

The Knowledge Revealed by an Allocation Process and the Informational Size of the Message Space

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We consider resource allocation processes in which each agent starts with certain fragmentary knowledge of the environment (in important cases, an agent knows his own economic characteristics, e.g., admissible holdings set, production set, preferences, etc.) and in which actions are chosen after a process of interchange of messages has been completed. Such allocation processes may be characterized by a correspondence from the space of environments to the message space, and by a function, called the outcome function, from the message space to the space of joint actions (Definition 1 of [7]; see also [2] and [3]). In evaluating such an allocation process, it is of interest to know the "deduction" from gross performance due to the "costs" of operating the process [3]. However, the function relating these costs to the process to which they attach is generally unknown. One line of approach to this problem is to look for properties of these processes which capture at least some features of information processing which seem important to an informal intuitive view. In this the hope is to find a property which orders the class of processes (at least partially) in agreement with the ordering that would be determined by the true but unknown "cost" function. Two such ideas are examined in this paper.

In [8] the knowledge of the environment and joint action acquired by an agent as a result of communicating according to the rules of a process was studied. This was formalized by means of a correspondence from the space of environments to the (product) space of environments and joint actions (Definition 1 below). This correspondence characterizes what the agent knows after communication stops about the environment and action, for each possible environment. For example, in a centrally planned economy in which each agent transmits his environmental

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component to a central agent who computes an "optimal" joint action, the central agent acquires precise knowledge of the environment and the joint action. In the Malinvaud planning procedure [6] the central agent does not acquire precise knowledge of the environment (in that case, the production sets of individual firms) but does acquire precise knowledge of the joint action (the input-output vectors chosen). In certain versions of the competitive process no agent acquires precise knowledge either of the environment or of the joint action but knows precisely only his own component of them.

A second idea relates to the size or complexity of the messages used by a process. It seems intuitively clear that "larger" or "more complex" messages can carry more information than "smaller" or "less complex" ones. A precise concept of the informational size of a message space has been given in Definition 9 of [7], which attempts to capture this notion.¹ According to that concept, one message space (assumed to be a topological space) is informationally larger than another (topological space) if every message of the first space can be translated by a fixed continuous function into a message of the second space in such a way that every message of the second space can be decoded in a locally continuous way. It is natural to expect that the knowledge transmitted should be related to the information-carrying capacity (informational size) of the message space used for communication. We are thus led to study the relationship between the knowledge of the environment and joint action revealed by a process to its participating agents and the informational size of the message space of the process.

SUMMARY

To make this paper more nearly self-contained, we restate from [8] the definition of the knowledge revealed by a process (Definition 2) and the concept of a condensation of a process (Definition 3). One process, say τ , is a condensation of another, π , if τ uses an informationally smaller message space than π does, in such a way that the message used by τ (the condensed process) in a given environment is a translation of the message used by π in the same environment. Two main results are established. First, if τ is a condensation of π , then π reveals more than τ (Theorem 1). Second, if π is a process realizing a function f (Definition 1 of [7]) and if the message correspondence of π is a locally sectioned

continuous function, then the message space of π has minimal informational size in the class of all processes whose message correspondences are upper semicontinuous and which reveal more than π (Theorem 2). Theorem 3 establishes sufficient conditions that one process reveals more than another. Specifically, if the performance function varies enough on the class of environments (Definition 4), then any privacy-preserving message correspondence is forced to distinguish environments (Lemma 2). Hence, a privacy-preserving process realizing the given performance function reveals more than any process realizing that function (Theorem 3). Finally, the informational minimality of the message space of the competitive process, first established by Theorem 31 of [7], is shown to follow as a special case from Theorems 2 and 3.

NOTATION AND DEFINITIONS

Let us define the following: $\{1, \dots, n\}$ as the set of agents; E^i as a topological space, the set of characteristics of the i th agent, for $i = 1, \dots, n$; $E \equiv \prod_{i=1}^n E^i$; Z as a topological space, the space of joint actions; $f: E \rightarrow Z$ as a given continuous locally sectioned function, the performance standard [we interpret f as designating the joint action $f(e) \in Z$, which is to be taken when e is the environment]; X and Y as topological spaces (usually message spaces).

Following [8], we characterize the state of knowledge of an agent after communication has taken place according to the rules of a given allocation process. We suppose that initially an agent knows the space E of joint environments and the space Z of joint actions; in addition we suppose that agent i knows directly "his" component e^i of the joint environment $e = (e^1, \dots, e^i, \dots, e^n) \in \prod_{i=1}^n E^i = E$. We suppose further that each agent knows the process—i.e., the message correspondence and the choice function—and that he receives the full message. Thus, if the process is $\pi = (\mu, \hat{f})$, then agent i knows $\mu: E \rightarrow X$ and $\hat{f}: X \rightarrow Z$ and receives the message $\mu(e)$ when the environment is $e \in E$.

