Envelope Theorems for Parametrized Choice Sets

Antoine LOEPER[1] and Paul MILGROM[2]

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[1]Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, a-loeper@kellogg.northwestern.edu

[2]Economics Department, Stanford University, paul@milgrom.net

Abstract

Existing envelope theorems apply to fixed choice sets or to convex maximization programs (Milgrom and Segal 2002). We derive envelope theorems for parametrized choice sets without imposing any convex or topological structure on the choice sets. We show that the traditional envelope theorem formula hold at any point where the generalized Lagrange multipliers and the derivative of the constraint are continuous. We provide conditions under which the value function is differentiable or absolutely continuous. We apply these theorems to topological spaces and mechanism design problems.

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1 Introduction

In Milgrom and Segal 2002, it was shown that to obtain the envelope theorem formula, no structure had to be imposed on the choice set nor continuity of the maximand with respect to the choice variable. More precisely, the differentiability of the parametrized program

$$V(t) = \sup_{x \in X} f(s, x), \tag{1}$$

was established without condition on the structure of X nor on the differentiability of $x \to f(s, x)$. The intuition underlying this results is that the supremum of a function is independent of the labelling of the choice set. With parametrized choice sets, the problem becomes

$$V(s) = \sup_{x \in X(s)} f(x). \tag{2}$$

where $X(s)$ is usually defined as $X(s) = \{x \in X : g(s, x) \leq 0\}$ for some function g. The main purpose of this paper is to generalize the conditions imposed by the existing envelope theorems for the optimization programs of type (2).

Contrary to the program with fixed choice set, one cannot simply dispense with conditions on the structure of the choice set or on the regularity of $f(x)$. The reason is that the program (2) can be re-written via an onto mapping $\Psi(s, .) : X(s_o) \to X(s)$ as

$$V(s) = \sup_{x \in X(s_o)} f(\Psi(s, x)).$$

This formulation is similar to the program (1) with a fixed choice set with the important difference that the maximand depends on s indirectly through the choice variable. The case of a discrete choice set X illustrates clearly the difference between the two cases: in the program (1), V is typically continuous and piecewise differentiable whenever f is differentiable in t. In the program (2), V will be piecewise constant and discontinuous.

The envelope theorem for parametrized choice set has been established for convex programs only. In this case, the envelope formula state that $V_s(s_o) = \lambda g_s(x_o, s_o)$ where x_o is a solution of (2) at $s = s_o$, and λ is the Lagrange multiplier of the program, i.e. the saddle point of the associated Lagrangian. Without convexity condition, the Lagrangian may not have a saddle point. However, λ can be defined more generally via the following family of programs:

$$W(s,c) = \sup_{g(s,x)+c\leq 0} f(x).$$

More precisely, whenever the Lagrangian has a saddle point (λ_o, x_o) at $s = s_o$, $\lambda_o = \frac{\partial W}{\partial c}(s_o, c = 0)$ and $V_s(s_o) = \frac{\partial W}{\partial c}(s_o, 0)\frac{\partial g}{\partial s}(s_o)$. Hence, the differentiability of W with respect to c is clearly a necessary condition for the envelope formula to hold. One of the main conclusion of this paper is that the existence of $\frac{\partial W}{\partial c}$ and $\frac{\partial g}{\partial s}$, together with some continuity condition, are sufficient.

In particular, we do not necessarily need to impose a metric or topological structure on X. Of course, such a structure may help to establish the regularity of W is sufficiently smooth in c but it is not needed if the underlying economic problem guarantees the regularity of W, as in the case in some applications as we shall argue.

Section 2 lays out the notations and introduce the generalized lagrange multiplier. Section 3 states general results on the differentiability and absolute continuity of V and section 4 derives some applications.

2 Notations

Unless stated otherwise, the set of admissible parameters S is an open bounded interval of $\mathbb{R}$. X is the domain of the maximand f. We are interested in the regularity of the parametrized program:

$$V(s) = \sup_{x \in X(s)} f(x), \tag{3}$$

where $X(s)$ is defined as the lower contour sets of a function g:

$$X(s) = \{x \in X; g(s,x) \leq 0\}. \tag{4}$$

Whenever they exist, we denote the solutions of (3) as follows:

$$\begin{aligned} X^*(s) &= \{x \in X(s) : f(x) = V(s)\}, \\ X^* &= \cup_{s \in S} X^*(s). \end{aligned}$$

2.1 Differentiability: definitions

For any map $\Phi : T \times Y$ where $T \subset \mathbb{R}$ and Y is some arbitrary space, we denote

$$\begin{aligned} \Phi_t^{\inf}(t+, y) &= \liminf_{h \searrow 0} \frac{\Phi(t+h, y) - \Phi(t, y)}{h}, \\ \Phi_t^{\sup}(t+, y) &= \limsup_{h \searrow 0} \frac{\Phi(t+h, y) - \Phi(t, y)}{h}, \end{aligned}$$

and likewise for $\Phi_t^{\inf}(t-, y)$ and $\Phi_t^{\sup}(t-, y)$. Whenever they exist, $\Phi_t(t, y)$ will denote the partial derivative of Φ w.r.t. t at (t, y) and $\Phi_t(t+, y)$ (resp. $\Phi_i(t-, y)$) its right-hand (resp. left-hand) directional derivative.

