Definition 2.1 is met, namely that the selection  $\gamma$  is locally threaded on  $E$ .

**DEFINITION 2.2:** The pair  $(\Gamma, g)$  consisting of the family  $\Gamma$  and the function  $g$  is a *mechanism* provided  $(\gamma, g)$  is a resource allocation process for all  $\gamma \in \Gamma$ .

Let  $\mathcal{P}: E \rightarrow Z$  be a correspondence. We interpret  $\mathcal{P}$  as the optimality criterion. Thus, for each  $e \in E$ ,  $\mathcal{P}(e) \subset Z$  is the set of points in  $Z$  which are considered optimal for  $e$ . (We do not here specify whether  $Z$  is the space of allocations, trades or other outcomes.) In the cases we will consider below  $\mathcal{P}$  is the Pareto correspondence, i.e.,  $\mathcal{P}(e)$  is the set of Pareto optima of  $e$ .

We next consider redistributions of the initial endowment. In the conventional pure trade case, for example, such redistributions leave the Edgeworth box and the contract curve fixed, varying only the initial endowment point in the box. We shall, in addition, require redistribution to be a continuous process. We may as well think of it as a particular kind of continuous transformation of the environment by a process not part of the economic mechanism and which does not alter the optimal set  $\mathcal{P}(e)$ .

**DEFINITION 2.3:** Let  $\mathcal{D}$  be the class of continuous functions  $d: E \rightarrow E$  such that  $\mathcal{P}(d(e)) = \mathcal{P}(e)$  for all  $e \in E$ . We call the function  $d$  a *redistribution rule*.

We now may define  $\mathcal{P}$  satisfactoriness of mechanisms.

**DEFINITION 2.4:** A mechanism  $(\Gamma, g)$  is  $\mathcal{P}$  satisfactory on  $E$  if (i) for every  $\gamma \in \Gamma$ , and  $e \in E$ ,  $g \circ \gamma(e) \in \mathcal{P}(e)$ ; (ii) for every  $e \in E$  and  $z \in \mathcal{P}(e)$ , there exist  $\gamma \in \Gamma$  and  $d \in \mathcal{D}$  such that  $g \circ \gamma \circ d(e) = z$ .

Condition (i) of Definition 2.4 (nonwastefulness) requires that every outcome of the mechanism be a  $\mathcal{P}$  optimum.

Condition (ii) (unbiasedness) requires that any outcome  $z$  which is  $\mathcal{P}$  optimal for an environment  $e$  be a possible outcome of the mechanism for some choice of equilibrium, allowing for an admissible transformation of the environment, e.g., a redistribution of initial endowment.<sup>4</sup>

<sup>4</sup> Hurwicz, taking  $\mathcal{P}$  to be the Pareto criterion, required only that an outcome Pareto equivalent to the given alternative  $z$  be found by the mechanism [7]. Our requirement is slightly stronger if  $Z$  is the space of allocations or trades, and the same if  $Z$  is the space of utility values. The stronger requirement avoids some complications which do not seem to contribute anything essential to our understanding. The weaker form of condition (ii) would require replacing the Pareto point  $z_0$  in Theorems 2.1 and 2.2 by a Pareto equivalent point.

We next note that the performance of a resource allocation process is continuous.

LEMMA 2.1: *Let  $(\gamma, g)$  be a resource allocation process on  $E$  with message space  $M$ , and let  $f = g \cdot \gamma: E \rightarrow Z$ . Then  $f$  is continuous.*

PROOF: Let  $U$  be open in  $Z$ . We shall show that  $f^{-1}(U)$  is open in  $E$ . Now  $f^{-1}(U) = \gamma^{-1} \cdot g^{-1}(U) = \gamma^{-1}(V)$ , where  $V$  is open in  $M$  since  $g$  is continuous, and where  $\gamma^{-1}(V) \equiv \{e \in E \mid \gamma(e) \in V\}$ . Let  $p \in \gamma^{-1}(V)$ . We shall show that  $\gamma^{-1}(V)$  contains a neighborhood of  $p$  in  $E$  and, hence, is open in  $E$ . Since  $\gamma$  is locally threaded, there exists an open set  $\bar{E}$  containing  $p$  and a continuous function  $s: \bar{E} \rightarrow M$  such that  $s(e) \in \gamma(e)$  for  $e \in \bar{E}$ .

Since  $s(p) \in V$  and  $V$  is open there exists an open neighborhood  $W$  of  $p$  in  $\bar{E}$  such that  $s(W) \subset V$ . But this implies  $W \subset \gamma^{-1}(V)$ , which since  $p \in W$ , suffices to show that  $\gamma^{-1}(V)$  is open in  $E$ . Q.E.D.

Thus, the fact that the message correspondence is locally threaded imposes a substantial degree of regularity on the performance of a resource allocation process. This has the effect of restricting the class of environments capable of being satisfactorily coordinated by means of mechanisms. Theorem 2.1 establishes necessary conditions on the correspondence  $\mathcal{P}$  so that a  $\mathcal{P}$  satisfactory mechanism exists.

THEOREM 2.1: *Let  $E$  be a class of environments and  $\mathcal{P}: E \rightarrow Z$  a correspondence. If  $(\Gamma, g)$  is a  $\mathcal{P}$  satisfactory mechanism on  $E$  then for each  $e_0 \in E$  and  $z_0 \in \mathcal{P}(e_0)$  there exists a (global) thread  $s_0: E \rightarrow Z$  of  $\mathcal{P}$  which passes through  $z_0$  at  $e_0$ .*

PROOF: Let  $e_0 \in E$  and  $z_0 \in \mathcal{P}(e_0)$ . Since  $(\Gamma, g)$  is  $\mathcal{P}$  satisfactory on  $E$ , by (ii) of Definition 2.4 there exist  $\gamma \in \Gamma$  and  $d \in \mathcal{D}$  such that  $g \cdot \gamma \cdot d(e_0) = z_0$ . Since  $\gamma$  is locally threaded, it follows from Lemma 2.1 that  $f = g \cdot \gamma$  is a continuous function on  $E$ . Take  $s_0 = f \cdot d$ . Since  $d: E \rightarrow E$  is continuous, it follows that  $s_0: E \rightarrow Z$  is continuous on  $E$ . By construction  $s_0(e_0) = z_0$ . Finally,  $s_0(e) \in \mathcal{P}(e)$  for all  $e \in E$ , since by (i) of Definition 2.4,  $g \cdot \gamma(e) \in \mathcal{P}(e)$  for all  $e \in E$ , and consequently  $d(e) = e'$  implies  $g \cdot \gamma(d(e)) \in \mathcal{P}(e') \cap \mathcal{P}(e)$ , since it follows from Definition 2.3 that  $d(e) = e'$  implies  $\mathcal{P}(e) = \mathcal{P}(e')$ . Q.E.D.

We next consider whether the condition that the correspondence  $\mathcal{P}$  be suitably threaded is sufficient for the existence of a  $\mathcal{P}$  satisfactory mechanism. It is possible to establish the converse of Theorem 2.1. However, because of our interest in informationally decentralized organization it is more interesting to state a ‘‘converse’’ to Theorem 2.1 with the condition that the mechanism preserve privacy. Thus, we take  $E$  to be the class of decomposable environments, i.e.,  $E = \prod_{i=1}^N E^i$ , where  $E^i$  is the space of agent  $i$ ’s environmental characteristics (his preferences, endowment, technology, etc.) and require the mechanism to preserve privacy [16, Definition 3]. Theorem 2.2 establishes the result that if the correspondence  $\mathcal{P}$  is

completely threaded, then there exists a privacy-preserving  $\mathcal{P}$  satisfactory mechanism on  $E$ . The converse of Theorem 2.1 can easily be obtained as a corollary.

We introduce the class of privacy-preserving processes. We reproduce from [16] the definition of a coordinate correspondence and of a privacy-preserving process.

**DEFINITION 2.5:** Let  $E = \prod_{i=1}^N E^i$ ; we say that the correspondence  $\mu: E \rightarrow M$  is a coordinate correspondence if there exist  $\mu^i: E^i \rightarrow M$  such that  $\mu(e) = \bigcap_{i=1}^N \mu^i(e)$  for all  $e \in E$ .

**DEFINITION 2.6:** Let  $E = E^1 \times \dots \times E^N$ . The resource allocation process  $(\gamma, g)$  preserves privacy if  $\gamma$  is a coordinate correspondence.

**DEFINITION 2.7:** The mechanism  $(\Gamma, g)$  preserves privacy on  $E = E^1 \times \dots \times E^N$  if  $\gamma \in \Gamma$  implies  $\gamma$  is a coordinate correspondence.

