Proof of Theorem 2 in "Stochastic Evolutionary Stability in Extensive Form Games of Perfect Information"

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To prove Theorem 2 I establish interim results for the new limiting conditions, which are analogous to the ones needed for the proof of Theorem 1.

Lemma 6 *Let $i \in N$ be an arbitrary node and $x \in A(i)$ an arbitrary action available to individuals at node i. Then there is a $\kappa \in \mathbb{R}_+$ and there is a $\bar{\sigma}$ such that for all (μ, σ, m_i) with $\sigma < \bar{\sigma}$ with $\mu < \sigma$ and $\mu m_i \geq \delta$ where $\delta \in \mathbb{R}_+$:*

$$\pi^m_{\mu,\sigma}\left(\Lambda^{i,x}_0\right) \leq \kappa\sigma \tag{39}$$

Proof: In the proof of Lemma 1 I established inequality (23), which is given again here:

$$\pi^m_{\mu,\sigma}\left(\Lambda^{i,x}_0\right) \leq \frac{1}{1+\frac{1-p_{00}}{\max_{k\geq 1}\{p_{k0}\}}}, \tag{40}$$

while for all $k < m_i$

$$\frac{p_{k+1,0}}{p_{k0}} = \frac{\mu\lambda_x + \sigma(1-\mu)}{1-\mu(1-\lambda_x)}, \tag{41}$$

which is less than 1 for small μ provided $\sigma < 1$. Hence, $\max_{k\geq 1}\{p_{k0}\} = p_{10}$ under these conditions.

By equation (18) we have

$$p_{00} = (1-\mu(1-\lambda_x))^{m_i} \tag{42}$$

and

$$p_{10} = (1-\mu(1-\lambda_x))^{m_i}\frac{\mu\lambda_x + \sigma(1-\mu)}{1-\mu(1-\lambda_x)}. \tag{43}$$

Given $\mu m_i \geq \delta$ we have

$$p_{00} = \left[(1-\mu(1-\lambda_x))^{\frac{1}{\mu}}\right]^{\mu m_i} \tag{44}$$
$$\leq \left[(1-\mu(1-\lambda_x))^{\frac{1}{\mu}}\right]^{\delta} \tag{45}$$
$$\leq \left[e^{-(1-\lambda_x)}\right]^{\delta} \tag{46}$$
$$\leq e^{-\delta(1-\lambda_x)} < 1. \tag{47}$$

Given $\mu < \sigma$ there is an $\alpha \in \mathbb{R}_+$ such that

$$p_{10} \leq \alpha\sigma \tag{48}$$

provided $\sigma < \bar{\sigma}$ and $\bar{\sigma}$ is small enough.

Hence, there is a $\kappa \in \mathbb{R}_+$ and there is a $\bar{\sigma}$ such that for all (μ, σ, m_i) with $\sigma < \bar{\sigma}$ with $\mu < \sigma$ and $\mu m_i \geq \delta$ where $\delta \in \mathbb{R}_+$

$$\pi^m_{\mu,\sigma}\left(\Lambda^{i,x}_0\right) \leq \kappa\sigma. \tag{49}$$

QED

A few corollaries, analogous to Corollaries 1, 2, and 3, follow immediately.

Corollary 4 *Let $i \in N$ be an arbitrary node and $x \in A(i)$ an arbitrary action available to individuals at node i. Let $\delta \in \mathbb{R}_+$.*

$$\lim_{\frac{\mu}{\sigma}\to 0, \sigma\to 0, m_i\mu\geq\delta} \pi^m_{\mu,\sigma}\left(\Lambda^{i,x}_0\right) = 0. \tag{50}$$

Corollary 5 *Let Ψ be defined as in Corollary 2. Then*

$$\lim_{\frac{\mu}{\sigma}\to 0, \sigma\to 0, m_i\mu\geq\delta\ \forall i\in N} \pi^m_{\mu,\sigma}\left(\Psi\right) = 0. \tag{51}$$

Corollary 6 *Let $i \in N$ be a final decision node. Then*

$$\lim_{\frac{\mu}{\sigma}\to 0, \sigma\to 0, m_i\mu\geq\delta\ \forall i\in N} \pi^m_{\mu,\sigma}\left(C^i_{b^i}\right) = 0. \tag{52}$$

The next lemma is the one additional piece of help I need to proof Lemma 8 below and hence Theorem 2.

