

## Informational Efficiency of Iterative Processes and the Size of Message Spaces

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Received December 12, 1973

1. The informational requirements imposed on a resource allocation process by the necessity of meeting a given performance standard have been studied in [3] and [5] and the broader motivations for undertaking such a study was discussed in the introduction to [5]. In [5] attention was directed to the equilibrium messages. Communication was represented by a correspondence from the space of environments to the joint message space, characterizing equilibrium messages. One interpretation of such a (privacy preserving) message correspondence is that it models a one-step communication process in which each agent sends the set of all joint messages that are acceptable to him, given his environmental component. In such a process the message consists in effect of the entire (equilibrium) message strategy of each agent for the environment in which he finds himself, i.e., he sends what he would respond to any message received from the other agents.

It is natural to consider processes in which communication takes place by an iterative exchange of messages, each round based on the messages received in the preceding one. Hurwicz's original formulation of adjustment processes was of this type [2]. Iterative communication opens the possibility of reaching the equilibrium set via a sequence of messages, each consisting of a single point of the message space rather than a set.<sup>2</sup> Thus, iteration may trade-off an increase in the number of messages for a decrease of the complexity of messages. A long, possibly infinite, sequence of point-messages may substitute for a single set-message. In this paper we study the possible advantages in terms of informational size opened

<sup>1</sup> I am indebted to Leonid Hurwicz for helpful discussions of this paper. This research was partly supported by The National Science Foundation (GS 31346X) and a grant from the General Electric Company.

<sup>2</sup> In Hurwicz's formulation the message space (language) was arbitrary, so that the restriction to point messages was not a consideration.

up by the use of iteration. We may consider this to be a step in the direction of a full analysis of the trade-off just referred to.

The possible informational advantages of iteration seem to be of two kinds. First, suppose that the message space is of minimal informational size sufficient for a given performance by a (one-step) process using a message correspondence. Can the size of the message space be further reduced by using an iterative process? In Section 3 below it is shown that no such reduction is possible. Thus, an iterative process must use *points* of a space having no less information than the minimal space sufficient for the given performance using a message correspondence.

Second, it may be possible to calculate the next message at each iteration on the basis of less information than is contained in the message space of minimal size sufficient for the given performance, even though the message calculated is itself a point of that space. Such reduction of the information used is indeed possible, but there is, as might be expected, a lower bound on the information needed. Lemma 2, which is an application of Lemma 10 of [5] to the case of iteration, characterizes the bound on the possible reduction of information needed to calculate the next message. Theorem 4 gives a bound for the Euclidean case.

The concept of informational efficiency was proposed by Hurwicz [2, p. 44] to study the "fineness of perception" (an indicator of information) needed to calculate the next message according to a given response or iteration rule. Here we study the relationship between the concept of informational efficiency proposed by Hurwicz and that of the informational size of message spaces given by Mount and Reiter [5].

Hurwicz's definition of informational efficiency is stated in terms of partitions and is set-theoretic, while the concept of informational size is stated in terms of continuous mappings and is topological. It is therefore necessary to augment the set-theoretic definition with a topological condition (see Definitions 4 and 5) which has the effect of restricting the scope of the definition somewhat. We refer to the concept so augmented as *R-efficiency* (*R* for *regular*). The formalization of an iteration process involves two message spaces, one in which the incoming messages lie and another in which the messages emitted lie. The concepts of *informational efficiency* (Definition 3) and of *R-efficiency* (Definition 5) relate to the dependence of the message emitted on the message received. These concepts are stated in terms of properties of the first of those two message spaces. It is also desirable to be able to state relationships of refinement among regular partitions in terms of information decreasing maps. Lemma 1 states such a result. Using the equivalence relation on messages implied by the response rule of a process, a familiar construction permits the given process to be replaced by one whose message space (referred

to as the quotient message space) is a quotient of the given one. Theorem 1 shows that if an iteration rule  $\eta$  is more *R-efficient* than another,  $\theta$ , then the quotient message space corresponding to  $\theta$  has more information than that corresponding to  $\eta$ .

It is interesting to know whether comparison of informational size of message spaces is sufficient to ensure comparison according to *R-efficiency*. Such a proposition would, if true, be the converse of Theorem 1. An example is given to show that this is not in general the case. Thus, the (partial) ordering of iteration procedures according to informational size of the space of perceived messages is a proper extension of that according to *R-efficiency*. However, in the case of finite sets such a converse holds, as is stated in Theorem 2. Theorem 3, a partial converse of Theorem 1, shows that comparison of informational size of quotient message spaces implies comparison according to *R-efficiency* of certain iteration rules which have those quotient spaces.

These results lend further support to the view that the concept of informational size introduced in [5] is capable of unifying apparently diverse informational properties of resource allocation processes.

