

The Common and Specific Components of Dynamic Volatility*

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Abstract

This paper develops a dynamic approximate factor model in which returns are time-series heteroskedastic. The heteroskedasticity has three components: a factor-related component, a common asset-specific component, and a purely asset-specific component. We develop a new multivariate GARCH model for the factor-related component. We develop a univariate stochastic volatility model linked to a cross-sectional series of individual GARCH models for the common asset-specific component and the purely asset-specific component. We apply the analysis to monthly US equity returns for the period January 1926 to December 2000. We find that all three components contribute to the heteroskedasticity of individual equity returns. Factor volatility and the common component in asset-specific volatility have long-term secular trends as well as short-term autocorrelation. Factor volatility has correlation with interest rates and the business cycle.

Keywords: APT; ARCH; Factor Models; Principal Components; Volatility

JEL Classification: C13, C23, G12

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1 Introduction

In the approximate factor model of asset returns developed by Chamberlain and Rothschild (1983), the random return on each of n assets is a linear combination of k common factors plus an asset-specific random return, where n is large and k is small. The asset-specific returns are only weakly correlated, in the sense that the largest eigenvalue of the covariance matrix of asset-specific returns is bounded above for all n . This implies that the risk in portfolios with holdings spread thinly over many assets comes only from the common factor returns, not from the asset-specific returns. The factor returns capture nondiversifiable risks, which arise from economy-wide shocks, whereas the asset-specific returns capture diversifiable risks, which arise from the idiosyncratic movements of individual security prices.

Connor and Korajczyk (1986, 1988) develop and apply the asymptotic principal components (APC) method to estimate approximate factor models. They show that, given that the average variance of asset-specific returns is constant through time, the first k eigenvectors of the cross-product matrix of asset returns are a consistent estimate of the k common factors. Scott (1988) and Jones (2001) provide evidence that the cross-sectional average asset-specific variance has considerable time variation. They generalize the APC technique to allow for time-series heteroskedasticity in asset specific returns.

Neither Scott (1988) nor Jones (2001) model the source or nature of the heteroskedasticity in returns since an explicit model is not required for application of their techniques. In this paper, we develop such a model, estimate it, and examine the implications of our findings for asset pricing theory.

We describe a dynamic approximate factor model which includes a three-component model of the dynamic heteroskedasticity in asset returns. One component comes from the dynamic heteroskedasticity in factor returns, one from common heteroskedasticity in asset-specific returns, and one from purely asset-specific heteroskedasticity in asset-specific returns. We use the model to decompose the dynamic heteroskedasticity of individual asset returns into factor-related, common asset-specific, and purely asset-specific components. We apply the techniques to monthly US equity returns for the 900-month period January 1926 to December 2000.

We find that all three components contribute to the dynamic heteroskedasticity of individual asset returns. An interesting aspect of the findings is that a large part comes from the common asset-specific component. Given that asset-specific returns capture the idiosyncratic movements of individual security prices, it is interesting that the dynamic volatility of asset-specific return has a large common component. Something about the random technology generating firm-specific cash flows, or the dynamic flow of information about these firm-specific cash flows, or investor's changing

reaction to firm-specific news, must underlie this common component in the volatility of asset-specific returns.

We find a strong secular component to common volatility of asset-specific returns. It rises sharply during the Great Depression, declines sharply after the Second World War, and then trends slowly upward (with variable slope) in the Post-War period. This is related to the findings of Campbell, Lettau, Malkiel, and Xu (2001) concerning secular trends in individual-asset volatility. We find that the typical asset's factor volatility is high during the depression, declines in the early 1940's and (consistent with Campbell, et al. (2001) is relatively stable in the Post-War period. We also relate the volatility components to macroeconomic variates. Factor volatility is strongly anti-cyclical (higher during recessions) and positively correlated with the term spread and default spread in bond markets. The innovation to common asset-specific volatility is not strongly correlated with the macroeconomic variates.

