

RISK SHARING AMONG CONSTANT RISK-TOLERANT INVESTORS

Here we study examples to illustrate two basic facts about optimal risk sharing among individuals who have constant risk tolerance. (1) When such individuals can form an investing partnership, they should share their risky returns in proportion to their individual risk tolerances, because such a proportional sharing rule will maximize the sum of their individual certainty equivalents. (2) When such partners share a risky investment in this optimal way, the sum of their certainty equivalents will be the same as the certainty equivalent of the whole investment for a person whose risk tolerance equals the sum of the partners' risk tolerances.

Example 1: sharing a Normal lottery.

Let us start with a specific example. Suppose that there are two individuals, numbered 1 and 2, who have constant risk tolerance indexes $RT_1 = \$4000$ and $RT_2 = \$6000$ respectively. Together, these two individuals can buy an asset that will pay a monetary return which will be drawn from a normal probability distribution with mean $\mu = \$17,000$ and standard deviation $\sigma = \$9000$. They must decide whether to buy the asset, and if so, how to divide the returns among themselves. Recall that, when an individual with constant risk tolerance RT gets monetary returns drawn from a Normal distribution, his risk premium is $0.5/RT$ times the square of the standard deviation of his monetary returns. As always, the individual's certainty equivalent is his expected monetary value minus his risk premium.

If individual 1 were to hold the asset himself, then he would not be willing to pay more than his certainty equivalent:

$$\mu - (.5/RT_1)*(\sigma^2) = 17,000 - (.5/6000)*(9000^2) = 17,000 - 10,125 = \$6875.$$

If 2 were to hold the asset herself, then she would not be willing to pay more than her certainty equivalent:

$$\mu - (.5/RT_2)*(\sigma^2) = 17,000 - (.5/4000)*(9000^2) = 17,000 - 6750 = \$10,250.$$

On the other hand, if they each took 50% of the returns from the asset, then each individual would anticipate a payment drawn from a normal distribution with mean $.50*17,000 = \$8500$ and standard deviation $.50*9000 = \$4500$. Thus individual 1 would be willing to pay up to

$$8500 - (.5/4000)*(4500^2) = 8500 - 2531.25 = \$5968.75,$$

individual 2 would be willing to pay up to

$$8500 - (.5/6000)*(4500^2) = 8500 - 1687.50 = \$6812.50,$$

and so they would jointly be willing to pay up to $5968.75 + 6812.50 = \$12,781.75$ to buy the asset when they share it equally. The asset is worth more to them when it is shared equally than when either of them owns it completely, because halving an individual's share halves the standard deviation of his monetary returns by a factor of .50, which in turn reduces his risk premium to a quarter of its former value.

The total value of the lottery to these two individuals can actually be increased even more if they share the monetary returns in proportion to their risk tolerances; that is, 40% = $4000/(4000+6000)$ for individual 1 and 60% for individual 2. When individual 1 takes a 40% share, his expected monetary value is $.4*17,000 = \$6800$ and his standard deviation is $.4*9000 = \$3600$, and so his certainty equivalent is

$$CE(1) = 6800 - (.5/4000)*(3600^2) = 6800 - 1620 = \$5180.$$

When individual 2 takes a 60% share, her expected monetary value is $.6*17,000 = \$10,200$ and her standard deviation is $.6*9000 = \$5400$, and so her certainty equivalent is

$$CE(2) = 10,200 - (.5/6000)*(5400^2) = 10,200 - 2430 = \$7770.$$

When they plan to share the risks in this way, the two individuals are willing to pay up to

$$5180 + 7770 = \$12,950$$

to buy this asset. Using Solver as shown in Figure 1, it can be shown that no other sharing rule yield a higher combined value of this asset.

