Individual and Group Selection in Symmetric 2-Player Games

Christoph Kuzmics*

J. L. Kellogg School of Management
Northwestern University
2001 Sheridan Road
Evanston, IL 60208-2009
email: c-kuzmics@kellogg.northwestern.edu
phone: (847) 491 2978
fax: (847) 467 1220

5 November 2003

*I am grateful to Hamid Sabourian, Jörgen Weibull, Larry Samuelson, Carlos Rodriguez, Florian Herold, Robert Evans and Daniel Sgroi for helpful comments and suggestions.

Abstract

A population of individuals is divided into groups. Individuals are recurrently randomly matched with individuals from their group to play a generic symmetric 2-player game. Deterministic inter- and intra-group dynamics are derived from a model of individual imitation within groups and individual migration between groups. Conditions are identified under which subsets (components) of the set of stationary states are (interior) asymptotically stable. The results are then applied to generic coordination games and the prisoners' dilemma. The unique asymptotically stable set of stationary states in coordination games is such that every individual in non-extinct groups plays the Pareto-optimal equilibrium. In the multi-group prisoners' dilemma there is no asymptotically stable subset of the set of stationary states.

JEL classification: C62, C72, C73
Keywords: equilibrium selection, payoff monotonic dynamics, inter- and intra-group dynamics, local interaction, hierarchical evolution, multilevel evolution, imitation, experimentation

1 Introduction

Many cases of strategic interaction between agents take place in a broader context than is often appreciated. The game played between employer and employee in one firm, for instance, is typically set in the context of firm competition. A fact which, in general, will have an impact on the possible emergence and stability of different equilibria. The equilibrium moral code, or set of social norms, in a given society will, in general, depend on the type of moral codes which have established themselves in neighboring societies. In biology, evolution of specific types of behavior in one species will, in general, influence the evolution of behavior in another species, especially when two species compete for the same resources, habitat, or niche. Indeed, evolution often takes place on two different levels*, on one level within a group of agents (a firm, a city, an entire civilization, a species, etc.) and on another level between these groups. While mutation, in form of experimentation or mistakes in human (economic) behavior or in form of pure chance mutation in genes in animals or plants, works only on the individual, natural or economic selection may work on both groups of individuals and on the individuals themselves.

The treatment of group selection in the biology literature has focussed on the question of whether altruism, whatever its definition, can evolve in a multi-level evolutionary process. Sober and Wilson (1998) provide a review and discussion of this branch of research. The problem is often abstracted to the question of whether cooperation can evolve in a prisoners' dilemma game. Sjöström and Weitzman (1996), in a model of group interaction where each group (island/firm) engages in a multi-person prisoners' dilemma game, show that whether group selection can counteract individual tendencies to

*Indeed one can imagine evolution to be active on more than two levels.

defection, is a matter of the relative speeds of adjustment. Only if the speed at which groups compete is of higher order than the strategy-adjustment process within a group, cooperation will survive in the long-run.

Boyd and Richerson (1990) were the first to explore the impact of group selection (in biology) on equilibrium selection in games with multiple evolutionary stable strategies. The hope here is that group selection favors Pareto-optimal evolutionary stable strategies. Canals and Vega-Redondo (1998) provide another example of equilibrium selection due to group selection. They analyze a stochastic model of individual and group selection in 2×2-coordination games in the spirit of Kandori, Mailath, and Rob (1993). While Kandori, Mailath, and Rob (1993), as well as, in a similar model, Young (1993) show that the risk-dominant equilibrium in the sense of Harsanyi and Selten (1988) is the unique stochastically stable outcome in the one-group dynamics, Canals and Vega-Redondo (1998) show that in a sufficiently multi-group environment only Pareto-efficient equilibria are stochastically stable.

Ely (2002) and Oechssler (1999) construct another stochastic model with finite populations where individuals revise both their strategies and their choice of location. Oechssler's (1999) dynamics start from specific initial conditions (a convention is already established in every group) and lead to Pareto-efficient predictions in a fairly large class of symmetric 2-player games. Ely (2002) builds on the models of Kandori, Mailath, and Rob (1993) and Young (1993) by creating an ergodic process on the space of demographies (strategy and location), allowing for random mutations on top of a sticky best-reply dynamics (in both strategy and location). For coordination games Ely (2002) finds that for fairly general neighborhood structures only, Pareto-efficient conventions will carry positive weight in the limiting invariant distribution when the mutation rate tends to zero.

This paper differs from the above in three major aspects. First, it uses a deterministic 'law of large numbers' approximation to a stochastic large population model, with payoff-monotonic dynamics. Second, it investigates general generic symmetric 2-player games. Finally, it allows for groups vanishing and re-emerging. Two different scenarios of what I allow a re-emerging group to look like are distinguished indirectly, by investigating asymptotic stability and interior asymptotic stability, a concept by Cressman and Schlag (1998), of sets of states.

In Section 2 deterministic inter- and intra-group dynamics are derived from a micro-model of individual imitation within groups and migration between groups. The same dynamics are derived from a biological model of group selection in Section 3. Standard definitions of the stability concepts used to analyze the derived dynamics are provided in Section 4. Section 5 gives a brief account of the well known one-group case. The main results for the multi-group setting are derived in Section 6, and applied to a simple coordination game and the prisoners' dilemma in Section 7. Section 8 offers a discussion of the results from the previous sections and relates them to the literature, particularly on cheap-talk games.

2 A Model of Imitation and Migration

A large (essentially infinite) population of individuals is split up into m groups (firms, villages, cities, states, countries). Repeatedly within each group two random individuals are drawn to play a symmetric 2-player game Γ. Every individual in every group at any given plays one of a set S of pure strategies available. There are $s = |S|$ pure strategies available to the players.

For $t \geq 2$ let $\Delta_t = \left\{\alpha \in \mathbb{R}^t | \sum_{i=1}^{t} \alpha_i = 1 \wedge \alpha_i \geq 0 \ \forall i\right\}$ denote the $t-1$-

dimensional unit simplex. Let $y \in \Delta_m$ denote the vector of relative group-sizes, i.e. y_k is the proportion of individuals who are in group k. For all $k \in \{1, 2, ..., m\}$ let $x_k \in \Delta_s$ be such that x_{ki} is the proportion of individuals in group k who play pure strategy $i \in S$. Let $x = (x_1, ..., x_m)$. Obviously it must be true that $\sum_{k=1}^{m} y_k = 1$ and, for all groups k, $\sum_{i=1}^{s} x_{ki} = 1$. One such pair (x, y) is then a state this world could be in. Let the set of all possible states, the state space, be denoted by $\Theta = (\Delta_s)^m \times \Delta_m$.

Let a_{ij} denote the payoff (fitness) an individual playing pure strategy $i \in S$ receives when playing against a person playing pure strategy j. Let $A = (a_{ij})$ be the matrix of pure strategy payoffs. Then $u(v, w) = v^t A w$ is the expected payoff to an individual playing mixed strategy $v \in \Delta_s$ when meeting an opponent playing mixed strategy $w \in \Delta_s$.

In the following few paragraphs I provide a model of imitation within groups and migration between groups in the spirit of Björnerstedt and Weibull (1996).