2. The following notations and structure will be used, except where alternatives are explicitly given.

Let  $E = \prod_{i=1}^n E^i$ ,  $X = \prod_{i=1}^n X^i$ ,  $Z$ ,  $U$ , and  $V$  be topological spaces. We shall use the same notation for the set  $X$  and the topological space  $X$  except where explicit reference to the topology is necessary. Let  $f: E \rightarrow Z$  be a locally sectioned continuous function. Let  $\pi = (\mu, f)$  be a privacy-preserving allocation process with message space  $X$  realizing  $f$  on  $E$ ; i.e.,  $\mu: E \rightarrow X$  is a locally sliced coordinate correspondence and  $f: X \rightarrow Z$  a continuous function such that  $\mu \circ f = f$ . Since  $\pi$  is privacy preserving,  $\mu$  is a coordinate correspondence, i.e., there exist locally sliced correspondences  $\mu^i: E^i \rightarrow X$  such that  $\mu = \bigcap_{i=1}^n \mu^i$ .<sup>3</sup>

DEFINITION 1.<sup>4</sup> A function  $\eta: E \times X \rightarrow X$ ,  $\eta = (\eta^1, \dots, \eta^n)$ ,  $\eta^i: E^i \times X \rightarrow X^i$  is an *iteration rule* on (the product space)  $X$  if and only if

- (i)  $\eta^i$  is a locally sectioned continuous function for  $i = 1, \dots, n$ , and
  - (ii)  $\{x \in X \mid x = \eta(e, x)\} \neq \emptyset$  for all  $e \in E$ .
- We say  $\eta$  is an *iteration rule* for  $\mu: E \rightarrow X$  if
- (iii)  $\mu(e) = \{x \in X \mid x = \eta(e, x)\}$  for all  $e \in E$ .

<sup>3</sup> The relevant definitions from [5] are reproduced here for convenience of the reader. Footnote and Lemma Numbers are as shown in [5].

<sup>4</sup> We use the term "iteration rule" rather than "response rule" to emphasize the restriction to point messages.

The message space of an allocation process, especially a minimal space, is not necessarily a product; however, in what follows, wherever the space  $X$  is assumed to be minimal, it is to be understood as minimal in the class of product spaces.<sup>5</sup>

DEFINITION 2.

- (i) Let  $\eta = (\eta^1, \dots, \eta^n)$  be an iteration rule on  $X$ . Let  $K_\eta$  denote the equivalence relation:  $x \equiv \bar{x} \pmod{\eta}$  if and only if  $\eta^i(e^i, x) = \eta^i(e^i, \bar{x})$  for all  $e^i \in E^i$  and  $i = 1, \dots, n$ . Let  $\mathcal{P}_\eta$  denote the partition of  $X$  determined by  $K_\eta$ . We refer to  $\mathcal{P}_\eta$  as the partition of  $X$  determined by  $\eta$ .
- (ii) If  $\alpha: X \rightarrow W$  is a function, we write  $K_\alpha$  for the equivalence kernel of  $\alpha$ ; i.e.,  $K_\alpha$  is the equivalence relation  $x \equiv \bar{x} \pmod{\alpha}$  if and only if  $\alpha(x) = \alpha(\bar{x})$ . We write  $\mathcal{P}_\alpha$  for the partition of  $X$  determined by  $K_\alpha$ .

If  $\alpha: X \rightarrow W$  is a function then its equivalence kernel  $K_\alpha$  determines a quotient set  $X/\alpha$  and a mapping  $\rho_\alpha: X \rightarrow X/\alpha$  which associates to each  $x$  in  $X$  its equivalence class in  $X/\alpha$ . We shall say that  $\rho_\alpha$  represents  $\alpha$ . There is also a unique mapping  $\tilde{\alpha}: X/\alpha \rightarrow W$  such that  $\alpha = \tilde{\alpha} \circ \rho_\alpha$  (See [4], pp. 20-23).

When  $X$  and  $X/\alpha$  are topological spaces the so-called natural map  $\rho_\alpha: X \rightarrow X/\alpha$  may or may not be continuous, depending on the topology of  $X/\alpha$ ;  $\rho_\alpha$  is continuous in the quotient topology for  $X/\alpha$ . In situations of interest, the space  $X/\alpha$  comes with its own topology, not necessarily related to the quotient topology. However, even with the quotient topology for  $X/\alpha$ , the function  $\rho_\alpha$  is not necessarily locally sectioned.

EXAMPLE. Let  $P: [0, 1] \rightarrow [0, 1] \times [0, 1]$  be the Peano function. Then  $K_P$  is the equivalence relation on  $[0, 1]$  given by  $r \equiv r' \pmod{K_P}$  iff  $P(r) = P(r')$ . We see that  $[0, 1] \times [0, 1]$  is  $[0, 1]/P$  and hence that  $\rho_P = P$ . Hence  $\rho_P$  is continuous (in the usual topology for  $[0, 1] \times [0, 1]$ ) but not locally sectioned. (See [5], p. 15 for a proof that  $P$  is not locally sectioned.)

DEFINITION 3 ([2], p. 44). Let  $\eta = (\eta^1, \dots, \eta^n)$  and  $\theta = (\theta^1, \dots, \theta^n)$  be iteration rules on  $X$ . The iteration rule  $\eta$  is *informationally more efficient* than  $\theta$  if  $\mathcal{P}_\theta$  is a refinement of  $\mathcal{P}_\eta$ .