2 Econometric Methodology

Using the Center for Research in Security Prices (CRSP) monthly dataset of US equity returns, we have a very large cross-section n (21,598 assets including all births and deaths) and a quite large time series T (900 months). We will therefore consider both large n asymptotics and large T asymptotics. Specifically, we will take $n \rightarrow \infty$ and then $T \rightarrow \infty$.¹ We estimate an approximate factor model by the modification of the Connor and Korajczyk (1986) APC method due to Jones (2001) that allows for time series heteroskedasticity assumed away in Connor and Korajczyk (1986). Bai and Ng (2002) and Bai (2003) give primitive conditions, including restrictions on the rates of n and T , under which the estimated factors are consistent in various senses suitable for our purposes, and we shall comment on this further in a subsequent section.

We then define models for heteroskedasticity in both the common factor and the idiosyncratic components of return. We use the estimated factors and idiosyncratic returns to define empirical counterparts of our model and to estimate the heteroskedasticity parameters.

There is a sizeable literature on factor ARCH/GARCH models starting with Engle (1987) and Diebold and Nerlove (1989), and recently reviewed in Sentana (2001). These models are directed

¹Given the large cross-section and moderately large time series, it seems reasonable to employ asymptotic approximations that assume $n \rightarrow \infty$ and $T \rightarrow \infty$ in such a way that $n/T \rightarrow \infty$ [although for some purposes we do not need large T]. This amounts to a pathwise limit approach where $T = T(n)$ for some function T . Under certain additional conditions, that are likely to be satisfied in our case, the pathwise limit is the same as the sequential limit in which: first $n \rightarrow \infty$ and then $T \rightarrow \infty$. The sort of conditions required include restrictions on the rate at which $n/T \rightarrow \infty$ and a so-called tightness condition. This sort of multi-index asymptotics are discussed in Phillips and Moon (1999). See also Bai and Ng (2002).

purely at heteroskedasticity in the common factor, while we also model the idiosyncratic heteroskedasticity, identification being achieved by invoking the large cross-section assumption. We comment below on the differences between our model and these models.

There have been a number of other advances in the estimation and specification of dynamic factor models in recent years. Forni and Reichlin (1998), and Forni, Hallin, Lippi, and Reichlin (2000) have developed and applied dynamic factor models, where the factors have an autoregressive structure in the mean. This type of dynamic factor model is particularly useful in macroeconomic applications where factors have strong predictability in their means. We shall not model any dependence in the mean of the factors but will focus on the volatility structure. This is more relevant for our financial application since return factors have little or no predictability in their means, due to efficient markets, but can have strong predictability in their volatilities.

Ahn, Lee, and Schmidt (2001) and Pesaran (2002) have discussed estimation of a factor model in which some factors are directly observed and others are not. They both propose a more general class of methods that uses both first and second moment information. They focus on the case where n is large and T is small. Nevertheless, their estimation strategy is valid in our case.

2.1 Review of Connor-Korajczyk and Jones

Let r_t denote the n -vector of excess returns on n assets at time t . We assume that returns follow an approximate factor model with fixed $n \times k$ exposure matrix B , k random factors f_t and n asset-specific returns ε_t :

$$r_t = Bf_t + \varepsilon_t. \quad (1)$$

Let $\|X\|$ denote the operator norm of any matrix X (equal to the maximum eigenvalue for a symmetric positive definite matrix). We do not assume that f_t, ε_t are i.i.d., but impose the following weaker conditions on (1):

$$E[\varepsilon_t | f_\tau] = 0 \text{ for all } t, \tau. \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} B' B = M \text{ a nonsingular matrix.} \quad (3)$$

$$\lim_{n \rightarrow \infty} \|E[\varepsilon_t \varepsilon_t']\| = c < \infty. \quad (4)$$

The estimation method we use for the factors $\{f_t\}_{t=1}^T$, APC, relies on the law of large numbers applied to $n^{-1} \varepsilon_t' \varepsilon_\tau$ where n is large. Sufficient conditions for this to apply are given in Connor and Korajczyk (1993): these include restrictions, i.e., mixing conditions, on the cross-sectional dependence and moment conditions. In the temporally heteroskedastic case of this paper, we require these conditions to hold uniformly over t . Here, we simply assume that the relevant law of large numbers

holds, i.e.,:

$$p \lim_{n \rightarrow \infty} \frac{1}{n} \varepsilon_t' \varepsilon_\tau = \begin{cases} 0 & \text{for } \tau \neq t \\ \phi_t > 0 & \text{for } \tau = t. \end{cases} \quad (5)$$

