The Informational Size of Message Spaces

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I. INTRODUCTION

Problems of economic policy may be grouped in two broad classes which may be loosely described as those involving choice of the value of a "parameter" within a given system of economic institutions and those involving choice among institutions. Familiar examples of problems of the first type include choice of tax rates, rates of government expenditures, and size of the money supply. Examples of the second type include the design of "new" economic systems, such as was embodied in the Yugoslavian economic reform of 1968, or the choice of economic institutions confronting a developing country, as well as more limited problems such as design of regulatory mechanisms, or structuring of the system of financial institutions, such as is embodied in the Federal Reserve Act of 1933.

In order to analyze and compare alternative economic systems so as to permit more enlightened choice among them, we seek to identify those properties of such systems on which choice should turn and to study their counterparts in a formal model. Among such properties are those

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The concept of a space having sufficient information for a certain purpose involves a careful consideration of the information required for the purpose. In order to identify the possible outcomes of a process, it is necessary to consider only those processes that have sufficient information. This leads to the following definitions:

**Definition:** Let $X$ and $Y$ be topological spaces, and let $f: X \to Y$ be a function. We say that $f$ is a strict inclusion of $X$ into $Y$ if $f$ is a homeomorphism and $X$ has more information than $Y$.

**Definition:** Let $X$ and $Y$ be topological spaces, and let $f: X \to Y$ be a function. We say that $f$ is a weak inclusion of $X$ into $Y$ if $f$ is a surjective function and $X$ has more information than $Y$.

**Example:** Let $X$ be a topological space, and let $f: X \to X$ be a function. If $f$ is a strict inclusion, then $X$ has more information than itself. Conversely, if $X$ has more information than itself, then $f$ is a weak inclusion.

We now consider the following problem: Given a function $f: X \to Y$, determine whether $f$ is a strict or weak inclusion. This problem is equivalent to determining whether $X$ has more information than $Y$.

**Theorem:** Let $X$ and $Y$ be topological spaces, and let $f: X \to Y$ be a function. Then $f$ is a strict inclusion if and only if $X$ has more information than $Y$.

**Proof:** Suppose that $f$ is a strict inclusion. Then $f$ is a homeomorphism, and $X$ has more information than $Y$. Conversely, suppose that $X$ has more information than $Y$. Then there exists a function $g: Y \to X$ such that $g \circ f = id_X$. Since $f$ is a surjective function, $f$ is a weak inclusion.

We now consider the following problem: Given a topological space $X$, determine whether $X$ has more information than $Y$. This problem is equivalent to determining whether $X$ is a strict or weak inclusion of $Y$.

**Theorem:** Let $X$ and $Y$ be topological spaces, and let $f: X \to Y$ be a function. Then $X$ has more information than $Y$ if and only if $f$ is a surjective function.

**Proof:** Suppose that $X$ has more information than $Y$. Then there exists a function $g: Y \to X$ such that $g \circ f = id_X$. Since $g$ is a surjective function, $f$ is a weak inclusion. Conversely, suppose that $f$ is a weak inclusion. Then there exists a function $g: Y \to X$ such that $g \circ f = id_X$. Since $g$ is a surjective function, $X$ has more information than $Y$.

**Table 1**

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**Table 2**

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\[
\begin{align*}
\mathbb{R} & = \{x \in \mathbb{R} \mid x > 0\}, \\
\mathbb{N} & = \{1, 2, 3, 4, \ldots\}, \\
\mathbb{Z} & = \{\ldots, -2, -1, 0, 1, 2, \ldots\}, \\
\mathbb{Q} & = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}, \\
\mathbb{R} & = \mathbb{Q} \cup \mathbb{R}^+.
\end{align*}
\]