Lecture 2: Equilibrium

Investments

Overview

1. Introduction
2. Assumptions
3. The Market Portfolio
4. The Capital Market Line
5. The Security Market Line
6. Conclusions

Introduction: Markowitz vs. the CAPM

- Mean-Variance Analysis selects a portfolio, *given* the expected returns and covariances.
- To operationalize MVA we need estimates.
- Expected returns are very hard to measure, so a model of what returns *should* be will be useful.
- The CAPM is an *equilibrium* model specifying a relation between expected rates of return and covariances for all assets.
  - ↪ *Equilibrium* is an economics term that characterizes a situation where no investor wants to do anything differently.
  - ↪ Note that the Markowitz portfolio problem is relevant for each investor, regardless of whether the *equilibrium* argument, and the CAPM, is correct or not.

Equilibrium Pricing

- The approach we will take is to ask:
  if everyone in the economy holds an efficient portfolio, then how should securities be priced so that they are actually bought 100% in equilibrium?
  - ↪ For example, if based on the prices/expected returns our model comes up with, we found that no maximizing investor would like to buy IBM, then something is wrong.
  - ↪ IBM would be priced too high (offer too low an expected rate of return).
  - ↪ The price of IBM would have to fall to the point where, in aggregate, investors want to hold exactly the number of IBM shares outstanding.
- So, what sort of prices (risk/return relationships) are feasible in equilibrium? The CAPM will give an answer.

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CAPM and the data

- Whether the CAPM holds in the data is a horly debated topic among academics.
- Even if the CAPM is wrong empirically, there are several reasons we want to spend some time with it:
  1. If it is wrong, this means that we can ”do better” than the market portfolio (assuming that expected return and standard deviation are what matter for us).
  2. MV Analysis and the CAPM gives you a *framework* to think about risk and return. Even though CAPM may not hold, because it is too simple, the intuition it is very powerful and will form the basis for the more advanced asset pricing models that we will look at.

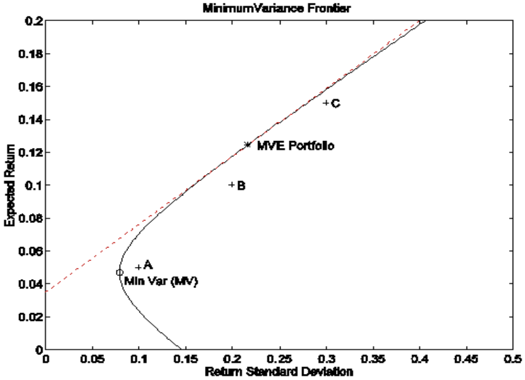
The CAPM Assumptions

- A number of assumptions are necessary to formally derive the CAPM relationship. Some of these can be relaxed without too-much effect on the results.
  1. No transaction costs.
  2. Assets are all tradable and are all infinitely divisible.
  3. No taxes.
  4. No individual can effect security prices (perfect competition).
  5. Investors care only about expected returns and variances:
     - returns are normally distributed.
     - all investors have a quadratic utility function.
  6. Unlimited short sales and borrowing and lending.
  7. Homogeneous expectations.
- Assumptions 5 - 7 imply everyone solves the passive portfolio problem we just finished, and they all see the same efficient frontier!

Two fund seperation

What do we know from the passive portfolio problem?

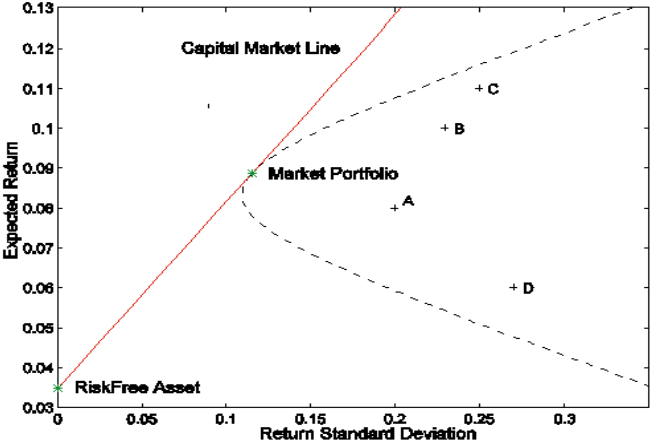
1. Everyone holds a linear combination of two portfolios
   - ↪ the risk-free security
   - ↪ the tangency portfolio
2. If everyone sees the same CAL, then everyone has the same tangency portfolio.