This maximal combined value could be computed another way. The total risk tolerance of a partnership is defined to be the sum of the risk tolerances of the individual partners. So in this case, the partnership of these two individuals has a total risk tolerance of $4000+6000 = \$10,000$. For this partnership treated as a corporate person with this risk tolerance, a Normal lottery with mean \$17,000 and standard deviation \$9000 would have certainty equivalent

$$17000 - (.5/10000)*(9000^2) = \$12,950.$$

	A	B	C	D	E	F	G	H	I
1	Suppose \$returns will be drawn from a Normal distribution								
2	Mean	17000							
3	Stdev	9000							
4									
5	Returns can be shared by investors 1 and 2								
6	Investor	RiskTol\$	%share	Mean\$	Stdev\$	CE\$		CE(1)+CE(2)	
7	1	4000	0.4	6800	3600	5180		12950	
8	2	6000	0.6	10200	5400	7770			
9								12950	
10	FORMULAS FROM RANGE A1:H8								
11	D7.	=C7*\$B\$2			SOLVER:				
12	E7.	=C7*\$B\$3			Maximize H7 by changing C7.				
13	F7.	=D7-(0.5/B7)*E7^2							
14	H7.	=F7+F8							
15	C8.	=1-C7							
16	D8.	=C8*\$B\$2							
17	E8.	=C8*\$B\$3							
18	F8.	=D8-(0.5/B8)*E8^2							
19	H9.	=B2-(0.5/SUM(B7:B8))*B3^2							

Figure 1. Sharing a Normal lottery.

Notice that buying and optimally sharing the asset (so as to maximize the total certainty equivalent) makes sense even if one individual could profitably buy it alone, or even if the asset is already owned by one of these individuals. For example, suppose that the asset is already owned by individual 1. As we have already seen, his certainty equivalent is \$6875 when he holds this investment alone. But individual 2 would be willing to pay up to \$7770 for a 60% share. After selling 60% of the investment to individual 2 for this maximal price, individual 1 would have \$7770 cash plus a risky investment that is worth \$5180 to him (his CE for a 40% share). Thus, selling 60% to individual 2 can increase individual 1's total certainty equivalent from \$6875 to $5180+7770 = \$12,950$.

Of course, 2 would prefer to pay less than \$7770 for a 60% share, and she might try to negotiate for a lower price. When the alternative is owning the risky asset himself, individual 1 would be willing to sell 60% of the asset to 2 for any price above $6875-5180 = \$1695$.

So regardless of who initially owns the asset, the individuals can maximize the sum of their certainty equivalents by sharing the risky returns in proportion to their risk tolerances. How this maximal value is divided among the investors is a bargaining problem. If one of them initially

owns a larger share, there will exist a range of transfer prices at which the individuals could both gain by changing to their optimal shares.

Example 2: sharing a discrete lottery

These results can be extended to investments offering monetary returns that will drawn from any distribution. For example, consider an investment that will offer total returns drawn from the following discrete distribution

<u>Returns</u>	<u>Probability</u>
\$30,000	.2
\$20,000	.4
\$10,000	.3
\$0	.1

(The expected value and standard deviation are \$17,000 and \$9000 here, as in Example 1, but the Normality assumption has been dropped.)

As before, we consider a two-person partnership in which individuals 1 and 2 have constant risk tolerances \$4000 and \$6000 respectively. Figure 2 shows a spreadsheet to find the plan for sharing these returns that would be best for individual 1. For each possible total return listed in the range A6:A9, the planned net monetary payments to the two partners are shown in cells D6:E9. For example, if the investment yields a total return of \$30,000 (the value of cell A6), then the net payments to individuals 1 and 2 are to be the amounts shown in cells D6 and E6 respectively. Using expected utility calculations in cells G6:H12, the partners' certainty equivalents of their individual shares are computed in cells D11 and E11. Of course, the total of the net monetary payments to the partners, computed in cells F6:F9, must equal the available investment returns given in cells A6:A9. So partner 1 wants Solver to maximize his certainty equivalent in cell D11 by changing the planned monetary shares in cells D6:E9, subject to the constraints that cells F6:F9 must equal cells A6:A9 respectively, and partner 2's certainty equivalent in cell E12 must be at least \$0 (or else individual 2 will quit the partnership).