Lemma 7 *Let $A \subset \Omega$ be such that $\pi^m_{\mu,\sigma}(A) < \kappa\sigma$ for some $\kappa \in \mathbb{R}_+$ and for all $\sigma < \bar{\sigma}$ for some $\bar{\sigma} > 0$. Let $(\Omega \times \Omega^l, P)$ be a probability space with $P(\omega_0, \omega_1, ..., \omega_l) = \pi^m_{\mu,\sigma}(\omega_0) \prod_{t=1}^{l} \left(Q^m_{\mu,\sigma}\right)_{\omega_{t-1},\omega_t}$, where $l = \lfloor \frac{1}{\sigma} \rfloor$. Let $N_l(A)$ denote the number of periods $t \in \{1, ..., l\}$ such that $\omega_t \in A$. Then for all $\eta \in (0,1)$:*

$$P\left(\frac{N_l(A)}{l} \geq \eta\right) \leq \frac{\kappa\sigma}{\eta} \tag{53}$$

Proof: For a positive integer T let $\left(\Omega \times \left(\Omega^l\right)^T, P\right)$ be a probability space with $P(\omega_0, \omega_1, ..., \omega_{lT}) = \pi^m_{\mu,\sigma}(\omega_0) \prod_{t=1}^{lT} \left(Q^m_{\mu,\sigma}\right)_{\omega_{t-1},\omega_t}$. Let N_l^τ denote the number of periods $t \in \{l(\tau-1)+1, ..., \tau l\}$ such that $\omega_t \in A$. Let $N_0 = \sum_{\tau=1}^{T} N_l^\tau$, i.e. N_0 denotes the number of periods in which the state is in A for the whole process up to period lT. By the properties of ergodic Markov chains we know that $\frac{X}{l\tau} \to \pi^m_{\mu,\sigma}(A)$ in probability when $\tau \to \infty$. As $\frac{X}{l\tau} \in [0,1]$ it must be true that $E\left(\frac{X}{l\tau}\right) < \kappa\sigma$ for all $\tau \geq \bar{\tau}$ for some $\bar{\tau}$. Of course, $E\left(\frac{N_0}{lT}\right) = \frac{1}{T}\sum_{\tau=1}^{T} E\left(\frac{N_l^\tau}{l}\right)$. By the properties of the invariant distribution, however, the distribution of N_l^τ must be the same as the distribution of $N_l^{\tau'}$ for all $\tau, \tau' \in \{1, ..., T\}$. Hence, $E\left(\frac{N_0}{lT}\right) = E\left(\frac{N_l(A)}{l}\right)$ with $N_l(A)$ defined as in the statement of the lemma. Hence, we have $E\left(\frac{N_l(A)}{l}\right) < \kappa\sigma$. By Markov's inequality we then get the result. QED

Lemma 8 *Let $i \in N$ be a node such that*

$$\lim_{\frac{\mu}{\sigma}\to 0, \sigma \to 0, m_i\mu \geq \delta \ \forall i \in N} \pi^m_{\mu,\sigma}\left(C^i_{b^i}\right) = 0.$$

Then for any $\epsilon \in (0,1)$:

$$\lim_{\frac{\mu}{\sigma}\to 0, \sigma \to 0, m_i\mu \geq \delta \ \forall i \in N} \pi^m_{\mu,\sigma}\left(B^i_{\epsilon,m}\right) = 1. \tag{54}$$

