

# Lecture 6: Arbitrage Pricing Theory

## Investments

1. Introduction
2. Multi-Factor Models
3. The Arbitrage Pricing Theory

- The empirical failure of the CAPM is not that surprising:
  - ↪ We had to make a number of pretty unrealistic assumptions to prove the CAPM.
  - ↪ For example, we made the assumption that investors had identical preferences, had the same information, and hold the same portfolio (the market).
  - ↪ Also, there is the problem that identifying and measuring the market return is difficult, if not impossible (the Roll Critique)
- In this lecture we will study a different approach to asset pricing called the **Arbitrage Pricing Theory** or **APT**.
- The APT specifies a pricing relationship with a number of “systematic” factors.

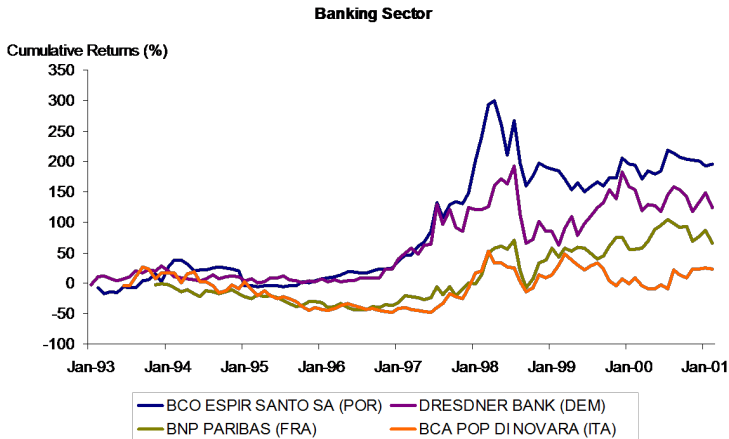
Barr Rosenberg, “Extra Market Components of Covariance in Security Markets,” Journal of Financial and Quantitative Analysis, 1974:

*“Companies possessing similar characteristics may, in a given month, show returns that are different from the other companies. The pattern of differing shows up as the factor relation.”*

- A set of common factors - not just a monolithic market - influence returns

# Comovement

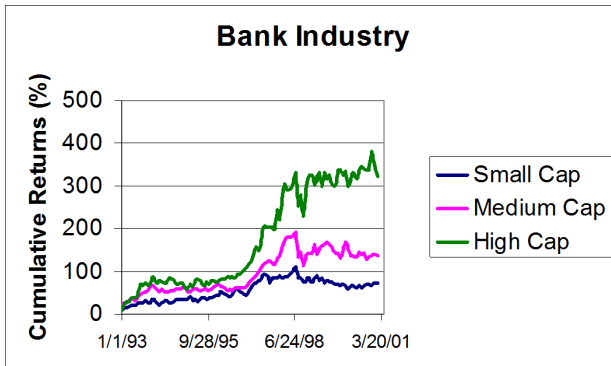
- Stocks in the same industry tend to move together.
- Example: European Banks



Source: BARRA

# Comovement

- However, there are other common factors that simultaneously affect returns.
- Within the banking industry, the size factor is at work



Source: BARRA

# Multiple factors and the CAPM

- The presence of multiple factors makes the CAPM a much more restrictive theory.
- Consider two sources of risk, “technology” and “monetary policy”.
  - ↪ Do all stocks respond the same way to technological innovation?
  - ↪ Do all stocks respond the same way to changes in interest rates?
- Assume that  $R_M = R_T + R_I$ ,
  - ↪ CAPM then implies that

$$E[R_i] = r_f + (E(R_m) - r_f) \beta_i$$

where

$$\beta_i \equiv \frac{\text{cov}(R_i, R_m)}{\text{var}(R_m)} = \frac{\text{cov}(R_i, R_T) + \text{cov}(R_i, R_I)}{\text{var}(R_m)}$$

- CAPM says that both covariances are priced exactly the same.

- The APT is an approach to determining asset values based on law of one price and no arbitrage.
- It is a multi-factor model of asset pricing.
- The APT is derived from a *statistical* model whereas the CAPM is an *equilibrium* asset pricing model.
  - ↪ To get the APT, we don't have to assume that *everyone* is optimizing.
  - ↪ Recall that we did need this assumption to get the CAPM.
  - ↪ This makes the APT a much more “reasonable” theory.
- Unlike the CAPM, we need very few assumptions to get the APT.