	A	B	C	D	E	F	G	H
1				Partner 1	Partner 2			
2		Risk Tolerance		4000	6000			
3								
4	TABLE OF POSSIBLE OUTCOMES							
5	\$Returns	Proby		\$Share(1)	\$Share(2)	\$Total	U(1)	U(2)
6	30000	0.2		19758	10242	30000	-0.0072	-0.1814
7	20000	0.4		15758	4242	20000	-0.0195	-0.4931
8	10000	0.3		11758	-1758	10000	-0.0529	-1.3404
9	0	0.1		7758	-7758	0	-0.1438	-3.6436
10								
11				CE(1)	CE(2)		EU(1)	EU(2)
12				12930	0		-0.0395	-1
13	1 wants SOLVER to							
14	maximize D12 by changing D6:E9				Sum of CEs			
15	subject to F6:F9 = A6:A9,				12930			
16	and E12 >= 0.							
17					Sum of individual RTs			
18	Partnership Analysis with RT =				10000			
19	\$Returns	Proby	Utility	EU	CE			
20	30000	0.2	-0.0498	-0.274455	12930			
21	20000	0.4	-0.1353					
22	10000	0.3	-0.3679					
23	0	0.1	-1					
24								
25	FORMULAS FROM RANGE A1:H23							
26	F6.	=D6+E6			A20.	=A6		
27	F6 copied to F6:F9				A20 copied to A20:B23			
28	G6.	=UTIL(D6,D\$2)			C20.	=UTIL(A20,\$E\$18)		
29	G6 copied to G6:H9				C20 copied to C20:C23			
30	G12.	=SUMPRODUCT(G6:G9,\$B\$6:\$B\$9)			D20.	=SUMPRODUCT(C20:C23,B20:B23)		
31	H12.	=SUMPRODUCT(H6:H9,\$B\$6:\$B\$9)			E20.	=UINV(D20,E18)		
32	D12.	=UINV(G12,D2)						
33	E12.	=UINV(H12,E2)						
34	E15.	=SUM(D12:E12)						
35	E18.	=SUM(D2:E2)						

Figure 2. Optimal risk sharing of a discrete lottery.

Solver's solution to this optimization problem is shown in Figure 2. In fact, this sharing plan is exactly equivalent to having individual 2 pay \$7758 to individual 1 for a 60% share in the returns from the investment. That is,

$$\begin{aligned}
 .6 * 30000 - 7758 &= \$10242, \\
 .6 * 20000 - 7758 &= \$4242, \\
 .6 * 10000 - 7758 &= -\$1758, \\
 .6 * 0 - 7758 &= -\$7758.
 \end{aligned}$$

Individual 2 gets a certainty equivalent of \$0 under this plan, while individual 1 gets a certainty equivalent of \$12,930 (from the 40% of the risky returns plus the fixed payment of \$7758).

If instead we asked Solver to choose D6:E9 so as to maximize the total of the two individual's certainty equivalents (computed in cell E15) subject to the total-returns constraints $F6:F9 = A6:A9$, then Solver may have given us another plan in which individual 2 and individual 1 would have gotten less by some constant amount. But the total of their certainty equivalents would have still been \$12,930, and the optimal plan could still be characterized as a 40%-60% division of the investment returns in cells A6:A9 plus a fixed monetary transfer between the investors. (For example, the optimal sharing plan for individual 2 would be to have individual 1 pay \$5178 for a 40% share in the returns from the investment, so that individual 2 would get this \$5178 plus the remaining 60% of the risky returns.)

The total of the two investor's risk tolerances is $4000 + 6000 = \$10,000$ in this example. Calculations in cells A19:E23 show that a person with constant risk tolerance \$10,000 would have a certainty equivalent of \$12,930 for the investment that is being shared by this partnership. Thus, we again find that a partnership should evaluate risky investments as if it were a single person whose risk tolerance is equal to the sum of the partners' actual risk tolerances.

Example 3: three partners

These results generalize to more than two partners. For example, consider a case where there are three partners, one who has risk tolerance \$4000 and two others who each have risk tolerance \$6000. These three partners together have a total risk tolerance of \$16,000. For a person with risk tolerance \$16,000, the discrete lottery that we discussed in the preceding section would have a certainty equivalent of \$14,428. (See the analysis in the range A18:E23 of Figure 3.) So allocating this lottery among themselves, the three partners should be able to attain a maximal sum of certainty equivalents equal to \$14,428 by allocating this lottery among themselves in the proportions 25% ($=4000/16000$) for the first partner and 37.5% ($=6000/16000$)

for each of the other two partners, plus any additional fix payments among the partners. These calculations are illustrated in Figure 3.