- What is the tangency portfolio?
  1. Markowitz: Investors *should* hold the tangency portfolio.
  2. Equilibrium Theory (market clearing):
     - The risk-free asset is in zero supply:
       Borrowing and Lending must cancel out.
     - Average Investor *must* want to hold the market portfolio.
  3. CAPM: the tangency portfolio **must be the market portfolio**.

- What do we mean by the "market" portfolio?

  ↪ **Definition:** The "market" or total wealth portfolio is a portfolio of **all** risky securities held in proportion to their market value. This must be the sum over all securities, i.e. stocks, bonds, real-estate, human capital, etc.



Our assumptions and results from the last section imply that, in equilibrium, every investor faces the same CAL.

What about Individual Assets?

- This CAL is called the *Capital Market Line* (CML). This line gives us the set of efficient or optimal risk-return combinations

$$E(\tilde{r}_e) = r_f + \left(\frac{E(\tilde{r}_m) - r_f}{\sigma_m} \right) \sigma_e$$

where $\tilde{r}_e$ is the return on any *efficient* portfolio.

  ↪ What is an efficient portfolio?

- Note that this says that all investors should only hold combinations of the market and the risk-free asset.

  ↪ How does this relate to the increased popularity of index funds?

- Now let's ask the question of what we can say about *inefficient portfolios* (or individual assets). Can we say anything about their expected returns in equilibrium?

What about Individual Assets?

- Investors will only want to hold a security in their portfolio if it provides a reasonable amount of extra reward (*i.e.*, expected return) in return for the risk (or variance) it adds to the portfolio
- **Preview:** For all securities, what a security adds to the risk of a portfolio will be just offset by what it adds in terms of expected return. The ratio of *marginal return* to *marginal variance* must be the same for all assets.
  - ↪ What it adds in expected return is its expected excess return.
  - ↪ What it adds in risk is proportional to its *covariance* with the portfolio.
  - ↪ This is the intuition for the standard form of the CAPM, which relates β to expected return.

The Security Market Line

- How does adding a small amount of security to the market portfolio affect its variance?

$$
\begin{aligned}
\sigma_m^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov(\tilde{r}_i, \tilde{r}_j) \\
&= \sum_{i=1}^{N} w_i \left[\sum_{j=1}^{N} w_j cov(\tilde{r}_i, \tilde{r}_j)\right] \\
&= \sum_{i=1}^{N} w_i cov\left(\tilde{r}_i, \underbrace{\left[\sum_{j=1}^{N} w_j \tilde{r}_j\right]}_{\text{return on the market}}\right)
\end{aligned}
$$

↪ What matters in determining the *marginal* increase in risk when you change the amount of a security in your portfolio is the **covariance with the portfolio return**.

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A formal derivation of the CAPM

Let's do this more formally:

1. Remember that we showed that, under our assumptions, all investors must hold the market portfolio.
2. Based on our notion of equilibrium, every investor must be content with their portfolio holdings; if this were not the case than the prices of the securities would have to change.
   - ↪ This is just a supply and demand argument; if some investors want to buy IBM, and no one wants to sell, prices will have to change (move up).
   - ↪ In equilibrium, everyone must be optimally invested.
3. This must mean that, in equilibrium no one can do anything to increase the Sharpe-ratio of their portfolio.

A formal derivation of the CAPM

Suppose you currently hold the market portfolio decide to invest a small additional fraction δ_{GM} of your wealth in GM, which you finance by borrowing at the risk free rate.

1. The return will become:

$$\tilde{r}_c = \tilde{r}_m - \delta_{GM} \cdot r_f + \delta_{GM} \cdot \tilde{r}_{GM}$$

2. So the expected return and variance will be:

$$\begin{aligned} E(\tilde{r}_c) &= E(\tilde{r}_m) + \delta_{GM} \cdot (E(\tilde{r}_{GM}) - r_f) \\ \sigma_c^2 &= \sigma_m^2 + \delta_{GM}^2 \cdot \sigma_{GM}^2 + 2 \cdot \delta_{GM} \cdot cov(\tilde{r}_{GM}, \tilde{r}_m) \end{aligned}$$

3. The changes in each of these are:

$$\begin{aligned} \Delta E(\tilde{r}_c) &= \delta_{GM} \cdot E(\tilde{r}_{GM} - r_f) \\ \Delta\sigma_c^2 &= 2 \cdot \delta_{GM} \cdot cov(\tilde{r}_{GM}, \tilde{r}_m) \end{aligned}$$

   where Δ denotes *the change in*. We ignore the δ_{GM}^2 term in the variance equation because, if δ is small (say 0.001), δ^2 must be so small that we can ignore it (0.000001).

