APPENDIX A

Numerical Methods

This appendix describes the numerical method used to compute the transition paths discussed in the main text. For all the models considered in this paper characterizing the competitive equilibrium amounts to solving a two point boundary value problem, i.e. a system of difference equations with boundary conditions specified at two different points in time. We will use the basic neoclassical model of section 2 to illustrate the operation of the algorithm. The competitive equilibrium for that economy is characterized by a system of two first order difference equations:

$$\gamma_X \gamma_N \lambda_t = \lambda_{t+1} \beta^* [D_1 F(k_{t+1}, n) + (1-\delta)] \tag{A.1}$$

$$\gamma_X \gamma_N k_{t+1} = F(k_t, n) + (1-\delta) k_t - \lambda_t^{-1/\sigma} \tag{A.2}$$

where $\beta^* = \beta \gamma_N^{\eta} \gamma_X^{1-\sigma}$ is the discount factor modified for steady growth in consumption and population; λ_t is the current-valued Lagrange multiplier associated with the resource constraint; and k_t is the per capita capital stock deflated by X_t (i.e. $k_t = K_t/(M_t X_t)$). This system of difference equations has two boundary conditions, one at time zero (the initial value of k, $k_o = K_o/(X_o M_o)$) and the other at infinity (the transversality condition, $\lim_{t\to\infty} (\beta^*)^t \lambda_t k_{t+1} = 0$).

To solve this problem we employed a shooting method that relies on knowledge of the near-steady-state dynamics of this system of equations. By linearizing the system around the steady state it is possible to show that, depending of the value selected for λ_o three types of paths may arise: (i) paths along which the capital stocks always grows, eventually overshooting the steady state and continuing to grow at an accelerating rate; (ii) paths along which the capital stock decreases or increases initially and then decreases; (iii) one path along which the capital stock increases converging to the steady state. Paths type (i) and (ii) violate the transversality condition so only (iii) is the desired solution. We denote the value of λ_o associated with (iii) by λ_o^*. Paths type (i) occur for values of $\lambda_o > \lambda_o^*$ while paths type (ii) correspond to $\lambda_o < \lambda_o^*$. This suggests a simple algorithm to search for λ_o^*:

<u>Step 1</u>: find a value of λ_o that generates a path type (i); denote it by $\overline{\lambda}_o$

<u>Step 2</u>: find a value of λ_o that generates a path type (ii); denote it by $\underline{\lambda}_o$ ($\underline{\lambda}_o$ = 0 will always work).

<u>Step 3</u>: Compute $\lambda_o = (\underline{\lambda}_o+\overline{\lambda}_o)/2$ and use it as initial condition to solve the system of difference equations . Set $\underline{\lambda}_o = \lambda_o$ if a path type (ii) is obtained and $\overline{\lambda}_o = \lambda_o$ otherwise. Repeat step 3 until $\underline{\lambda}_o-\overline{\lambda}_o$ is lower than a chosen tolerance error (usually the smallest number recognized by the computer as different from zero).

The number of iterations needed for convergence is given by the first integer j such that $j > \ln(\Delta/tol)/\ln 2$, where tol is the chosen tolerance and Δ the initial value of $\overline{\lambda}_o-\underline{\lambda}_o$.

This method is different from "simple" and "multiple" shooting which are the standard algorithms used to solve this type of problem. The advantage of both of these algorithms is that they require no knowledge of the dynamics of the system. A detailed discussion of these methods can be found in Roberts and Shipman (1972) and in Lipton et al (1982) but we provide here a brief description to contrast them with the shooting method that we employed.

The basic idea underlying simple shooting is that a system of equation such as (A.1) - (A.2) can be viewed as defining a function $Z_T = f(\lambda_o)$, where λ_o is an arbitrary guess and Z_T is the difference between the value of the boundary condition at T associated with λ_o and the desired value for that boundary condition. A numerical method for finding zeros of equations (e.g. Newton-Raphson) is then used to generate a new guess for λ_o with the process being repeated iteratively until $Z_T = 0$. In our example the second boundary condition is at infinity so it is usually approximated by choosing T to be a large number (say, 200 years). Z_T can be defined as $(\beta^*)^T \lambda_T k_{T+1} - 0$ or as $k_T - k_s$ since paths that satisfy the transversality condition converge to the steady state. Simple shooting does not usually work because arbitrary guesses for λ_o can generate paths for the capital stock along which k_t becomes negative leading to nonsensical complex values for k_T and λ_T. To avoid this it is often necessary to split the path into various parts and apply the method to each part (e.g. compute the path for the first five years, then use k_5 as an initial condition to compute the path for the following five years, etc.), a technique that is known as multiple shooting.

The numerical results that we obtained for the models described in section IV using multiple shooting were very similar to the paths computed with our shooting algorithm.