DEFINITION 4. Let  $\eta = (\eta^1, \dots, \eta^n)$  be an iteration rule on  $X$  and let  $\mathcal{P}_\eta$  denote the partition of  $X$  determined by  $\eta$ .

<sup>5</sup> See Definition 1' below for the case in which  $X$  is not assumed to be a product. The results obtained using Definition 1 apply with suitable reinterpretations when Definition 1' is used.

Let  $\rho = \rho_\eta: X \rightarrow X/\eta$  be the natural map of  $X$  onto  $X/\eta$ , and let  $U = (X/\eta, \mathcal{S})$  be a topological space with set  $X/\eta$  and topology  $\mathcal{S}$ . We say that  $\rho$  *represents*  $\eta$  in  $U$ .

We say  $\mathcal{P}_\eta$  is *regular* with respect to  $U$  if  $\rho$  is continuous and locally sectioned.

We now introduce the concept of *regular informational efficiency* in place of *informational efficiency*, and refer to it as *R-efficiency*.

DEFINITION 5. Let  $\eta$  and  $\theta$  be iteration rules on  $X$ . The iteration rule  $\eta$  is (at least) as *R-efficient* as  $\theta$  if  $\mathcal{P}_\eta$  and  $\mathcal{P}_\theta$  are regular with respect to  $X/\eta$  and  $X/\theta$ , respectively and  $\mathcal{P}_\theta$  is a refinement of  $\mathcal{P}_\eta$ . (It is understood that the topologies on  $X/\eta$  and  $X/\theta$  are given.)

Remark. When  $E, X, U$  and  $V$  have the discrete topology, all functions are continuous and locally sectioned, so that R-efficiency and informational efficiency coincide.

LEMMA 1. Let  $\eta: E \times X \rightarrow X$  and  $\theta: E \times X \rightarrow X$  be iteration rules on  $X$  and let  $\rho: X \rightarrow U$  represent  $\eta$ , and  $\tau: X \rightarrow V$  represent  $\theta$ .  $\tau(x) = \tau(\bar{x})$  implies  $\rho(x) = \rho(\bar{x})$  for all  $x$  and  $\bar{x}$  in  $X$ , if and only if there exists a mapping  $\sigma$  of  $V$  onto  $U$  such that  $\rho = \sigma \circ \tau$ .

Proof. We show "necessity" first. Construct  $\sigma: V \rightarrow U$  by  $\sigma = \rho \circ \tau^{-1}$ . For any  $v \in V$ , and for  $x, \bar{x} \in \tau^{-1}(v)$  it follows from the hypothesis that  $\rho(x) = \rho(\bar{x})$  since  $x, \bar{x} \in \tau^{-1}(v)$  implies  $\tau(x) = \tau(\bar{x})$ . Thus  $\rho$  is constant on  $\tau^{-1}(v)$ ; hence  $\sigma = \rho \circ \tau^{-1}$  is single-valued. Further  $\sigma$  is onto, since  $\rho$  is onto  $U$  and the domain of  $\tau$  is  $X$ .

We now show sufficiency.

Suppose there exists  $\sigma: V \rightarrow U$  such that  $\sigma \circ \tau = \rho$ . Let  $x, \bar{x} \in X$  such that  $\tau(x) = \tau(\bar{x})$ . Then  $\sigma(\tau(x)) = \sigma(\tau(\bar{x}))$  since  $\sigma$  is single-valued. It follows that  $\rho^{-1}(\sigma(\tau(x))) = \rho^{-1}(\sigma(\tau(\bar{x})))$ . But,  $\rho = \sigma \circ \tau$  implies that  $y \in \rho^{-1}(\sigma(\tau(y)))$  for all  $y \in X$ . Hence,  $x \in \rho^{-1}(\sigma(\tau(x)))$  and  $\bar{x} \in \rho^{-1}(\sigma(\tau(\bar{x})))$ . Since  $\rho^{-1}(\sigma(\tau(x))) = \rho^{-1}(\sigma(\tau(\bar{x})))$ , it follows that  $\rho(x) = \rho(\bar{x})$ . ■

THEOREM 1. Let  $\eta$  and  $\theta$  be iteration rules on  $X$  and let  $\rho: X \rightarrow U$  and  $\tau: X \rightarrow V$  represent  $\eta$  and  $\theta$  respectively. If  $\eta$  is as R-efficient as  $\theta$ , then  $V$  has as much information as  $U$ .

Proof. Define  $\sigma: V \rightarrow U$  by  $\sigma = \rho \circ \tau^{-1}$ . By Lemma 1,  $\sigma$  is single-valued and maps  $V$  onto  $U$ . Further,  $\sigma$  is continuous, since  $\tau$  is continuous and continuity of  $\tau$  implies that the graph of  $\tau^{-1}$  is closed. Hence  $\tau^{-1}$  is an upper semicontinuous correspondence and, since  $\rho$  is continuous,  $\rho \circ \tau^{-1}$  is upper semicontinuous. Since  $\sigma = \rho \circ \tau^{-1}$  is a function, it is

continuous. Finally, we show that  $\sigma$  is locally sectioned. Since  $\rho$  is locally sectioned, given  $u \in U$ , there exists  $N(u)$ , an open neighborhood of  $u$  in  $U$  and a function  $r_u: N(u) \rightarrow X$  such that  $\rho \circ r_u$  is the identity on  $N(u)$ . Hence  $\rho \circ \tau^{-1} \circ \tau \circ r_u$  is the identity on  $N(u)$ ; i.e.,  $s_u = \tau \circ r_u$  is a local section for  $\sigma$ .