Individuals are assumed generally to stick to their strategy choice for a while before reviewing it. The review times for each individual are generated by a Poisson process. The processes are statistically independent from each other. Let $r_{lj}(x, y)$ denote the average review rate of an individual in group l playing pure strategy $j \in S$. Given a j-strategist in group l reviews her strategy, she decides to either remain in group l, with probability $\gamma_{lj}(x, y)$, or to migrate to another group (possibly also to group l as well), with probability $1 - \gamma_{lj}(x, y)$. In case the individual migrates, she does migrate to group k, there playing strategy $i \in S$, with probability $\pi_{lj}^{ki}(x, y)$. For convenience I allow migration from group l to l as well, also strategy switches from j to j. In case the individual decides not to migrate she switches to strategy $i \in S$ with probability $p_{lj}^{i}(x, y)$.

For all groups $k \in \{1, 2, ..., m\}$ and al strategies $i \in S$ let $z_{ki} = y_k x_{ki}$.

Hence, the aggregate arrival time of an individual switching from group l and strategy $j \in S$ to group k and strategy $i \in S$ is given by

$$\begin{cases} z_{lj} r_{lj}(x,y) \left[\gamma_{lj}(x,y) p^i_{lj}(x,y) + (1 - \gamma_{lj}(x,y))\, \pi^{li}_{lj}(x,y) \right] & \text{if } k = l \\ z_{lj} r_{lj}(x,y)\, (1 - \gamma_{lj}(x,y))\, \pi^{ki}_{lj}(x,y) & \text{otherwise} \end{cases} \tag{1}$$

By appealing to the law of large numbers due to Benaim and Weibull (2003), the evolution of the population state is then given by the following dynamics

$$\begin{aligned} \dot{z}_{ki} &= \sum_{l=1}^{m} \sum_{j=1}^{s} z_{lj} r_{lj}(x,y)\, (1 - \gamma_{lj}(x,y))\, \pi^{ki}_{lj}(x,y) \\ &+ \sum_{j=1}^{s} z_{kj} r_{kj}(x,y) \gamma_{kj}(x,y) p^i_{kj}(x,y) \\ &- z_{ki} r_{ki}(x,y). \end{aligned} \tag{2}$$

The first line in equation (2) captures the movement of all individuals in all groups (including k itself), who decide to migrate to group k, and then playing strategy $i \in S$. The second line is an account of the strategy changes of all individuals in group k (including i-strategists) who decide not to migrate but change their strategy to $i \in S$. The remaining line in (2) comprises the movement of all individuals currently in group k playing strategy i, who review and then either move away from group k or remain in group k and change strategy to some $j \in S$ (including 'changes' to $i \in S$).

There are many promising ways of specializing the dynamics in (2). For a variety of these specialized dynamics see Section 4.4 in Weibull (1995). Here, I will take one approach and defer discussion of other possible approaches to Section 8. I want to concentrate on the dynamics which results from what Björnerstedt and Weibull (1996) term "imitation of successful individuals". I will assume for simplicity that the review rates are identical for all individuals and normalized to one, i.e.

$$r_{lj}(x,y) = 1 \tag{3}$$

for all x, y, for all groups l, and for all strategies $j \in S$. I furthermore assume that the probability of migrating is independent of the group an individual is in and the strategy it plays, i.e.

$$\gamma_{lj}(x, y) = \gamma \tag{4}$$

for all x, y, for all groups l, and for all strategies $j \in S$. The probability of migrating to a particular group k, conditional on the individual migrating in the first place, only depends on the average payoff, $u(x_k, x_k)$, which is currently realized in group k, and on the current vector of population sizes, $y \in \Delta_m$. This is to say that individuals, when migrating, only look at summary statistics of the various groups. Individuals migrating between countries, for instance, might look at per capita GDP or similar aggregate data. In the context of firms as groups, individuals may base their decision of migrating to another firm on the mean salary of people in a firm or on the firm's performance, measured by e.g. the performance of it's shares. The probability of then adopting strategy $i \in S$ shall be x_{ki}. This is achieved if a migrant adopts the strategy of the first (random) person she or he meets, a behavior which has been termed "cultural conformism" by Boyd and Richerson (1985) and "pure imitation" by Björnerstedt and Weibull (1996). In sum

$$\pi_{lj}^{ki}(x, y) = \frac{\alpha_G\left(u(x_k, x_k)\right) y_k}{\sum_{k'=1}^{m} \alpha_G\left(u(x_{k'}, x_{k'})\right) y_{k'}} x_{ki}, \tag{5}$$

where α_G is a strictly positive and strictly increasing function. Note that this probability is the same for all individuals, regardless of which group they are in and which strategy they play. Note also that the probability of migrating to a large group with low average payoff may well be higher than the probability of migrating to a small group with large average payoff. However, relative to the group size higher payoff groups are chosen with relatively more probability.

Similarly the probability that a reviewing j-strategist who remains in his current group k switches to using strategy $i \in S$ is assumed to depend only on the payoff strategy i currently yields against the population mix at group k, and on the current population composition in group k, $x_k \in \Delta_s$.

$$p^i_{kj}(x,y) = \frac{\alpha_I\left(u(e^i, x_k)\right) x_{ki}}{\sum_{i' \in S} \alpha_I\left(u(e^{i'}, x_k)\right) x_{ki'}}, \tag{6}$$

where α_I is also a strictly positive and strictly increasing function. In sum, individuals either imitate "successful agents" within their group or migrate to "successful" groups, where they then imitate "purely", i.e. the first random person they meet.

The general dynamics in equation (2) under restrictions (3), (4), (5), and (6) reduce to

$$\begin{aligned} \dot{z}_{ki} = y_k x_{ki} \quad &\cdot \quad \left\{ (1-\gamma) \left[\frac{\alpha_G\left(u(x_k, x_k)\right)}{\sum_{l=1}^{m} \alpha_G\left(u(x_l, x_l)\right) y_l} - 1 \right] + \right. \\ &+ \quad \left. \gamma \left[\frac{\alpha_I\left(u(e^i, x_k)\right)}{\sum_{j \in S} \alpha_I\left(u(e^j, x_k)\right) x_{kj}} - 1 \right] \right\} \end{aligned} \tag{7}$$

for all groups k, all $i \in S$, and for all $(x,y) \in \Theta$. Note that, if the functions α_G and α_I were constant, the above dynamics would reduce to the trivial dynamics $\dot{z}_{ki} = 0$ for all groups k and all strategies i. This is not because individuals do not change strategy or migrate but because their aggregate (average) behavior exactly cancels. I.e. if a certain fraction of people leave group k exactly the same fraction will migrate to group k on average. Therefore, by assuming that the functions α_G and α_I are strictly increasing in their arguments, the above dynamics will resemble a payoff-monotonic dynamics, which generally leads to the growth of successful groups and successful strategies within groups.

As the group size, as a proportion of the total population, is given by $y_k = \sum_{i=1}^{s} z_{ki}$, we obtain a dynamics for the evolution of the group sizes simply by aggregating the dynamics for the various strategies played in the respective groups, i.e. $\dot{y}_k = \sum_{i=1}^{s} \dot{z}_{ki}$. Here this yields

$$\dot{y}_k = (1-\gamma) y_k \left[\frac{\alpha_G\left(u(x_k, x_k)\right)}{\sum_{l=1}^{m} \alpha_G\left(u(x_l, x_l)\right) y_l} - 1 \right] \tag{8}$$

for all groups k. Now, as $z_{ki} = x_{ki} y_k$ it is true that $\dot{z}_{ki} = \dot{x}_{ki} y_k + x_{ki} \dot{y}_k$. Inspection of (7) and (8) reveals that, for all groups k, and for all $i \in S$, and for any relatively interior state $(x, y) \in \Theta$, such that $y_k > 0$,

$$\dot{x}_{ki} = \gamma x_{ki} \left[\frac{\alpha_I\left(u(e^i, x_k)\right)}{\sum_{j \in S} \alpha_I\left(u(e^j, x_k)\right) x_{kj}} - 1 \right]. \tag{9}$$

For states where $y_k = 0$ for some groups k I choose

$$\dot{x}_{ki} = 0 \tag{10}$$

for convenience. Note that the dynamics of the group sizes only depends on the group-average payoffs and the vector of group sizes. Also the dynamics of the group-k proportions of different strategies does neither depend on the group sizes nor on the population composition in groups other than k. Indeed the population dynamics for one group, provided it has a positive population share, is as if it was derived without regard of the other existing groups and is therefore the same, except for the γ-rescaling, as the dynamics in equation (8) in Björnerstedt and Weibull (1996).