**LEMMA 2.2:** Let  $E = E^1 \times \dots \times E^N$  be a class of (decomposable) environments and let  $\mathcal{P}: E \rightarrow Z$  be a correspondence on  $E$ . If for each  $e_0 \in E$  and  $z_0 \in \mathcal{P}(e_0)$  there exists a continuous function  $s_0: E \rightarrow Z$  such that  $s_0(e) \in \mathcal{P}(e)$  for all  $e \in E$  and  $s_0(e_0) = z_0$ , then there exists a privacy-preserving resource allocation process  $(\gamma, g)$  such that  $g \cdot \gamma(e) \in \mathcal{P}(e)$  for all  $e \in E$  and  $g \cdot \gamma(e_0) = z_0$ .

**PROOF:** Take  $M = E$ , and for  $e^i \in E^i$  and  $i = 1, \dots, N$  define

$$\gamma^i(e^i) = E^1 \times \dots \times \{e^i\} \times \dots \times E^N = \{(\bar{e}^1, \dots, \bar{e}^N) \in E \mid \bar{e}^i = e^i\},$$

and let

$$\gamma(e) = \bigcap_{i=1}^N \gamma^i(e^i) = \{e\} \quad \text{for } e \in E.$$

Finally, let  $g = s_0$ .

Since  $\gamma$  is the identity function on  $E$ , it is a locally threaded correspondence. It follows from  $g = s_0$  that  $g_0 \gamma(e) \in \mathcal{P}(e)$  and  $g \cdot \gamma(e_0) = z_0$ . *Q.E.D.*

**THEOREM 2.2:** Let  $E = E^1 \times \dots \times E^N$  be a class of (decomposable) environments and let  $\mathcal{P}: E \rightarrow Z$  be a correspondence on  $E$ . If for each  $e \in E$  and  $z \in \mathcal{P}(e)$  there exists a thread of  $\mathcal{P}$  passing through  $z$  at  $e$ , then there exists a privacy-preserving mechanism which is  $\mathcal{P}$  satisfactory on  $E$ .

**PROOF:** Let  $\mathcal{S}$  be the set of continuous functions from  $E$  to  $Z$ , with the topology of pointwise convergence. Then  $\mathcal{S}$  contains the (continuous) threads of the correspondence  $\mathcal{P}$ .

Let  $M = E \times \mathcal{S}$ , and let the outcome function  $g$  be given by,

$$g(e, s) = s(e),$$

and note that  $g$  is continuous on  $E \times \mathcal{S}$  [13, p. 218].

For  $e_0 \in E$  and  $z_0 \in \mathcal{P}(e_0)$ , there exists a thread  $s_0: E \rightarrow Z$  such that  $s_0(e) \in \mathcal{P}(e)$  for all  $e \in E$  and  $s_0(e_0) = z_0$ . For each  $i = 1, \dots, N$  and  $e \in E$ , define

$$\gamma_0^i(e^i) = E^1 \times \dots \times \{e^i\} \times \dots \times E^N \times \{s_0\} \subset E \times \mathcal{S}$$

and

$$\gamma_0(e) = \bigcap_{i=1}^N \gamma_0^i(e^i) = (\{e\}, \{s_0\}).$$

Clearly,  $\gamma_0$  is threaded, since for fixed  $s_0$  it is a continuous function on  $E$ . Finally,

$$g \cdot \gamma_0(e) = s_0(e) \in \mathcal{P}(e) \quad \text{for all } e \in E,$$

and

$$g \cdot \gamma_0(e_0) = s_0(e_0) = z_0, \quad \text{by construction.}$$

Take  $\Gamma$  to be the collection of all  $\gamma_0$  generated as  $s_0$  varies over all the threads of  $\mathcal{P}$ . Then  $(\Gamma, g)$  is the required mechanism. Q.E.D.

Theorems 2.1 and 2.2 together tell us that it is necessary and sufficient for a class of (decomposable) economies to admit of being  $\mathcal{P}$  satisfactorily organized by means of a (privacy-preserving) mechanism, that the correspondence  $\mathcal{P}$  be a union of threads.

Examples of Pareto satisfactory and privacy-preserving processes may be found in [7, 9, and 16] and the references cited in those papers.

### 3. NOTATIONS AND ASSUMPTIONS

In what follows we shall make use of certain notations and assumptions which we list here for convenient reference:  $R^l$  is a euclidean space of  $l$  dimensions, the commodity space and  $R_+^l$  is the nonnegative orthant of  $R^l$ .  $X^i \subset R^l$  is the admissible consumption set of agent  $i$ ,  $i = 1, \dots, N$ .  $\succeq_i$  is the preference preordering of  $X^i$ ;  $w^i \in X^i$  is the initial endowment of agent  $i$ ,  $i = 1, \dots, N$ ;  $Y^i$  is the set of admissible trades for agent  $i$ . In the pure trade case  $Y^i = X^i - \{w^i\}$ .

$E = E^1 \times \dots \times E^N$  is the set of environments, sometimes called economies. In the pure trade case,  $E^i = \{e^i | e^i = (X^i, \Pi^i, w^i)\}$ , where  $\Pi^i \subset X^i \times X^i$  is the graph of  $\succeq_i$ . We give  $E^i$  the product topology using the topology of closed convergence for spaces of subsets and the euclidean topology for  $R^l$ .

$$X = X^1 \times \dots \times X^N,$$

$$Y = Y^1 \times \dots \times Y^N,$$

$\mathcal{F}: E \rightarrow X$ , where  $\mathcal{F}(e) \subset X$  is the set of allocations feasible for  $e$ ;

$F: E \rightarrow Y$ , where  $F(e) \subset Y$  is the set of trades feasible for  $e$ ;

$$\mathcal{G}^i(e^i, x^i) = \{\bar{x}^i \in X^i | \bar{x}^i \succeq_i x^i\}, \quad \text{for } x^i \in X^i, \quad \text{and}$$

$$G^i(e^i, y^i) = \{\bar{y}^i \in Y^i | \bar{y}^i \succeq_i y^i\} \quad \text{for } y^i \in Y^i.$$

We understand the symbol  $\succeq_i$  between elements  $\bar{y}^i$  and  $y^i$  of  $Y^i$  to mean

$$w^i + \bar{y}^i \succeq_i w^i + y^i.$$

We shall also write  $\bar{x} \succeq_i x$  for  $\bar{x}, x \in X$  if and only if  $\bar{x}^i \succeq_i x^i$ , where

$$\bar{x} = (\bar{x}^1 \dots \bar{x}^i \dots \bar{x}^N) \quad \text{and} \quad x = (x^1 \dots x^i \dots x^N).$$

Thus,  $\mathcal{G}^i : E^i \times X^i \rightarrow X^i$ , and  $G^i : E^i \times Y^i \rightarrow Y^i$  for  $i = 1, \dots, N$ . We write  $\mathcal{G} = \mathcal{G}^1 \times \dots \times \mathcal{G}^N$ , and  $G = G^1 \times \dots \times G^N$ , and hence  $\mathcal{G} : E \times X \rightarrow X$ , and  $G : E \times Y \rightarrow Y$ .

For  $A$  and  $B$  subsets of  $R^m$ ,  $d(A, B)$  denotes the distance between  $A$  and  $B$  in the metric topology of closed convergence. We shall denote the norm in  $R^m$  by  $|\cdot|$ .

$\mathcal{P} : E \rightarrow X$  denotes the Pareto correspondence, i.e.,  $\mathcal{P}(e)$  is the ‘‘contract curve’’ in  $X$ .

$P : E \rightarrow Y$  is the Pareto correspondence in terms of trades.

If  $u(e, \cdot) = (u^1(e^1, \cdot), \dots, u_N(e^N, \cdot))$  for  $e \in E$  is a vector of utility functions on  $X$  then  $U : E \rightarrow R^N$ , where  $U(e)$  is the image of  $\mathcal{T}(e)$  under  $u(e, \cdot)$ , is the feasible utility correspondence and  $\hat{U} : E \rightarrow R^N$ , where  $\hat{U}(e)$  is the image of  $\mathcal{P}(e)$  under  $u(e, \cdot)$ , is the Pareto frontier correspondence.