- The assumptions necessary for the APT are:
  - a) All securities have finite expected values and variances.
  - b) *Some* agents can form well diversified portfolios
  - c) There are no taxes
  - d) There are no transaction costs
- Notice that we have considerably fewer than with the CAPM!
- The central idea behind the APT will be that we can price some assets *relative* to other assets.
  - a) We will derive restrictions on the price of assets based on no-arbitrage.
  - b) We will be able to say something stronger if we exclude “near-arbitrage” or extremely good deals.

# Specifying Uncertainty

- We need a systematic way of thinking about uncertainty or risk in a dynamic setting.
- We will assume that investors know exactly which *states of nature* can occur, and exactly what will happen in each of the states.
- We know which states are possible, but not which will actually occur:
  - ↪ We only know what the probabilities of each of the states are.
  - ↪ You can think of this as meaning that we can always draw a multi-branched *tree* for the world.
- One implication is that investors know what the price of each and every security will be if the economy evolves in a particular way.
- With standard factor models, there are an infinite number of states of nature, since the factors can take on *any* values.

- Pricing restrictions in the APT come from the *Absence of Arbitrage*.
- Absence of Arbitrage in financial markets means that ***NO SECURITY EXISTS WHICH HAS A NEGATIVE PRICE AND A NON-NEGATIVE PAYOFF.***
  - ↪ Also, no security can be created which has this property.
- This rule also implies that:
  - a) Two securities that always have the same payoff must have the same price.
  - b) No security exists which has a zero price and a strictly positive payoff.
- This is the same idea that has been applied to option pricing!

# No Arbitrage

1. In an efficiently functioning financial market arbitrage opportunities should not exist (for very long).
2. Unlike *equilibrium* rules such as the CAPM, arbitrage rules require only that there just one intelligent investor in the economy.
  - ↪ This is why derivatives security pricing models do a better job of predicting prices than equilibrium based models.
  - ↪ In this lecture, we will see how to apply the same concept to pricing portfolios of assets.
3. If arbitrage rules are violated, then unlimited risk-free profits are possible.

## Arbitrage - Example

- Suppose that there are only two possible **states** for Inflation and Interest Rates: high or low
- We know *exactly* how four securities will perform in each of the possible states:

State/ Stock	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
<i>Int. Rate</i>	5%	5%	0%	0%
<i>Inflation</i>	10%	0%	10%	0%
<i>Prob.</i>	0.25	0.25	0.25	0.25
Apex (A)	-20	20	40	60
Bull (B)	0	70	30	-20
Crush (C)	90	-20	-10	70
Dreck (D)	15	23	15	36

## Arbitrage - Example

- Let's assume the prices of each of the four securities are \$100 and calculate expected returns, standard deviations and correlations:
- Everything looks normal here, but there is a simple **arbitrage opportunity** lurking in these numbers!

Stock	Current Price	Expect. Return(%)	Standard Dev. (%)	Correlation Matrix			
				A	B	C	D
A	100	25.00	29.58	1.00	-0.15	-0.29	0.68
B	100	20.00	33.91	-0.15	1.00	-0.87	-0.38
C	100	32.50	48.15	-0.29	-0.87	1.00	0.22
D	100	22.25	8.58	0.68	-0.38	0.22	1.00

## Arbitrage - Example

- Consider the return of an Equal-Weighted portfolio of A, B and C, and compare this with the return of D:

<i>State/ Port.</i>	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
EW Port. of A,B,C	23.33	23.33	20.00	36.67
Dreck (D)	15	23	15	36

- This table shows that the return of the EW portfolio is higher in all states, this means that there is an *arbitrage opportunity*.
- What would happen to the price of D in a well functioning market?

- Based on the arguments presented earlier, we will take the position that arbitrage opportunities cannot exist (for very long!). That means that there is something wrong with these prices.
- Our goal in this lecture is to come up with a model of security prices where:
  - a) If prices/returns obey this model, there is no arbitrage.
  - b) If prices/returns *fail to* obey this model, there *is* arbitrage.