Given the assumptions I made about the functions α_G and α_I, both dynamics are payoff-monotonic. This is to say that

$$\dot{y}_k = y_k g_k^{\text{ter}}(y, \bar{u}(x)), \tag{11}$$

with $\bar{u}(x) = (u(x_1, x_1), u(x_2, x_2), ..., u(x_m, x_m))$ denoting the vector of group-average payoffs, where $g_k^{\text{ter}}(y, \bar{u}(x))$ is such that:

$$g_k^{\text{ter}}(y, \bar{u}(x)) > g_l^{\text{ter}}(y, \bar{u}(x)) \Longleftrightarrow u(x_k, x_k) > u(x_l, x_l). \tag{12}$$

Equivalently,

$$\dot{x}_{ki} = x_{ki} g_{ki}^{\text{tra}}(x_k) \mathbf{1}_{y_k > 0}, \tag{13}$$

where $g_{ki}^{\mathrm{tra}}(x_k)$ is such that

$$g_{ki}^{\mathrm{tra}}(x_k) > g_{kj}^{\mathrm{tra}}(x_k) \Longleftrightarrow u(e^i, x_k) > u(e^j, x_k), \tag{14}$$

and where $\mathbf{1}_{y_k>0}$ is one if $y_k > 0$ and zero otherwise.

I will call (11) the inter-group dynamics and (13) the intra-group dynamics.

Similar to what Björnerstedt and Weibull (1996) observe about the one-group dynamics, it is true that the inter- and intra-group dynamics (11) and (13) reduce to a rescaling of the replicator dynamics of Taylor and Jonker (1978) if the functions α_I and α_G are affine.

3 A biological model of group selection

Suppose multiple species share the same habitat. Animals only interact with other members of their own species. This interaction is depicted by a symmetric 2-player game Γ, just as in the previous section. As the multiple species share the same habitat, however, a strategy used in one species grows if its current payoff (against the population mix in this same species) does better than the total average payoff, averaged over all individuals. Imagine the two species of lions and cheetahs feeding on the same herd of antelopes. The success of a particular strategy used by lions (in the game with other lions to catch prey) relative to the success of cheetahs is not independent of the strategies used by cheetahs in the same game. Formally,

$$\dot{z}_{ki} = \left[u(e^i, x_k) - \sum_{l=1}^{m} y_l u(x_l, x_l) \right] z_{ki}. \tag{15}$$

The evolution of group sizes is then given by $\dot{y}_k = \sum_{i=1}^{s} \dot{z}_{ki}$, i.e.

$$\dot{y}_k = \left[u(x_k, x_k) - \sum_{l=1}^{m} y_l u(x_l, x_l) \right] y_k. \tag{16}$$

Now, as $\dot{z}_{ki} = \dot{x}_{ki} y_k + x_{ki} \dot{y}_k$, the within group dynamics is found to be

$$\dot{x}_{ki} = \left[u(e^i, x_k) - u(x_k, x_k) \right] x_{ki} \mathbf{1}_{y_k > 0}. \tag{17}$$

The combined dynamics given by (16) and (17) is just a special case of the inter- and intra-group dynamics given by (11) and (13).

4 Definitions of Stability

Before I analyze the model, I want to give standard definitions of Lyapunov and asymptotic stability of a state and of sets of states.

The following definitions of Lyapunov and asymptotic stability of a state are taken from Definition 6.5 in Weibull (1995). Let $\Omega \subset \mathbb{R}^n$ denote the state space and let the state evolve according to a regular dynamics, i.e. $\dot{\omega}_i = g_i(\omega)\omega_i$ for $i = 1, ..., n$ with all growth-rate functions g_i Lipschitz continuous and such that every solution trajectory stays in Ω all the time. Then for any starting point is ω^0 there is a unique solution trajectory, which shall be denoted by $\zeta(t, \omega^0)$.

Definition 1 *A state $\omega \in \Omega$ is Lyapunov stable if every neighborhood B of ω contains a neighborhood B^0 of ω such that $\zeta(t, \omega^0) \in B$ for all $\omega^0 \in B^0 \cap \Omega$ and $t \geq 0$. A state $\omega \in \Omega$ is asymptotically stable if it is Lyapunov stable and there exists a neighborhood B^* such that $\lim_{t\to\infty} \zeta(t, \omega^0) = \omega$ for all $\omega^0 \in B^* \cap \Omega$.*

The following definitions of Lyapunov and asymptotic stability of sets of states are taken from Definition 6.6 in Weibull (1995). For a subset $A \subset \Omega$ of the state space, the forward orbit is defined as

$$\gamma^+(A) = \{\omega \in \Omega | \omega = \zeta(t, \omega^0) \text{ for some } \omega^0 \in A \text{ and for some } t \geq 0\}.$$

We say that $\zeta(t, \omega^0)$ converges to a set A if the distance[†] between the state $\zeta(t, \omega^0)$ and the set A converges to zero.

Definition 2 *A closed set $A \subset \Omega$ is Lyapunov stable if every neighborhood B of A contains a neighborhood B^0 of A such that $\gamma^+(B^0 \cap \Omega) \subset B$. A closed set $A \subset \Omega$ is asymptotically stable if it is Lyapunov stable and if there exists a neighborhood B^* of A such that $\zeta(t, \omega^0)_{t\to\infty} \to A$ for all $\omega^0 \in B^* \cap \Omega$.*

Another stability concept will be of interest to us, the concept of interior asymptotic stability as defined in Cressman and Schlag (1998). This is a slightly less demanding stability concept than asymptotic stability. It requires states or sets to be robust only against perturbations, which are in the strict interior of the state space Ω.

Definition 3 *A closed set $A \subset \Omega$ is interior asymptotically stable if it is Lyapunov stable and if there exists a neighborhood B^* of A such that $\zeta(t, \omega^0)_{t\to\infty} \to A$ for all $\omega^0 \in B^* \cap int(\Omega)$.*

5 One Group

In this section I define sets of asymptotically stable states and various other sets of states for the well-known one-group case. This notation will prove useful when I establish the link between the multi-group case and the one-group case. The dynamics in the one-group case, obviously, reduces to the intra-group dynamics as given in equation (13).