If Y is a topological space, we will say that Φ_t is continuous at (t_o, y_o) if Φ is differentiable w.r.t. t in a neighborhood of (t_o, y_o) and Φ_t is jointly continuous in (t, y) at (t_o, y_o). We will say that Φ_t is continuous at (t_o-, y_o) (resp. (t_o+, y_o)) if the restriction Φ^- of Φ to $(T \cap]-\infty, t_o]) \times Y$ (resp. $(T \cap [t_o, +\infty[) \times Y$) is such that Φ_t^- is continuous at (t_o, y_o).

On top of absolute continuity (see e.g. Royden 1988), we will use the following notions which were introduced in Milgrom and Segal 2002:

Definition 1 *A family of function* $(\Phi(., y))_{y \in Y}$ *is equidifferentiable in* t *at* t_o *if* $\left(\frac{\Phi_i(t,y)-\Phi_i(t_o,y)}{t-t_o}\right)_{y \in Y}$ *converges uniformly as* $t \to t_o$*. It is equidifferentiable on* T *if it is equidifferentiable for all* $t \in T$*.*
A family of function $(\Phi(., y))_{y \in Y}$ *is uniformly absolutely continuous if for all* $y \in Y$*,* $\Phi(., y)$ *is absolutely continuous and there exists a measurable function* $b : T \to \mathbb{R}$ *such that for all* y *and almost all* t*,* $|\Phi_t(t, y)| \leq b(t)$*.*

As shown in Milgrom and Segal 2002, equidifferentiability is implied by the equicontinuity of $\{\Phi_s(., y)\}_{y \in Y}$ and uniform absolute continuity is the key ingredient to ensure the absolute continuity of the program (1).

2.2 Generalized Lagrange Multipliers

Traditional envelope theorems factor in the variations of the choice set $X(s)$ in the value function via the Lagrange multipliers, i.e. the saddle points of the Lagrangian. However, when the choice sets $X(s)$ are not convex and the function f is not quasi-concave, the min-max theorem does not apply and the Lagrange multipliers may not be well defined. To extend the notion of Lagrange multiplier to non convex programs, we define the following family of maximization program:

$$W(s,c) = \sup_{g(s,x)+c\leq 0} f(x), \tag{5}$$

and the choice sets and solutions of (5) are denoted

$$\begin{aligned} X(s,c) &= \{x : g(s,x) + c \leq 0\}, \\ X^*(s,c) &= \{x \in X(s,c) : f(x) = W(s,c)\}. \end{aligned}$$

By definition, $V(s) = W(s,0)$ and W is non increasing in c. As such, it is differentiable in c almost everywhere.

When the maximization problem is convex and the Lagrangean has a unique saddle point (x,λ), W is differentiable in c and $W_c(s,0) = \lambda$. Existing envelope theorems state that when the Lagrangian has a unique saddle point, g_s is continuous in both argument and $x \in X^*(s)$, then $V'(s) = \lambda g_s(s,x)$ or equivalently, $V'(s) = W_c(s,0)\, g_s(s,x)$.[1] Hence, the existence of the derivatives W_c and g_s are clearly necessary for the envelope theorem to hold. Our results will show that the existence and smoothness of W_c and g_s are actually sufficient. Even though no convexity or norm structure need to be imposed on the set X for our general results, we will show in section 4 that the regularity conditions easier to check in well behaved spaces.

Although the map W is not a primitive of the problem, we will see in subsection 4.3 that in some cases, W has an economic interpretation and the regularity conditions that we will impose on W to establish our results have an intuitive meaning. When X is a linear space, one can show that whenever the following derivatives exist, $W_c = \frac{\nabla_x f(x)}{\nabla_x g(s,x)}$. In general, there is no compact

[1] See e.g. Milgrom and Segal 2002, Corollary 5.

formulation of W_c as a function of the primitives f and g. Nevertheless, using the following notations

$$\begin{aligned} X^{\mathbb{N}}(s, c-) &= \left\{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : -g(s, x_n) \nearrow c\right\}, \\ X^{\mathbb{N}}(s, c+) &= \left\{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : -g(s, x_n) \searrow c\right\}, \end{aligned} \tag{6}$$

we have:

Remark 1 *If there exists $x_o \in X^*(s_o, c_o)$ such that $g(s_o, x_0) = c_o$, then*

$$W_c^{\sup}(s_o, c_o+) = \max_{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}(s_o, c_o+)} \left(\limsup_{n\to\infty} -\frac{f(x_n) - f(x_0)}{g(s_o, x_n) - g(s_o, x_0)}\right). \tag{7}$$

If there exists $x_o \in X^*(s_o, c_o)$ such that $g(s_o, x_0) = c_o$, then either $W(s_o, .)$ is locally constant to the left of c_o or

$$W_c^{\inf}(s_o, c_o-) = \min_{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}(s_o, c_o-)} \left(\liminf_{n\to\infty} -\frac{f(x_n) - f(x_0)}{g(s_o, x_n) - g(s_o, x_0)}\right). \tag{8}$$

Proof. See appendix. ■

3 General Results

3.1 Directional Derivatives

Lemma 1 *Let $s, s' \in S$, $x \in X^*(s)$ and $x' \in X^*(s')$ then*

$$V(s') - V(s) \leq W(s, -g(s, x')) - W(s, -g(s', x')), \tag{9}$$

and

$$V(s') - V(s) \geq W(s', -g(s, x)) - W(s', -g(s', x)). \tag{10}$$

Proof. By definition of V and W,

$$V(s') \leq \sup_{g(s,y)\leq g(s,x')} f(y) = W(s, -g(s,x')), \tag{11}$$

and since $g(s', x') \leq 0$,

$$V(s) \geq \sup_{g(s,y)\leq g(s',x')} f(y) = W(s, -g(s',x')). \tag{12}$$

Substracting (12) to (11), we get (9). By switching the role of (s, x) and (s', x') we get (10). ■

The next two propositions prove that the envelope theorem formula hold at any point where V_s, W_c and g_s exist. The first proposition imposes an additional regularity condition on W_c.