- The previous example here is too restrictive. In reality:
  1. An infinite number of states are possible
  2. There are generally a large number of *factors*, and a *continuum* of possible factor realizations.
- The factor model framework gives us a systematic way of describing how expected security returns must relate to their comovement with the economy.
- If there is to no arbitrage in the economy, then we can also price assets relative to one another based on their comovement with these factors. *This is the basis of the APT.*

- First we need a **Factor Model** (or *Return Generating Process (RGP)*), which is a mathematical expression for how the security returns move with economic factors:
  - ↪ We will call the sources of movement **factors**
  - ↪ We will call the stock sensitivities to the factors **factor loadings** (or, equivalently, *factor betas* or *factor sensitivities*).
- We have already seen an example of such a model. It is the single-index model that we used to simplify the correlation structure between securities.

The **Factor Model** (or RGP) is:

$$\tilde{r}_i - E(\tilde{r}_i) = b_{i,1}\tilde{f}_1 + \dots + b_{i,K}\tilde{f}_K + \tilde{\epsilon}_i$$

- The  $f_i$ 's are common factors that affect most securities. Examples are economic growth, interest rates, and inflation.
- We require that, for each of the factors,  $E(\tilde{f}_i) = 0$ . This means that, instead of defining an  $f$  directly as economic growth, we would have to define it as the deviation of economic growth from what was expected.
- Often we assume  $cov(\tilde{f}_i, \tilde{f}_j) = 0$  for  $i \neq j$ .

The **Factor Model** (or RGP) can be written as:

$$\tilde{r}_i = E(\tilde{r}_i) + b_{i,1}\tilde{f}_1 + \dots + b_{i,K}\tilde{f}_K + \tilde{\epsilon}_i$$

- $b_{i,j}$  denotes the **loading** of the  $i$ 'th asset on the  $j$ 'th factor. This tells you how much the asset's return goes up when the factor is one unit higher than expected.
- The  $\tilde{\epsilon}_i$  in this equation is *idiosyncratic* risk.
  - ↪ For example,  $\tilde{\epsilon}_i$  will be negative when a firm's president dies, or a firm loses a big contract.
  - ↪ We will assume that  $cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$  for all securities  $i$  and  $j$ .

An example using this return generating process (*BKM*, chapter 10):

*Suppose that two factors have been identified for the U.S. economy: the growth rate of industrial production,  $IP$ , and the inflation rate,  $IR$ .  $IP$  is expected to be 4%, and  $IR$  6%. A stock with a beta of 1.0 on  $IP$  and 0.4 on  $IR$  currently is expected to provide a rate of return of 14%. If industrial production actually grows by 5%, while the inflation rate turns out to be 7%, what is your revised estimate of the realized return on the stock?*

## Factor Model - Example

- We know  $E(IP) = 4\%$  and  $b_{IP} = 1$ ,  $E(IR) = 6\%$ ,  $b_{IR} = .4$ , and  $E(r_i) = 14\%$
- The two factors are therefore:

$$f_{IP} = (0.05 - 0.04) = 0.01$$

$$f_{IR} = (0.07 - 0.06) = 0.01$$

- Plug these into the return generating process gives the expected return **conditional on** these realization of the industrial production growth rate (IP) and the inflation rate (IR):

$$\begin{aligned} E(\tilde{r}_i | f_{IP}, f_{IR}) &= 0.14 + 1 \cdot 0.01 + 0.4 \cdot 0.01 \\ &= 15.4\% \end{aligned}$$

↪ What about the idiosyncratic risk ( $e_i$ )?

- Using the return generating process we can calculate the variance of this portfolio as (for two factors):

$$\begin{aligned} \text{var}(R_i) &= \text{var}(b_{i,1}\tilde{f}_1 + b_{i,2}\tilde{f}_2 + \tilde{e}_i) \\ &= b_{i,1}^2 \text{var}(f_1) + b_{i,2}^2 \text{var}(f_2) + 2 \cdot b_{i,1} \cdot b_{i,2} \cdot \text{cov}(\tilde{f}_1, \tilde{f}_2) + \sigma_{e,i}^2 \\ &= b_{i,1}^2 \text{var}(f_1) + b_{i,2}^2 \text{var}(f_2) + \sigma_{e,i}^2 \quad (\text{if factors are uncorrelated}) \end{aligned}$$

the general formula for  $n$  factors:

$$\sigma_i^2 = \sum_{j=1}^n \sum_{k=1}^n b_{i,j} \cdot b_{i,k} \cdot \sigma_{j,k} + \sigma_{e,i}^2$$

where  $\sigma_{j,k}$  denotes the covariance between the  $j$ 'th and  $k$ 'th factors. (What is  $\sigma_{2,2}$ ?)