Let R denote the $n \times T$ matrix of excess returns on the n assets over a time period of length T . Let F denote the $k \times T$ matrix of common factor returns and E be the $n \times T$ matrix of asset specific returns for the same sample. Let $\widehat{\Omega} = R'R/n$ denote the $T \times T$ cross-product matrix of returns. We use $Diag[\cdot]$ to denote the function that transforms a T -vector into a $T \times T$ diagonal matrix and $diag[\cdot]$ (with a lower case d) for the function that changes the diagonal components of a $T \times T$ diagonal matrix into a T -vector. Taking the probability limit of the cross-product matrix as n goes to infinity, and using (1)-(5):

$$p \lim_{n \rightarrow \infty} \widehat{\Omega} = \Omega = F'MF + Diag[\phi_1, \dots, \phi_T]. \quad (6)$$

In the special case considered by Connor and Korajczyk where asset-specific returns are identically distributed through time we have $p \lim_{n \rightarrow \infty} E'E/n = \phi I_T$, where ϕ is a constant and I_T denotes the $T \times T$ identity matrix. Note that $F'MF$ has k eigenvectors corresponding to non-zero eigenvalues, and that these eigenvectors are equal to LF for some nonsingular $k \times k$ matrix L . Also, note that the eigenvectors of $F'MF + \phi I_T$ equal the eigenvectors of $F'MF$.² Using the fact that the eigenvector function is a smooth function of a nonsingular matrix this gives, for the case $\phi_t = \phi$ for all t :

$$p \lim_{n \rightarrow \infty} eigvec_k[\widehat{\Omega}] = eigvec_k[p \lim_{n \rightarrow \infty} \widehat{\Omega}] = LF. \quad (7)$$

Equation (7) is the basic result from Connor and Korajczyk: under their assumptions, the first k eigenvectors of the cross-product matrix converge to a rotation of the factor returns. Bai (2003, Theorem 1) derives conditions (including restrictions on the relative growth rates of n and T) under which the eigenvectors provide consistent estimates of LF . Jones (2001) generalizes to the heteroskedastic case by letting the limit of the cross-product matrix of asset-specific returns, $p \lim_{n \rightarrow \infty} E'E/n$, be diagonal rather than scalar. In particular, suppose that we observe this diagonal (rather than scalar) matrix of cross-sectional mean asset-specific variances $Diag[\Phi]$, where $\Phi = [\phi_1, \dots, \phi_T]$, then given (6) we

²To see this, just write out the eigendecomposition $F'MF = Q\Lambda Q'$, where Q, Λ are $T \times T$ matrices with $QQ' = I_T$ and Λ being diagonal with all but the first k diagonal elements being zero. Then, $F'MF + \phi I_T = Q[\Lambda + \phi I_T]Q'$, i.e., the eigenvectors of $F'MF + \phi I_T$ are the same as those of $F'MF$ by the uniqueness of the eigendecomposition. If we normalize F so that F' is the first k columns of Q , corresponding to non-zero eigenvalues, we are implicitly normalizing M as the sub-block of non-zero elements of Λ .

have:

$$p \lim_{n \rightarrow \infty} \text{eigvec}_k \{ \text{Diag}[\Phi]^{-1/2} \widehat{\Omega} \text{Diag}[\Phi]^{-1/2} \} \text{Diag}[\Phi]^{1/2} = \quad (8)$$

$$\text{eigvec}_k \{ \text{Diag}[\Phi]^{-1/2} F' M F \text{Diag}[\Phi]^{-1/2} + I_T \} \text{Diag}[\Phi]^{1/2} = LF. \quad (9)$$

We follow Jones in using (8) to estimate the factors and asset-specific returns. Our methodological contribution is to develop an explicit model for the heteroskedasticity in returns, and to estimate this three-component model of heteroskedasticity together with the approximate factor model.