A formal derivation of the CAPM

Now what if we invest δ more in GM, and invest just enough less in the IBM so that our portfolio variance stays the same.

1. The change in the variance is:

$$\Delta\sigma_c^2 = 2 \cdot (\delta_{GM} \cdot cov(\tilde{r}_{GM}, \tilde{r}_m) + \delta_{IBM} \cdot cov(\tilde{r}_{IBM}, \tilde{r}_m))$$

2. To make this zero, it must be the case that:

$$\delta_{IBM} = -\delta_{GM} \left(\frac{cov(\tilde{r}_{GM}, \tilde{r}_m)}{cov(\tilde{r}_{IBM}, \tilde{r}_m)} \right)$$

3. The change in the expected return of the portfolio will be:

$$\begin{aligned} \Delta E(\tilde{r}_c) &= \delta_{GM} \cdot E(\tilde{r}_{GM} - r_f) + \delta_{IBM} \cdot E(\tilde{r}_{IBM} - r_f) \\ &= \delta_{GM} \left[E(\tilde{r}_{GM} - r_f) - E(\tilde{r}_{IBM} - r_f) \left(\frac{cov(\tilde{r}_{GM}, \tilde{r}_m)}{cov(\tilde{r}_{IBM}, \tilde{r}_m)} \right) \right] \end{aligned}$$

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A formal derivation of the CAPM

- Remember that, we are holding the market portfolio, which is also the tangency portfolio.
- This portfolio has the highest Sharpe Ratio of *all* portfolios.
- Therefore, by definition, we **cannot** increase its expected return while keeping the variance constant.
- For this to be true it must be that:

$$\frac{E(\tilde{r}_{GM}) - r_f}{cov(\tilde{r}_{GM}, \tilde{r}_m)} = \frac{E(\tilde{r}_{IBM}) - r_f}{cov(\tilde{r}_{IBM}, \tilde{r}_m)} = \lambda$$

- λ is the ratio of the marginal benefit to the marginal cost.

- Note that this also holds for portfolios of assets as well.
- We can use the market portfolio in place of IBM:

$$\frac{E(\tilde{r}_{GM})-r_f}{cov(\tilde{r}_{GM},\tilde{r}_m)}=\frac{E(\tilde{r}_m-r_f)}{cov(\tilde{r}_m,\tilde{r}_m)}=\frac{E(\tilde{r}_m-r_f)}{\sigma_m^2}=\lambda$$

which means that:

$$\begin{aligned} E(\tilde{r}_{GM})-r_f &= \frac{E(\tilde{r}_m-r_f)}{\sigma_m^2}cov(\tilde{r}_{GM},\tilde{r}_m) \\ &= E(\tilde{r}_m-r_f)\underbrace{\frac{cov(\tilde{r}_{GM},\tilde{r}_m)}{\sigma_m^2}}_{\beta_{GM}} \end{aligned}$$

- This characterizes the **Security Market Line(SML)**.

Summary

- By the definition of the tangent portfolio, investors should not be able to achieve a higher return/risk tradeoff (Sharpe Ratio) by combining the tangent portfolio with *any* other asset.
- This restriction implies a linear relationship between an asset's equilibrium return and its beta with the tangent portfolio:

$$E(\tilde{r}_i) - r_f = E(\tilde{r}_T - r_f)\beta_i$$

- The CAPM is the statement, that **in equilibrium**, the tangent portfolio is the market portfolio ($\tilde{r}_T = \tilde{r}_M$.
- One way to interpret this equation is as saying that the reward ($E(\tilde{r}_i) - r_f$) must equal the amount of risk that is priced (β), times its price ($E\tilde{r}_M - r_f$)

SML - Example

Asset	$E(r)$	σ
A	8%	20%
B	10%	23%
C	11%	25%
D	6%	27%

Correlations				
Assets	A	B	C	D
A	1.0	0.0	0.3	-0.2
B	0.0	1.0	0.2	-0.2
C	0.3	0.2	1.0	-0.2
D	-0.2	-0.2	-0.2	1.0

- We also assume that $r_f = 3.5\%$.
- The resulting MVE portfolio has weights on the risky assets of:

$$w_{MVE} = \begin{bmatrix} 0.2515 \\ 0.3053 \\ 0.2270 \\ 0.2161 \end{bmatrix}$$

- CAPM: MVE portfolio is the market.