Proof: Let $\{\Omega \times \Omega^l, P\}$ denote a probability space, where P is such that $P(\tilde{\omega}) = \pi^m_{\mu,\sigma}(\omega_0) \prod_{t=1}^{l} \left(Q^m_{\mu,\sigma}\right)_{\omega_{t-1},\omega_t}$ for all $\tilde{\omega} = (\omega_0, \omega_1 ..., \omega_l) \in \Omega \times \Omega^l$ where $l = \lfloor \frac{1}{\sigma} \rfloor$. Define $U : \Omega \times \Omega^l \rightarrow \{0, 1, ..., m_i\}$ such that $U(\tilde{\omega})$ is the number of individuals at population $M(i)$ who play b^i in state ω_0. Similarly let $V : \Omega \times \Omega^l \rightarrow \{0, 1, ..., m_i\}$ be a random variable such that $V(\tilde{\omega})$ is the number of individuals at population $M(i)$ who play b^i in state ω_l. Note that

$$\{\omega_0 \in \Omega | U(\omega_0, ..., \omega_l) = k\} = \{\omega_l \in \Omega | V(\omega_0, ..., \omega_l) = k\} = \Lambda_k^{i,b^i}.$$

Let $Z : \Omega \times \Omega^l \rightarrow \{-m_i, -m_i + 1, ..., -1, 0, 1, ..., m_i\}$ denote a third random variable such that $Z(\tilde{\omega})$ is the "addition" of b^i-players at population $M(i)$ in the transition from state ω_0 to ω_l. Obviously $Z(\tilde{\omega}) = V(\tilde{\omega}) - U(\tilde{\omega})$.

As in the proof of Lemma 4 we have $E(U) = E(V)$ and, hence, $E(Z) = 0$. Let $z = \frac{Z}{m_i}$. Then

$$\begin{aligned} E(z) &= P\left(\frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) E\left(z \middle| \frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) + \qquad (55) \\ &+ \sum_{k=0}^{m_i} P\left(U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} < \eta\right) E\left(z \middle| U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} < \eta\right) \\ &= P\left(\frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) E\left(z \middle| \frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) + \qquad (56) \\ &+ \sum_{k=0}^{m_i} P\left(U = k\right) E\left(z \middle| U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} < \eta\right) - \\ &- \sum_{k=0}^{m_i} P\left(U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) E\left(z \middle| U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} < \eta\right). \end{aligned}$$

Now, $1 \geq E\left(z|\cdot\right) \geq -1$ whatever the conditioning is on. Of course, $P\left(U = k\right) = \pi^m_{\mu,\sigma}\left(\lambda_k^{i,b^i}\right)$. We then have

$$\begin{aligned} 0 = E(z) &\geq -2P\left(\frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) + \\ &+ \sum_{k=0}^{m_i} \pi^m_{\mu,\sigma}\left(\Lambda_k^{i,b^i}\right) E\left(z \middle| U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} < \eta\right). \qquad (57) \end{aligned}$$

Let $\alpha_k = E\left(z \middle| U = k \wedge \frac{N_l\left(C^i_{b^i}\right)}{l} < \eta\right)$. Rearranging then yields

$$2P\left(\frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) \geq \sum_{k=0}^{m_i} \pi^m_\mu\left(\Lambda^{i,b^i}_k\right)\alpha_k. \tag{58}$$

By Lemma 7 $P\left(\frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right) \to 0$ under our limiting conditions. Hence the right hand side of inequality (58) must tend to zero as well.

Let $k_* = \lfloor (1-\epsilon)m_i \rfloor$. By Lemma 9 below there is an $\bar{\alpha} > 0$, an $\underline{\alpha} > 0$, a $\bar{\sigma} > 0$, and a $\bar{\phi} > 0$ such that for all $k < k_*$, for all $\sigma < \bar{\sigma}$, and for all $\frac{\mu}{\sigma} < \bar{\phi}$ we have that $\alpha_k \geq \bar{\alpha} - \underline{\alpha}\eta$. Also then $\alpha_k \geq -\underline{\alpha}\eta$ for all k, in particular also for all $k \geq k_*$.