Thus, for  $e \in E$

$$U(e) = \{r \in R^N \mid r = u(e, x) \text{ for some } x \in \mathcal{T}(e)\}$$

and

$$\hat{U}(e) = \{r \in R^N \mid r \in U(e), \text{ and } (r' \succeq r, r' \neq r) \text{ imply } r' \notin U(e)\}.$$

Thus  $U(e)$  is the set of utility values attainable for the economy  $e$  with utility functions  $u(e, \cdot)$ , and  $\hat{U}(e)$  is the Pareto frontier of utility values.

For  $a$  and  $b$  vectors, we write  $a \succeq b$  if  $a_i \succeq b_i$  for all  $i$  but  $a \neq b$ , and  $a > b$  if  $a_i > b_i$  for all  $i$ .

We shall list the following assumptions. Not all assumptions will be in force simultaneously, but only as indicated. We shall, however, always assume continuity of preferences (Assumption 2).

ASSUMPTION 1a: For each  $i = 1, \dots, N$ ,  $X^i$  is an arc wise connected subset of  $R^I$ .

ASSUMPTION 1b: For each  $i = 1, \dots, N$ ,  $X^i = R^I_+$ .

ASSUMPTION 2: For each  $i = 1, \dots, N$ , the preference relation  $\succeq_i$  is continuous. That is, for every  $x^i \in X^i$  the sets  $\{\bar{x}^i \in X^i \mid \bar{x}^i \succeq_i x^i\}$  and  $\{\bar{x}^i \in X^i \mid x^i \succeq_i \bar{x}^i\}$  are closed.

ASSUMPTION 3a (Local nonsatiation): For each  $i = 1, \dots, N$ , each  $x^i \in X^i$  and for every open set  $\mathcal{O}^i(x^i)$  containing  $x^i$  there exists  $\bar{x}^i \in \mathcal{O}^i(x^i)$  such that  $\bar{x}^i >_i x^i$ .

ASSUMPTION 3b (Strict monotonicity): For each  $i = 1, \dots, N$ ,  $\bar{x}^i \geq x^i$  implies  $\bar{x} >_i x^i$ .

ASSUMPTION 4 (Strict dominance): For  $e \in E$  if  $x \in \mathcal{F}(e)$  and  $x \notin \mathcal{P}(e)$ , then there exists  $\bar{x} \in \mathcal{T}(e)$  such that  $\bar{x} >_i x$  for  $i = 1, \dots, N$ .

Stated in terms of utilities, the strict dominance assumption is: For  $e \in E$ , if  $p \in U(e)$  and  $p \notin \hat{U}(e)$ , then there exists  $q \in U(e)$  such that  $q > p$ .

ASSUMPTION 5 (Pointedness): For  $e \in E$  and  $x \in \mathcal{P}(e)$ ,  $\mathcal{G}(e, x) \cap \mathcal{F}(e) = \{x\}$ .

ASSUMPTION 6a: The correspondence  $\mathcal{F}: E \rightarrow X$  is (i) upper hemi-continuous, (ii) lower hemi-continuous, and (iii) such that its image sets each contain more than one point.

ASSUMPTION 6b: For  $e \in E$ , if  $x \in X$ ,  $y \in X$ ,  $x \in \mathcal{F}(e)$  and  $\sum_{i=1}^N x^i = \sum_{i=1}^N y^i$ , then  $y \in \mathcal{T}(e)$ .

The foregoing assumptions are clearly not all independent. We note some of the relationships among them, without proof: (i) Assumption 1b implies Assumption 1a; (ii) Assumptions 1b and 3b imply Assumption 3a; (iii) Assumptions 1b, 2, 3b, and 6b imply Assumption 4. We also note that Assumptions 2 and 3a imply a weaker form of Assumption 4, namely that each non-optimal feasible point which is dominated by a point interior to the feasible set is strictly dominated.

#### 4. THE FEASIBLE UTILITY CORRESPONDENCE AND THE PARETO FRONTIER

In this section we study the continuity of the utility possibility set and its Pareto utility frontier.

We note first the following lemmas:

LEMMA 4.1: *If  $\mathcal{F}: E \rightarrow X$  is continuous on  $E$  (upper and lower hemi-continuous) and  $u: E \times X \Rightarrow \mathbb{R}^N$  is continuous on  $E \times X$ , then  $U: E \Rightarrow \mathbb{R}^N$ ,  $U(e) = u(e, \mathcal{F}(e))$  is a continuous correspondence on  $E$  [1, p. 113]. It is clear that if  $\mathcal{T}$  is compact valued and  $u$  continuous, then  $U$  is compact valued.*

LEMMA 4.2: *Let  $A: X \rightarrow \mathbb{R}^N$  be an upper hemi-continuous correspondence on a topological space  $X$  and let  $x_0 \in X$ . Then there exists a compact set  $K \subset \mathbb{R}^N$  and an open neighborhood  $\mathcal{O}(x_0)$  of  $x_0$  such that  $x \in \mathcal{O}(x_0)$  implies  $A(x) \subset K$ .*

PROOF: The upper hemi-continuity of  $A$  implies  $A(x_0)$  is compact. [1, p. 110, Theorem 2]. It follows that  $\text{cl } A_\varepsilon(x_0)$  is also compact, where, for  $\varepsilon > 0$ ,

$$A_\varepsilon(x_0) = \{y \in \mathbb{R}^N \mid |y - z| < \varepsilon \text{ for some } z \in A(x_0)\}.$$

Since  $A_\varepsilon(x_0)$  is open, and  $A$  is upper hemi-continuous, there exists an open neighborhood  $\mathcal{O}(x_0)$  of  $x_0$  such that  $x \in \mathcal{O}(x_0)$  implies  $A(x) \subset A_\varepsilon(x_0)$  [1, p. 109]. Hence,  $x \in \mathcal{O}(x_0)$  implies  $A(x) \subset \text{cl } A_\varepsilon(x_0)$ . Q.E.D.

LEMMA 4.3: *If  $\mathcal{F}: E \rightarrow R^{IN}$  is continuous and compact valued (Assumption 6a), if  $u: E \times R^{IN} \rightarrow R^N$  is continuous and satisfies Assumption 4 (strict dominance) on  $\mathcal{F}$ , then the correspondence  $\hat{U}: E \rightarrow R^N$  is upper hemi-continuous on  $E$ .*

PROOF: The correspondence  $U: E \rightarrow R^N$  is continuous and, hence, upper hemi-continuous (Lemma 4.1). Since  $\hat{U}(e) \subset U(e)$  for all  $e \in E$ , we may write  $\hat{U} = \hat{U} \cap U$ . If  $\hat{U}$  is a closed correspondence it follows that  $\hat{U}$  is upper hemi-continuous, since the intersection of a closed correspondence with an upper hemi-continuous correspondence is upper hemi-continuous [1, p. 112, Theorem 7]. We shall show that  $\hat{U}$  is a closed correspondence.

Let  $e_0 \in E$ ,  $e_j \rightarrow e_0$ ,  $r_j \rightarrow r_0$  and let  $r_j \in \hat{U}(e_j)$  for  $j = 1, 2, \dots$ . Since  $\hat{U}(e_j) \subset U(e_j)$ ,  $r_j \in \hat{U}(e_j)$  if and only if there exists  $x_j \in \mathcal{F}(e_j)$  such that (i)  $u(e_j, x_j) = r_j$ , and (ii)  $\{x \in \mathcal{F}(e_j) | u(e_j, x) \geq u(e_j, x_j)\} = \emptyset$ . Since  $\mathcal{F}(e_0)$  is compact (Assumption 6a(ii)) and  $\mathcal{F}$  upper hemi-continuous (Assumption 6a(i)) by Lemma 4.2, for every  $\varepsilon > 0$ , there exists an integer  $J(\varepsilon)$  such that  $j > J(\varepsilon)$  implies  $\mathcal{F}(e_j) \subset \text{cl } \mathcal{F}_\varepsilon(e_0)$ , a compact set. Hence, the infinite set  $\{x_j | j > J(\varepsilon)\}$  has a point of accumulation  $x_0$ , and consequently a subsequence  $x_{j_k} \rightarrow x_0$ . By continuity of  $u$  on  $E \times R^{IN}$ ,

$$u(e_{j_k}, x_{j_k}) \rightarrow u(e_0, x_0).$$

But  $u(e_{j_k}, x_{j_k}) = r_{j_k}$  and by hypothesis  $r_{j_k} \rightarrow r_0$ . It follows that  $u(e_0, x_0) = r_0$ .

Now suppose  $r_0 \notin \hat{U}(e_0)$ . Since  $r_0 \in U(e_0)$  and  $r_0 \notin \hat{U}(e_0)$  it follows from the strict dominance assumption applied to  $r_0$  (Assumption 4) that there exists  $\bar{r} \in U(e_0)$  such that  $\bar{r} > r_0$  and correspondingly  $\bar{x} \in \mathcal{F}(e_0)$  such that  $\bar{r} = u(e_0, \bar{x}) > u(e_0, x_0)$ .