2.2 A Two-component Model of Heteroskedasticity in Asset-Specific Returns

As mentioned in the introduction, the presence of heteroskedasticity in asset-specific returns is not surprising, but the *nature* of the heteroskedasticity is interesting. In particular, there is a strong *commonality* in the heteroskedasticity, so that the average across n assets of squared asset-specific return varies through time.

We capture the commonality in asset-specific volatilities with a univariate time-series model for the cross-sectional mean-square asset-specific return ϕ_t (hereafter called common asset-specific variance). By construction, this variate is always positive. We assume that its logarithm follows a nonparametric local trend model with stationary innovations:

$$\log \phi_t = g_\phi(t/T) + u_t, \quad a(L)u_t = b(L)\psi_t, \quad (10)$$

where ψ_t is a stationary process, and the lag polynomial $a(L) = 1 - a_1L - \dots - a_pL^p$ has roots outside the unit circle. The ‘local’ trend term $g_\phi(t/T)$ is motivated by the graphical evidence for secular trends shown later (Figure 1a). We do not wish to restrict the functional form of $g_\phi(\cdot)$, and use semiparametric methods to determine its shape. At the same time we allow for short run dynamics through the polynomials $a(L), b(L)$. The central version of this model has ψ_t being a martingale difference sequence, i.e., $E(\psi_t | I_{t-1}) = 0$, where I_{t-1} is lagged information. We also consider the case where ψ_t is identified off a quantile restriction, so that $\mathbf{quant}_\vartheta(\psi_t | I_{t-1}) = 0$ for some $\vartheta \in (0, 1)$, where \mathbf{quant}_ϑ is the ϑ -quantile function. The motivation to consider the quantile model is its robustness with respect to large observations, which we certainly have during the Great Depression period. In both specifications we allow for conditional heteroskedasticity in ψ_t . We give some more detail below on how the function $g_\phi(\cdot)$ was estimated and which orders of a, b were chosen.

In either the mean-identified or quantile-identified model, the dominant part is the nonparametric trend function in terms of interpretation and statistical difficulty. We conducted some unit root tests below and find against a stochastic trend [unit root] alternative, and in favour of our deterministic

trend specification. We have tried a number of other specifications for ϕ_t , including: specifications in levels, including monthly dummy variables, and fitting the evident heteroskedasticity in ϕ_t with volatility models, but the basic features of the data remain in these other specifications.

We note that the particular form of the time-series model for ϕ_t has no effect on the estimates for the other components of volatility.

Next we model the purely asset-specific volatility. Let η_t denote the n -vector time-series of volatility-specific, asset-specific returns. The n -vector of asset-specific returns at time t is the product of the scalar common volatility and the n -vector of volatility-specific, asset-specific returns:

$$\varepsilon_t = \phi_t^{1/2} \eta_t. \quad (11)$$

Since the scale of the two components in the product (11) is indeterminate we normalize η_t to have unit expected inner-product

$$\frac{1}{n} E[\eta_t' \eta_t] = 1. \quad (12)$$

Equations (10)-(12) are a type of stochastic volatility model, where ϕ_t is the state variable generating time-varying common volatility. One difference from fixed-dimensional stochastic volatility models is that the state variable in this stochastic volatility model is approximately observable, by taking a probability limit for large n .

Although, by construction, η_{it} has no common volatility across assets, it can still have time-series autoregressive heteroskedasticity on an individual-asset basis. To capture this, we assume that η_{it} follows a standard univariate volatility model for each i

$$\eta_{it} = h_{it}^{1/2} z_{it}, \quad (13)$$

where h_{it} is time-varying variance. We consider three models for this purely asset-specific volatility. The first is a GARCH(1,1) model (Bollerslev (1986)):

$$h_{it} = \omega_{0i} + \omega_{1i} h_{it-1} + \omega_{2i} \eta_{it-1}^2, \quad (14)$$

where $(\omega_{0i}, \omega_{1i}, \omega_{2i})$ are unknown parameters varying freely from asset to asset.