To summarize,  $\sigma: V \rightarrow U$  is a locally sectioned continuous map of  $V$  onto  $U$ . Hence  $V$  has as much information as  $U$ . ■

The concept of informational efficiency registers every "aggregation" of the information needed to calculate the next message according to one iteration rule as compared to another. Thus, if  $\mathcal{P}_\theta$  is strictly finer than  $\mathcal{P}_\eta$  anywhere in the space  $X$ , and as fine everywhere,  $\eta$  is classified as strictly more efficient than  $\theta$ . On the other hand, the comparison of  $X/\eta$  and  $X/\theta$  according to informational size recognizes only certain sufficiently large "aggregations" of the information needed to calculate the next message. Thus, according to Theorem 1,  $\mathcal{P}_\eta$  could be strictly coarser than  $\mathcal{P}_\theta$  (for regular partitions  $\mathcal{P}_\eta$  and  $\mathcal{P}_\theta$ ) while  $U$  and  $V$  have the same information size. However  $U$  could not have strictly less information than  $V$  when  $\mathcal{P}_\theta$  is coarser than  $\mathcal{P}_\eta$ .

Theorem 1 tells us that the partial ordering of (regular) iteration rules according to informational size of the space of perceived messages does not contradict the partial ordering of (regular) iteration rules according to R-efficiency. The converse of Theorem 1 would, if true, tell us that the partial ordering according to R-efficiency includes that according to informational size. In particular, this would say that the same comparisons are made according to either concept. However, the converse of Theorem 1 is in general false, as the following example shows.

EXAMPLE. Let  $V = U = R$ ,  $X = R^2$ . Let  $\tau(x_1, x_2) = (x_1 + x_2)/2$

$$\rho(x_1, x_2) = x_2$$

Note that  $\tau: X \rightarrow V$  and,  $\rho: X \rightarrow U$  are continuous and locally sectioned, and onto. Then  $\sigma: V \rightarrow U$  is the identity, i.e.,  $\sigma(v) = v$ , for  $v \in R$ .

But  $\tau(x) = \tau(\bar{x})$   $(x_1 + x_2)/2 = (\bar{x}_1 + \bar{x}_2)/2$ , while  $\rho(x) = \rho(\bar{x}) \Leftrightarrow x_2 = \bar{x}_2$ . But  $x = (1, 1)$ ,  $\bar{x} = (1\frac{1}{2}, \frac{1}{2})$  satisfies  $(1 + 1)/2 = 1 = (1\frac{1}{2} + \frac{1}{2})/2$ , while

$$x_2 = 1 \neq \frac{1}{2} = \bar{x}_2.$$

$V$  has as much information as  $U$ , but  $\tau(x) = \tau(\bar{x})$  does not imply  $\rho(x) = \rho(\bar{x})$ . Hence  $\eta$  is not more R-efficient than  $\theta$ .

Theorem 1 is also related to a comment made by Balassa in his discussion of Hurwicz's paper ([1], p. 533). From Balassa's comments, he would find the concept of informational efficiency a more appealing basis of comparison than the dimension of the (Euclidean) message space, which

Hurwicz used in the paper Balassa's comments refer to for the definition of informational decentralization. If the class of spaces eligible to be message spaces is Euclidean, then informational size and dimension agree, (Lemma 2). In that case the contra-positive form of Theorem 1 assures us that comparison according to dimension cannot contradict comparison according to R-efficiency.

While the converse of Theorem 1 is not in general true, it is true when the message spaces are finite sets (with the discrete topology). Theorem 2 states this result.

THEOREM 2.: Let  $\eta$  and  $\theta$  be iteration rules on  $X$ , let  $\rho: X \rightarrow U$  and  $\tau: X \rightarrow V$  represent  $\eta$  and  $\theta$ , respectively, and let  $X$ ,  $U$  and  $V$  be finite sets with the discrete topology. If  $V$  has as much information as  $U$ , then  $\theta$  is not strictly more R-efficient than  $\eta$ .

Proof. We have already noted that when  $X$ ,  $U$  and  $V$  have the discrete topology,  $\rho$  and  $\tau$  are continuous and locally sectioned. Hence in the present case  $\mathcal{P}_\eta$  and  $\mathcal{P}_\theta$  are regular. If  $\theta$  is not more R-efficient than  $\eta$ , then it is not strictly more R-efficient than  $\eta$ .

If  $\theta$  is as R-efficient as  $\eta$  then, by Theorem 1,  $U$  has as much information as  $V$ . Therefore it suffices to consider the case in which  $U$  and  $V$  have the same information. In that case there exist functions  $\sigma: V \rightarrow U$ , and  $\sigma^*: U \rightarrow V$ , which map  $V$  onto  $U$  and  $U$  onto  $V$ , respectively. Since  $U$  and  $V$  are finite, it follows that  $\|U\| = \|V\|$ , where  $\|\cdot\|$  denotes cardinality of the enclosed set.