For $x \in \Delta_s$ let $C(x) = \{i \in S | x_i > 0\}$ denote the support of x. Let

$$\Delta^{NE} = \{x \in \Delta_s | u(e^i, x) = \max_{z \in \Delta_s} u(z, x) \ \forall i \in C(x)\}$$

[†]The distance between a set and a state shall be defined as the minimum distance between the state and any state in the set. Let the distance between states be e.g. Euclidean distance or the maximum absolute difference between any coordinates in the two states.

denote the set of symmetric Nash equilibria of the game Γ. Let

$$\Delta^{\circ} = \{x \in \Delta_s | u(e^i, x) = u(x,x) \ \forall i \in C(x)\}$$

denote the set of stationary states. Let Δ^{LS} denote the set of Lyapunov stable states, Δ^{iAS} the set of interior asymptotically stable states, and Δ^{AS} the set of asymptotically stable states. For any symmetric 2-player game and for any payoff-monotonic dynamics the following is true:

$$\Delta^{AS} \subset \Delta^{iAS} \subset \Delta^{LS} \subset \Delta^{NE} \subset \Delta^{\circ} \tag{18}$$

The only non-trivial set inclusion $\Delta^{LS} \subset \Delta^{NE}$ was first shown by Bomze (1986). See also Weibull (1995). It is well know that there are games (and payoff-monotonic dynamics) with no Lyapunov stable states, hence no interior asymptotically stable states and no asymptotically stable states.

6 Multiple Groups

Let Θ° be the set of stationary states under the combination of both the intra- and inter-group dynamics given in (13) and (11).

$$\Theta^{\circ} = \left\{ (x,y) \in \Theta \;\middle|\; \begin{array}{cc} u(e^i, x_k) = u(x_k, x_k) & \forall i \in C(x_k) \ \forall k \in C(y) \\ \wedge\ u(x_k, x_k) = \sum_j y_j u(x_j, x_j) & \forall k \in C(y) \end{array} \right\}$$

The set of stationary states is the same for all payoff-monotonic dynamics.

Proposition 1 *Let $(x,y) \in \Theta^{\circ}$ be Lyapunov stable. Then $x_k \in \Delta^{NE}$ for all $k \in C(y)$.*

Proof: For the case $m = 1$ this proposition has been proved by Bomze (1986) for the replicator dynamics and was shown to hold for payoff-monotonic dynamics by Nachbar (1990). See also Proposition 4.8 in Weibull (1995). The extension of the proof to $m \geq 2$ is trivial: Let $(x,y) \in \Theta^{\circ}$ be Lyapunov stable. Let k be such that $y_k > 0$. Then by the fact that the proposition holds for $m = 1$ and that Lyapunov stability requires that x_k is Lyapunov stable relative to the intra-group dynamics, it must be true that $x_k \in \Delta^{NE}$. QED

Theorem 1 *Let Γ be any symmetric 2-player game. Under the combined intra- and inter-group dynamics of (13) and (11) with at least two groups ($m \geq 2$) there is no asymptotically stable state.*

Proof: This theorem is a straightforward corollary of Lemma 3 in the appendix, which states that no stationary state is isolated, and the fact that every Lyapunov stable state (and hence every asymptotically stable state) is stationary (see Proposition 6.4 in Weibull (1995)). Let $(x, y) \in \Theta^\circ$. Then, by Lemma 3, in any neighborhood of (x, y) there is another stationary point. QED

The above theorem tells us that there are no asymptotically stable states in our model of group selection. Typically, many stationary states, on the other hand, are Lyapunov stable. The drawback of Lyapunov stability as a long-run evolutionary solution concept is that perpetual perturbations might gradually lead quite far away from a Lyapunov stable state. Given the non-existence of asymptotically stable states, the possible multitude of Lyapunov stable states, and the fact that many stationary states of interest in our model are payoff-equivalent[‡], I turn to set-wise concepts of stability.

Below are definitions of a few candidate sets, which may or may not be Lyapunov or asymptotically stable. For $x^* \in \Delta_s$, let

$$\Phi^{x^*} = \{(x, y) \in \Theta | x_k = x^* \ \forall k\} .$$

[‡]Consider for instance two states in which every group plays the same strategy profile but the vector of group sizes differs in the two states.

The set Φ^{x^*} is the closed set of states, where all groups have the same strategy profile x^*, regardless of whether they have a positive or vanishing population share. For $x^* \in \Delta_s$, let

$$\Psi^{x^*} = \{(x, y) \in \Theta | x_k = x^* \ \forall k \in C(y)\}.$$

The set Ψ^{x^*} is the closed set of states, in which all non-vanishing groups have the same strategy profile x^*. All states in one set Ψ^{x^*} (or Φ^{x^*}) are, therefore, in some sense outcome equivalent. Before I make stability statements about these sets, I make a few additional observations.

First note that obviously for every $x^* \in \Delta_s$,

$$\Phi^{x^*} \subset \Psi^{x^*}. \tag{19}$$

Also note that both Φ^{x^*} as well as Ψ^{x^*} are connected sets for any $x^* \in \Delta_s$. Note furthermore that the set Φ^{x^*} is convex for any $x^* \in \Delta_s$, which is not the case for Ψ^{x^*} unless $m = 1$, in which case both sets are identical.

Proposition 2 *The following two statements are equivalent.*

1. $x^* \in \Delta^\circ$ *(stationary in the intra-group dynamics alone)*
2. $\Psi^{x^*} \subset \Theta^\circ$

Proof: "$\Rightarrow$": Let $x^* \in \Delta^\circ$. For any $(x, y) \in \Theta^{x^*}$ both the intra- and inter-group dynamics (13) and (11) vanish. "$\Leftarrow$": Let $\Psi^{x^*} \subset \Theta^\circ$ and suppose $x^* \notin \Delta^\circ$. But then there is a k with $y_k > 0$ and $x_k = x^*$. As $x^* \notin \Delta^\circ$ the intra-group dynamics (13) for group k does not vanish. We thus arrive at a contradiction. QED

It is easy to see that for $x^*, \bar{x} \in \Delta_s$ with $x^* \neq \bar{x}$ it is true that $\Psi^{x^*} \cap \Psi^{\bar{x}} = \emptyset$ (see also Lemma 5 for a stronger result). Also

$$\bigcup_{x^* \in \Delta^\circ} \Psi^{x^*} \subset \Theta^\circ. \tag{20}$$

In general there are other sets, which are subsets of Θ° and do not intersect with any set Ψ^{x^*} for any $x^* \in \Delta_s$. Consider the case where there are two groups and the game is given by payoff matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every state in the set X of states $(x, y) \in \Theta$ such that $x_{k1} = 1$ for some groups k with $y_k > 0$ and $x_{l2} = 1$ for the remaining non-vanishing groups l is also stationary. Also the set Y of states $(x, y) \in \Theta$, where $x_{k1} = x_{k2} = \frac{1}{2}$ for some groups k with $y_k > 0$ and $x_{l3} = 1$ for the remaining non-vanishing groups l is a subset of Θ°.

I make the following genericity assumption to eliminate these sets of stationary states.

Definition 4 *A game Γ is generic if for all $x^*, \tilde{x} \in \Delta^\circ$ with $x^* \neq \tilde{x}$ it is true that $u(x^*, x^*) \neq u(\tilde{x}, \tilde{x})$.*

An example that meets this genericity assumption is the 2×2 coordination game with payoff matrix

$$A = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}.$$

This game has three stationary states, which correspond to the three symmetric Nash equilibria, two of them are pure and one is mixed. Each of them yields a different payoff against itself. Also the PD-game meets this genericity assumption.

For games that meet this genericity assumption the sets Ψ^{x^*}, for $x^* \in \Delta^\circ$, constitute a partition of the whole set of stationary states, Θ° by Lemma 1 and equation (20). In this case, one might call the sets Ψ^{x^*}, for $x^* \in \Delta^\circ$, the components of Θ°.