Since  $\mathcal{F}$  is lower hemi-continuous (Assumption 5a), it follows from  $e_{j_k} \rightarrow e_0$  and  $\bar{x} \in \mathcal{F}(e_0)$  that there exists  $\bar{x}_{j_k} \in \mathcal{F}(e_{j_k})$  such that  $\bar{x}_{j_k} \rightarrow \bar{x}$ . Since  $u$  is continuous, for  $\bar{\varepsilon} = |u(e_0, \bar{x}) - u(e_0, x_0)| > 0$ , there exists an integer  $K(\bar{\varepsilon})$  such that  $k > K(\bar{\varepsilon})$  implies

$$|u(e_{j_k}, \bar{x}_{j_k}) - u(e_0, \bar{x})| < \frac{\bar{\varepsilon}}{2}$$

and

$$|u(e_{j_k}, x_{j_k}) - u(e_0, x_0)| < \frac{\bar{\varepsilon}}{2}.$$

It follows that for  $k > K(\bar{\varepsilon})$ ,

$$\bar{r}_{j_k} = u(e_{j_k}, \bar{x}_{j_k}) > u(e_{j_k}, x_{j_k}) = r_{j_k}$$

Thus,  $\bar{r}_{j_k} \in U(e_{j_k})$ ,  $\bar{r}_{j_k} > r_{j_k}$ , and hence  $r_{j_k} \notin \hat{U}(e_{j_k})$ , contrary to construction. Thus  $r_0 \notin \hat{U}(e_0)$  must be false. This establishes the upper hemi-continuity of  $\hat{U}$ .

*Q.E.D.*

LEMMA 4.4: *If  $U: E \rightarrow R^n$  is continuous and compact valued on  $E$  then  $\hat{U}$  is lower hemi-continuous on  $E$ .*

PROOF:<sup>5</sup> Assume  $e_0 \in E$  and  $r_0 \in \hat{U}(e_0)$  and suppose  $e_j \rightarrow e_0$ , where  $\{e_j\}$  is a convergent sequence of environments  $e_j \in E$ .

Since  $\hat{U}(e) \subset U(e)$  for all  $e \in E$ , it follows that  $r_0 \in U(e_0)$ . Since  $U$  is lower hemi-continuous on  $E$ , and since  $e_j \rightarrow e_0$ , it follows that there exist  $r_j \in U(e_j)$  such that  $r_j \rightarrow r_0$ .

Let

$$\Omega(r) = \{s \in R^N \mid s \succcurlyeq r\},$$

and let  $B_j = \{r \in R^N \mid r \text{ is maximal for } \succcurlyeq \text{ on } \Omega(r_j) \cap U(e_j)\}$ . By hypothesis,  $U(e_j)$  is compact and hence so is  $U(e_j) \cap \Omega(r_j)$ . Hence,  $B_j$  is non-empty for all  $j$ .

Further,  $B_j \subset \hat{U}(e_j)$  for all  $j$ . To see this, let  $r \in B_j$  and  $r' \in U(e_j)$  with  $r' \neq r$  and  $r' \succcurlyeq r$ . Then  $r' \succcurlyeq r \succcurlyeq r_j$ , thus  $r' \in U(e_j) \cap \Omega(r_j)$ . It follows that  $r$  is not maximal on that set, which contradicts  $r \in B_j$ .

Given  $\varepsilon > 0$ , it follows from the hypothesis that  $U$  is compact valued, that  $\text{cl } U_\varepsilon(e_0)$  is compact. Since  $U$  is upper hemi-continuous there exists an integer  $K$  such that  $j > K$  implies

$$U(e_j) \subset \text{cl } U_\varepsilon(e_0).$$

Thus,

$$B_j \in U(e_j) \subset \text{cl } U_\varepsilon(e_0) \quad \text{for } j > K.$$

Take  $z_j \in B_j$  for  $j > K$ , which may be done, since  $B_j \neq \emptyset$  for all  $j$ .

Compactness of  $\text{cl } U_\varepsilon(e_0)$  guarantees that the infinite subset  $\{z_j\}$  has at least one accumulation point, say  $\bar{z}$ , and that if  $\bar{z}$  is the *only* accumulation point of  $\{z_j\}$ , then it is the limit of  $\{z_j\}$ . Let  $z_0$  be an accumulation point of  $\{z_j\}$ , and let  $\{z_{j_k}\}$  be a subsequence which converges to  $z_0$ . We have  $z_{j_k} \succcurlyeq r_{j_k}$ , for all  $j_k > K$  (because  $z_j \in B_j \subseteq \Omega(r_j)$ ), and  $r_{j_k} \rightarrow r_0$ ; hence,  $z_0 \succcurlyeq r_0$ .

But we also have  $e_{j_k} \rightarrow e_0$  and  $z_{j_k} \in U(e_{j_k})$ , which, since  $U$  is upper hemi-continuous, guarantees that  $z_0 \in U(e_0)$ . Since  $r_0 \in \hat{U}(e_0)$ , then we must have  $z_0 = r_0$ ; that is,  $r_0$  is the only accumulation point of  $\{z_j\}$ , and is consequently the limit of  $\{z_j\}$ . Since, for  $j > K$ ,  $z_j \in B_j \subseteq \hat{U}(e_j)$ , the proof is complete. Q.E.D.

### 5. THREADS OF THE PARETO FRONTIER

In this section we give conditions sufficient for the existence of a continuous thread of the Pareto correspondence passing through an arbitrarily chosen point of the graph of that correspondence. As we found in Theorem 2.1, the existence of such threads is necessary in order that a class of economies be susceptible of being organized by a Pareto-satisfactory mechanism satisfying the condition that the message correspondence is locally threaded. These conditions show that a class of economies for which the Pareto correspondence has such a thread consists of economies such that the consumption set of each agent is the nonnegative orthant of the Euclidean commodity space, the production sets permit a nonzero alloca-

<sup>5</sup> Mark Walker's suggestions have made this proof both shorter and clearer than it was before.

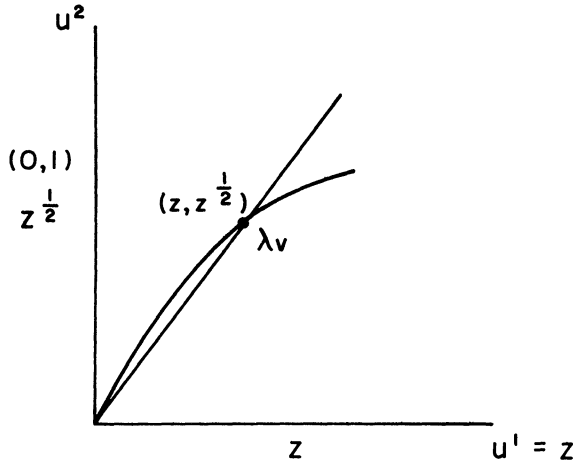


FIGURE 5.1

tion and are such that the set of attainable allocations is compact, and such that preferences are strictly monotonic for each agent. Convexity of either preferences or production sets is not required. In [10] this class of economies was shown to satisfy the assumption of openness [10, Lemma 5.9 and its corollary].

We show below in Section 6 that strict monotonicity cannot be replaced by a weaker form of monotonicity. This is not unexpected, since both local nonsatiation and the strict dominance properties might then fail. Furthermore, convexity of preference, implying local nonsatiation, is not sufficient to ensure that the Pareto frontier has no holes. Example 5.1 below shows this.

The property of strict dominance, which is indispensable for the Pareto frontier to be without holes, involves a relationship among the preferences of all the agents, as well as among their consumption sets (and production sets in the case of production). If we specify a class of characteristics of individuals (consumption sets, preferences, etc.) such that any  $N$ -tuple of agents drawn from that class will have the strict dominance property, then we conjecture that strict monotonicity of preference over the nonnegative orthants (or consumption sets that are suitable translates of them) is essentially the largest such class. However, as Lemma 3.2 above shows, continuity of preference and local nonsatiation are, when redistribution is feasible (Assumption 6b) sufficient for the strict dominance property to hold except on the set of non-optimal points dominated only by boundary points of the consumption set.