If  $\theta$  is as R-efficient as  $\eta$ , then  $\mathcal{P}_\eta$  is a refinement of  $\mathcal{P}_\theta$ . Applying Lemma 1, there exists a function  $\delta$  mapping  $U$  onto  $V$  such that  $\tau = \delta \circ \rho$ . Since  $\|U\| = \|V\|$ ,  $\delta$  is 1-1. Hence  $\delta^{-1}: V \rightarrow U$  is a function mapping  $V$  onto  $U$ . Further,  $\delta^{-1} \circ \tau = \delta^{-1} \circ \delta \circ \rho = \rho$ . Hence, applying the other half of Lemma 1, it follows that  $\mathcal{P}_\theta$  is a refinement of  $\mathcal{P}_\eta$ , and hence that  $\eta$  is as R-efficient as  $\theta$ . It follows that  $\theta$  is not strictly more R-efficient than  $\eta$ . Thus the theorem is established. ■

A second result, in the nature of a partial converse for Theorem 1, is given in Theorem 3, which states that if the quotient message space  $U$  of an iteration rule  $\eta$  has no more information than that,  $V$ , of  $\theta$ , then there is an iteration rule  $\eta^*$  with the same quotient message space as  $\eta$  and with the same responses as  $\tilde{\eta}$  to quotient messages, but which is as R-efficient as  $\theta$ .

THEOREM 3. Let  $\eta$  and  $\theta$  be iterative rules on  $X$ . Suppose  $\rho: X \rightarrow U$  and  $\tau: X \rightarrow V$  represent  $\eta$  and  $\theta$  respectively, and that  $\tilde{\eta}: E \times U \rightarrow X$  and  $\tilde{\theta}: E \times V \rightarrow X$  satisfy  $\eta = \tilde{\eta} \circ (I \times \rho)$  and  $\theta = \tilde{\theta} \circ (I \times \tau)$ . If  $V$  has as

much information as  $U$ , then there exists an iteration rule  $\eta^*$  on  $X$ , and a locally sectioned continuous function  $\varphi: X \rightarrow U$ , onto  $U$ , such that (i)  $\eta^* = \tilde{\eta} \circ (I \times \varphi)$  and (ii)  $\eta^*$  is as  $R$ -efficient as  $\theta$ .

*Proof.* Since  $V$  has as much information as  $U$ , there exists a locally sectioned continuous function  $\sigma: V \rightarrow U$  mapping  $V$  onto  $U$ . Let  $\varphi = \sigma \circ \tau$ . Then  $\varphi$  is locally sectioned continuous and onto, since  $\tau$  and  $\sigma$  have those properties.

Define  $\eta^*: E \times X \rightarrow X$  by  $\eta^* = \tilde{\eta} \circ (I \times \varphi)$ . It is immediate that  $\eta^*$  is an iteration rule on  $X$  since  $\varphi$  is locally sectioned and continuous.

By Lemma 1 it follows from  $\varphi = \sigma \circ \tau$  that  $\tau(x) = \tau(\bar{x})$  implies  $\varphi(x) = \varphi(\bar{x})$ . Hence  $\mathscr{P}_\tau$  is finer than  $\mathscr{P}_\sigma$ . By definition of  $\eta^*$ ,  $\mathscr{P}_\sigma$  is finer than  $\mathscr{P}_\eta$ . Hence  $\mathscr{P}_\tau$  is finer than  $\mathscr{P}_\eta$ . Since  $\tau$  and  $\varphi$  are locally sectioned continuous and onto, both  $\mathscr{P}_\tau$  and  $\mathscr{P}_\eta$  are regular. Hence  $\eta^*$  is as  $R$ -efficient as  $\theta$ . ■

*Remark.* We have noted above that when  $X, E, U$  and  $V$  have the discrete topology, all functions are continuous and locally sectioned, so that  $R$ -efficiency and informational efficiency coincide. In that case Theorem 1 tells us that if  $\eta$  is informationally as efficient as  $\theta$ , then  $X/\theta$  has as much information as  $X/\eta$ . Similarly, in Theorem 3 the conclusion may be stated in terms of informational efficiency as well as  $R$ -efficiency.

3. Consider a privacy preserving allocation process  $\pi = (\mu, f)$  which is sufficient for the function  $f: E \rightarrow Z$  and whose message space  $X$  has minimal informational size in the class of (product) spaces sufficient for  $f$ . While no allocation process with message space smaller than  $X$  is available, the use of an iteration rule for  $\mu$  offers other informational advantages which we shall now study. An iteration rule  $\eta$  on  $X$  for  $\mu$  permits each agent to receive a point of the message space  $X$  and to calculate and emit a point of  $X^i$  repeatedly, rather than to calculate and emit a subset of  $X$  once. Thus, the message correspondence  $\mu$  might require an agent to calculate and emit his excess demand function (or correspondence), while an iteration rule  $\eta$  for  $\mu$  might require him to receive a point of the "price-trade" space and to emit a "price-trade" offer several (perhaps infinitely many) times.