SML - Example

1. Calculate the βs for each of the four assets:

$$\beta_i = \frac{cov(r_i, r_m)}{\sigma_m^2}$$

   1.1 use the equation for the covariance of two portfolios

$$\begin{aligned} cov(r_A, r_m) &= cov(r_A, w_A r_A + w_B r_B + w_C r_C + w_D r_D) \\ &= w_A cov(r_A, r_A) + w_B cov(r_A, r_B) \\ &\quad + w_C cov(r_A, r_C) + w_D cov(r_A, r_D) \end{aligned}$$

   1.2 use the equation for the variance of a portfolio:

$$\begin{aligned} \sigma_m^2 &= \sum_{i=A}^{D} \sum_{j=A}^{D} w_i w_j cov(r_i, r_j) \\ &= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + w_C^2 \sigma_C^2 + w_D^2 \sigma_D^2 + \underbrace{2 w_A w_B cov(r_A, r_B) + \ldots}_{\text{6 covariance terms}} \end{aligned}$$

2. Plot expected excess returns vs βs (SML).

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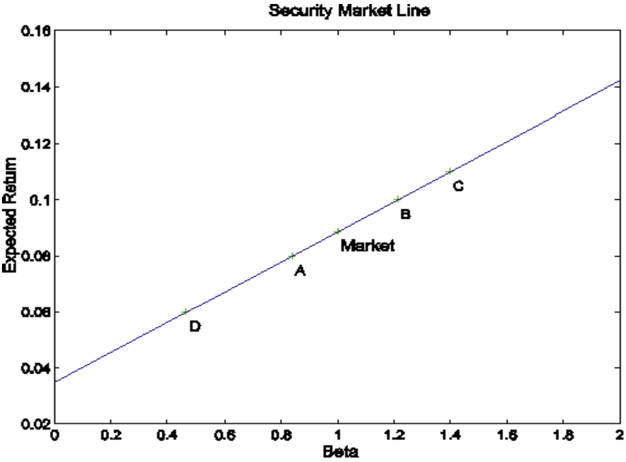
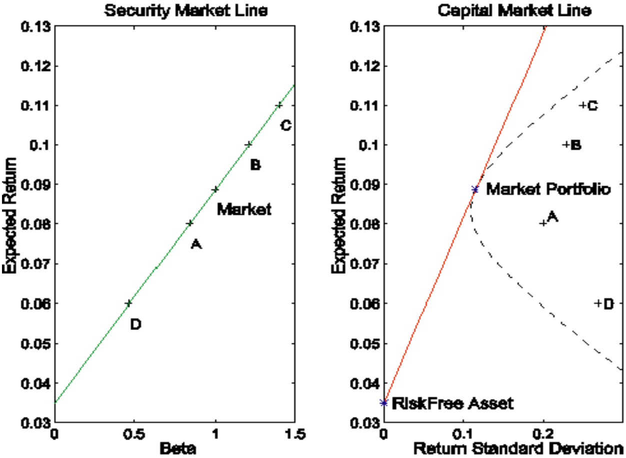
SML - Example



- Expected returns of all assets lie on the SML!
- What is the difference with the CAL?

SML vs CML



1. Every security lies on the SML
2. **Only** the market and the risk-free asset lie on the CML.
3. SML plots rewards vs systematic risk.
4. CML plots rewards vs total risk (systematic + unsystematic).

Systematic vs Idiosyncratic Risk

Interpreting β as a Regression Coefficient

- Consider the regression equation:

$$\tilde{r}_i^e = \alpha_i + \beta_i \tilde{r}_m^e + \tilde{\varepsilon}_{i,t}$$

- The OLS regression coefficient for this regression is

$$\beta_i = \frac{cov(\tilde{r}_i^e, \tilde{r}_m^e)}{\sigma_m^2},$$

  the same as the definition of β_i we developed earlier!