	A	B	C	D	E	F	G	H	I	J
1				Partner1	Partner2	Partner3				
2		Risk Tolerance		4000	6000	6000				
3										
4	POSSIBLE OUTCOMES									
5	\$Returns	Proby		\$Share1	\$Share2	\$Share3	\$Total	U(1)	U(2)	U(3)
6	30000	0.2		7500	11250	11250	30000	-0.15	-0.15	-0.15
7	20000	0.4		5000	7500	7500	20000	-0.29	-0.29	-0.29
8	10000	0.3		2500	3750	3750	10000	-0.54	-0.54	-0.54
9	0	0.1		0	0	0	0	-1	-1	-1
10										
11				CE(1)	CE(2)	CE(3)		EU(1)	EU(2)	EU(3)
12				3607	5411	5411		-0.41	-0.41	-0.41
13										
14	SOLVER: maximize E15				Sum of CEs					
15	by changing E6:F9				14428					
16	subject to G6:G9 = A6:A9.									
17				Sum of individual RTs						
18	Partnership Analysis with RT =				16000					
19	\$Returns	Proby	U	EU	CE					
20	30000	0.2	-0.15	-0.40585	14428					
21	20000	0.4	-0.29							
22	10000	0.3	-0.54							
23	0	0.1	-1							
24										
25	FORMULAS FROM RANGE A1:J23									
26	G6. =SUM(D6:F6)				E18. =SUM(D2:F2)					
27	G6 copied to G6:G9				A20. =A6					
28	H6. =UTIL(D6,D\$2)				A20 copied to A20:B23					
29	H6 copied to H6:J9				C20. =UTIL(A20,\$E\$18)					
30	H12. =SUMPRODUCT(H6:H9,\$B\$6:\$B\$9)				C20 copied to C20:C23					
31	H12 copied to H12:J12				D20. =SUMPRODUCT(C20:C23,B20:B23)					
32	D12. =UINV(H12,D2)				E20. =UINV(D20,E18)					
33	D12 copied to D12:F12									
34	E15. =SUM(D12:F12)									

Figure 3. Allocating risks among three partners.

Example 4: a Lognormal lottery

Knowing that optimal risk shares are proportional to individual risk tolerances enables us

to study risk-sharing of lotteries with more complicated distributions by simulation. As in Examples 1 and 2 above, suppose that an individual 1 who has risk tolerance \$4000 can sell a share of some investment that he owns to another individual who has risk tolerance \$6000, and suppose again that total returns from this investment have expected value \$17,000 and standard deviation \$9000. But now suppose that these returns are drawn from a Lognormal distribution. We cannot apply the quadratic formula (which is only for Normal lotteries), and the infinite number of possible outcomes makes computing expected utilities difficult. But we can still use the fact that optimal shares should be 40% for individual 1 and 60% for individual 2, after some nonrandom payment from 2 to 1. To estimate the amount of money that individual 2 should be willing to pay for a 60% share of 1's investment, we can make a large table of simulated values of a random variable computed by the formula $=0.6*LNORMINV(RAND(),17000,9000)$. Then we can compute the certainty equivalent of these simulated incomes by the Simtools CE function, with risk tolerance 6000. From such analysis, shown in Figure 4, we can estimate that individual 2 would be willing to pay over \$8300 for a 60% share in this Lognormal lottery.

	A	B	C	D	E	F	G	H
1	Allocating returns from a Lognormal distribution							
2	with	Mean	Stdev					
3		17000	9000		FORMULAS FROM RANGE A1:C15			
4					A6. =LNORMINV(RAND(),B3,C3)			
5	Simulated returns				B10. =B9/SUM(\$B\$9:\$C\$9)			
6	6138.682				C10. =C9/SUM(\$B\$9:\$C\$9)			
7					B15. =B10*\$A\$6			
8		Partner1	Partner2		C15. =C10*\$A\$6			
9	RiskTols	4000	6000		B12. =CE(B16:B1015,B9)			
10	%Shares	40%	60%		C12. =CE(C16:C1015,C9)			
11								
12	CE	5589.792	8384.687					
13								
14		\$Share(1)	\$Share(2)					
15	SimTable	2455.473	3683.209					
16	0	5203.919	7805.879					
17	0.001	6601.209	9901.813					
18	0.002	19941.24	29911.86	[data continues to row 1015]				

Figure 4. Estimating certainty equivalents of optimal shares of a Lognormal lottery.