Since  $X$  is of minimal size sufficient for  $f$ , it is not possible to use spaces informationally smaller than  $X^i$ ,  $i = 1, \dots, n$ , for the messages of an iteration rule without impairing performance. (The correspondence  $\nu$  defined by the stationary messages of such an iteration rule would lie in an informationally smaller message space than  $X$ , resulting in a contradiction.) While the use of a sequence of single points of  $X$ , rather than a subset, may in itself be an advantage in some cases, the use of an iteration

rule  $\eta$  offers the further possibility of reducing the space of messages perceived by the individual agents. As we have shown, this may be expressed in terms of informational size of the space  $U$  in which the partition  $\mathscr{P}_\eta$  is represented. A natural question concerns the possible reduction in size of  $U$  as compared to that of  $X$  and  $X^i$ . We consider each agent separately and examine the internal computation involved in calculating his "next" message on the basis of his information. Alternatively, Lemma 2 (the main result used to characterize the possible reduction of the space of perceived messages used to compute iteration rule  $\eta$  on  $X$ ) also follows from the assumption that  $X^i = \eta^i(E^i \times X)$ ,  $i = 1, \dots, n$ , i.e., that  $\eta^i$  is onto  $X^i$ , rather than that  $X$  is minimal. The following diagram portrays this situation.

As shown in Fig. 1, agent  $i$  can "process" his information, a point of  $E^i \times X$ , via a locally sliced correspondence  $\nu^i$  to a topological space  $W^i$

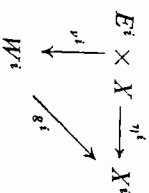


Figure 1

and then, via the continuous function  $g^i$  to  $X^i$ . The space  $W^i$  is sufficient for  $\eta^i$  via the process  $(\nu^i, g^i)$  if  $\eta^i = g^i \circ \nu^i$ . From Lemma 10 of [5], which is applicable to  $(\nu^i, g^i)$ , we see that:

LEMMA 2. If  $W^i$  is sufficient for  $\eta^i$ , then  $W^i$  has as much information as  $X^i$ , ( $\cong \eta^i(E^i \times X)$ ).

*A Special Case.* The significance of Lemma 2 emerges more clearly in the special case in which the space of environments and the message space  $X^i$  are Euclidean spaces and in which environmental information is carried fully to  $W^i$ . We first show that for Euclidean spaces dimension measures informational size. This case arises, for example, if message are  $k$ -tuples of real numbers and the correspondence  $\nu^i$  factors into the identity and a representation of  $\eta$  in  $U$ , as is shown in the diagram in Fig. 2.

LEMMA 3. Let  $X$  and  $Y$  be Euclidean spaces.  $X$  has as much information as  $Y$  if and only if  $\dim X \geq \dim Y$ , with equality if and only if  $Y$  also has as much information as  $X$ .

*Proof.* If  $\dim X = \dim Y$ , then  $X$  and  $Y$  are homeomorphic. In that case  $X$  has as much information as  $Y$  and  $Y$  has as much information as  $X$ .

$$\begin{array}{ccc} E^i \times X & \xrightarrow{\eta^i} & X^i \\ \downarrow r^i & \searrow \eta^i & \\ E^i \times U^i & & \end{array}$$

FIGURE 2

If  $\dim X > \dim Y$ , the projection of  $X$  onto  $Y$  is onto, continuous and locally sectioned. That it is locally sectioned is established as follows.  $X$  is homeomorphic by  $h: X \rightarrow \bar{X}$  to a space  $\bar{X}$  whose dimension is that of  $X$  and which contains  $Y$  as a subspace. We may define a local section  $s_y$  for the projection  $p$  of  $\bar{X}$  onto  $Y$  by taking an arbitrary element  $\bar{x}$  of  $\bar{X} - Y$  and defining  $s_y(\bar{y}) = (\bar{y}, \bar{x}) \in \bar{X}$ . Thus,  $p \circ h: X \rightarrow Y$  is onto, continuous and has local section  $h^{-1} \circ s_y: N(y) \rightarrow X$  at  $y$ , when  $N(y)$  is an open set containing  $y$ . This establishes that  $X$  has as much information as  $Y$ .

$Y$  does not have as much information as  $X$ , since if it did, there would be a function  $\alpha: Y \rightarrow X$ , which is onto  $X$ , continuous and locally sectioned. Let  $\beta_x$  be a local section for  $\alpha$  at  $x \in X$ . Then there is an open set  $N(x) \subset X$  such that  $\beta_x(N(x)) \subset Y$  is homeomorphic to  $N(x)$ , since  $\beta_x$  is continuous on  $N(x)$  with continuous inverse  $\alpha$ . But, this is impossible, since  $N(x)$  is open in  $X$  and  $\dim Y < \dim X$ .