Hence, for these parameter values,

$$\begin{aligned}
\sum_{k=0}^{m_i} \alpha_k \pi^m_{\mu,\sigma}\left(\Lambda^{i,b^i}_k\right) &\geq \sum_{k=0}^{k_*} \left(\bar{\alpha} - \underline{\alpha}\eta\right)\pi^m_{\mu,\sigma}\left(\Lambda^{i,b^i}_k\right) + \sum_{k=k_*+1}^{m_i} -\underline{\alpha}\eta\pi^m_{\mu,\sigma}\left(\Lambda^{i,b^i}_k\right) \\
&\geq \left(\bar{\alpha} - \underline{\alpha}\eta\right)\pi^m_\mu\left(B^{i,c}_{\epsilon,m}\right) - \underline{\alpha}\eta\pi^m_\mu\left(B^i_{\epsilon,m}\right) &(59)\\
&\geq \left(\bar{\alpha} - \underline{\alpha}\eta\right)\pi^m_\mu\left(B^{i,c}_{\epsilon,m}\right) - \underline{\alpha}\eta\left(1 - \pi^m_\mu\left(B^{i,c}_{\epsilon,m}\right)\right) &(60)\\
&\geq \bar{\alpha}\pi^m_\mu\left(B^{i,c}_{\epsilon,m}\right) - \underline{\alpha}\eta &(61)
\end{aligned}$$

Combining inequalities (58) and (61), we obtain

$$\bar{\alpha}\pi^m_{\mu,\sigma}\left(B^{i,c}_{\epsilon,m}\right) - \underline{\alpha}\eta \leq \sum_{k=0}^{m_i} \alpha_k \pi^m_{\mu,\sigma}\left(\Lambda^{i,b^i}_k\right) \leq 2P\left(\frac{N_l\left(C^i_{b^i}\right)}{l} \geq \eta\right). \tag{62}$$

Taking $\sigma \to 0$ while $\frac{\mu}{\sigma} \to 0$ and $m_i\mu \geq \delta > 0$ in inequality (62), we obtain

$$\lim_{\frac{\mu}{\sigma}\to 0, \sigma\to 0, m_i\mu\geq\delta\ \forall i\in N} \pi^m_{\mu,\sigma}\left(B^{i,c}_{\epsilon,m}\right) \leq \eta\underline{\alpha}/\bar{\alpha}. \tag{63}$$

This must be true for all $\eta \in (0,1)$. Hence, $\pi^m_{\mu,\sigma}\left(B^{i,c}_{\epsilon,m}\right) \to 0$. QED

Lemma 9 *Let $k_* = \lfloor (1-\epsilon)m_i \rfloor$. There is an $\bar{\alpha} > 0$, an $\underline{\alpha} > 0$, a $\bar{\sigma} > 0$, and a $\bar{\phi} > 0$ such that for all $k < k_*$, for all $\sigma < \bar{\sigma}$, and for all $\frac{\mu}{\sigma} < \bar{\phi}$ we have that $\alpha_k \geq \bar{\alpha} - \underline{\alpha}\eta$. Also then $\alpha_k \geq -\underline{\alpha}\eta$ for all k.*

Proof: Let $Z^t(\tilde{\omega})$ denote the addition in b^i-players at node i in the transition from ω_{t-1} to ω_t. Then $Z = \sum_{t=1}^{l} Z^t$. Let $z^t = \frac{Z^t}{m_i}$. Let $h^t = \frac{U}{m_i} + \sum_{j=1}^{t} z^t$. By the law of iterated expectations we get:

$$\alpha_k = E\left(\sum_{t=1}^{l-1} z^t + E\left(z^l \left| h^{l-1} \wedge \frac{N_l\left(C_{b^i}^i\right)}{l} < \eta\right.\right) \left| U = k \wedge \frac{N_l\left(C_{b^i}^i\right)}{l} < \eta\right.\right). \tag{64}$$

In fact the law of iterated expectations can be applied $l-1$ times so that α_k is the sum of l expected one-step net increases in the proportion of b^i-players each conditional on the fraction of b^i-players in the previous period. In each period t, now, the state ω_t is either in $C_{b^i}^i$ or its complement. By the fact that $\frac{N_l\left(C_{b^i}^i\right)}{l} < \eta$ we know that the proportion of periods in which the state is in $C_{b^i}^i$ is less than η. The question now is for which sequence of states in $C_{b^i}^i$ or its complement that satisfies $\frac{N_l\left(C_{b^i}^i\right)}{l} < \eta$ is the expected proportional increase of b^i-players minimal for $k < k_*$.