The second is the GJR model (Glosten, Jagannathan, and Runkle (1993)):

$$h_{it} = \omega_{0i} + \omega_{1i} h_{it-1} + \omega_{2i} \eta_{it-1}^2 1(\eta_{it-1} < 0) + \omega_{3i} \eta_{it-1}^2 1(\eta_{it-1} \geq 0). \quad (15)$$

where $1(x)$ has the value 1 if the condition x is true and zero otherwise. The third alternative is the EGARCH model (Nelson (1991)):

$$\log(h_{it}) = \omega_{0i} + \omega_{1i} \log(h_{it-1}) + \omega_{2i} (|\eta_{it-1}|/h_{it-1} - \omega_{3i} \eta_{it-1}/h_{it-1}). \quad (16)$$

For all three models, we suppose that z_{it} is an innovation sequence that has mean zero and variance one, conditional on past information. We can allow for general cross-section dependence in z_{it} and some temporal dependence and heterogeneity (semi-strong GARCH), although the central model here is i.i.d. across time and asset. We are primarily interested in weakly stationary processes because only in this case does the unconditional variance of r_t exist. This entails certain restrictions on the parameters of (14)-(16). In (14) we suppose that $\omega_{0i} > 0, \omega_{1i}, \omega_{2i} \geq 0$ for positivity of h_{it} , and $\omega_{1i} + \omega_{2i} \leq 1$ for weak stationarity. In the GJR model, the positivity restrictions are the same as GARCH(1,1) with the addition of $\omega_{3i} \geq 0$. The stationarity restrictions are more complicated and depend on the distribution of z_{it} : the central case is where z_{it} is symmetric about zero in which case it is necessary and sufficient that $\omega_{1i} + (\omega_{2i} + \omega_{3i})/2 \leq 1$ for weak stationarity. In the EGARCH model it is not necessary to impose any restrictions on $\omega_{0i}, \omega_{2i}, \omega_{3i}$ for positivity of variance, and for both weak stationarity and strong stationarity it suffices that $|\omega_{1i}| < 1$ [under moment conditions on z_{it}]. See Carrasco and Chen (2002) for more detailed discussion.

2.3 A One-Component Model of Heteroskedasticity in Common Factors

The Diebold and Nerlove (1989) factor ARCH model specifies an ARCH process directly for each of the unobserved factors f_t .³ The fit of the factor model of returns (1) is unaffected if a k -vector of factor variates f_t is replaced with any nonsingular rotation Lf_t as long as the exposure matrix B is replaced by BL^{-1} . However the dynamic volatility process followed by any individual factor is *not* invariant with respect to a rotation. Hence, volatility models which apply to the individual factors are problematic, unless a particular, economically meaningful, rotation is chosen to define the individual factors.

We develop a univariate measure of factor volatility that is invariant to factor rotations. In particular, note that the quadratic product $f_t'(\frac{1}{n}B'B)f_t$ is rotation invariant and aggregates in a useful way the volatility of all the factors. Using the assumed convergence of $\frac{1}{n}B'B$ we have $\lim_{n \rightarrow \infty} f_t'(\frac{1}{n}B'B)f_t = f_t'Mf_t$. This quadratic product has a useful relationship to ϕ_t defined above, in particular,

$$p \lim_{n \rightarrow \infty} \frac{1}{n} r_t' r_t = f_t' M f_t + \phi_t = \gamma_t + \phi_t.$$

In words, cross-sectional mean-square excess return approximately equals the sum of cross-sectional mean-square common-factor-related return and cross-sectional mean-square asset-specific return. This approximation ignores any observed cross-sectional covariance between common factor and asset-specific returns, since this covariance will be small for large n . We suppose that

³Engle (1987) instead replaces f_t by observed portfolios.

$$\log \gamma_t = g_\gamma(t/T) + v_t, \quad c(L)v_t = d(L)\xi_t, \quad (17)$$

where ξ_t is a stationary process, and the lag polynomial $c(L) = 1 - a_1L - \dots - a_pL^p$ has roots outside the unit circle. We allow the function $g_\gamma(\cdot)$ to have a general shape. Similar considerations apply to the time series process for γ_t as they did to the time series process for ϕ_t .