1. The *systematic* variance is  $\sum_{j=1}^n \sum_{k=1}^n b_{i,j} \cdot b_{i,k} \cdot \sigma_{j,k}$ .
2. The *idiosyncratic* variance is  $\sigma_{e,i}^2$ .

- Under the factor model, the covariance between two stocks,  $i$  and  $j$  (for two factors)

$$\begin{aligned} \text{cov}(R_i, R_j) &= \text{cov}(b_{i,1}f_1 + b_{i,2}f_2 + e_i, b_{j,1}f_1 + b_{j,2}f_2 + e_j) \\ &= b_{i,1}b_{j,1}\text{var}(f_1) + b_{i,2}b_{j,2}\text{var}(f_2) + \\ &\quad + (b_{i,1}b_{j,2} + b_{j,1}b_{i,2})\text{cov}(f_1, f_2) \\ &= b_{i,1}b_{j,1}\text{var}(f_1) + b_{i,2}b_{j,2}\text{var}(f_2) \\ &\quad \text{(if factors are uncorrelated)} \end{aligned}$$

- Examples of factor models:
  - a) The Fama-French 3-factor model of returns.
  - b) The 3-factor model of the term-structure (level, slope, curvature).
- The APT will give a *theoretical* justification for the use of empirical factor models in determining the “fair” rate of return.
- Multi-Factor Models vs the CAPM
  - ↪ Does a well-diversified portfolio have any idiosyncratic risk?
  - ↪ How do our “APT” definitions of a systematic and idiosyncratic risk differ from the CAPM definitions?
  - ↪ How does our “APT” definition of a well-diversified portfolio differ from the CAPM definition?

# Definitions

- In the context of the APT, a diversified portfolio is a portfolio that carries no idiosyncratic risk:

$$\tilde{r}_p - E(\tilde{r}_p) = b_{p,1}\tilde{f}_1 + \dots + b_{p,K}\tilde{f}_K$$

- ↪ This is defined *relative to a specific factor model*.
- ↪ We will assume that investors can form such diversified portfolios.

- In the context of the CAPM, you may see different definitions:

1. A diversified portfolio will have the *least* variance for a given level of expected returns.
  - ▶ ALL minimum-variance efficient portfolios are diversified according to this definition.
2. A diversified portfolio will have zero idiosyncratic risk, i.e. it will have an  $R^2$  with the market.
  - ▶ ONLY the market portfolio (and combinations of this with the riskless asset, i.e. portfolios on the CAL) will be diversified according to this definition.

- The APT Pricing Equation is:

$$E(\tilde{r}_i) = \lambda_0 + \lambda_1 b_{i,1} + \dots + \lambda_K b_{i,K}$$

- This equation specifies that the relation between the expected return of a security and its factor loadings (*bs*) is *linear*.
  - ↪ The  $\lambda$ s or *factor risk premia* tell you how much extra return you get for each extra unit of risk your portfolio has.
  - ↪ Note that there is one  $\lambda$  for each factor in the economy, plus one extra:  $\lambda_0$ .
  - ↪ If there is a risk-free asset, then it must be the case that portfolios with no risk (with all *bs* equal to zero) have a return of the risk-free rate, *i.e.*, that  $\lambda_0 = r_f$ .
- How does it compare to the CAPM?

- Here we will use a simple example to understand how investors price risky securities.
- This will allow us to better understand what each of the the elements of the Arbitrage Pricing Theory.
- We will try to better understand what **Risk** means in the APT context, and to try to better define and understand the following terms:
  - ↪ **Factor**
  - ↪ **Factor Loading**
  - ↪ **Factor Risk-Premia**
  - ↪ **Arbitrage**
- To start out, let's look again at how we specify uncertainty in the APT setting.
  - ↪ Note that this is the specification used in most of finance and economics, including in derivatives securities pricing.

- Suppose that we know that IBM is going to pay a liquidating dividend in exactly one year, and this is the only payment that it will make.
- However, that dividend that IBM pays is uncertain – its size depends on how well the economy is doing.
  - ↪ If the economy is in an expansion, then IBM will pay a dividend of **140**.
  - ↪ However, if the economy is in a recession, then its dividend will be only **100**.
  - ↪ If the two states are equally likely, the expected cash flow from IBM is  $E(CF_1^{\text{IBM}}) = 120$ .
- Note that, consistent with our definition of uncertainty, we know exactly what will happen to IBM *in each scenario*, but we don't know which scenario will occur.