Suppose  $X$  has as much information as  $Y$ . Then the argument just given shows that  $\dim X \geq \dim Y$ . If in addition  $Y$  has as much information as  $X$ , then the same argument shows that  $\dim Y \geq \dim X$ , and hence that  $\dim Y = \dim X$ . ■

Here  $E^i \times X$  has as much information as  $E^i \times U^i$ , which in turn has as much information as  $X^i$ . Let  $\dim A$  denote the dimension of the Euclidean space  $A$ . Since  $\dim(E^i \times U^i) = \dim E^i + \dim U^i$ , and  $\dim X = \sum_{i=1}^n \dim X^i$ , it follows that

$$\dim U^i \geq \dim X^i - \dim E^i$$

Thus, the saving of informational size achieved by using the space  $U^i$  instead of  $X$  is bounded by the difference in dimension between  $X^i$  and  $E^i$ , i.e.,  $U^i$  can have smaller dimension than  $X^i$  by at most the dimension of  $E^i$ . If it is required that  $U^i = U$  for  $i = 1, \dots, n$ , it follows that

$$\dim U \geq \max_{\{1, \dots, n\}} [\dim X^i - \dim E^i]$$

If  $p: X \rightarrow U$  is continuous, onto, and locally sectioned, then  $f^i \times p$  is continuous, onto and locally sectioned, with local section given by  $f^i \circ r^i$  where  $r^i$  is a local section for  $p$ . Hence,  $E^i \times X$  has as much information as  $E^i \times U$ . Thus,

$$\dim(E^i \times X) = \dim E^i + \dim X \geq \dim E^i + \dim U = \dim(E^i \times U),$$

or,

$$\dim X \geq \dim U.$$

Thus, Theorem 4 below is established.

**THEOREM 4.** Let  $\eta = (\eta^1, \dots, \eta^n)$  be an iteration rule on  $X$ , where  $X^i$  and  $E^i$  are Euclidean spaces for  $i = 1, \dots, n$ . Let  $U$  be a Euclidean space; let  $p: X \rightarrow U$  be a continuous locally sectioned mapping of  $X$  onto  $U$ , and let  $g^i: U \rightarrow X^i$  be a continuous function such that the process  $(f^i \times p, g^i)$  realizes  $\eta^i$  on  $E^i \times X$ , for  $i = 1, \dots, n$ . Then,

$$\max_{\{1, \dots, n\}} [\dim X^i - \dim E^i] \leq \dim U \leq \dim X.$$

The message space  $X$  of an allocation process, especially the minimal space sufficient for a given performance, is not necessarily a product. We now give a definition of a (generalized) iteration rule for a process with a general message space  $X$ .

**DEFINITION 1'.** A function  $\eta = (\eta^1, \dots, \eta^n)$  is called a *generalized iteration rule* on  $X$  if:

- (i)  $\eta^i: E^i \times Y \rightarrow X$  is a locally sectioned continuous function, for  $i = 1, \dots, n$ , where  $Y = X \times \dots \times X$ , ( $n$  times).
  - (ii) for every  $e \in E$  there exists  $y \in Y$  such that  $\eta(e, y) = y$ .
  - (iii) if  $y = \eta(e, y)$  then  $y = (x, \dots, x)$  for some  $x \in X$ .
- We say  $\eta$  is a generalized iteration rule for  $\mu$ , if,
- (iv) If  $x \in \mu(e)$ , then  $\eta(e, (x, \dots, x)) = (x, \dots, x)$  and,
  - (v) if  $\eta(e, y) = y$ , then  $y = (x, \dots, x)$  for some  $x \in \mu(e)$ .

To relate this to the case in which  $X$  is a product, i.e.,  $X = \prod_{i=1}^n X^i$ , we note that the generalized iteration rule component  $\eta^i$  may depend on proposals of the  $j$ th agent referring to other agents. Thus,

$$\eta^i(e^i, y) = \eta^i(e^i, \eta^1(e^1, \bar{y}), \eta^2(e^2, \bar{y}), \dots, \eta^n(e^n, \bar{y}))$$

where  $\bar{y}$  is the array of messages at a preceding stage. In the formulation

given above we restricted attention to iteration rules in which each agent's message is a point of his own component space  $X^i$ . In that case the array  $\eta^1, \dots, \eta^n$  is a point of  $X$ . More formally, if  $\eta = (\eta^1, \dots, \eta^n)$  is a generalized iteration rule, we consider the derived rule  $\bar{\eta} = (\bar{\eta}^1, \dots, \bar{\eta}^n)$  where  $\bar{\eta}^i = P_i \circ \eta^i$  and  $P_i: X \rightarrow X^i$  is the projection on  $X^i$ . Thus  $\bar{\eta}: X \rightarrow X^i$ . I.e., at stage  $i$ , for  $y^i = \bar{y} \in Y$ ,

$$P_i \cdot \eta^i(e^i, P_i(\eta^1(e^1, \bar{y}))) P_2(\eta^2(e^2, \bar{y})), \dots, P_n(\eta^n(e^n, \bar{y}))) \\ = P_i \cdot \eta^i(e^i, (x_1^1, \dots, x_1^n)) = x_{i+1}^i$$

for  $i = 1, \dots, n$ .