Example 5: risk sharing with incentive constraints

Of course, people do not share all the risks in proportion to their risk tolerances. One basic reason is that people who are well insured against all risks sometimes do not work hard enough to avoid them. This problem is called moral hazard in the insurance industry. To avoid such moral hazard problems, workers and managers in an enterprise are often forced to bear more of the enterprise's risks than they would bear under an optimal risk-sharing system.

To illustrate how moral hazard can change the optimal allocation of risks, let us modify the discrete two-partner Example 2 by assuming that the distribution of returns listed in cells A5:B9 of Figures 2 and 5 can only be achieved if individual 1 attends to some managerial duties which his partner individual 2 cannot directly observe. Suppose that if individual 1 neglected these duties then the investment returns would instead be either \$10,000 or \$0, each with probability 1/2 (as shown in J5:K9 of Figure 5). Suppose also that neglecting these duties would enable individual 1 to take up another private project that would be worth \$5000 to him.

Under the sharing plan shown in Figure 2, if individual 1 neglected his duties to the partnership, he would be equally likely to get either \$4000 or \$0 from his 40% of the investment, plus the payment of \$7758 from individual 2, plus \$5000 from his private project, and his certainty equivalent for these combined monetary receipts would be \$14,278 (given that his risk tolerance is \$4000). But individual 1's certainty equivalent under the plan in Figure 2 is \$12,930, so he would prefer to neglect his duties!

Figure 5 shows the Solvers analysis of a revised optimization problem that takes account of this moral-hazard problem. The expected utility of individual 1's returns from the partnership when he neglects his managerial duties is computed in cell J13, using the revised probabilities that would apply in this case. Cell J15 computes 1's corresponding certainty equivalent when the \$5000 from the private project is included. Then the Solver problem in Figure 5 differs from the problem in Figure 2 by the inclusion of one additional constraint, called an incentive constraint, which requires that 1's certainty equivalent when he performs his duties (in cell D12) must be greater than or equal to his certainty equivalent when he neglects his duties (in cell J15). The resulting optimal sharing plan is shown in D6:E9 of Figure 5. Compared to Figure 2, individual 1 here gets more when the investment does well, but 2 pays him less (\$4894 instead of \$7758).

	A	B	C	D	E	F	G	H	I	J	K	L	M
1				Partner 1	Partner 2					MORAL-HAZARD INCENTIVE PROBLEM			
2		Risk Tolerance		4000	6000					Private\$ to 1 if he neglects duties			
3										5000			
4	TABLE OF POSSIBLE OUTCOMES									Probabilities when 1 neglects duties			
5	\$Returns	Proby		\$Share(1)	\$Share(2)	\$Total	U(1)	U(2)		\$Returns	Proby	\$Shares as in D6:E9	
6	30000	0.2		21431	8569	30000	-0.0047	-0.2398		30000	0		
7	20000	0.4		17432	2568	20000	-0.0128	-0.6518		20000	0		
8	10000	0.3		12633	-2633	10000	-0.0425	-1.5509		10000	0.5		
9	0	0.1		4894	-4894	0	-0.2942	-2.2606		0	0.5		
10													
11				CE(1)	CE(2)		EU(1)	EU(2)		So if 1 neglects duties, he gets			
12				12127	0		-0.0482	-1		EU for 1 from his share of \$returns			
13										-0.16836			
14					Sum of CEs					CE for 1, including private \$:			
15					12127					12127			
16													
17	Probabilities in A5:B9 assume 1 performs managerial duties									1's revised SOLVER problem:			
18	which 2 cannot directly observe.									Maximize D12			
19										by changing D6:E9 subject to			
20	FORMULAS FROM RANGE A1:J15									F6:F9 = A6:A9,			
21	F6. =D6+E6									E12 >= 0,			
22	F6 copied to F6:F9									D12 >= J15.			
23	G6. =UTIL(D6,D\$2)												
24	G6 copied to G6:H9												
25	G12. =SUMPRODUCT(G6:G9,\$B\$6:\$B\$9)												
26	H12. =SUMPRODUCT(H6:H9,\$B\$6:\$B\$9)												
27	D12. =UINV(G12,D2)												
28	E12. =UINV(H12,E2)												
29	E15. =SUM(D12:E12)												
30	J13. =SUMPRODUCT(K6:K9,G6:G9)												
31	J15. =UINV(J13,D2)+J3												