**DEFINITION 5.1:** Let  $e \in E$  be an economy with continuous utility indicators  $u = (u^1, \dots, u^N)$  such that for each  $i = 1, \dots, N$   $\inf u^i(x) \geq 0$ , and let  $\hat{U}(e) \subset \mathbb{R}_+^N$  denote  $u(\mathcal{P}(e))$ , the Pareto set of  $e$  in the space of utility values. We say that  $\hat{U}(e)$  has no holes if and only if for each nonzero vector  $p \in \mathbb{R}_+^N, \{tp | t \geq 0\} \cap \hat{U}(e) \neq \emptyset$ .

Thus, the set of Pareto utility values has no holes if every ray out of the origin in a nonnegative direction intersects the Pareto set. Lemma 5.1 gives sufficient conditions, involving the “no holes” property, that the Pareto correspondence has a thread through every point of its graph.

LEMMA 5.1: *If  $E$  is a class of economies such that (i) the Pareto correspondence  $\hat{U}$  is upper hemi-continuous on  $E$ , (ii)  $\hat{U}(e) \subset R_+^N$  for  $e \in E$ , and (iii) for each  $e \in E$ ,  $\hat{U}(e)$  has no holes, then for every  $e_0 \in E$ , and every point  $p_0 \in \hat{U}(e_0)$ , there exists a continuous thread  $s: E \rightarrow R_+^N$  of  $\hat{U}$  such that  $s(e_0) = p_0$ .*

PROOF:<sup>6</sup> Let  $e_0 \in E$ , let  $p_0 \in \hat{U}(e_0)$ , and for each  $e \in E$ , let  $s(e)$  be the set  $\{tp_0 | t \geq 0\} \cap \hat{U}(e)$ , the intersection of  $\hat{U}(e)$  with the ray through  $p_0$ . According to (ii) and (iii),  $s(e)$  is non-empty for each  $e \in E$ , i.e.,  $s$  is a correspondence from  $E$  to  $R_+^N$ . Since  $s$  is the intersection of the upper hemi-continuous correspondence  $\hat{U}$  with the constant (hence, continuous) correspondence  $\{tp_0 | t \geq 0\}$ , it is itself upper hemi-continuous [1, p. 114, Theorem 2']. Finally  $s(e)$  obviously consists of only a single point for each  $e$  (by definition of  $\hat{U}$ ) and, hence,  $s$  is a continuous function. Q.E.D.

DEFINITION 5.2: Let  $E^*$  be the class of economies satisfying Assumptions 1b, 2, 3b, 6a, and 6b. That is,  $E^*$  is the class of economies such that (i) for each agent  $i \in (1, \dots, N)$ : (a) the consumption set  $X^i = R_+^I$ ; (b) the preference relation  $\preceq_i$  is continuous and strictly monotone; and (ii): (c) the correspondence  $\mathcal{F}: E^* \rightarrow R^{IN}$  is continuous and compact-valued, and such that its image sets contain more than one point; (d) if  $x \in \mathcal{F}(e)$ ,  $x = (x^1, \dots, x^N)$ , and if  $y = (y^1, \dots, y^N)$   $y^i \geq 0$  for  $i = 1, \dots, N$  and  $\sum_{i=1}^N y^i = \sum_{i=1}^N x^i$ , then  $y \in \mathcal{F}(e)$ ; and (e) for each  $e \in E^*$  there exists  $x \in \mathcal{F}(e)$ ,  $x \geq 0$ .

LEMMA 5.2: *There exists a utility representation  $u^*: E^* \times R_+^{IN} \rightarrow R^N$  such that the correspondence  $U^*: E^* \rightarrow R^N$  given by  $U^*(e) = \{p \in R^N | u^*(e, x) = p \text{ for some } x \in \mathcal{F}(e)\}$  is (i) upper hemi-continuous and lower hemi-continuous, (ii)  $U^*(e) \subset R_+^N$  for all  $e \in E^*$ , and (iii)  $0 \in R_+^N$  is dominated for all  $e \in E^*$  (i.e., for all  $e \in E^*$ ,  $0$  is not a Pareto point of  $U^*(e)$ ).*

PROOF: If  $e \in E^*$ , under (i) and (ii) of Definition 5.2, it is well-known that there exists a vector of utility functions  $u: E^* \times R_+^{IN} \rightarrow R^N$  which is a jointly continuous representation of preferences. (Kannai [12, p. 798] established the existence of a utility function which is jointly continuous in preferences and commodities and which represents the class of continuous monotone preferences on  $R_+^I$ , where the topology on the space of preference relations is that of closed convergence.)

Let

$$m_e^i = \min_{x \in R_+^I} u^i(e, x).$$

<sup>6</sup> We owe this improved version of the proof to suggestions made by Mark Walker.

Since  $R_+^l$  contains its lower bound for  $\leq$ , namely, 0, and  $u^i(e, \cdot)$  is strictly increasing with respect to  $\leq$ ,  $m_e^i$  is well-defined. For  $e \in E^*$  define

$$u^*(e, \cdot) = A_u \cdot u(e, \cdot) \equiv u(e, \cdot) - (m_u^1, \dots, m_u^N).$$

Thus,

$$u^*: E^* \times R_+^{lN} \rightarrow R^N.$$

Then,

$$u^{*i}(e, x^i) \geq 0 \quad \text{for } i = 1, \dots, N \quad \text{and } x^i \in R_+^l,$$

$$u^{*i}(e, 0) = 0,$$

and  $u^{*i}$  is jointly continuous on  $E^* \times R_+^l$ , because  $A_u$  is continuous, since  $m_u^i$  is the minimum of a continuous function on a fixed set.

It follows immediately that  $U^*$  is continuous on  $E^*$ .

*Q.E.D.*

LEMMA 5.3:  $U^*$  satisfies Assumption 4 on  $E^*$ .

PROOF:<sup>7</sup> Let  $p \in U^*(e)$  and suppose  $p$  is not a Pareto point. Then there exists  $q \in U^*(e)$  such that  $q \geq p$ . Since  $p$  and  $q$  are in  $U^*(e)$  there exists  $x$  and  $y$  in  $\mathcal{F}(e)$  such that  $u^*(e, x) = p$  and  $u^*(e, y) = q$ . Since  $q \geq p \geq 0$ , there is at least one agent  $i_0 \in \{1, \dots, N\}$  such that  $q^{i_0} = u^{*i_0}(e, y^{i_0}) > u^{*i_0}(e, x^{i_0}) \geq 0$ . It follows from strict monotonicity of preference and from  $\min_{\mathcal{F}(e)} u^*(e, x) = 0$ , that there is at least one commodity  $j_0 \in \{1, \dots, l\}$  such that  $y_{j_0}^{i_0} > 0$ . Since  $u^{*i_0}(e, \cdot)$  is continuous and assumes the value, 0 at 0 and  $q^{i_0}$  at  $y^{i_0} \geq 0$ , there exists a scalar  $0 < \alpha^0 < 1$  such that

$$u^{*i_0}(e, \alpha^0 y^{i_0}) = \frac{1}{2}(u^{*i_0}(e, y^{i_0}) + p^{i_0}) > p^{i_0}.$$

We may define a new allocation  $z$ , by

$$z^i = \begin{cases} y^i + \frac{1 - \alpha^0}{N - 1} y^{i_0} & \text{for } i \in \{1, \dots, N\} \setminus \{i_0\}, \\ \alpha^0 y^{i_0} & \text{for } i = i_0. \end{cases}$$

Clearly  $z \in R_+^{lN}$ , and  $\sum_{i=1}^N z^i = \sum_{i=1}^N y^i$ ; hence, by Assumption 6b,  $z \in \mathcal{F}(e)$ . Furthermore  $z^i \geq y^i$  for  $i \neq i_0$ , which by strict monotonicity of  $u^{*i}(e, \cdot)$  implies

$$u^{*i}(e, z^i) > u^{*i}(e, y^i) \geq p^{i_0} \quad \text{for } i \neq i_0$$

while for  $i = i_0$ ,  $u^{*i_0}(e, z^i) = u^{*i_0}(e, \alpha^0 y^{i_0}) > p^{i_0}$ . Thus, it follows that

$$u^*(e, z) \in U^*(e)$$

and

$$u^*(e, z) > p.$$

*Q.E.D.*

<sup>7</sup> The proof of this lemma follows an argument in the proof of Theorem 5 in [3, p. 25].

LEMMA 5.4: *The Pareto correspondence  $\hat{U}^*: E^* \rightarrow R_+^N$ , given by  $\hat{U}^*(e) = \{p \in U^*(e) | q \in U^*(e), q \geq p \text{ implies } q = p\}$  is upper hemi-continuous on  $E^*$ .*

PROOF: It follows from Lemmas 5.2 and 5.3 that  $U^*$  satisfies all the hypotheses of Lemma 4.3. The conclusion follows.