*Remark.* We note that if  $X$  is not a product space and if the iteration rule  $\eta$  is replaced in the statement of Theorem 4 by a generalized iteration rule on  $X$ , then the conclusion of Theorem 4 becomes:

$$\max_{\{1, \dots, n\}} [\dim X - \dim E^i] \leq \dim U \leq \dim X.$$

**DEFINITION 1.** Suppose that  $X$ ,  $M$ , and  $Z$  are topological spaces, and suppose that  $f: X \rightarrow Z$  is a function. A pair which consists of a correspondence  $\mu: X \rightarrow M$  and a function  $\bar{f}: M \rightarrow Z$  is said to be *compatible with f* if and only if for each  $x \in X$ ,  $\bar{f}$  is constant on  $\mu(x)$  and has value  $f(x)$ . Thus if  $u \in \mu(x)$ , then  $\bar{f}(u) = f(x)$ . We shall say that  $M$  has *sufficient information* for the function  $f$  if there is a pair  $(\mu, \bar{f})$  such that  $\mu: X \rightarrow M$ ,  $\bar{f}: M \rightarrow Z$ ,  $(\mu, \bar{f})$  is compatible with  $f$ , and  $\mu$  is a locally sectioned correspondence (see Definition 6 below). We shall say then that  $(\mu, \bar{f})$  realizes  $f$ . We call the pair  $(\mu, \bar{f})$  a *resource allocation process*, (briefly, *process*) with message space  $M$ , and choice function  $\bar{f}$ .

**DEFINITION 2.** Suppose that  $X^1, \dots, X^n$  is a set of topological spaces and suppose that  $M$  is a topological space. A correspondence  $\mu: X^1 \times \dots \times X^n \rightarrow M$  is said to be a *coordinate correspondence* if and only if there are correspondences  $\mu_i: X^i \rightarrow M$  such that for each  $(x^1, \dots, x^n) \in X^1 \times \dots \times X^n$ ,  $\mu(x^1, \dots, x^n) = \mu_1(x^1) \cap \dots \cap \mu_n(x^n)$ .

**DEFINITION 3.** Let  $X = \prod_{i=1}^n X^i$ ,  $M$  and  $Z$  be topological spaces and let  $(\mu, \bar{f})$ , where  $\mu: X \rightarrow M$  and  $\bar{f}: M \rightarrow Z$ , be a resource allocation process (with message space  $M$  and choice function  $\bar{f}$ ). We say that  $(\mu, \bar{f})$  *preserves privacy* if and only if  $\mu$  is a coordinate correspondence.

**LEMMA 5.** Suppose that  $X^1, \dots, X^n$ ,  $M$  are topological spaces and suppose that  $\mu = X^1 \times \dots \times X^n \rightarrow M$  is a correspondence. A necessary

and sufficient condition (which we shall call the "crossing condition") that  $\mu$  be a coordinate correspondence is that for each pair of points  $x = (x_1, \dots, x_n)$  and  $x' = (x_1', \dots, x_n')$  in  $X^1 \times \dots \times X^n$  and each integer  $1 \leq i \leq n$ ,

$$(*) \quad \mu(x) \cap \mu(x') = \mu(x' \otimes_i x) \cap \mu(x \otimes_i x').$$

**DEFINITION 6.** Suppose the  $X$  and  $Y$  are topological spaces. If  $\mu: X \rightarrow Y$  is a correspondence from  $X$  to  $Y$ , then we shall say that  $\mu$  is *locally sliced* if the following condition is satisfied:

for each  $p \in X$ , there exists an open set  $U(p)$  which contains  $p$  and a function  $s: U(p) \rightarrow Y$  such that for each  $u \in U(p)$ ,  $s(u) \in \mu(u)$ .

The function  $s$  will be called a *local slice* or *slice* of  $\mu$ .

**DEFINITION 7.** If  $X$  and  $Y$  are topological spaces, then an onto function  $f: X \rightarrow Y$  is said to be *locally sectioned* if the correspondence  $f^{-1}$  from  $Y$  to  $X$  is locally sliced.

**DEFINITION 9.** Suppose that  $X$  and  $Y$  are topological spaces. We shall say that  $Y$  has *as much information as*  $X$  if and only if there exists a locally sectioned function from  $Y$  to  $X$ . We shall say that  $Y$  has *strictly more information than*  $X$ , if  $Y$  has as much information as  $X$ , but  $X$  does not have as much information as  $Y$ .

#### REFERENCES

1. B. BALASSA, *Discussion*, AER Papers and Proceedings, May 1969, Vol. LIX, No. 2, p. 533-537.
2. L. HURWICZ, "Optimality and Informational Efficiency in Resource Allocation Processes," *Mathematical Methods in the Social Sciences*, 1959, (K. J. Arrow, S. Karlin and P. Suppes, Eds), Stanford University Press, Stanford, California, 1960, pp. 27-48.
3. L. HURWICZ, "On the Dimensional Requirements of Informationally Decentralized Pareto-Satisfactory Processes," (manuscript) presented at the Conference Seminar on Decentralization, Northwestern University, Evanston, Illinois, February 1972.
4. S. MACLANE and G. BIRKHOFF, *Algebra*, The MacMillan Co., New York, 1967.
5. K. MOUNT and S. REITER, "The Informational Size of Message Spaces," *J. Econ. Theory* 8 (1974), 161-192.