Proposition 1 *Let $x_o \in X^*(s_o)$, if W_c is continuous at $(s, c) = (s_o, 0)$, then whenever the following derivatives exist, we have*

$$\begin{aligned} V_s(s_o-) &\leq W_c(s_o, 0) \times g_s(s_o, x_o), \\ V_s(s_o+) &\geq W_c(s_o, 0) \times g_s(s_o, x_o). \end{aligned}$$

If V is differentiable, the above inequalities hold with equality.

Proof. Suppose first $g(s_o, x_o) = 0$. Let $s_n \searrow s_o$. Using successively equation (10) in lemma 1 and Taylor's theorem, we get

$$\begin{aligned} \frac{V(s_n) - V(s_o)}{s_n - s_o} &\geq -\frac{W(s_n, -g(s_n, x_o)) - W(s_n, g(s, x_o))}{s_n - s_o} \\ &= W_c(s_n, -g_n) \frac{g(s_n, x_o) - g(s, x_o)}{s_n - s_o}. \end{aligned}$$

for some $g_n \in [0, g(s_n, x_o)]$. Taking the limit, we get the first inequality. The second inequality obtains by considering $s_n \nearrow s_o$.

Suppose now $g(s_o, x_o) < 0$. This implies that W is constant in c on $[0, -g(s_o, x_o)]$, so $W_c(s_o, 0) = 0$. Since g is continuous in s at (s_o, x_o), $x_o \in X(s)$ for s sufficiently close to s_o. Hence, whenever they exist, $V_s(s_o-) \leq 0$ and $V_s(s_o+) \geq 0$, so the inequalities of the proposition still hold. ■

The second proposition imposes an additional regularity condition only on the primitive g_s, but requires X to be a topological space.

Proposition 2 *If there exists $s_n \nearrow s_o$, $x_n \in X^*(s_n)$ such that $x_n \to x_o^-$ and g and g_s are continuous at (s_o-, x_o^-), then whenever the following derivatives exist,*

$$V_s(s_o-) \geq W_c(s_o, 0) \times g_s(s_o, x_o^-).$$

If there exists $s_n \searrow s_o$, $x_n \in X^*(s_n)$ such that $x_n \to x_o^+$ and g_s is continuous at (s_o+, x_o^+), then whenever the following derivatives exist

$$V_s(s_o+) \leq W_c(s_o, 0) \times g_s(s_o, x_o^+).$$

If V is differentiable and $g_s(s_o, x_o^-) = g_s(s_o, x_o^+)$, the above inequalities hold with equality.

Proof. Consider the case $s_n \searrow s_o$. Without loss of generality, suppose $g(s_o, x_n) \to l \in \mathbb{R} \cup \{\pm\infty\}$. Since g is continuous at (s_o, x), $g(s_n, x_n) \to l$, so $l \leq 0$. Suppose first $l = 0$. Using successively equation (9) in lemma 1 and Taylor's theorem, we get

$$\begin{aligned}\frac{V(s_n) - V(s_o)}{s_n - s_o} &\leq -\frac{W(s_o, -g(s_n, x_n)) - W(s_o, -g(s_o, x_n))}{g(s_n, x_n) - g(s_o, x_n)} \frac{g(s_n, x_n) - g(s_o, x_n)}{s_n - s_o} \\ &= -\frac{W(s_o, -g(s_n, x_n)) - W(s, -g(s, x_n))}{g(s_n, x_n) - g(s_o, x_n)} g_s(s_n', x_n),\end{aligned}$$

for some $s'_n \in [s_o, s_n]$. Since W_c exists and g_s is continuous at (s_o+, x), the right-hand side tends to $W_c(s_o, 0) \times g_s(s_o, x_o^+)$.[2]

Suppose $l < 0$. If $V(s_n)$ is not constant for $n \geq N$ for some $N \in \mathbb{N}$, necessarily there exists two subsequences $\rho, \sigma \in \mathbb{N}^{\mathbb{N}}$ such that $V(\rho(n)) < V(\sigma(n))$, and so $g\left(s_{\rho(n)}, x_{\sigma(n)}\right) > 0$ for all n. Since g is continuous at (s_o, x), $g\left(s_{\rho(n)}, x_{\sigma(n)}\right) \to l \geq 0$, a contradiction. So $V(s_n)$ is constant for $n \geq N$ and $V_s(s_o-) = 0$. Since $g(s_o, x_n) \to l$, $W(s_o, c)$ is constant in c on $[0, -l[$, so $W_c(s_o, 0) = 0$ and the equation of the proposition holds. ■

The next proposition combines regularity conditions on both g and W to establish the directional differentiability of V. It does not impose any structure on X nor on the convergence of $X^*(s)$ as $s \to s_o$.