THEOREM 5.1:<sup>8</sup> *If  $e \in E^*$ , then  $\hat{U}^*(e)$  has no holes.*

PROOF: For  $e \in E^*$  define  $\dot{U}^*(e) = \{p \in R_+^N | p \leq q \text{ for some } q \in U^*(e)\}$  and let  $\mathcal{B}^*(e)$  denote the relative boundary of  $\dot{U}^*(e)$  in  $R_+^N$ . It follows from compactness of  $\mathcal{F}(e)$  and continuity of  $u^*(e, \cdot)$  that  $U^*(e)$  is compact and, hence, that  $\dot{U}^*(e)$  is compact. Hence,

$$\mathcal{B}^*(e) \cap \dot{U}^*(e) = \mathcal{B}^*(e) \quad \text{for all } e \in E.$$

For each  $e \in E^*$  the hypotheses of Theorem 5 [3, p. 25] are satisfied; it follows that

$$\hat{U}^*(e) = \mathcal{B}^*(e).$$

This, in turn, implies that  $\hat{U}^*(e)$  has no holes. To see this, we must show that if  $p \in R_+^N, p \neq 0$ , then the half line  $\{tp | t \geq 0\}$  has a non-empty intersection with the Pareto set  $\hat{U}^*(e)$ . It is clear that  $\{tp | t \geq 0\} \cap \dot{U}^*(e) \neq \emptyset$ . Since  $\dot{U}^*(e)$  is compact and contains a point  $q \geq 0$ , there is a largest value of  $t$ , say  $t = \bar{t}, \bar{t} > 0$ , such that  $\bar{t}p \in \dot{U}^*(e)$ . It follows that  $\bar{t}p \in \mathcal{B}^*(e)$ , since  $t > \bar{t}$  implies  $tp$  is exterior to  $\dot{U}^*(e)$ .

By [3, Theorem 5],  $\bar{t}p \in \hat{U}^*(e)$ . That is, for each  $e \in E, \hat{U}^*(e)$  has no holes.

*Q.E.D.*

THEOREM 5.2: *If  $E^*$  is a class of economies satisfying Definition 5.2, and if  $u: E^* \times R_+^{IN} \rightarrow R^N$  is a jointly continuous representation of preferences and  $\hat{U}: E^* \rightarrow R_+^N$  the Pareto correspondence induced by  $u$ , then for every economy  $e_0 \in E^*$  and every Pareto point  $p_0 \in \hat{U}(e_0)$  there is a continuous thread  $s: E^* \rightarrow R_+^N$  which takes the value  $p_0$  at  $e_0$ .*

PROOF: Let  $A_u$  be the transformation which carries  $u$  to  $u^*$ , as in the proof of Lemma 5.2. Then  $A_u$  is orthogonal and nonsingular. Hence  $A_u^{-1}$  is well-defined, continuous, and carries Pareto points to Pareto points. That is, if

$$q \in \hat{U}^*(e), \quad \text{then } A_u^{-1}(q) \in \hat{U}(e).$$

It follows from Theorem 5.1 that for  $e \in E^*, \hat{U}^*(e)$  has no holes. Hence, Lemma 5.1 applies. We may conclude that there exists a thread  $s^*: E^* \rightarrow R_+^N$  such that  $s^*(e_0) = A_u \cdot (p_0)$ .

Then,

$$s: E^* \rightarrow R_+^N$$

<sup>8</sup> Chipman and Moore point out that their form of this result is implicit in [5].

given by

$$s(e) = A_u^{-1}(s^*(e))$$

is the required function.

*Q.E.D.*

We have assumed that the set of feasible allocations  $\mathcal{F}(e)$  is compact for  $e \in E^*$ . Conditions on the production and consumption sets which imply that the set of attainable allocations is bounded are given in [11, p. 581, Theorem 1]. These conditions do not require convexity of either production or consumption sets. It is straightforward to show that if the aggregate production set is closed, then  $\mathcal{F}(e)$  is closed.

It can be further shown (see [17, pp. 39 and 40] for proofs) that conditions on individual production sets which (in the presence of others) are sufficient for boundedness of the set of feasible allocations are also sufficient to insure that the aggregate production set is closed whenever the individual production sets are closed. These conditions are stated in Lemmas 5.5 and 5.6 below.

**LEMMA 5.5:** *Let  $Y^1, \dots, Y^N$  be subsets of  $R^1$  and let  $Y = \sum_{i=1}^N Y^i$ . If  $AY \cap (-AY) = \{0\}$ , then  $AY^1, \dots, AY^N$  are positively semi-independent.<sup>9</sup>*

**LEMMA 5.6:** *If  $Y^1, \dots, Y^N$  are closed subsets of  $R^1$  and if  $AY \cap (-AY) = \{0\}$ , where  $Y = \sum_{i=1}^N Y^i$ , then  $Y$  is closed.*

We next give an example showing that convexity of preferences, and “steepness”, implying local nonsatiation, are not sufficient to ensure that the Pareto frontier has no holes. Notice that in this example the utility functions are concave, but utility is not freely disposable. (Disposability of utility may be considered to be a form of monotonicity.)

**EXAMPLE 5.1** (2 commodities, 2 agents): Let  $\mathcal{F}(e)$  (the feasible set in the space of allocations) be the Edgeworth box

$$\{(x_1, x_2) \in R^2 | 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$$

so that Agent 2 holds  $(1 - x_1, 1 - x_2)$  if Agent 1 holds  $(x_1, x_2)$ . Let  $u^1(x_1, x_2) = x_1 + x_2$  ( $x_1, x_2 \geq 0$ ) and let  $u^2(y_1, y_2) = [u^1(1 - y_1, 1 - y_2)]^{\frac{1}{2}}$  ( $y_1, y_2 \geq 0$ ). Then, it follows that

$$u^2(y_1, y_2) = [u^1(1 - y_1, 1 - y_2)]^{\frac{1}{2}} = [u^1(x_1, x_2)]^{\frac{1}{2}}$$

and, hence, that the image of  $\mathcal{F}(e)$  is the curve  $(z, z^{\frac{1}{2}})$  in  $R^2$ , where  $z \in [0, 1]$  and, hence,  $z^{\frac{1}{2}} > z$ . The Pareto set consists of the point  $(1, 1)$ . Now clearly for  $v =$

<sup>9</sup>  $AX$  denotes the asymptotic cone of the set  $X$ .  $N$  cones with vertex 0,  $X^1, \dots, X^N$ , are positively semi-independent if  $\sum x^i = 0, x^i \in X^i$ , for  $i = 1, \dots, N$ , implies  $x^i = 0$  for  $i = 1, \dots, N$ . (See [4, p. 22, (1.9 m) and (1.9 n)] for definitions.)

$(9/16, 3/4)$ ,  $\bar{\lambda} = \max \{\lambda : \lambda v \text{ is feasible}\}$  satisfies  $\bar{\lambda} = 1$ . That is,  $\bar{\lambda}v = (9/16, 3/4)$  is feasible, but for  $\lambda > 1$ ,  $\lambda v$  is not feasible, since for  $\lambda > 1$

$$\lambda \frac{3}{4} > \lambda \frac{1}{4} = \left(\lambda \frac{9}{16}\right)^{\frac{1}{2}}.$$

But  $(9/16, 3/4)$  is not a Pareto point, since  $(9/16, 3/4) < (1, 1)$ .

6. THREADS OF THE CONTRACT CURVE

In this section we study the Pareto set in the space of allocations or of trades. We give conditions under which the utility map is a homeomorphism between the graph of the contract curve correspondence and the Pareto frontier correspondence (Lemma 6.2). This homeomorphism can be used to lift a thread of the Pareto correspondence in the utility space to a thread of the Pareto correspondence in the allocation space (Lemma 6.1).

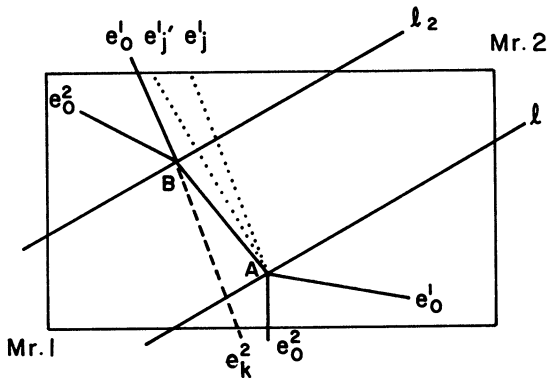


FIGURE 6.1

The conditions under which the utility function is a homeomorphism between the graphs of the Pareto correspondences include the so-called “pointedness” condition (Assumption 5), namely, that the upper contour set on a Pareto point contains only a single feasible point. Example 6.1 shows that this condition is indispensable for the existence of threads, for without it there is a set of economies and a point through which there is no local thread of the Pareto correspondence and, a fortiori, no thread.