DEFINITION 1. Let $\pi = (\mu, \hat{f})$ be an allocation process and let

$$\xi_{\pi^i}: E \rightarrow E \quad \text{and} \quad \eta_{\pi^i}: E \rightarrow Z, \quad i = 1, \dots, n,$$

be given by

$$\xi_{\pi^i}(e) \equiv \mu^{-1}(\mu(e)) \cap \left\{ e^i \right\} \times \prod_{j=1, j \neq i}^n E^j,$$

¹ Hurwicz has studied the dimension of Euclidean message spaces in this connection [5].

and

$$\eta_{\pi^i}(e) \equiv \bar{f}(\mu(e))$$

for $i = 1, \dots, n$.²

The correspondences ξ_{π^i} and η_{π^i} characterize what agent i knows about the environment and action as a result of direct knowledge and communication.

DEFINITION 2. Let $\pi = (\mu, \bar{f})$ and $\tau = (\nu, g)$ be allocation processes with the same space of environments E and actions Z . We say π reveals as much as τ if for every $i \in \{1, \dots, n\}$ $\xi_{\pi^i}(e) \subset \xi_{\tau^i}(e)$ and $\eta_{\pi^i}(e) \subset \eta_{\tau^i}(e)$ for all $e \in E$.³

DEFINITION 3. The allocation process $\tau = (\nu, g)$ with message space Y is a condensation of the allocation process $\pi = (\mu, \bar{f})$ with message space X , if there exists a mapping ρ of X onto Y which is continuous, locally sectioned, and satisfies

$$\rho(\mu(e)) \subset \nu(e) \quad \text{for all } e \in E.$$

A condensed process is a sort of "quotient" process. If τ is a condensation of π , then it is a quotient process whose message space Y has no more information than X .

We note in passing,

LEMMA 1. Let $\pi = (\mu, \bar{f})$ be an allocation process with message space X . If $\tau = (\nu, g)$ with message space Y is a condensation of π and τ realizes f , then there exists a process $\pi' = (\mu, h)$ such that τ is a condensation of π' and π' realizes f .

Proof. Let $h = g \circ \rho$, where ρ is the mapping of X onto Y according to Definition 3. Then h maps X onto Z . Since $\rho(\mu(e)) \subset \nu(e)$, $g \cdot \rho(\mu(e)) = g(\nu(e))$. Since τ realizes f , $g(\nu(e)) = f(e)$. Hence, $h(\mu(e)) = g \cdot \rho(\mu(e)) = f(e)$. Finally, h is continuous since ρ and g are continuous.

² In case agent i receives a function θ^i of the joint message $\mu(e)$, we would define ξ_{π^i} and η_{π^i} by

$$\xi_{\pi^i}(e) \equiv \mu^{-1}(\theta^{i-1}(\theta^i(\mu(e)))) \cap \left\{ e^i \right\} \times \prod_{j=1, j \neq i}^n E^j,$$

and

$$\eta_{\pi^i}(e) = f(\theta^{i-1}(\theta^i(\mu(e)))) \quad \text{for } i = 1, \dots, n.$$

³ This relation was called "more informative than" in [8].

THEOREM 1. Let $\pi = (\mu, \bar{f})$ and $\tau = (\nu, g)$ be allocation processes realizing f with message spaces X and Y , respectively, and let τ be a condensation of π . Then π reveals as much as τ .

Proof. The process π reveals as much as τ if for each $i = 1, \dots, n$,

$$\xi_{\pi^i} \times \eta_{\pi^i} \subset \xi_{\tau^i} \times \eta_{\tau^i}.$$

Since π and τ both realize f ,

$$\eta_{\pi^i}(e) = \bar{f} \circ \mu(e) = f(e) = g \circ \nu(e) = \eta_{\tau^i}(e), \quad \text{for } i = 1, \dots, n.$$

Thus,

$$\eta_{\pi^i} \subset \eta_{\tau^i} \quad \text{for } i = 1, \dots, n.$$

It remains to show

$$\xi_{\pi^i} \subset \xi_{\tau^i} \quad \text{for } i = 1, \dots, n.$$

Let $e \in E$ and let

$$\bar{e} \in \xi_{\pi^i}(e) = \mu^{-1}(\mu(e)) \cap \left\{ e^i \right\} \times \prod_{j=1, j \neq i}^n E^j, \quad i = 1, \dots, n.$$

Since

$$\xi_{\tau^i}(e) = \nu^{-1}(\nu(e)) \cap \left\{ e^i \right\} \times \prod_{j=1, j \neq i}^n E^j, \quad i = 1, \dots, n$$

to show $\bar{e} \in \xi_{\tau^i}(e)$ it suffices to show that $\bar{e} \in \nu^{-1}(\nu(e))$. Now, $\bar{e} \in \xi_{\pi^i}(e)$ implies $\bar{e} \in \mu^{-1}(\mu(e))$, which in turn implies $\mu(\bar{e}) \cap \mu(e) \neq \emptyset$. Let $\bar{x} \in \mu(\bar{e}) \cap \mu(e)$. Then by Definition 3, $\rho(\bar{x}) \in \rho(\mu(\bar{e})) \cap \rho(\mu(e)) \subset \nu(\bar{e}) \cap \nu(e)$. It follows that $\bar{e} \in \nu^{-1}(\nu(e))$.