Proposition 3 *If $X^*(s) \neq \emptyset$ for all $s \in S$, $g(s_o, .)$ and $g_s(s_0, .)$ are bounded, W_c is continuous at $(s_0, 0)$ and $(g(., x))_{x \in X^*}$ is equidifferentiable (see definition 1), then V is left-hand and right-hand differentiable at s_o and for any selection $x^*(s) \in X^*(s)$,*

$$\begin{aligned} V_s(s_0-) &= W_c(s_0, 0) \lim_{s \nearrow s_0} g_s(s_0, x^*(s)), \\ V_s(s_0+) &= W_c(s_0, 0) \lim_{s \searrow s_o} g_s(s_0, x^*(s)). \end{aligned}$$

Proof. Let $s'_n \searrow s$, $s''_n \searrow s$, $x'_n \in X^*(s'_n)$ and $x''_n \in X^*(s''_n)$. Suppose without loss of generality that $s'_n < s''_n$ for all n and either for all n, $g(s''_n, x''_n) \neq g(s'_n, x''_n)$ or for all n, $g(s''_n, x''_n) = g(s'_n, x''_n)$. Since $g(s_0, x'_n)$, $g(s_0, x''_n)$, $g_s(s_0, x'_n)$ and $g_s(s_0, x''_n)$ are bounded, we can restrict attention to converging subsequences. We denote their limit g', g'', g'_s and g''_s respectively.

[2]Note that if $g(s_n, x_n) = g(s, x_n)$ for all $n \geq N$ for some N, then by continuity, $g_s(s, x) = 0$. Moreover $g(s, x_n) \leq 0$ so $x_n \in X(s)$ for all $n \geq N$, which implies $V_s(s+) \leq 0$ and the inequality of the proposition still holds.

Suppose first that $g(s''_n, x''_n) \neq g(s'_n, x''_n)$ for all n. By equidifferentiability of g, $g(s'_n, x''_n)$ and $g(s''_n, x''_n)$ both converge to g''. Let us first consider the case $g'' = 0$. Since W_c is continuous at $(s_o, 0)$,

$$\frac{W(s'_n, -g(s'_n, x''_n)) - W(s'_n, -g(s''_n, x''_n))}{g(s''_n, x''_n) - g(s'_n, x''_n)} \to W_c(s, 0). \tag{13}$$

Since g is equidifferentiable,

$$\frac{g(s''_n, x''_n) - g(s'_n, x''_n)}{s''_n - s'_n} = g_s(s_o, x''_n) + o(|s''_n - s_o| + |s'_n - s_o|) \to g''_s. \tag{14}$$

Combining (13) and (14) with (9),

$$\limsup \frac{V(s''_n) - V(s'_n)}{s''_n - s'_n} \leq W_c(s, 0) \times g''_s. \tag{15}$$

Suppose now that $g'' < 0$. As argued in the proof of proposition 2, necessarily, $\lim \frac{V(s''_n) - V(s'_n)}{s''_n - s'_n} = 0$ and $W_c(s, 0) = 0$, so (15) still holds.

Finally, if $g(s''_n, x''_n) = g(s'_n, x''_n)$ for all n sufficiently large, then $x''_n \in X(s'_n)$ so $V(s'_n) \geq V(s''_n)$ and $\limsup \frac{V(s''_n) - V(s'_n)}{s''_n - s'_n} \leq 0$. Moreover, by equidifferentiability of g, $g''_s = 0$, so (15) still holds.

With the same reasoning as above, (10) implies that if $g(s'_n, x'_n) \neq g(s''_n, x'_n)$ for all n,

$$\frac{V(s''_n) - V(s'_n)}{s''_n - s'_n} \geq \frac{W(s''_n, -g(s'_n, x'_n)) - W(s''_n, -g(s''_n, x'_n))}{g(s''_n, x'_n) - g(s'_n, x'_n)} \frac{g(s''_n, x'_n) - g(s'_n, x'_n)}{s''_n - s'_n}.$$

If $g' = 0$, taking the limit we get

$$\liminf \frac{V(s''_n) - V(s'_n)}{s''_n - s'_n} \geq W_c(s, 0) \times g'_s. \tag{16}$$

If $g' < 0$, for the same reason as above, $\lim \frac{V(s''_n) - V(s'_n)}{s''_n - s'_n} = 0$ and $W_c(s, 0) = 0$, so (16) still holds.

Finally, if $g\left(s_n', x_n'\right) = g\left(s_n'', x_n'\right)$ for all n, then $x_n' \in X\left(s_n''\right)$ so $V\left(s_n''\right) \geq V\left(s_n'\right)$. Moreover, by equidifferentiability of g, $g_s' = 0$ so (16) still holds.

To conclude the proof, observe that by inverting the role of s_n' and s_n'', we get from (15) and (16) that

$$V_s(s+) = W_c\left(s, 0\right) \times g_s'' = W_c\left(s, 0\right) \times g_s'.$$

■

3.2 Absolute Continuity

In this subsection, we establish a set of conditions that guarantee the absolute continuity of V. The following lemma is a straightforward corollary of Theorem 2 in Milgrom and Segal 2002.

Lemma 2 *If there exists $Y \subset X$ and $\left(h\left(., x\right)\right)_{x \in Y}$ uniformly absolutely continuous such that for all $s \in S$,*

$$V\left(s\right) = \sup_{x \in Y} f\left(x\right) - h\left(s, x\right),$$

then V is absolutely continuous.

When the maximization program is convex, the function $g\left(s, x\right)$ itself, times its Lagrange multiplier $\lambda\left(s\right)$, can play the role of h in lemma 2. The next proposition gives a condition under which $h\left(s, x\right) = C \max\left(0, g\left(s, x\right)\right)$ does the job for some $C \geq 0$.