LEMMA 6.1: *Let  $u : E \times X \rightarrow R^N$  be a utility function on  $E \times X$ . Let  $\hat{u}$  denote the restriction of  $u$  to the graph  $\mathcal{Q}(\mathcal{P})$  of  $\mathcal{P}$ . Let  $\hat{u}$  be a homeomorphism of  $\mathcal{Q}(\mathcal{P})$  to the graph  $\mathcal{Q}(\hat{U})$  of  $\hat{U}$ . Let  $s : E \rightarrow R^N$  be a thread of  $\hat{U}$ , and let  $Pr_x : E \times X \rightarrow X$  be the projection of  $E \times X$  to  $X$ . Then,  $Pr_x \circ \hat{u}^{-1} \circ s : E \rightarrow X$  is a thread of  $\mathcal{P}$ .*

PROOF: The proof is immediate.

LEMMA 6.2: *Under Assumptions 1a, 2, 3a, 5, and 6a(i),  $\hat{u}$  is a homeomorphism of  $\mathcal{Q}(\hat{\mathcal{P}})$  and  $\mathcal{Q}(\hat{U})$ .*

PROOF: Under Assumptions 1a, 2, and 3a,  $u: E \times X \rightarrow R^N$  is jointly continuous. It follows that  $\hat{u}$  is continuous on  $\mathcal{Q}(\mathcal{P})$ .

It follows from Assumption 5 (pointedness) that  $\hat{u}^{-1}: \mathcal{Q}(\hat{U}) \rightarrow \mathcal{Q}(\mathcal{P})$  is single-valued. It remains only to show that  $\hat{u}^{-1}$  is continuous on  $\mathcal{Q}(\hat{U})$ . To show this let  $(e_0, p_0) \in \mathcal{Q}(\hat{U})$  and let  $(e_j, p_j) \rightarrow (e_0, p_0)$  where  $(e_j, p_j) \in \mathcal{Q}(\hat{U})$  for  $j = 1, 2, \dots$

Let  $\hat{u}^{-1}(e_j, p_j) = (e_j, x_j)$ , and  $\hat{u}^{-1}(e_0, p_0) = (e_0, x_0)$ . Since  $e_j \rightarrow e_0$  by assumption, we need only show that  $x_j \rightarrow x_0$ .

Since  $\mathcal{F}: E \rightarrow X$  is upper hemi-continuous (Assumption 6a(i)) it follows from 4.2 that there exists a compact set  $K \subset X$  and an integer  $J$  such that  $\mathcal{F}(e_j) \subset K$  for all  $j > J$ . Omitting the first  $J$  terms from the sequence  $x_j$ , it suffices to show that the remaining sequence converges to  $x_0$ . To establish this proposition, we shall show that every infinite subsequence of  $\{x'_j: j > J\}$  converges to  $x_0$ .

Since  $\{x'_j\} \subset K$ , it has a convergent subsequence  $x_{j_k}$  whose limit is  $x'$ . Since  $\mathcal{F}$  is upper hemi-continuous,  $e_{j_k} \rightarrow e_0$ ,  $x_{j_k} \rightarrow x'$  and  $x_{j_k} \in \mathcal{F}(e_{j_k})$  for all  $k$ , implies  $x' \in \mathcal{F}(e_0)$ . Since  $u$  is continuous, it follows that  $\hat{u}(e_{j_k}, x_{j_k}) \rightarrow \hat{u}(e_0, x')$ . But  $\hat{u}(e_{j_k}, x_{j_k}) = p_{j_k}$ , and  $p_{j_k} \rightarrow p_0$ , since  $p_{j_k}$  is a subsequence of  $p_j$ , and  $p_j \rightarrow p_0$ . It follows that  $u(e_0, x') = u(e_0, x_0)$ . But it follows from Assumption 5 that  $x_0 = x'$ .

Consider the terms of the sequence  $x_j$  which are not in the subsequence  $x_{j_k}$ . Either there is a finite number of them in which case,  $x_j \rightarrow x_0$ , or there is an infinite number in which case the same argument applied to the remaining infinite set of  $x_j$ 's leads to the conclusion that there is a convergent subsequence with the limit  $x_0$ . Hence, it follows that every infinite subsequence of  $\{x_j\}$  converges to  $x_0$  and, hence, that  $\{x_j\}$  converges to  $x$ .

This establishes the continuity of  $\hat{u}^{-1}$  on  $\mathcal{Q}(\hat{U})$ .

*Q.E.D.*

Chipman and Moore [3, Theorems 6 and 7] have given conditions under which the “contract curve” (the set of Pareto allocations) is homeomorphic to an  $(N - 1)$  simplex. These conditions include strict convexity of the preference relations, which implies that Assumption 5 (pointedness) is satisfied. While the pointedness assumption can be satisfied without strict convexity of preferences, it is not clear whether there is a (nontrivial) class of economies which satisfy all the assumptions of Theorem 6.1 without strictly convex preferences.

THEOREM 6.1: *Under Assumptions 1b, 2, 3b, 5, 6a, and 6b, there is for each  $e \in E$  and  $p \in \hat{U}(e)$ : (i) a thread of  $\hat{U}$   $s_p: E \rightarrow R^N$ , such that  $s_p(e) = p$ , and (ii) a corresponding thread  $pr_x \circ \hat{u}^{-1} \circ s_0: E \rightarrow X$  of  $\mathcal{P}$ .*

PROOF: (i) follows from Theorem 5.2; (ii) follows from (i) and Lemmas 6.1 and 6.2.

EXAMPLE 6.1: we next give an example in which the Pareto correspondence in the space of allocations is not locally threaded, although the Pareto utility

correspondence is threaded. We shall sketch this example rather than present it formally.

Let two lines  $l_1$  and  $l_2$  (in Figure 6.1) be given a fixed distance apart, say one. In the environment  $e_j = (e_j^1, e_j^2)$  indifference curves of Agent 1 consist of parallel translates of the dotted line labeled  $e_j^1$  in Figure 6.1, while the indifference map of Agent 2 consists of parallel translates of the solid curve labeled  $e_0^2$ . For  $j' > j$ , the indifference  $j'$  curves of Agent 1 are closer to the flat segment  $AB$ . As  $j \rightarrow \infty$ ,  $e_j^1 \rightarrow e_0^1$ . Thus, for all  $j$  the Pareto set for  $e_j = (e_j^1, e_0^2)$  is the line  $l_1$ , while in the limit  $e_0$ , the Pareto set is the strip (closed) included between the lines  $l_1$  and  $l_2$ . Similarly, for  $\bar{e}_k = (e_0^1, e_k^2)$  where the indifference curves of Agent 2 are parallel to the dashed curve labeled  $e_k^2$  in Figure 6.1, the Pareto set is the line  $l_2$ .

Now, every open set containing  $e_0$  contains  $e_j$  and  $\bar{e}_k$  for  $j$  and  $k$  sufficiently large.

A local thread for the Pareto correspondence  $P$  would have to satisfy the following conditions.

There exists an open set  $U$  containing  $e_0$  and a continuous function  $s: U \rightarrow P(U)$ ; (i.e., for each  $e \in U$ ,  $s(e) \in P(e)$ .)

Therefore, for  $e_j = (e_j^1, e_0^2) \in U$  and  $\bar{e}_k = (e_0^1, \bar{e}_k^2) \in U$ , we must have  $s(e_j) \in l_1$  and  $s(\bar{e}_k) \in l_2$ . But the distance of  $s(e_j)$  from  $s(\bar{e}_k)$  is one for all  $j, k$ . Hence,  $s$  is not continuous on  $U$ , for any open set  $U$  containing  $e_0$ .

The preferences in this example are continuous, convex, satisfy local nonsatiation, and (can be chosen) strictly monotone. We may choose a utility function to represent these preferences, say, by taking the distance from the respective origins to each indifference curve measured along the diagonal of the box. Then the Pareto utility image does have a thread, although the Pareto allocation correspondence does not.

Note that, while the example shows indifference curves with kinks, smoothly differentiable curves can yield the same result, since such indifference curves can be made to turn smoothly into flat segments, with suitable tangencies along  $l_1$  and  $l_2$ , respectively.