Figure 5. Risk sharing with a moral-hazard incentive constraint.

Risk-sharing questions

1. G. Washington owns a real-estate development which will pay returns drawn from a Normal distribution, with expected value \$50,000, and standard deviation \$12,000. Washington has constant risk tolerance, and his risk-tolerance index is \$8000. J. Madison's risk-tolerance index is \$5000. (The development does not require any significant unobservable effort from Washington himself.)

(a) What is Washington's certainty equivalent for the real-estate development, if he keeps the development for himself?

(b) Washington is planning to sell Madison a share in the development. To maximize their total certainty-equivalent values of their shares, what share should Washington sell to Madison? What is the minimum price that Washington should accept for selling this share? What is the maximum price that Madison would be willing to pay for this share?

(c) B. Arnold owns a real-estate development up river that will pay returns drawn from a Normal distribution with expected value \$64,000 and standard deviation \$20,000. Arnold has offered to exchange his up-river development for Washington's development. Washington expects to take Madison on as a partner in his real-estate deals in any case. Should Washington accept Arnold's offer to exchange developments? Explain your answer.

THE WILSON ESTATE

Daniel Wilson and Rebecca Wilson Tisler have just inherited their mother's real estate holdings, which consist of scattered commercial properties, some of which have substantial mortgages. Without their mother to manage the properties, they plan to sell the properties and split the returns equally.

A sudden offer by P. J. Cooney to buy all of their mother's properties for \$140,000 over the cost of repaying all mortgages has caused disagreement among the brother and sister, however. Daniel is eager to accept the offer, but Rebecca feels strongly that they should try to do better by selling the various properties separately over the coming year. You have been asked to help resolve their conflict.

After lengthy discussions with the Daniel and Rebecca, you have found that they essentially agree about their prospective risks if they turn down Cooney's offer. Appraisals of the properties by independent real-estate agents have suggest that the expected returns from selling the properties separately is \$200,000 over the cost of repaying all mortgages, although their actual realized returns could be substantially above or below that amount. So both Daniel and Rebecca agree that a Normal distribution with this mean of \$200,000 and a standard deviation of \$75,000 can accurately describe their beliefs about their combined returns from selling the properties separately after rejecting Cooney's offer.

Where the siblings differ is in their risk tolerance, however. Daniel, with children about to enter college, is substantially less risk-tolerant than Rebecca. By asking them about their willingness to take hypothetical risks, you have assessed Daniel's risk-tolerance index to be \$20,000, and you have assessed Rebecca's risk-tolerance index to be \$45,000. It seems reasonable to assume that they each have constant risk tolerance.

(a) What is Daniel's certainty equivalent for a 50% share of the returns from selling the properties after rejecting Cooney's offer?

What is Rebecca's certainty equivalent for a 50% share of the returns from selling the properties after rejecting Cooney's offer?

(b) Consider an alternative plan in which Rebecca pays her brother some amount of money now to buy a larger share of the returns from the properties. To increase the sum of their certainty equivalents as much as possible, what share should Rebecca take?

How much money would she have to pay Daniel now to buy this increased share, so that his certainty equivalent for his remaining share plus her payment to him should be equal to \$70,000 (the amount that he would get if they sold to Cooney now and divided the money equally)?

(c) If Rebecca paid her brother and increased her share of the returns from selling the properties as you described in part (b), then what would be her certainty equivalent for her share of the returns minus her payment to her brother?

(d) Make a chart showing the (inverse) cumulative distribution of the net returns (including the payment to or from the other sibling) that Daniel and Rebecca will each get under your plan from part (b).