As a second check on the algorithm that we employed we also verified that the paths computed numerically for the one and two-sector models replicated the analytical solutions that can be obtained for the cases of 100% depreciation and logarithmic momentary utility (for a discussion of these closed forms see Radner [1966] and Long and Plosser [1983]).

APPENDIX B

Embodied Technical Progress

This Appendix shows that modifying the model of section II to view technological progress as embodied, along the lines of Solow [1959], generates an economy that is basically observationally equivalent to the original model.

The technology of the Solow [1959] model translated to discrete time is comprised by the following equations:

$$\text{(B.1)} \qquad Y_{vt} = A\ (\gamma_E^v\ K_{vt})^{1-\alpha}\ N_{vt}^{\alpha}$$

$$\text{(B.2)} \qquad K_{v,t+1} = I_v\ (1-\delta)^{t-v}$$

$$\text{(B.3)} \qquad N_t = \sum_{v=o}^{t-1} N_{vt}$$

$$\text{(B.4)} \qquad Y_t = \sum_{v=o}^{t-1} Y_{vt}$$

$$\text{(B.5)} \qquad Y_t = C_t + I_t .$$

The first equation expresses the output at time t of a production technology of vintage v as a Cobb-Douglas function of the capital of that vintage in existence at time t and of the labor combined with that capital. The rate of embodied technical progress is denoted by γ_E. Equation (B.2) relates the stock of capital of vintage v existent at time t+1 to the original investment made in that vintage (I_v) and of the rate of depreciation. Equation (B.3) is the adding-up constraint on labor, (B.4) states that total output is the sum of the output produced by the various vintages and (B.5) that total output can be devoted to consumption or investment.

An efficient allocation of labor requires that its marginal product be equated across the different vintages. Solow [1959] showed that using this fact the vintage-specific capital stocks can be aggregated into a composite capital stock, J_t defined as:

$$(B.6) \qquad J_t = \sum_{v=o}^{t-1} \gamma_E^v K_{v,t}.$$

The advantage of defining this composite capital good is that total output can be expressed as a function of N_t and J_t:

$$(B.7) \qquad Y_t = A\ J_t^{1-\alpha}\ N_t^{\alpha}.$$

The law of motion for J_t (B.6) can also be expressed without reference to the vintage-specific capital stocks:

$$(B.8) \qquad J_{t+1} = J_t\ (1-\delta) + \gamma_E^t\ I_t.$$

In the steady state capital grows at rate $\gamma_J = \gamma_N\ \gamma_E{}^{1/\alpha}$, where γ_N is the growth rate of population, while output, consumption and investment grow at rate $\gamma_Y = \gamma_N\ \gamma_E^{(1-\alpha)/\alpha}$.

It is easy to show, using the description of technology given by (B.5), (B.7) and (B.8), that at any point in time the real interest rate is given by:

$$(B.9) \qquad r_t = [(1-\alpha)\ A\ (j_t)^{-\alpha}\ n^{\alpha} + (1-\delta)]/\gamma_E - 1.$$

where $j_t = J_t/(M_t\ \gamma_J^t)$, i.e. the per capita value of the composite capital stock detrended by its growth rate.

To study the model's implications for the Japanese real interest rate in 1950 we start by using (B.9) and the knowledge of the steady state real interest rate, r^*, to compute the steady state value of j_t, j^*. Next we use the fact that Japanese per capita output is 1950 was 19% of that of the U.S.:

$$\text{(B.10)} \qquad [A\ (J_o^J)^{1-\alpha}\ (N^J)^{\alpha}/M_o^J]/\ [A\ (J_o^{US})^{1-\alpha}\ (N^{US})^{\alpha}/M_o^{US}] = .19$$

Assuming that the number of hours worked per capita is the same in the two countries (in fact this number was higher in Japan so that this assumption biases the results toward finding a low interest rate), we can rewrite (B.10) in terms of j's as:

$$\text{(B.11)} \qquad j_o^J = j_o^{US}\ (.19)^{1/(1-\alpha)}.$$

Under the assumption that the U.S. was at the steady state in 1950, i.e., $j_o^{US} = j^*$, we can compute j_o^J an the associated real interest rate implied by (B.9). Using the parameter values employed in the main text, α=2/3, δ=.10, r_s=.065 and γ_N = 1.01 (the Japanese population grew at 1% in the post war period—see Barro [1987], page 296) and choosing γ_E so that the steady state growth rate of per capita output is 2% per year, the value of the Japanese interest rate implied by the model is 560%.

In terms of dynamics this model is almost identical to the baseline economy of section II. The system of Euler equations that governs the the competitive equilibrium for this economy is identical to (A.1), (A.2) with $(\gamma_X \gamma_N)$ replaced by γ_J and k_t replaced by j_t.