### 7. AN ECONOMY WHOSE PARETO FRONTIER HAS HOLES

In Section 5 we constructed a thread of the Pareto utility correspondence by intersecting a fixed ray out of the origin with the set of feasible utility values, and taking the point of maximum distance from the origin on that ray. Under suitable conditions, including strict monotonicity of utilities, the point so obtained was shown to be a Pareto point. This property is essential to the construction of threads. We have already seen in Example 5.1 that strict monotonicity of preferences is indispensable for this construction. In that example preferences have the local nonsatiation and convexity properties, but are not monotone. It is instructive to examine the borderline case in which preferences are monotone but not strictly monotone. Of course, in that case preferences may not satisfy the local nonsatiation assumption. For this borderline case, we construct a class of economies with two agents and one commodity for which there is a complete

description of the Pareto utility set. For this economy the feasible utility set is the line segment from  $(0, 1)$  to  $(1, 0)$  in  $R_+^2$ , and the Pareto utility frontier is an arbitrary closed subset of that segment containing the end points. (See [17, pp. 47–55] for details of this construction.)

We consider the case of two consumers and one commodity. The admissible consumption set of each agent will be the nonnegative real line, thus satisfying Assumption 1b, and the total endowment of the commodity will be a positive real number  $w = w_e$  in the environment  $e$ . Preferences of consumers will be described by a real valued continuous function  $u^i$ , taking nonnegative values on the nonnegative reals. Then, the set of feasible allocations consists of all  $x^1 \geq 0, x^2 \geq 0$  such that  $x^1 + x^2 = w$ . This is the usual Edgeworth box situation.

We note that Assumptions 6a and 6b are satisfied. We may without loss of generality take  $w = 1$ . Then the feasible set  $\mathcal{F}(e)$  consists of the line segment in  $R \times R$  joining the point  $(0, 1)$  to  $(1, 0)$ . Now, let  $f^i: [0, 1] \rightarrow R_+, i = 1, 2$  be continuous functions and let  $f(t) = (f^1(t), f^2(t))$  for  $t \in [0, 1]$ . Let  $\xi$  be the image of  $[0, 1]$  in  $R_+^2$  under  $f$ , so that  $\xi$  is a curve in  $R_+^2$ .

Now, for  $x \in [0, 1]$  set  $u^1(x) = f^1(x)$  and  $u^2(x) = f^2(1-x)$ . Then for a feasible allocation  $(x^1, x^2)$ , the utility image  $(u^1, u^2)$  of  $(x^1, 1-x^1)$  is  $(u^1(x^1), u^2(1-x^1)) = (f^1(x^1), f^2(x^1))$ , and hence the utility image  $U(e)$  of the feasible set  $\mathcal{F}(e)$  is precisely the curve  $\xi$ . Thus, in order to specify a possible utility image  $U(e)$  of an economy  $e$  (with continuous utility functions) we need only specify a continuous image of the unit interval.

We then construct an economy which has as the utility image of its feasible set the line segment  $L$  from the point  $(1, 0)$  to  $(0, 1)$  in  $R_+^2$  and whose Pareto frontier consists of the line segment from  $(1, 0)$  to  $(0, 1)$  with an open interval  $S$  deleted. This may be done by choosing  $u^1$  and  $u^2$  to be piecewise linear and weakly monotone, each being constant on a subinterval of  $[0, 1]$ . The line segment parameterized by  $g(t) = (t, 1-t)$  ( $0 \leq t \leq 1$ ) is easily seen to have as image of  $(0, 1)$  under  $(u^1, u^2)$ , the graph shown in Figure 7.1.

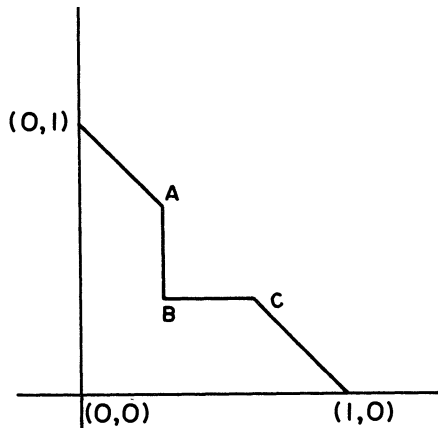


FIGURE 7.1

The basic idea of the construction is to use the method pictured in Figure 7.1 to eliminate an open interval from the Pareto set, defining a pair of functions on the unit interval which do the job. By composing such functions we can eliminate from the Pareto set all the open intervals of  $L - S$ . It remains only to take care of the technical details ensuring that the infinite composition of continuous functions yields a (pair of) continuous functions. We do this using the length of subintervals to be eliminated to index the functions, constructing a uniformly convergent sequence of continuous functions, thus ensuring that the limit function is continuous.

#### 8. A CLASS OF ECONOMIES WHOSE PARETO FRONTIER CORRESPONDENCE IS NOT THREADED THROUGH ALL POINTS

The example referred to in Section 7 shows that the assumption of strict monotonicity of preference is indispensable for the construction used in Lemma 5.1 to define a thread of the Pareto utility correspondence. However, failure of the construction used in our proof does not necessarily entail non-existence of a thread. It is clear, for instance, that if the utility functions in Example 5.1 are made continuous in the environments, the continuity and single-valuedness of the Pareto correspondence in that example would ensure the existence of a thread, despite the fact that the relative boundary of the feasible utility set and the Pareto frontier are not the same, and hence that the ray construction fails. The existence of threads means that for any economy  $e_0$  and any utility allocation  $p_0$  which is Pareto for  $e_0$ , there must exist a global thread  $s$  of the Pareto correspondence such that  $s(e_0) = p_0$ . However, we have constructed an example of a class of exchange economies with two agents and one commodity, with weakly monotone utilities, which contains an economy with a Pareto point through which no thread exists. (This example is presented in detail in [17, pp. 56–74].)

*Northwestern University*

*Manuscript received December, 1974; last revision received May, 1976.*

#### REFERENCES

- [1] BERGE, C.: *Topological Spaces*. New York: The MacMillan Co., 1963. (English translation of *Espaces, Topologiques, Fonctions Multivoques*. Paris: Dunod, 1959.)
- [2] BOAS, R. P., JR.: *A Primer of Real Functions*. Carus Monograph No. 13. New York: Mathematical Association of America, 1960.
- [3] CHIPMAN, JOHN S., AND JAMES C. MOORE: "The Compensation Principle in Welfare Economics," in *Papers in Quantitative Economics*, ed. by Arvid N. Zarely. The University of Kansas Press, 1971.
- [4] DEBREU, GERARD: *Theory of Value*. New York: Wiley, 1959.
- [5] GORMAN, T.: "Community Preference Fields," *Econometrica*, 21 (1953), 63–80.
- [6] HILDENBRAND, WERNER: "On Economies With Many Agents," *Journal of Economic Theory*, 2 (1970), 161–187.

- [7] HURWICZ, LEONID: "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences*, 1959, ed. by K. J. Arrow, S. Karlin, and P. Suppes. Stanford, California: Stanford University Press, 1969.
- [8] ———: "On Decentralizability in the Presence of Externalities," presented at the 1966 Econometric Society Meeting, San Francisco, December, 1966.
- [9] ———: "On Informationally Decentralized Systems," in *Decision and Organization*, ed. by C. B. McGuire and Roy Radner. New York: North-Holland/American Elsevier, 1972.
- [10] HURWICZ, L., R. RADNER, AND S. REITER: "A Stochastic Decentralized Resource Allocation Process," *Econometrica*, forthcoming.
- [11] HURWICZ, L., AND S. REITER: "On the Boundedness of the Feasible Set Without Convexity Assumptions," *International Economic Review*, 14 (1973), 580–586.
- [12] KANNAI, YAKAR: "Continuity Properties of the Core of a Market," *Econometrica*, 38 (1970), 791–815.
- [13] KELLEY, J.: *General Topology*. New York: Van Nostrand Co., Inc., 1955.
- [14] MOORE, JAMES C.: "The Existence of 'Compensated Equilibrium' and the Structure of the Pareto Efficiency Frontier," unpublished manuscript, revised, 1974.
- [15] MOUNT, K., AND S. REITER: "Continuous Representation of Preferences," Discussion Paper No. 81, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, 1974.
- [16] ———: "The Informational Size of Message Spaces," *Journal of Economic Theory*, 8 (1974), 161–192.
- [17] ———: "Economic Environments for which there are Pareto Satisfactory Mechanisms," Discussion Paper No. 124, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, 1974.