I claim that for all $k < k_*$ this expected increase is minimal if all states ω_t for the initial $l_* = \lfloor (1-\eta) l \rfloor$ periods are in $C_{b^i}^i$, while all the remaining states are in its complement. Let $\Upsilon_j \subset \Omega \times \Omega^l$ be the set of states in which $U = k \wedge \omega_t \in C_{b^i}^{i,c}\ \forall t \in \{0, ..., j-1\}$. Then I claim that

$$\alpha_k \geq E\left(\sum_{t=1}^{l_*}\left(z^t \left| \Upsilon_{l_*}\right.\right)\right) + \tag{65}$$

$$+E\left[E\left(\sum_{t=l_*+1}^{l}\left(z^t \left| h^{l_*} \wedge \omega_t \in C_{b^i}^i\ \forall t \in \{l_*, ..., l-1\}\right.\right)\right) \left| \Upsilon_{l_*}\right.\right].$$

To prove this claim note that

$$E\left(z^t | h^{t-1} = h \wedge \omega_{t-1} \in C_{b^i}^{i,c}\right) = \left[\sigma(1-\mu) + \mu(1-\lambda_{b^i})\right](1-h) - h\mu\lambda_{b^i},$$

as in the proof of Lemma 5, or alternatively

$$E\left(z^t | h^{t-1} = h \wedge \omega_{t-1} \in C_{b^i}^{i,c}\right) = \beta_0 - \beta_1 h, \tag{66}$$

where $\beta_0 = \sigma(1-\mu) + \mu(1-\lambda_{b^i})$ and $\beta_1 = \sigma(1-\mu) + \mu$. Similarly

$$E\left(z^t | h^{t-1} = h \wedge \omega_{t-1} \in C^i_{b^i}\right) \geq \mu(1-\lambda_{b^i})(1-h) - h\left[\sigma(1-\mu) + \mu\lambda_{b^i}\right].$$

The right hand side of the above inequality is the expected number of additional b^i-players when b^i is not a best reply. For some states $\omega_{t-1} \in C^i_{b^i}$, however, b^i is a best reply, but not the unique one. In this case the expected number of additional b^i-players can only be greater. Hence, the inequality.

Note that the expected one-step proportional increase of b^i-players in both cases is the smaller the greater h. Also as in the proof of Lemma 5 $E\left(z^t | h^{t-1} = h \wedge \omega_{t-1} \in C^{i,c}_{b^i}\right) > 0$ if and only if

$$h < \frac{\sigma(1-\mu) + \mu(1-\lambda_{b^i})}{\sigma(1-\mu) + \mu}. \tag{67}$$

Hence, provided that after the first l_* periods the new proportion of b^i-players, denoted by h', satisfies $h' < \frac{\sigma(1-\mu)+\mu(1-\lambda_{b^i})}{\sigma(1-\mu)+\mu}$, then the claim must be true. It remains to be shown, therefore, that indeed $h' < \frac{\sigma(1-\mu)+\mu(1-\lambda_{b^i})}{\sigma(1-\mu)+\mu}$. To show this I need to calculate

$$E\left(\sum_{t=1}^{l_*}\left(z^t \left| \Upsilon_{l_*}\right.\right)\right).$$

For any $j \in \{1, ..., l_*\}$ let

$$e_k(j) = E\left(\sum_{t=1}^{j}\left(z^t \left| \Upsilon_j\right.\right)\right). \tag{68}$$