Proposition 4 *If $\sup_{s \in S}\left|\frac{W(s,c) - W(s,0)}{c}\right|$ is bounded as $c \nearrow 0$ and $\left(g\left(., x\right)\right)_{x \in X}$ is uniformly absolutely continuous (see definition 1), then V is absolutely continuous.*

Proof. Let $h(s,x) = \max(0, g(s,x))$. Since 0 and $g(s,x)$ are absolutely continuous for all x, so is $h(s,x)$. The integrable bound condition of definition 1 clearly holds, so h is uniformly absolutely continuous. We denote

$$\Omega_\lambda(s,c) = \sup_{x \in X(s,c)} f(x) - \lambda h(s,x).$$

By construction, for all $\lambda \geq 0, c \leq 0$ and all $s \in S$, $\Omega_\lambda(s,c) \geq V(s)$. Suppose first that there exists $\lambda_o \geq 0, c_o < 0$ such that for all s, $\Omega_{\lambda_o}(s,c_o) = V(s)$. Since g is uniformly absolutely continuous, there exists $\varepsilon > 0$ such that $|s' - s| \leq \varepsilon$ implies that for all $x \in X$, $|g(s',x) - g(s,x)| \leq -c_o$. This shows that for all $s_o \in S$, and all $s \in [s_o - \varepsilon, s_o + \varepsilon]$,

$$X(s) \subset \cup_{s' \in [s_o - \varepsilon, s_o + \varepsilon]} X(s') \subset \cap_{s' \in [s_o - \varepsilon, s_o + \varepsilon]} X(s', c_o). \tag{17}$$

If we denote $Y = \cup_{s' \in [s_o - \varepsilon, s_o + \varepsilon]} X(s')$, (17) implies that for all $s \in [s_o - \varepsilon, s_o + \varepsilon]$,

$$V(s) \leq \sup_{x \in Y} f(x) - \lambda_o h(s,x) \leq \Omega_{\lambda_o}(s,c_o).$$

Since $V(s) = \Omega_{\lambda_o}(s,c_o)$, the above inequality holds with equality and lemma 2 implies that V is absolutely continuous on $[s_o - \varepsilon, s_o + \varepsilon]$ for all s_o, and therefore on S.

Suppose now that for all $\lambda \geq 0, c < 0$, there exists $s \in S$ such that $\Omega_\lambda(s,c) > V(c)$. Then there exists two sequences s_n and x_n such that $g(s,x_n) \searrow 0$ and $f(x_n) - ng(s_n,x_n) > V(s_n)$. Therefore, for all n

$$W(s_n, -g(s_n,x_n)) - W(s_n, 0) > -ng(s_n,x_n),$$

which is impossible under our assumptions. ■

Since absolute continuity implies differentiability almost everywhere, by combining proposition 1 and 4 we get the integral form of the envelope theorem:

Corollary 1 *Let $x^*(s)$ be a selection of $X^*(s)$ for almost all s. If W_c is continuous on $S \times [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, $g_s(s, x^*(s))$ exists for almost all $s \in S$ and $(g(.,x))_{x \in X}$ is uniformly absolutely continuous, then V is absolutely continuous and for all $s, s' \in S$,*

$$V(s') - V(s) = \int_s^{s'} W_c(s, 0) \times g_s(s, x^*(s)) ds.$$

Proof. If W_c is continuous on $S \times [-\varepsilon, 0]$, then $\left| \frac{W(s,c) - W(s,0)}{c} \right|$ is bounded on $C \times [-\varepsilon, 0]$ for any compact subset C of S. From proposition 4, for all $s, s' \in S$, V is absolutely continuous on $[s, s']$ and as such, almost everywhere differentiable. From proposition 1, at any such point s, $V_s(s) \leq W_c(s, 0) \times g_s(s, x^*(s))$. ■

4 Applications

4.1 Topological Spaces

In this subsection, we derive an envelope theorem when the choice set is a second countable topological space.[3]

Lemma 3 *Suppose T is a separable topological space[4] and T' is second countable, and S is a subset of $T \times T'$, then the set of isolated points of S (for the product topology) is at most countable.*

[3] A topological space is second countable if it has a countable base. In particular, any separable metric space is second countable. Examples include Euclidean spaces and the space of continuous functions on a compact space.

[4] A topological space is separable if it contains a countable dense subset.

Proof. Let $(B_n)_{n\in\mathbb{N}}$ be a countable base of T' and let $\mathcal{I} = \{(x_i, x_i') : i \in I\}$ be a set of isolated points of S. For all $i \in I$ there are open sets $O_i \subset T$ and $O_i' \subset T'$ such that $O_i \times O_i' \cap \mathcal{I} = \{(x_i, x_i')\}$. Since B is a base, there exists $n(i) \in \mathbb{N}$ such that $x_i' \in B_{n(i)}$ and $B_{n(i)} \subset O_i'$. Thus $O_i \times B_{n(i)} \cap \mathcal{I} = \{(x_i, x_i')\}$. Let $I(n) = \{i \in I : n(i) = n\}$. By construction, the points $\{x_i : i \in I(n)\}$ are isolated in T. Since T is separable, $I(n)$ must be countable, and so does I as the countable union of countable sets.[5] ■

Proposition 5 *If X is second countable and $x^*(s)$ is a selection of $X^*(s)$ such that g and g_s are continuous at $(s, x^*(s))$ for almost all s, then at almost all points where these derivatives exists, $V_s(s) = W_c(s, 0) \times g_s(s, x)$.*

Proof. S endowed with the lower limit topology[6] is separable. Let $G = \{(s, x^*(s)) : s \in S\}$ be the graph of a selection $x^*(.) \in X^*(.)$. From the previous lemma, for almost all s, $(s, x^*(s))$ is not isolated in G, i.e. there exists $s_n \nearrow s$ and $x^*(s_n) \to x^*(s)$. A similar argument holds for the upper limit topology. Proposition 2 completes the proof. ■

4.2 Continuous Functions on Compact Sets

In this section, X is a compact set and S is a closed bounded interval.