- Going back to regression analysis, the part of $\tilde{r}_i^e$ that is "explained" by the market return is $\beta_i \tilde{r}_m^e$. This part is the *systematic* or *market* risk of the asset.
- The part of $\tilde{r}_i^e$ that is not explained by the market return is $\tilde{\varepsilon}_{i,t}$. This part provides the *idiosyncratic* risk of the asset.

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Systematic vs Idiosyncratic Risk

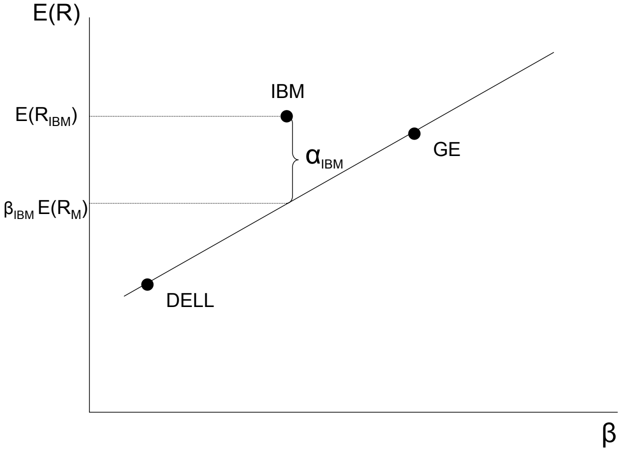
- We can decompose the total variance:
  1. The systematic variance is $\beta_i^2\sigma_m^2$
  2. The unsystematic variance is σ_ε^2
- The CAPM implies that only the systematic component is *priced*.
  - $\hookrightarrow$ Note that the total return variance of i is $\beta_i^2\sigma_m^2 + \sigma_\varepsilon^2$
  - $\hookrightarrow$ (Why is the correlation between the two parts equal to zero?)
- Why don't investors care about unsystematic risk?
- What is the R^2 of the regression?

Definitions

- In the context of the CAPM, diversification and idiosyncratic risk can be confused. In different textbooks or courses you might see two separate definitions:
  1. A diversified portfolio will have the *least* variance for a given level of expected returns.
     - ALL minimum-variance frontier portfolios are diversified according to this definition.
     - These portfolios also contain "idiosyncratic" risk, defined as risk that is uncorrelated with the market.
  2. A diversified portfolio will have zero "idiosyncratic risk", i.e. $\sigma_\varepsilon^2 = 0$ with the market.
     - ONLY the market portfolio (and combinations of this with the riskless asset, i.e. portfolios on the CAL) will be diversified according to this definition.
- Bottom Line: Make sure you define first what you mean when you refer to diversified portfolios.

Deviations from the CAPM

- What must α_i be, according to the CAPM?
- α_i denotes the deviation of a security from the SML.

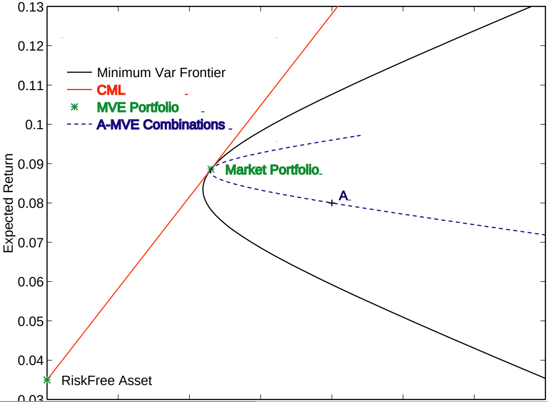


Summary

- The CAPM is an equilibrium model that states what an asset's expected return should be in *equilibrium*.
- The CAPM is a useful first model. In the next lecture we will see how to use the CAPM to aid our asset allocation decision.
- The CAPM need not be literally true to be useful. Even if it does not perform very well empirically, there are extensions to it that do. These extensions are based on the same intuition.