Lemma 1 *Let* x^* *and* $\tilde{x}$ *denote two elements in* Δ°*. Then*

1. $\Psi^{x^*} \cap \Psi^{\tilde{x}} = \emptyset$ *and*

2. $\Theta^\circ \subset \bigcup_{x^* \in \Delta^\circ} \Psi^{x^*}$

Proof: Part 1 is a corollary to Lemma 5, which states that any two non-identical sets Ψ^{x^*} and $\Psi^{\tilde{x}}$ have disjoint neighborhoods. To prove part 2 let $(x, y) \in \Theta^\circ$. Then $x_k \in \Delta^\circ$ for all $k \in C(y)$. Suppose there are $k, l \in C(y)$ such that $x_k \neq x_l$. But then, by the genericity assumption, $u(x_k, x_k) \neq u(x_l, x_l)$, which implies that $g_k^{\text{ter}}(y, \bar{u}(x)) \neq g_l^{\text{ter}}(y, \bar{u}(x))$. Hence, at least one of the two growth rates must be nonzero. We thus arrive at a contradiction. QED

In the following I restrict the analysis to generic games according to Definition 4. Before I turn to the main results I need one more definition.

Definition 5 *A strategy profile* $x^* \in \Delta_s$ *symmetrically payoff-dominates another strategy profile* $\bar{x} \in \Delta_s$ *if* $u(x^*, x^*) > u(\bar{x}, \bar{x})$*. Let* $A \subset \Delta_s$*. A strategy profile* $x^* \in A$ *is symmetrically payoff-dominant relative to the set* A *if it symmetrically payoff-dominates all* $\bar{x} \in A$ *with* $\bar{x} \neq x^*$*.*

Note that, for generic games, a symmetrically payoff-dominant strategy profile relative to the set Δ^{AS} exists (and is then unique) if and only if Δ^{AS} is non-empty.

Lemma 2 *Let* $x^* \in \Delta_s$ *be symmetrically payoff-dominated by some* $\tilde{x} \in \Delta_s$*. Then* Ψ^{x^*} *is unstable (not Lyapunov stable).*

Proof: There is a state $(x, y) \in \Psi^{x^*}$ such that there is a group k such that $y_k = 0$, $x_k = \tilde{x}$, and $x_j = x^*$ for all $j \neq k$. Let d denote the distance according to the maximum norm. For all $\epsilon > 0$ there is $(x', y') \in \Theta$ such that $y'_k > 0$, $d(x'_k, \tilde{x}) < \epsilon$, and $d(x'_j, x^*) < \epsilon$ for all $j \neq k$. For ϵ small enough, $u(x'_k, x'_k) > u(x'_j, x'_j)$ for all $j \neq k$, by continuity of u and the fact that $\tilde{x}$ symmetrically payoff-dominates x^*. By the payoff monotonicity of g^{ter} we have $g_k^{\text{ter}}(y', \bar{u}(x')) > g_j^{\text{ter}}(y', \bar{u}(x'))$ for all $j \neq k$ and, hence, $g_k^{\text{ter}}(y', \bar{u}(x')) > \delta$ for all such (x', y'). Hence, the group k grows at least initially until its size is greater than ϵ. QED

Theorem 2 *Let Γ be a generic symmetric 2-player game with two or more groups ($m \geq 2$). Let $X \subset \Theta^\circ$. Then X is asymptotically stable if and only if $X = \Psi^{x^*}$, where $x^* \in \Delta^{AS}$ and x^* is symmetrically payoff-dominant relative to Δ_s.*

Proof: "$\Rightarrow$": Suppose $X \subset \Theta^\circ$ is asymptotically stable. We know that $\Theta^\circ = \bigcup_{\tilde{x} \in \Delta^\circ} \Psi^{\tilde{x}}$ by equation (20) and Lemma 1. By lemma 2 we know that $\Psi^{\tilde{x}}$ is unstable for all symmetrically payoff-dominated $\tilde{x} \in \Delta^\circ$. Lemma 5 states that any two non-identical sets $\Psi^{\bar{x}}$ and $\Psi^{\tilde{x}}$ are separable. It is easy to see that a union of separable unstable sets of states must be unstable. Hence, $X \subset \Psi^{x^*}$, where x^* is symmetrically payoff-dominant relative to Δ_s. Also asymptotic stability of Ψ^{x^*} requires asymptotic stability of x^* relative to the intra-group dynamics. Hence, $x^* \in \Delta^{AS}$. Finally $X = \Psi^{x^*}$ by the fact that Ψ^{x^*} is connected and the fact that no proper subset of a connected set of stationary states can be asymptotically stable by lemma 4, given in the appendix.

"$\Leftarrow$": Let $x^* \in \Delta^{AS}$ be symmetrically payoff-dominant relative to Δ_s. Let $(x, y) \in \Psi^{x^*}$. Define $K = \{k \in \{1, ..., m\} | x_k = x^*\}$. Obviously $C(y) \subset K$ be definition of Ψ^{x^*}. Hence, $K \neq \emptyset$. Note that by stationarity of (x, y) it must be that $g_k^{\text{ter}}(y, \bar{u}(x)) = 0$ for all $k \in C(y)$. By payoff monotonicity also $g_k^{\text{ter}}(y, \bar{u}(x)) = 0$ for all $k \in K$ and $g_k^{\text{ter}}(y, \bar{u}(x)) < 0$ for all $k \notin K$. By continuity of the growth rate functions g^{ter} and by the fact that $x^* \in \Delta^{AS}$ there is a neighborhood B of (x, y) such that $g_k^{\text{ter}}(y', \bar{u}(x')) < 0$ for all $k \notin K$

and for all $(x', y') \in B$ and all trajectories are such that in groups $k \in K$ the strategy profile converges back to x^*. Hence trajectories approach Ψ^{x^*}. QED

The above theorem reveals that, for generic games, the only candidate for an asymptotically stable subset of the set of stationary states is the component generated by the symmetrically payoff-dominant asymptotically stable state of Γ under the intra-group dynamics (13). Furthermore the component, Ψ^{x*}, of Θ° is indeed asymptotically stable provided x^* is not only symmetrically payoff-dominant among all asymptotically stable points but also among all other possible vectors $\bar{x} \in \Delta_s$ as well.

There may well be other asymptotically stable sets, which are not subsets of the set of stationary states. An example is the whole state space. I will not explore the possible existence of more such sets any further, however, at this point.

An immediate corollary to the above theorem is that if there is a state, which symmetrically payoff-dominates all asymptotically stable states of the intra-group dynamics (13) no set of stationary states can be asymptotically stable.

Corollary 1 *Let* Γ *be a generic 2-player game with two or more groups* $(m \geq 2)$*. If there is a vector* $\tilde{x} \in \Delta_s$ *which symmetrically payoff-dominates all* $x^* \in \Delta^{AS}$ *then there is no asymptotically stable subset of* Θ°*.*

In the following I investigate sets of stationary states with respect to interior asymptotic stability.

Theorem 3 *Let* Γ *be a generic 2-player game with two or more groups* $(m \geq 2)$*. Then the set* Φ^{x^*} *is interior asymptotically stable for every asymptotically stable state* $x^* \in \Delta^{AS}$ *of* Γ *under the intra-group dynamics.*

Proof: Let $x^* \in \Delta^{AS}$ be an asymptotically stable state of Γ under the intra-group dynamics (13). Take an arbitrary $(x, y) \in \Phi^{x^*}$. By definition $x_k = x^*$ for all groups k and all $y \in \Delta_m$ (i.e. even for extinct groups k with $y_k = 0$). Let $(x', y') \in \text{int}(\Theta)$ be such that it is within an ϵ-ball around (x, y). As $x^* \in \Delta^{AS}$ and by the fact that all groups k must have $y'_k > 0$, for ϵ small enough it must be true that trajectories staring in this ϵ-ball around (x, y) must approach Φ^{x^*}. QED

In the next few paragraphs I apply the above theorems to generic coordination games.