THEOREM 2. Let $\pi = (\mu, \bar{f})$ be an allocation process with message space X which realizes f , and let μ be a locally sectioned continuous function. If $\tau = (\nu, g)$ is an allocation process with message space Y such that τ realizes f , where ν is an upper-semicontinuous correspondence, and if τ reveals as much as π , then Y has as much information as X .

*Proof.*⁴ To show that Y has as much information as X , it suffices to exhibit a locally sectioned continuous map of Y onto X . We shall show that $\varphi \equiv \mu \circ \nu^{-1}$ is such a map, where $\nu^{-1}(x) = \{e \in E \mid x \in \nu(e)\}$.

⁴ This proof follows that of Theorem 31 in [7], except for the first part.

We show first that φ is a function. If, for $y \in Y$, \bar{e} and \bar{e} belong to $\nu^{-1}(y)$, then $\mu(\bar{e}) = \mu(\bar{e})$. This is established as follows. Since τ reveals as much as π , it follows that

$$\nu^{-1}(\nu(e)) \subset \mu^{-1}(\mu(e)) \quad \text{for all } e \in E. \quad (***)$$

Since \bar{e} and \bar{e} in $\nu^{-1}(y)$ implies $y \in \nu(\bar{e}) \cap \nu(\bar{e})$, which can be written $\bar{e} \in \nu^{-1}(\nu(\bar{e}))$, it follows from (***) that $\bar{e} \in \mu^{-1}(\mu(\bar{e}))$. Since μ is a function, it follows that $\mu(\bar{e}) = \mu(\bar{e})$. Hence, μ is constant on the sets $\nu^{-1}(y)$ for $y \in Y$. Since μ is a function, so is $\varphi = \mu \circ \nu^{-1}$.

We show next that φ is continuous. Because ν is upper semicontinuous, so is the correspondence ν^{-1} , since they have the same (closed) graph. Regarded as a correspondence, μ is upper semicontinuous, since it is continuous. Hence the composition $\mu \cdot \nu^{-1} = \varphi$ is continuous (see [1, pp. 109-111]).

It is immediate that φ is onto X since ν^{-1} is onto E and μ is onto X .

We show finally that φ is locally sectioned. Let $x \in X$ and let U be an open neighborhood of x in X , such that $t_x: V \rightarrow E$ is a local section of μ . Since τ is an allocation process, ν is a locally sliced correspondence. Therefore, given $e \in E$, there exists an open set H containing e and a continuous function $v: H \rightarrow Y$ such that $v(\bar{e}) \in \nu(\bar{e})$ for $\bar{e} \in H$. Given H , by continuity of t_x , there exists an open subset $V \subset U$ such that $t_x(V) \subset H$. The function $\Psi \equiv v \cdot t_x: V \rightarrow Y$ is a local section for φ , since it is continuous and satisfies $\Psi \cdot \varphi = \text{Id}_V$. The last equality is established as follows: Let $\bar{x} \in V$ and let $\bar{e} = t_x(\bar{x})$. Then $v(\bar{e}) \in \nu(\bar{e})$, which implies $\bar{e} \in \nu^{-1}(\nu(\bar{e}))$. Hence $\Psi \cdot \varphi(\bar{x}) = \mu(\bar{e}) = \bar{x}$. Thus $\Psi \cdot \varphi = \text{Id}_V$.

Thus, according to Theorem 2, if a process π which realizes the performance function f has a message correspondence which is a locally sectioned continuous function, then any other process which also realizes f , which has an upper-semicontinuous message correspondence, and which reveals as much as does π , must use a message space which is informationally as large as that of the process π .

In order to apply Theorem 2, it is of interest to know when a process reveals as much as another. Theorem 3 gives sufficient conditions for this, which are of particular interest because they cover the case of the competitive process.

DEFINITION 4. The performance function $f: E \rightarrow Z$ is said to be personally sensitive on E (or, briefly, sensitive on E) if, for every pair of points e and \bar{e} in E , there exist points of the "cube" $\{e, \bar{e}, e \otimes_1 \bar{e}, \dots, e \otimes_n \bar{e}, \bar{e} \otimes_1 e, \dots, \bar{e} \otimes_n e\}$ generated by e and \bar{e} , at which f takes distinct values.