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[5] This is true only under the axiom of countable choice. Alternatively, if T is the real line, the projection of I on T has Lebesque measure 0, which is all we need to derive our results.

[6] The lower limit topology is the topology generated by the intervals of the form $]a, b[$ and $]a, b]$ for $a < b$. In particular, a sequence x_n is converging if and only if $x_n \nearrow l$.

Lemma 4 *Suppose f is upper semi-continuous and g is continuous, then if V is continuous at s, $X^*(.)$ is compact valued and upper hemi-continuous at s.*

Proof. Compactness is immediate. To show upper hemi-continuity, let $s_n \to s$, $x_n \in X^*(s_n)$ and $x_n \to x$. By continuity of g, $g(s,x) \leq 0$. So by definition of V, $f(x) \leq V(s)$. By upper hemi-continuity of f, $\limsup V(s_n) \leq f(x)$. By continuity of V, $V(s_n) \to V(s)$. Therefore, $V(s) = f(x)$ and $x \in X^*(s)$. ■

Proposition 6 *If g and g_s are continuous on $S \times X$ and W_c is continuous on $S \times [-\epsilon, 0]$ for some $\varepsilon > 0$, then V is absolutely continuous. If furthermore f is upper semi-continuous, for all $s \in S$,*

$$\begin{aligned} V_s(s+) &= W_c(s,0) \min_{x \in X^*(s)} g_s(s,x), \\ V_s(s-) &= W_c(s,0) \max_{x \in X^*(s)} g_s(s,x). \end{aligned}$$

V is differentiable if and only if $\{g_s(s,x) : x \in X^*(s)\}$ is a singleton.

Proof. Since $X \times S$ is compact, by continuity g_s is bounded on $X \times S$ and thus absolutely continuous in s for all x. Likewise, since $S \times [-\epsilon, 0]$ is compact, W_c is bounded. Hence the condition of proposition 4 are satisfied and V is absolutely continuous.

As noted in subsection 2.1, by continuity of g_s, $(g(.,x))_{x \in X^*}$ is equidifferentiable. Since f is upper semi continuous, $X^*(s)$ is non empty and compact. Therefore, from proposition 3, the directional derivatives exist.

Since V is absolutely continuous, it is continuous. From lemma 4, the compactness of X and the continuity of g_s, for any selection $x^*(s) \in X^*(s)$, and any $s_o \in S$,

$$\liminf_{s \searrow s_o} g_s\left(s, x^*\left(s\right)\right) \geq \min_{x \in X^*\left(s_o\right)} g_s\left(s_o, x\right),$$

and since $W_c \leq 0$, from proposition 3,

$$V_s\left(s_o+\right) \leq W_c\left(s_o, 0\right) \min_{x \in X^*\left(s_o\right)} g_s\left(s_o, x\right).$$

From proposition 1, the opposite inequality holds, which proves the first equality. The second equality is proven similarly. The last points follows immediately. ■

4.3 Mechanism Design

Establishing the integral representation of the agents utility in principal agent problems is a key step in characterizing the optimal mechanism. This can be done by applying the envelope theorem to the maximization program of the agent (Mirless 1971, Myerson 1981). The previous theorems can be applied to mechanism design problems in which the type of the agent enters in the constraint of his utility maximization program.

To illustrate the applicability of our results, we will use them to characterize group strategy-proof mechanisms for two players. Let $N = \{1, 2\}$ denote the set of players, X the set of possible alternatives, $\{U_i\left(\theta_i, .\right) : \forall i \in N, \theta_i \in \Theta_i\}$ the set of admissible utility functions of the agents. We will say that a set of pay-off profiles $\{U_i\left(\theta_i, .\right) : \forall i \in N, \theta_i \in \Theta_i\}$ is regular if for all $i \in N$, Θ_i is an open interval of $\mathbb{R}$, for all $\theta_i \in \Theta_i$, $U_i\left(\theta_i, x\right)$ is uniformly absolutely continuous in θ_i. A mechanism Φ maps a profile of type θ into an outcome $\Phi\left(\theta\right)$ for all $\theta \in \Theta$. The range of the mechanism is denoted R_Φ.

Definition 2 *A mechanism Φ is strategy-proof if for all $i \in N$, $\theta_i, \theta_i' \in \Theta_i$, and $\theta_{-i} \in \Theta_{-i}$,*

$$U_i\left(\theta_i, \Phi\left(\theta_i, \theta_{-i}\right)\right) \geq U_i\left(\theta_i, \Phi\left(\theta_i', \theta_{-i}\right)\right).$$

It is group strategy-proof if for all $\theta_1, \theta_1' \in \Theta_1$, $\theta_2, \theta_2' \in \Theta_2$,

$$\begin{aligned} U_1\left(\theta_1, \Phi\left(\theta_1, \theta_2\right)\right) &\geq U_1\left(\theta_1, \Phi\left(\theta_1', \theta_2'\right)\right), \\ \text{or } U_2\left(\theta_2, \Phi\left(\theta_1, \theta_2\right)\right) &\geq U_2\left(\theta_2, \Phi\left(\theta_1', \theta_2'\right)\right). \end{aligned}$$

It has a smooth range if for all $\varepsilon > 0$, the Pareto frontier on the range of the mechanism:

$$U_i^P\left(\theta, U_j\right) = \max_{x \in R_\Phi : U_j(\theta_j, x) \geq U_j} U_i\left(\theta_i, x\right)$$

is differentiable w.r.t. U_j and its derivative is bounded and continuous in $\left(\theta_j, U_j\right)$ on $\left\{U_j : U_j < \sup_{x \in R_\Phi} U_j\left(\theta_j, x\right) - \varepsilon\right\}$.