## APT - Example

	IBM
Boom Payoff (Pr=0.5)	140
Bust Payoff (Pr=0.5)	100
$E(CF_1)$	120
Time 0 Price	100
Discount Rate	20%

- Assuming that the price of IBM is \$100, we see that investors in this economy are applying a discount rate of 20% to IBM's expected cash-flows.
- The way they come up with the price of IBM is to take the expected cash flow at time 1 from IBM,  $E(CF_1^{IBM}) = \$120$ , and discount this cash-flow back to the present at the "appropriate" discount rate, which is apparently 20%.
- Equivalently, we can say that the expected return of IBM is 20%.

## APT - Example

- Now let's consider a second security, DELL which, like IBM, will pay a liquidating dividend in one year, and which has only two possible cash flows, depending on whether the economy booms or goes into a recession over the next year.

	IBM	DELL
Boom Payoff (Pr=0.5)	140	160
Bust Payoff (Pr=0.5)	100	80
$E(CF_1)$	120	120
Time 0 Price	100	?
Discount Rate	20%	?

- Now let's consider how investors will "price" DELL.
- Note the IBM and DELL have the same expected cash-flow, so one might guess that a reasonable price for DELL might also be \$100.

- However, even though the expected cash-flows for DELL are the same, we see that the pattern of cash flows across the two states are probably worse for DELL:
- DELL's payoff is lower in the bust/recession state, which is when we are more likely to need the money. It only does better when things are good.
  - ↪ The recession is when the rest of our portfolio is more likely to do poorly, and when we are more likely to lose our job, and our consulting income is likely to be lower. We “need” the cash more in a recession
  - ↪ In the boom/expansion, our portfolio will probably do better; we're more likely to have a good job; we'll probably have more consulting income.
- This means that if DELL were the same price as IBM, we would buy IBM.

## APT - Example

- Therefore, to induce investors to buy all of the outstanding DELL shares, it will have to be the case that DELL's price is lower than \$100.
- Also, note that the three statements are equivalent:
  - ↪ The price investors will pay for DELL will be less than \$100, even though DELL's expected cash flows are the same as IBM's.
  - ↪ The discount rate investors will apply to DELL's cash flows will be higher than the 20% applied to IBM's cash flows.
  - ↪ The expected return investors will require from DELL will be higher than the IBM's expected return of 20%.
- Let's assume that investors are only willing to buy up all of DELL's shares if the price of DELL is \$90, or, equivalently, that the discount rate that they will apply to DELL is 33.33%:

$$E(R_{DELL}) = \frac{E(CF_1^{DELL}) - P_{DELL}}{P_{DELL}} = \frac{120 - 90}{90} = \frac{30}{90} = 0.3333$$

## APT - Example

	IBM	DELL
Boom Payoff (Pr=0.5)	140	160
Bust Payoff (Pr=0.5)	100	80
$E(CF_1)$	120	120
Time 0 Price	\$100	\$90
Discount Rate	20%	33.33%

- The way to think about expected returns is that this is something that investors determine.
- After looking at the pattern of cash-flows from any investment, and deciding whether they *like* or *dislike* this pattern, investors determine what rate they will discount these cash flows at (to determine the price.)
- This rate is what we then call the “expected return.”
- Since investors accurately calculate the expected cash-flows, the average return they realize on the investment *will be* the discount rate.

Now let's see how all of this relates to the APT equations:

1. Calculating the business cycle factor  $f_{BC}$  in the two states:

- ↪ First, assume that we use the NBER (National Bureau of Economic Research) business cycle indicator to construct our factor.
  - ▶ The indicator is one (at the end of the next year) if the economy is in an expansion, and zero if the economy is in a recession.
- ↪ However, remember that for the factor, we need the *unexpected component* of the business cycle.
- ↪ Assuming there is a 50%/50% chance that we will be in an expansion/recession, the expected value of the indicator is 0.5.
- ↪ This means that the business-cycle factor has a value of  $0.5 = 1 - 0.5$  if the economy booms, and  $-0.5 = 0 - 0.5$  if the economy goes bust.