⁵ Following [7], if $e = (e_1, \dots, e_n)$ and $\bar{e} = (\bar{e}_1, \dots, \bar{e}_n)$, then $e \otimes_j \bar{e} = (\bar{e}_1, \dots, e_j, \dots, \bar{e}_n)$.

LEMMA 2. Let $f: E \rightarrow Z$ be personally sensitive on E , and let $\pi = (\mu, f)$ be an allocation process realizing f such that μ is a coordinate correspondence. If e and \bar{e} are distinct elements of E , then $\mu(e) \cap \mu(\bar{e}) = \emptyset$.

Proof. Suppose e and \bar{e} are distinct elements of E and $\mu(e) \cap \mu(\bar{e}) \neq \emptyset$. Since μ is a coordinate correspondence, it satisfies the crossing condition for e and \bar{e} , i.e., $\mu(e) \cap \mu(\bar{e}) = \mu(e \otimes_j \bar{e}) \cap \mu(\bar{e} \otimes_j e)$ for all $j \in \{1, \dots, n\}$. Hence, $f \circ \mu(e) = f \circ \mu(\bar{e}) = f \circ \mu(e \otimes_j \bar{e}) = f \circ \mu(\bar{e} \otimes_j e)$ for all $j \in \{1, \dots, n\}$. Since π realizes f on E , it follows that $f(e) = f(\bar{e}) = f(e \otimes_j \bar{e}) = f(\bar{e} \otimes_j e)$ for all $j \in \{1, \dots, n\}$; i.e., f is constant on the "cube" generated by e and \bar{e} . This contradicts the hypothesis that f is sensitive on E .

COROLLARY TO LEMMA 2. Under the hypotheses of Lemma 2, if μ is a function, then μ is one to one.

THEOREM 3. Let $f: E \rightarrow Z$ be sensitive on E , and let $\pi = (\mu, f)$ and $\tau = (\nu, g)$ be allocation processes which realize f , such that ν is a coordinate correspondence. Then τ reveals as much as π .

Proof. The process τ reveals more than π if $\nu^{-1}(\nu(e)) \subset \mu^{-1}(\mu(e))$ for all $e \in E$. This is equivalent to

$$\bar{e} \in \nu^{-1}(\nu(e)) \Rightarrow \bar{e} \in \mu^{-1}(\mu(e)),$$

which may be written

$$\nu(\bar{e}) \cap \nu(e) \neq \emptyset \Rightarrow \mu(\bar{e}) \cap \mu(e) \neq \emptyset.$$

This in turn is equivalent to

$$\mu(\bar{e}) \cap \mu(e) = \emptyset \Rightarrow \nu(\bar{e}) \cap \nu(e) = \emptyset. \quad (***)$$

By hypothesis, f is sensitive on E . Since $\tau = (\nu, g)$ satisfies the other hypotheses of Lemma 2, it follows that $\nu(\bar{e}) \cap \nu(e) = \emptyset$. Hence the implication (***) is true, and τ reveals as much as π .

If μ also preserves privacy, then it follows from Lemma 2 that $\nu(\bar{e}) \cap \nu(e) = \emptyset$ implies $\mu(\bar{e}) \cap \mu(e) = \emptyset$. Hence π reveals as much as τ , and τ reveals as much as π . If μ does not preserve privacy, then we may not conclude that π reveals as much as τ .⁶

Theorem 31 of [7] established the informational minimality of the message space of the competitive process in the class of processes which are Pareto satisfactory on a class of pure trade environments, including the Cobb-Douglas ones, and which have upper semicontinuous privacy-preserving message correspondences. This result is also established by

Theorems 2 and 3, of which the result in [7] is an application. This may be seen as follows: Let $\pi = (\mu, f)$ denote the competitive process with message space X and E as the class of Cobb-Douglas environments, and let $\tau = (\nu, g)$ be a process which realizes f [7, Definition 22], such that ν is an upper semicontinuous coordinate correspondence with message space Y . Lemma 26 of [7] establishes that μ is a locally sectioned continuous function on E . Lemma 29 of [7] establishes that f is personally sensitive on E . It follows from Theorem 3 that τ reveals as much as π . It then follows from Theorem 2 that the message space Y of τ is informationally as large as X .

Interpreting these results a bit more, Lemma 2 tells us that, if the performance function is sensitive on the class of environments, a message correspondence which is part of a process realizing f will be required to distinguish points of E ; if the message correspondence preserves privacy, it is forced to distinguish all points of E . Hence all such processes reveal the same knowledge of E to an agent who receives the full (terminal) message. In the presence of regularity conditions, a message *function* which separates all those environments uses a message space of minimal informational size; i.e., such a function performs the required separation in an "efficient" way.

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