Definition 6 *A 2-player game Γ with payoff matrix A is a coordination game if*

$$a_{ii} > a_{ji} \ \forall j \in S.$$

Note that a generic coordination game not only has to satisfy that $a_{ii} \neq a_{jj}$ for all $i \neq j$, but also that all mixed symmetric Nash equilibria yield different payoffs when played against itself. Note furthermore that in generic coordination games there is a unique Pareto-optimal symmetric Nash equilibrium.

Corollary 2 *Let Γ be a generic 2-player coordination game with two or more groups ($m \geq 2$). Let $X \subset \Theta^\circ$. Then X is asymptotically stable if and only if $X = \Psi^{x^*}$ where $x^* \in \Delta^{NE}$ is the unique Pareto-optimal symmetric Nash equilibrium of Γ.*

This follows directly from Theorem 2.

Corollary 3 *Let Γ be a generic 2-player coordination game with two or more groups ($m \geq 2$). Then the set Φ^{x^*} is interior asymptotically stable for every pure symmetric Nash equilibrium strategy $x^* \in \Delta^{NE}$.*

This follows directly from Theorem 3 and the fact that all pure symmetric Nash equilibrium strategies x^* in generic 2-player coordination games are asymptotically stable in the one-group dynamics (i.e. are in Δ^{AS}).

7 Examples

7.1 A Coordination Game

Let Γ be the 2×2 coordination game already mentioned with payoff matrix,

$$A = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}. \tag{21}$$

Let there be $m = 2$ groups. I.e. there are two pure strategies A and B such that the normal form of the game is given by:

1\2	A	B
A	4,4	0,2
B	2,0	3,3

The strategy pair (A,A) is the Pareto-dominant equilibrium and (B,B) the risk-dominant equilibrium in the sense of Harsanyi and Selten (1988).

Figure 1 depicts the state space with the two stationary components corresponding to the two pure Nash equilibria of Γ, Ψ^A and Ψ^B. Observe that each component is connected and that there is some distance between the two components. The stationary component corresponding to the mixed equilibrium is ignored, since it does not contain even a single Lyapunov stable state. Also depicted are the two sets $\Phi^A \subset \Psi^A$ and $\Phi^B \subset \Psi^B$. The unique asymptotically stable set of stationary states is the set Ψ^A by force of Corollary 2. By Corollary 3 both sets Φ^A and Φ^B are interior asymptotically stable.

7.2 The Prisoners' Dilemma

Let Γ be the Prisoners' Dilemma game with payoff matrix

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}. \tag{22}$$

Let there be $m \geq 2$ groups. I.e. there are two pure strategies C and D such that the normal form of the game is given by:

1\2	C	D
C	2,2	0,3
D	3,0	1,1

This game has only one Nash equilibrium, (D,D), which is asymptotically stable under any payoff-monotonic dynamics ($m = 1$). By force of Theorem 2 the only candidate component of Θ° to be asymptotically stable is Ψ^D, the component where every individual in non-extinct groups plays pure strategy D. By force of the same Theorem 2, however, this set fails to be asymptotically stable, as any strategy profile which involves any fraction of C-strategists symmetrically payoff dominates pure strategy D. Indeed the set Ψ^D can be invaded by the emergence of any group with some proportion of C-strategists (see Figure 2 for a phase diagram with 2 groups, where x_{2C} is fixed at zero). Let $\tilde{x} \in \Delta_2$ be such that $\tilde{x}_1 = \gamma > 0$, hence $\tilde{x}_2 = 1 - \gamma$, i.e. a proportion γ of individuals plays C in this group. The payoff which $\tilde{x}$ gets against itself is $u(\tilde{x}, \tilde{x}) = 1 + \gamma$, which is strictly greater than 1, the payoff which strategy D receives when playing against itself. Thus, any group that emerges with an arbitrary proportion of C-strategists will grow. Eventually evolution will favor D-strategists within this group. A small deviation from the set Ψ^D, therefore, initially leads away from it – hence, Ψ^D is not even Lyapunov stable – but will ultimately lead back to another section of Ψ^D.

Indeed the only asymptotically stable set is the whole state space. Of course, by force of theorem 3, the set Φ^D is interior asymptotically stable (and, implicitly, Lyapunov stable).

8 Discussion

Theorem 2 reveals that if an asymptotically stable set (among the set of stationary states) exists it must be that component of the set of stationary states where the strategy profile in every non-extinct group coincides with the symmetrically payoff-dominant asymptotically stable state of the one-group dynamics. If any other state of the one-group dynamics symmetrically payoff-dominates all asymptotically stable states of this dynamics, then no such set exists.

Asymptotic stability requires a set of states to be robust against any kind of perturbation, be it in the interior of the state space or on the boundary. In some sense, therefore, it requires a set to be robust against the emergence of a high-efficiency group. The only stationary component to satisfy this is the symmetrically payoff-dominant component. A symmetrically payoff-inferior component can be invaded by the emergence of a small group of individuals all playing the symmetrically payoff-dominant strategy.

Theorem 3, on the other hand, gives multiple predictions. Indeed it fails to provide a refinement of asymptotic stability of states under the one-group dynamics. The reason is that in this case perturbations cannot be on the boundary of the state space. In particular, a perturbation cannot be such that a latent group (i.e. extinct) gradually moves from playing a certain strategy profile to one where everyone plays the symmetrically payoff-dominant equilibrium strategy. Thus if a new group emerges it must have a strategy profile close to the ones observed in the state the system is in.

The result that in coordination games Pareto-efficiency can be achieved through group selection coincides with the result in Ely's (2002) stochastic model of evolution, when individuals choose strategy as well as location. As Ely (2002) points out, this result is closely related to the result that in a cheap-talk coordination Pareto-efficiency is the only evolutionary stable outcome (see Robson, 1990, Matsui, 1991, Wärneryd, 1993, and Kim and Sobel, 1995).

In a cheap talk coordination game, an equilibrium with everyone playing a Pareto-inferior equilibrium strategy can be invaded by the emergence of a group of specific mutants. These mutants send a previously unsent message and play the Pareto-inferior equilibrium strategy if the randomly met opponent sends a different signal, but play the Pareto-optimal equilibrium strategy if the opponents sends the same signal. The mutants earn higher payoffs than the incumbents and cannot be invaded again. Crucial to this argument is the existence of a previously unsent signal, which can then be used to coordinate on. If every available message is sent by a fraction of individuals in the population, and if every such message is answered by playing the inefficient equilibrium strategy, then no mutant type playing the efficient equilibrium strategy can invade the population. Kim and Sobel (1995) argue that evolutionary drift ultimately leads to at least one message not being used by anyone, and a subsequent invasion by a mutant type sending this message and playing the efficient equilibrium strategy upon receiving it will be successful. This process may take quite a long time, as argued e.g. in Bhaskar (1998), who makes communication noisy (i.e. sent messages do not always coincide with received messages) to avoid this problem.

In the case of group selection the group can be viewed as a message: being in a specific group is like sending a particular message. Only that set of states where all individuals are in groups, where the Pareto-optimal equilibrium strategy is played, is non-invadable. As with the cheap-talk literature, this result relies on evolutionary drift. Here drift must first make at least one group disappear, which then latently drifts towards efficiency and subsequently bursts back into existence. Yet another interpretation circumvents the drift argument: rather than a latent drifting group jumping into existence, it could just be a new highly efficient group, which will then gain a foothold rather quickly.