## 2. Calculating the factor loadings for the two securities:

- ↪ The way of doing this is to run a time-series regression of the returns of IBM on the factor. Let's do this first for IBM:

$$r_{IBM,t} = E(r_{IBM}) + b_{IBM,BC} \cdot f_{BC,t} + e_{IBM,t}$$

where  $E(r_{IBM})$  is the intercept and  $b_{IBM,BC}$  is the slope coefficient.

- ↪ Here, we have only two points, so we can fit a line exactly:

$$0.40 = E(r_{IBM}) + b_{IBM,BC} \cdot 0.5 \quad (\text{boom})$$

$$0.00 = E(r_{IBM}) + b_{IBM,BC} \cdot -0.5 \quad (\text{bust})$$

- ↪ Solving these gives  $E(r_{IBM}) = 0.20$  (which we already knew) and  $b_{IBM,BC} = 0.4$ .
- ↪ Similar calculations for DELL gives  $E(r_{DELL}) = 0.3333$  and  $b_{DELL,BC} = 0.8889$ .

### 3. Interpreting the factor loadings.

- ↪ The factor loadings  $b_{IBM,BC}$  and  $b_{DELL,BC}$  tell us how much *risk* IBM and DELL have.
- ↪ Risk means that the security moves up or down when the factor (the business cycle) moves up and down.
- ↪ Based on the way that we have characterized uncertainty, the expected return and the factor loading  $b_{i,BC}$  of each security tell us everything that we need to know to calculate all of the payoff, per dollar, in every state of nature.
  - a) This will be true for every well-diversified portfolio, in every factor model.
  - b) One way of thinking about this is that the expected return tells us what the reward is, and the  $b$ 's tell us what the risk of the security is.

## 4. Calculating the factor risk premia ( $\lambda$ 's)

- ↪ The Factor Model tells us nothing about why investors are discounting the cash flows from the different securities at different rates.
- ↪ To determine how investors view the risks associated with each of the risks in the economy, we have to evaluate the **APT Pricing Equation**.
- ↪ Now that we have the factor loadings, we can determine the factor risk-premia (or  $\lambda$ 's), by regressing expected returns  $E r_i$  on factor loadings  $b_{ik}$ .
- ↪ Since there is only one factor and two stocks

$$\begin{aligned} E(r_{IBM}) &= \lambda_0 + \lambda_{BC} \cdot b_{IBM,BC} \\ E(r_{DELL}) &= \lambda_0 + \lambda_{BC} \cdot b_{DELL,BC} \end{aligned}$$

to which the solutions are  $\lambda_0 = 0.0909$  and  $\lambda_{BC} = 0.2727$ .

## 5. Interpreting the risk-premia ( $\lambda$ s):

- ↪ DELL is considered to be a riskier security than IBM, in the sense that investors discount DELL's cashflows at a higher rate than IBM.
- ↪ This is reflected in DELL's higher loading on the BC factor.
- ↪ The  $\lambda_{BC}$  is a measure of how much more investors discount a stock as a result of having one extra unit of risk relating to the BC factor.
- ↪  $\lambda_0$  tells you how high a return investors require if a security has no risk.

## APT - Example

- Using the set of securities that we used to calculate the  $\lambda$ 's for the pricing equations, we can always construct a portfolio  $p$  with *any* set of factor loadings ( $b_{p,k}$ ).
- The pricing equation then tells us what the expected return (or discount rate) for the portfolio of securities is.
- Alternatively, this equation tells us, if we find a (well-diversified) portfolio with certain factor loadings, what discount-rate that portfolio must have for there to be no arbitrage.
- Finally, if we add a third security to the mix and calculate its factor loadings, the APT will tell us what its expected return **must be** in order to avoid arbitrage opportunities.
- Essentially what we are doing is pricing securities *relative* to other securities, in much the same way we are pricing derivatives relative to the underlying security.

- Arbitrage arises when the price of risk differs across securities.
  1. If investors are pricing risk inconsistently across securities, then, assuming these securities are well diversified, arbitrage will be possible.
  2. To see this, let's extend the example to include a risk-free asset which we can buy or sell at a rate of 5%.
    - ▶ That is the rate for borrowing or lending is 5%.
  3. Since  $\lambda_0 = 0.0909$ , we know that we can combine DELL and IBM in such a way that we can create a *synthetic* risk-free asset with a return of 9.09%.
  4. So we borrow money at 5% (by selling the risk-free asset) and lend money at 9.09%

# Building the Arbitrage Portfolio

- To determine how much of IBM and DELL we buy/sell, we solve the equation for the weights on IBM and DELL in a portfolio which is risk-free:

$$w_{IBM} \cdot b_{IBM,BC} + (1 - w_{IBM}) \cdot b_{DELL,BC} = b_{p,BC} = 0$$

or

$$w_{IBM} = \frac{b_{DELL,BC}}{b_{DELL,BC} - b_{IBM,BC}} = 1.8182$$

and  $w_{DELL} = (1 - w_{IBM}) = -0.8182$ .