Geometrically, the smooth range condition means that the Pareto frontier on the range of the mechanism has a continuous and bounded slope at any interior point in the utility space (U_1, U_2). The usual envelope theorem allows to characterize strategy-proof mechanism. It shows that the value function $V_i(\theta) \equiv U_i\left(\theta_i, \Phi\left(\theta_i, \theta_{-i}\right)\right)$ under a strategy-proof mechanism Φ on a regular set of payoff profile is absolutely continuous in θ_i and

$$V_i(\theta) = V_i\left(\underline{\theta}_i, \theta_{-i}\right) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial U_i}{\partial \theta_i}\left(t, \Phi\left(t, \theta_{-i}\right)\right) dt. \tag{18}$$

Our envelope theorems allow to characterize group strategy-proof mechanism with a smooth range. By comparison to the traditional approach, we show that the value function V_i is absolutely continuous in the other players' types.

Proposition 7 *Let Φ be a group strategy-proof mechanism with a smooth range defined on a regular set of admissible pay-off profiles, then for all $i \neq j$, V_i is absolutely continuous in θ_j and either V_i is locally constant in θ_j or j is a dictator at θ, i.e. $V_j(\theta) = \sup_{x \in R_\Phi} U_j(\theta_j, x)$.*

Proof. Let $i = 1, j = 2$ and fix θ_1. Since $\Phi(\theta)$ is group strategy-proof, it solves the following program:

$$V_1(\theta) = \max_{x \in R_\Phi : V_2(\theta) - U_2(\theta_2, x) \leq 0} U_1(\theta_1, x), \tag{19}$$

By construction, there exists a solution for all θ. Since Φ is strategy-proof, using (18), for almost all θ_2, the constraint of (19) is differentiable w.r.t. θ_2.

For all (θ_2, c) such that $\{x \in R_\Phi : V_2(\theta) - U_2(\theta_2, x) + c \leq 0\} \neq \emptyset$, we define

$$W_1(\theta, c) = \max_{x \in R_\Phi : V_2(\theta) - U_2(\theta_2, x) + c \leq 0} U_1(\theta_1, x),$$

This program is well-defined (i.e. has a non empty choice set) in a neighborhood of $(\theta_2, 0)$ if player 2 is not a dictator at θ_2. In this case, since Φ has a smooth range, $\frac{\partial W_1}{\partial c}$ is continuous at $(\theta_2, 0)$. From what precedes, proposition 1 implies that at any θ_2 such that player 2 is not a dictator and $\frac{\partial V_1}{\partial \theta_2}(\theta)$ and $\frac{\partial V_2}{\partial \theta_2}(\theta)$ exists, we have:

$$\frac{\partial V_1}{\partial \theta_2}(\theta) = \frac{\partial W_1}{\partial c}(\theta, 0) \times \left(\frac{\partial V_2}{\partial \theta_2}(\theta) - \frac{\partial U_2}{\partial \theta_2}(\theta_2, x = \Phi(\theta)) \right).$$

From (18), the second term is zero whenever it is defined. Since $\frac{\partial V_2}{\partial \theta_2}$ exists for almost all θ_2, we have proven that for almost all θ_2, if player 2 is not a dictator and $\frac{\partial V_1}{\partial \theta_2}$ exists at θ_2, then $\frac{\partial V_1}{\partial \theta_2}(\theta) = 0$.

Let $[\underline{\theta}_2, \overline{\theta}_2] \subset \Theta_2$ be such that for all $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$, player 2 is not a dictator. Since Φ has a smooth range, $\frac{\partial W_1}{\partial c}$ is bounded on $[\underline{\theta}_2, \overline{\theta}_2] \times \mathbb{R}^-$

so from Taylor's theorem, $\sup_{\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]} \frac{W_1(\theta,c)-W_1(\theta,0)}{c}$ is bounded as $c \nearrow 0$. From (18) V_2 is absolutely continuous and $U_2(\theta_2, x)$ is uniformly absolutely continuous, so the constraint of (19) is uniformly absolutely continuous as well. From theorem 4, $V_1(\theta)$ is absolutely continuous in θ_2. In particular, for almost all θ_2, $\frac{\partial V_1}{\partial \theta_2}$ exists and from what precedes, $\frac{\partial V_1}{\partial \theta_2} = 0$. Since V_1 is absolutely continuous, it must be constant on $[\underline{\theta}_2, \overline{\theta}_2]$. ■

In words, proposition 7 shows that for a mechanism to be group strategy-proof, at any point of its domain, it must be either constant or some player must be a local dictator. To characterize the set of group strategy-proof rules, it suffices then to fix a (smooth) range R_Φ and determine the regions of the type space in which player 1 or 2 are dictators or the mechanism is constant. The border of these region in the type space must be compatible with strategy-proofness.