A graphical approach to understanding the CAPM

- Our derivation of the CAPM implies that a combination of any asset and the market must be efficient.
- We should be able to see this graphically. First, let's look at the possible combinations of the MVE portfolio (the market if the CAPM is true) and Asset A



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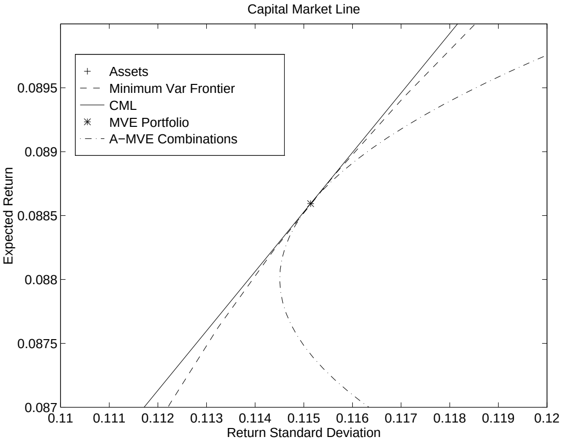
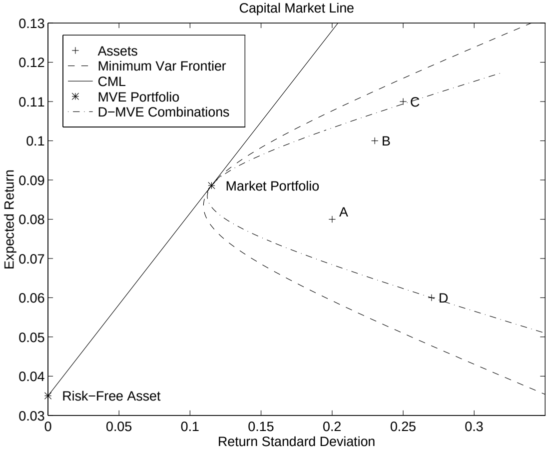
A graphical approach to understanding the CAPM

- It looks like all combinations fall inside the MVE frontier. The only way this can be the case is if the marginal return/variance is the same as for the market, that is if the combination line is tangent to the CML.



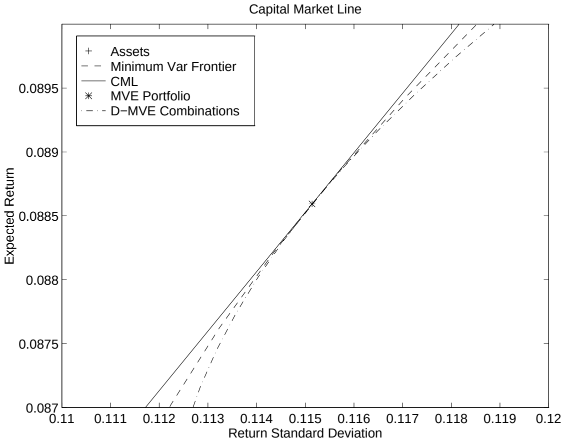
A graphical approach to understanding the CAPM

- These plots show the possible combinations of the MVE portfolio and asset D. We get the same tangency condition.
- The “combination” curve must be tangent to the CML for every asset.



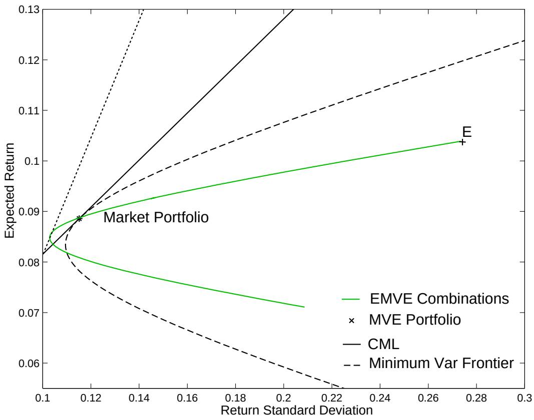
A graphical approach to understanding the CAPM

- Unless the ratio of marginal return to marginal variance is identical for all securities (and the market), investors will be unhappy holding the market portfolio, and prices will have to adjust to get to equilibrium.



A graphical approach to understanding the CAPM

- What will these curves look like if the CAPM does not hold?
- If the expected return on new asset E is less than predicted by the CAPM, then the possible portfolios are:



A graphical approach to understanding the CAPM

- Alternatively, if the expected return on new asset E is greater than predicted by the CAPM,