- This means that, to create an arbitrage portfolio, we can invest \$1.8182 in IBM, short \$0.8182 worth of DELL, and short \$1 worth of the risk-free asset.
- This portfolio requires *zero* initial investment and has a positive payoff in *all* states.

# Building the Arbitrage Portfolio

- To verify that this works, let's look at the payoff to the IBM/DELL risk-free portfolio in the two states:

$$\text{Payoff(boom)} = w_{IBM} \cdot (140/100) + (1 - w_{IBM}) \cdot (160/90) = 1.0909$$

$$\text{Payoff(bust)} = w_{IBM} \cdot (100/100) + (1 - w_{IBM}) \cdot (80/90) = 1.0909$$

This means that the payoff from the zero-investment portfolio is **\$0.0409** = 1.0909 – 1.05 in both the boom and bust states. So this portfolio (in this simple economy) is risk free.

- Of course, we can scale this up as much as we like. For \$1 million investment in the long and short portfolios, we would get a risk-free payoff of \$40,900 (assuming prices did not move with our trades)

# Interpreting the Arbitrage

- What is going on here is that investors are pricing risk inconsistently across securities.
- With any pair of securities, we can calculate  $\lambda_0$  and  $\lambda_{BC}$ , however, each pair of securities will give you a different set of  $\lambda$ 's:

Security 1	Security 2	$\lambda_0$	$\lambda_{BC}$
IBM	DELL	0.0909	0.2727
IBM	RF	0.05	0.3750 (= (0.2 - 0.05)/0.4)
DELL	RF	0.05	0.3187 (= (0.3333 - 0.05)/0.8889)

- This means that, to find the arbitrage, we could have
  1. calculated the  $\lambda$ 's using any pair of securities, and then
  2. calculated the expected return (or discount rate) for the third security
  3. bought the high return and sold the low return.

# Interpreting the Arbitrage

- Whenever this happens, there will be an arbitrage.
- If there are arbitrageurs in the economy, they will move prices until the arbitrage disappears, and risk is priced consistently.
- This is the argument behind the APT.
- Note that this assumes however that the arbitrageurs have unlimited capital and patience. We will see later that in some cases that this may be an erroneous assumption and in fact, there may be *limits to arbitrage*.

What have we learned?

1. Here, we have used a simple example to understand how investors price risky securities.
2. The idea behind the APT is that investors require different rates of return from different securities, depending on the riskiness of the securities.
3. If, however, risk is priced inconsistently across securities, then there will be arbitrage opportunities.
4. Arbitrageurs will take advantage of these arbitrage opportunities until prices are pushed back into line, risk is priced consistently across securities, and arbitrage disappears.

We have defined the following terms:

- **Factors** ( $\tilde{f}$ 's) move up and down with the economy, and affect the future cash flows of securities.
- **Factor loadings** ( $b$ 's) for each security, for each factor, tell you how much the security moves (on average, in percent) when the factor moves by 1%.
- **Factor risk-premia:** ( $\lambda$ 's) for each factor, tell you how much higher a discount-rate investors apply to a security if its factor loading on a particular factor is higher by one.
- **Arbitrage** arises if risk is priced inconsistently across (well-diversified) securities.
  - ↪ One way of thinking about this is that we can find two well diversified securities/portfolios, with exactly the same risk/factor-loadings, which have their cash flows discounted at different discount rates.

- The APT is an alternative model to the CAPM.
- It makes fewer assumptions, and as a result gets weaker predictions.
- In particular, the APT *does not say* what the systematic factors are, whereas the CAPM says that the market portfolio is the only systematic source of risk.
- Later: How to specify the factors?
  - a) Factors can be specified a priori: they could be macroeconomic variables (ex inflation, output) that capture the systematic risk in the economy or portfolios proxying for these risks.
  - b) Factors can be extracted via Principal Components or Factor Analysis.