The analogue between group-selection and cheap-talk can also be established for the PD-game. Robson (1990) notes that given a situation where everyone plays D, a group of mutants could be successful in the short term if they coordinate and play C conditional on observing (and sending) a previously unsent signal as well. Robson (1990) notes, however, that a new type of mutant could appear who would send the same signal and play D regardless of all the signals sent. This new type would then successfully invade a population consisting of some proportion of conditional or unconditional C-strategists.

In group selection a new group of C-strategists can also emerge, who are successful not because they send messages and coordinate on them, but because they happen to be in the same group. The effect, however, will be temporary, since newly emerging D-strategists within this group will dominate this group in the long-run. This temporary effect is the reason for the non-existence of a sensible asymptotically stable set in the multiple-group PD-game.

In the last paragraph I discuss the case where populations are finite and dynamics are truly stochastic. In stochastic models with finite populations whose limit is a deterministic dynamics as in Benaim and Weibull (2001) it is the size of the basin of attraction of the various components which is crucial. If boundaries are reflecting (here in the sense that extinct groups will emerge again) then the relative sizes of the various basins of attraction determine the long-run prediction of the evolutionary model. Essentially, the component with the largest basin of attraction will be the long-run outcome. Let us go back to the generic 2×2-coordination game given in the example. Figure 2 is a phase diagram of the two intra-group dynamics. The inter-group dynamics is not sketched. Nevertheless, it is easy to see that in the upper-left and lower-right areas one of the groups will eventually vanish in the limit. Note that the basin of attraction of the stationary component corresponding to the Pareto-inferior pure strategy is relatively smaller than in the 1-group case. Indeed the relative size of the basin of attraction of the Pareto-inferior component is α^m, where m is the number of groups and α is the probability attached to the Pareto-inferior pure strategy in the mixed equilibrium. Hence, in the ultra-long run every survivor in all groups will play the pure strategy corresponding to the Pareto-optimal Nash equilibrium.

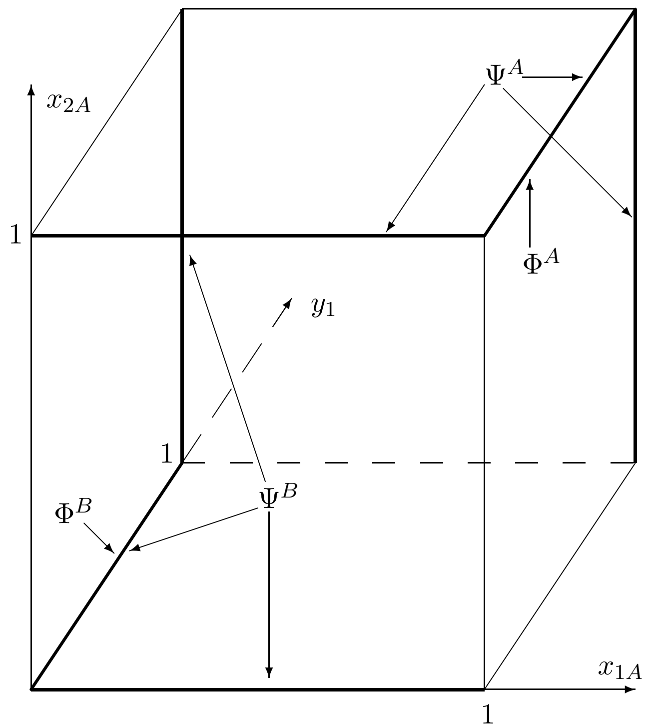

Figure 1: State space Θ for the 2×2-coordination game with payoff matrix given by (21) when there are 2 groups. x_{1A} and x_{2A} denote the proportion of A-strategists in group 1 and 2, respectively; y_1 denotes the proportion of individuals in group 1. Ψ^A and Ψ^B are the components of the set of stationary states, where every individual in non-extinct groups plays A and B, respectively. Similarly Φ^A and Φ^B are the set of stationary states, where every individual in every groups, extinct or not, plays A and B, respectively.

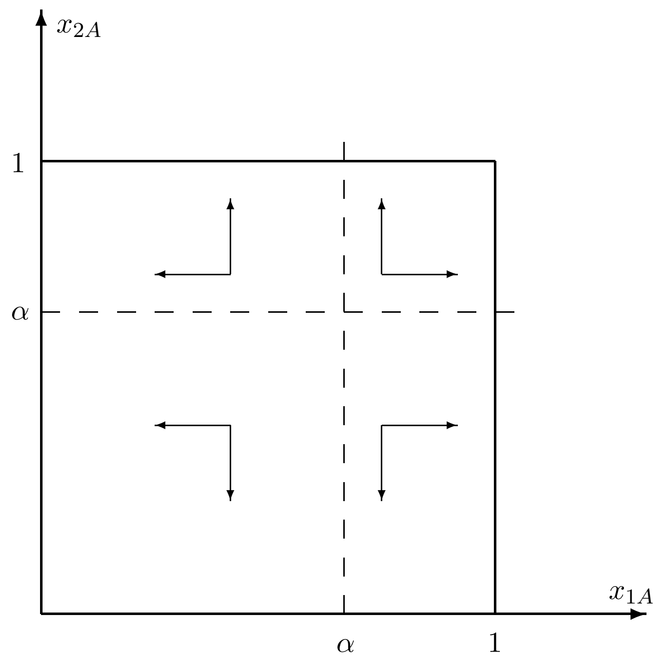

Figure 2: Phase diagram for the intra-group dynamics in the 2×2-coordination game with payoff matrix given by (21) when there are 2 groups. x_{1A} and x_{2A} denote the proportion of A-strategists in group 1 and 2, respectively. α is the probability put on strategy A in the mixed NE in this game.

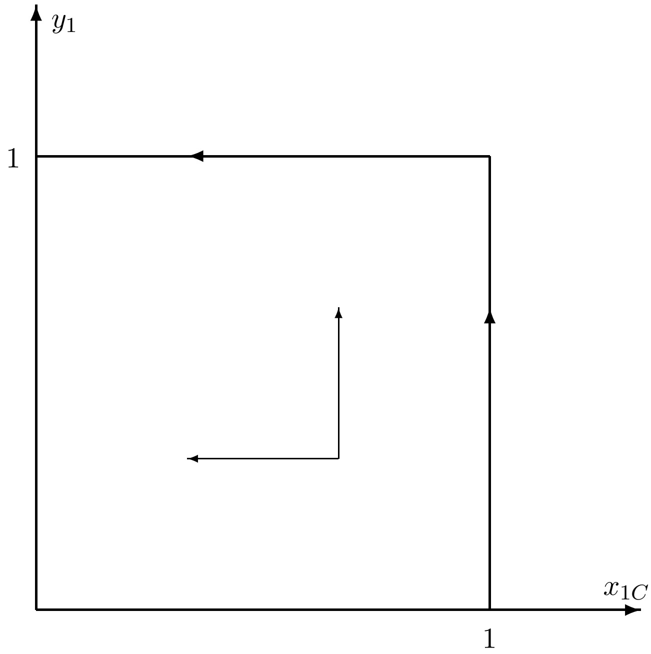

Figure 3: Phase diagram for the inter- and intra-group dynamics in the PD-game with payoff matrix given by (22) when there are 2 groups and where the proportion of C-strategists in group 2 is zero, i.e. $x_{2C} = 0$. x_{1C} denotes the proportion of C-strategists in group 1. y_1 denotes the proportion of individuals in group 1.