Let $e_k(0) = 0$. By the law of iterated expectation

$$e_k(j) = E\left[\sum_{t=1}^{j-1}\left(z^t + E\left(z^j \left| h^{j-1} \wedge \omega_{j-1} \in C^{i,c}_{b^i}\right.\right) \Big| \Upsilon_j\right)\right], \tag{69}$$

which, by equation (66), is given by

$$e_k(j) = \beta_0 - \beta_1 \frac{k}{m_i} + (1-\beta_1)e_k(j-1). \tag{70}$$

Hence,

$$e_k(l_*) = \left(\beta_0 - \beta_1 \frac{k}{m_i}\right) \sum_{j=0}^{l_*-1} (1-\beta_1)^j = \left(\beta_0 - \beta_1 \frac{k}{m_i}\right) \frac{1-(1-\beta_1)^{l_*}}{\beta_1}, \quad (71)$$

or in terms of the original parameters,

$$e_k(l_*) = \left[1-(1-\sigma(1-\mu)+\mu)^{l_*}\right]\left[\left(1-\frac{k}{m_i}\right) - \frac{\mu\lambda_{b^i}}{\sigma(1-\mu)+\mu}\right]. \quad (72)$$

Note that for all $k \leq k_*$ we have that $e_k(l_*) \geq e_{k_*}(l_*)$. As $l_* = \lfloor \frac{1-\eta}{\sigma} \rfloor$ and $k_* = \lfloor (1-\epsilon)m_i \rfloor$ we have that

$$e_{k_*}(l_*) \to \left(1 - e^{-(1-\eta)}\right)\epsilon. \quad (73)$$

Note that $h' = \frac{k+e_{l_*}(k)}{m_i} \leq \left(1 - \epsilon e^{-(1-\eta)}\right)$ for all $k < k_*$, and indeed $h' < \frac{\sigma(1-\mu)+\mu(1-\lambda_{b^i})}{\sigma(1-\mu)+\mu}$ in the limit as anticipated.

Hence, for all $k \leq k_*$ there is a $\bar{\alpha} > 0$, an $\underline{\alpha} > 0$, a $\bar{\sigma} > 0$, and a $\bar{\phi} > 0$ such that for all $k < k_*$, for all $\sigma < \bar{\sigma}$, and for all $\frac{\mu}{\sigma} < \bar{\phi}$ we have that the first term in the upper bound for α_k in inequality (65) satisfies,

$$E\left(\sum_{t=1}^{l_*}\left(z^t \,|\Upsilon_{l_*}\right)\right) \geq \bar{\alpha}.$$

We could pick $\bar{\alpha} = (1-2e^{-1})\epsilon$ for instance.

The second term of inequality (65) is easily found to be bounded by $-\eta l\left[\sigma(1-\mu)+\mu\lambda_{b^i}\right]$. As $l = \lfloor \frac{1}{\sigma} \rfloor$ there is an $\underline{\alpha} > 0$ such that $-\eta l\left[\sigma(1-\mu)+\mu\lambda_{b^i}\right] \geq -\underline{\alpha}\eta$. This proves the first part of Lemma 9. To prove the second note that for all k

$$\alpha_k \geq \frac{\sigma(1-\mu)+\mu(1-\lambda_{b^i})}{\sigma(1-\mu)+\mu} - 1 - \mu\lambda_{b^i} - \eta l\left[\sigma(1-\mu)+\mu\lambda_{b^i}\right], \quad (74)$$

and hence $\alpha_k \geq -\underline{\alpha}\eta$ for the same $\underline{\alpha}$ as before. Inequality (74) is due to the fact that the worst case scenario is when $U = m_i$ and the state is in $C_{b^i}^{i,c}$ for the first $(1-\eta)l$ periods, and in $C_{b^i}^{i}$ thereafter. In this case the fraction of b^i-players at most may drop from 1 to $\frac{\sigma(1-\mu)+\mu(1-\lambda_{b^i})}{\sigma(1-\mu)+\mu}$, below which expected increases in b^i-players are again positive and from there will never drop by more than $\mu\lambda_{b^i}$. In the remaining periods where b^i is not necessarily a best reply the fraction will never decrease more than by another $\eta l\left[\sigma(1-\mu)+\mu\lambda_{b^i}\right]$. QED

With all these ingredients the proof of Theorem 2 is now exactly the same as the proof of Theorem 1. QED