As an immediate corollary, proposition 7 characterizes strategy-proof Pareto efficient mechanisms for two players on regular sets of pay-off profiles such that the unrestricted Pareto frontier has a smooth frontier. Proposition 7 can also be generalized to characterize pairwise strategy-proof mechanisms for more than two players by assuming that the option set of two player i and j:

$$R_\Phi^{i,j}(\theta) = \left\{x \in X : x = \Phi\left(\theta_i', \theta_j', \theta_{-\{i,j\}}\right), \theta_i' \in \Theta_i, \theta_j' \in \Theta_j\right\}$$

has a smooth Pareto frontier in the sense of definition 2.

5 Appendix

In this appendix, we prove remark 1. We first prove (7). Let $x \in X^*(s,c)$ be such that $g(s,x)+c=0$ and let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}(s,c_o+)$. By definition of W, $W(s,-g(s,x_n)) \geq f(x_n)$. Moreover, $-g(s,x_n)+g(s,x)>0$ so

$$\frac{W(s,-g(s,x_n))-f(x)}{-g(s,x_n)+g(s,x)} \geq \frac{f(x_n)-f(x)}{-g(s,x_n)+g(s,x)}.$$

Since $-g(s,x_n) \searrow c$, $x \in X^*(s,c)$ and $g(s,x)+c=0$, taking the limit, we have

$$W_c^{\sup}(s,c+) \geq \limsup -\frac{f(x_n)-f(x)}{g(s,x_n)-g(s,x)}. \tag{20}$$

We now prove the opposite inequality. Let $c_n \searrow c$ and $x_n \in X(s,c_n)$ such that

$$\frac{f(x_n)-W(s,c)}{c_n-c} \to W_c^{\sup}(s,c+). \tag{21}$$

Since $0<c_n-c \leq -g(s,x_n)-c$ and $f(x_n) \leq W(s,-g(s,x_n))$,

$$\frac{f(x_n)-W(s,c)}{c_n-c} \leq \frac{f(x_n)-W(s,c)}{-g(s,x_n)-c} \leq \frac{W(s,-g(s,x_n))-W(s,c)}{-g(s,x_n)-c}. \tag{22}$$

If $\limsup g(s,x_n) < -c$, then $W(s_o,.)$ must be constant on $]c_o, c_o+\varepsilon[$ for some $\varepsilon>0$ and it has a downward jump to the right of c_o. In this case, $W_c^{\sup}(s_o,c_o+)=-\infty$ and (7) holds. Now suppose that $g(s,x_n) \nearrow -c$, by definition of $W_c^{\sup}(s,c+)$, the limit of the left hand-side of (22) is bounded by $W_c^{\sup}(s,c+)$ while from (21), the limit of the right hand-side of (22) is $W_c^{\sup}(s,c+)$. Therefore, (22) implies

$$\frac{f(x_n)-W(s,c)}{-g(s,x_n)-c} = -\frac{f(x_n)-f(x)}{g(s,x_n)-g(s,x)} \to W_c^{\sup}(s,c+), \tag{23}$$

which proves (7).

We now prove (8). Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}(s, c-)$. By definition of W, $W(s, -g(s, x_n)) \geq f(x_n)$. Moreover, $-g(s, x_n) + g(s, x) < 0$ so,

$$\frac{W(s, -g(s, x_n)) - f(x)}{-g(s, x_n) + g(s, x)} \leq \frac{f(x_n) - f(x)}{-g(s, x_n) + g(s, x)}.$$

Since $-g(s, x_n) \nearrow c$, $x \in X^*(s, c)$ and $g(s, x) + c = 0$, taking the limit, we have

$$W_c^{\inf}(s, c-) \leq \liminf -\frac{f(x_n) - f(x)}{g(s, x_n) - g(s, x)}. \tag{24}$$

We now prove the opposite inequality. Let $c_n \nearrow c$ and $x_n \in X^*(s, c_n)$ be such that

$$\frac{f(x_n) - W(s, c)}{c_n - c} \to W_c^{\inf}(s, c-). \tag{25}$$

Since W is weakly decreasing in c, one can choose x_n such that $W(s, c) \leq f(x_n)$. Observe that if W is not locally constant to the left of c at (s, c), then for all n, $-c_n \leq g(s, x_n) < -c$ so $g(s, x_n) \searrow -c$. Moreover, $c_n - c \leq -g(s, x_n) - c < 0$ and $W(s, c) \leq f(x_n) \leq W(s, -g(s, x_n))$, so

$$\frac{f(x_n) - W(s, c)}{c_n - c} \geq \frac{f(x_n) - W(s, c)}{-g(s, x_n) - c} \geq \frac{W(s, -g(s, x_n)) - W(s, c)}{-g(s, x_n) - c}. \tag{26}$$

The limit of the right hand-side of (26) is bounded below by $W_c^{\inf}(s, c-)$ while from (25), the left hand-side of (26) tends to $W_c^{\inf}(s, c-)$. Therefore, (26) implies

$$\frac{f(x_n) - W(s, c)}{-g(s, x_n) - c} = -\frac{f(x_n) - f(x)}{g(s, x_n) - g(s, x)} \to W_c^{\inf}(s, c-). \tag{27}$$

References

[1] Milgrom, P. and I. Segal, 2002, "Envelope Theorems for Arbitrary Choice Sets", Econometrica, Vol. 20, No. 2, pp. 583-601.

[2] Myerson, R., 1981. Optimal auction design. Mathematics of Operation Research 6, vol. 1, pp. 58–73

[3] Mirlees, James A, 1971. "An Exploration in the Theory of Optimum Income Taxation," Review of Economic Studies 38, 175–208

[4] Royden, H. L. (1988): Real Analysis, Third Edition. Englewood Cliffs: Prentice-Hall