9 Appendix

9.1 Additional Lemmas

Lemma 3 states that if there are at least two groups ($m \geq 2$) no stationary state is isolated. To state the lemma we need to define some distance function $d(\cdot,\cdot)$ between points in $\mathbb{R}^n$ for any n. For $z, z' \in \mathbb{R}^n$ let $d(z,z') = \max_{1\leq i\leq n} |z_i - z'_i|$.

Lemma 3 *For all states $(x,y) \in \Theta^\circ$ and for all $\epsilon > 0$ there is another state $(x',y') \in \Theta^\circ$ with $(x,y) \neq (x',y')$ such that*

$$d\left((x,y),(x',y')\right) < \epsilon. \tag{23}$$

Proof: Let $(x,y) \in \Theta^\circ$. Consider the first of two cases.

1. Let $y \in \text{int}(\Delta_m)$. Then $u(x_k,x_k) = \sum_j y_j u(x_j,x_j)$ for all $k = 1,...,m$. Hence $u(x_k,x_k) = u(x_j,x_j)$ for all $k,j = 1,...,m$. Let $y' \neq y$ be such that $\max_k |y'_k - y_k| < \epsilon$ (e.g. move $\frac{\epsilon}{2}$ from a group j with $y_j > \frac{\epsilon}{2}$ to any other group) and let $x' = x$. Then $(x',y') \neq (x,y)$ and still $u(x'_k,x'_k) = \sum_j y'_j u(x'_j,x'_j)$, while $y'_k > 0$ for all $k = 1,...,m$. Hence (x',y') is stationary and $d\left((x,y),(x',y')\right) < \epsilon$.

2. Suppose there is a group k such that $y_k = 0$. Let $x'_k \neq x_k$ be such that $\max_i |x'_{ki} - x_{ki}| < \epsilon$ (in a similar fashion as in the first case for y'), while $x'_j = x_j$ for all $j \neq k$ and $y' = y$. Then (x',y') is a rest point of both the intra- and the inter-group dynamics (13) and (11) and, hence, stationary, while it is true that $d\left((x,y),(x',y')\right) < \epsilon$.

QED

The set $\text{int}(\Delta_m) = \{\alpha \in \Delta_m | \alpha_i > 0 \; \forall i\}$ denotes the set of all strictly interior points in the m-simplex.

Lemma 4 *Let $X \subset \Theta^\circ$ be connected. Then no subset $Y \subset X$ is asymptotically stable.*

Proof: Any neighborhood of Y will contain a point $z \in X$ which is not in Y. From there the combined dynamics will not lead back to Y. QED

Lemma 5 *Let $x^*, \bar{x} \in \Delta_s$ such that $x^* \neq \bar{x}$. Then there are open sets B^* and $\bar{B}$ with the property that $\Psi^{x^*} \subset B^*$ and $\Psi^{\bar{x}} \subset \bar{B}$ such that $B^* \cap \bar{B} = \emptyset$.*

Proof: As $x^* \neq \bar{x}$ there is an $\epsilon > 0$ such that $d(x^*, \bar{x}) > 2\epsilon$. Let

$$B^* = \left\{(x', y') \in \Theta | d\left((x', y'), \Psi^{x^*}\right) < \epsilon\right\}$$

and let

$$\bar{B} = \left\{(x', y') \in \Theta | d\left((x', y'), \Psi^{\bar{x}}\right) < \epsilon\right\}.$$

Let (x', y') be an arbitrary state in B^*. For any such (x', y') there is a group k such that $d(x'_k, x^*) < \epsilon$ and $y'_k > \epsilon$. Suppose $(x', y') \in \bar{B}$. As $d(x^*, \bar{x}) > 2\epsilon$, by the triangular inequality it must be true that $d(x'_k, \bar{x}) > \epsilon$. Hence, the only way (x', y') is in $\bar{B}$ is if it is a perturbation from a state $(x, y) \in \Psi^{\bar{x}}$ with $y_k = 0$ and $d(x_k, x^*) < \epsilon$. In this case, however, it must be true that $y'_k < \epsilon$, which provides a contradiction. QED

References

[1] Benaim, M. and Weibull, J. W. (2001). Deterministic approximation of stochastic evolution in games. *mimeo.*

[2] Bhaskar, V. (1998). Noisy communication and the evolution of cooperation. *Journal of Economic Theory*, **82**, 110-31.

[3] Björnerstedt, J. and Weibull, J. W. (1996). Nash equilibrium and evolution by imitation. In: *The Rational Foundations of Economic Behavior.* Proceedings of the IEA Conference held in Turin, Italy. Kenneth J. Arrow *et al.* (Eds.). London: MacMillan Press Ltd, pp 155-71.

[4] Bomze, I. (1986). Non-cooperative two-person games in biology: A classification. *International Journal of Game Theory*, **15**, 31-57.

[5] Boyd, R. and Richerson, P. J. (1985). *Culture and the Evolutionary Process*, University of Chicago Press, Chicago, Illinois.

[6] Boyd, R. and Richerson, P. J. (1990). Group selection among alternative evolutionary stable strategies. *Journal of Theoretical Biology*, **145**, 331-42.

[7] Boylan, R. (1992). Laws of large numbers for dynamical systems with randomly matched individuals. *Journal of Economic Theory*, **57**, 473-504.

[8] Canals, J. and Vega-Redondo, F. (1998). Multi-level evolution in population games. *International Journal of Game Theory*, **27**, 21-35.

[9] Cressman, R. and Schlag, K. H. (1998). The dynamic (in)stability of backwards induction. *Journal of Economic Theory*, **83**, 260-85.

[10] Ely, J. C. (2002). Local conventions. *Advances in Theoretical Economics*, **2**, 1, 1.

[11] Harsanyi, J. and Selten, R. (1988). *A general theory of equilibrium selection in games*, MIT Press, Cambridge, Massachusetts.

[12] Kandori, M., Mailath, G. and Rob, R. (1993). Learning, mutation, and long-run equilibria in games. *Econometrica*, **61**, 29-56.

[13] Kim, Y.-G., and Sobel, J. (1995). An evolutionary approach to pre-play communication. *Econometrica*, **63**, 1181-93.

[14] Matsui, A. (1991). Cheap talk and cooperation in a society. *Journal of Economic Theory*, **54**, 245-58.

[15] Nachbar, J. (1990). "Evolutionary" selection dynamics in games: Convergence and limit properties. *International Journal of Game Theory*, **19**, 59-89.

[16] Oechssler, J. (1999). Competition among conventions. *Computational & Mathematical Organization Theory*, **5**, 31-44.

[17] Robson, A. J. (1990). Efficiency in evolutionary games: Darwin, Nash and the secret handshake. *Journal of Theoretical Biology*, **144**, 379-96.

[18] Sjöström, T. and Weitzman, M. L. (1996). Competition and the evolution of efficiency. *Journal of Economic Behavior and Organization*, **30**, 25-43.

[19] Sober, E. and Wilson, D. S. (1998). *Unto others: The Evolution and Psychology of Unselfish Behavior*, Harvard University Press, Cambridge, Massachusetts.

[20] Taylor, P. and Jonker, L. (1978). Evolutionary stable strategies and game dynamics. *Mathematical Biosciences*, **40**, 145-56.

[21] Wärneryd, K. (1993). Cheap talk, coordination, and evolutionary stability. *Games and Economic Behavior*, **5**, 532-46.

[22] Weibull, J. W. (1995). *Evolutionary Game Theory*, Cambridge, MIT Press.

[23] Young, H. P. (1993). The evolution of conventions. *Econometrica*, **61**, 57-84.