2.4 Overview

In this section we give a discussion of the full model and the relevant asymptotic theory for our estimates. The full model is (1), (10),(11), (13), (14), (17). The basic unmodelled innovation processes are ψ_t, ξ_t , and the parameters to be determined include the factors f_t , the dynamic parameters in the lag polynomials a, b, c, d , the functions g_γ, g_ϕ , and the parameters of the individual volatility models, $\{\omega_{ji}\}$. Also to be determined are the number of factors to include and the orders of the various dynamic process and the shapes of the trend functions. In view of the enormous sample size at our disposal and the large number of parameters to be estimated we do estimation of the separate parts of our model one at a time rather than try to jointly optimize. It may be possible to improve the efficiency of some of our procedures, but a more complicated procedure would carry some risks due to the consequences of misspecification.

Under (1)-(12) we have that $p \lim_{n \rightarrow \infty} \widehat{F} = F$ and $p \lim_{n \rightarrow \infty} \widehat{\Phi} = \Phi$ by the continuous mapping theorem for each T . This result can be extended to $\max_{1 \leq t \leq T} \|\widehat{f}_t - f_t\| = o_p(1)$ and $\max_{1 \leq t \leq T} \|\widehat{\phi}_t - \phi_t\| = o_p(1)$ when T is allowed to grow with n , under some restrictions on the growth of T, n ; see Bai (2003). Regarding the parameters of our time series models, for example the parameters of $g_\phi(\cdot), a(L)$, and $b(L)$, we will compute standard errors based on a standard root-T consistency theory. In order for the estimation error at the factor estimation stage not to affect the subsequent time series estimation, we require the stronger uniform convergence results:

$$\max_{1 \leq t \leq T} \|\widehat{f}_t - f_t\| = o_p(T(n)^{-1/2}) \quad \text{and} \quad \max_{1 \leq t \leq T} \|\widehat{\phi}_t - \phi_t\| = o_p(T(n)^{-1/2}). \quad (18)$$

These conditions can be satisfied under strong restrictions on the rates at which n, T increase, see Bai and Ng (2002) and Bai (2003). Specifically, a necessary condition is that $n/T^2 \rightarrow \infty$. Given the very large cross-section we have, this condition seems quite reasonable. When the required regularity conditions are satisfied, our time series estimates are \sqrt{T} consistent and the usual distribution theory holds.

3 Data and Estimation Details

We use 900 months of excess returns data on 21,598 (total) US equities for the period January 1926 to December 2000 from the Center for Research in Security Prices (CRSP) monthly returns data file. For the estimation of the factor model, we divide the period into fifteen, 60-month subperiods. Connor and Korajczyk (1993) find between three and six pervasive factors for the CRSP monthly data whereas Bai and Ng (2002), relying on a different test but similar CRSP data, argue for only two pervasive factors. Jones (2001) used five factors. After experimenting with five, ten and fifteen factors, we decided to use five as the number of pervasive factors.

We use the Connor and Korajczyk (1987) definition of the cross-product matrix to account for missing observations:

$$\Omega_{t\tau} = \frac{1}{n_{t\tau}} \sum_{i=1}^{n_{t\tau}} r_{it} r_{i\tau},$$

where $n_{t\tau}$ is the number of firms with returns in both months t and τ and the index runs over these firms. Table 1 shows the minimum, maximum and average of n_{tt} and $n_{t\tau}$ $t \neq \tau$ within each 60-month subperiod.

The first estimation step produces n -consistent estimates of the $k \times T$ matrix of factor returns F and common asset-specific variances $\Phi = (\phi_1, \dots, \phi_T)'$ on each of the 60-month subperiods. The second-stage estimates come from essentially the same procedure as in Jones (2001). Let $\hat{\Lambda}$ denote the $k \times k$ diagonal matrix of the first k eigenvalues of the cross-product matrix $\hat{\Omega}$. We have:

$$\hat{F} = \text{eigvec}_k \{ \text{Diag}[\hat{\Phi}]^{-1/2} \hat{\Omega} \text{Diag}[\hat{\Phi}]^{-1/2} \} \text{Diag}[\hat{\Phi}]^{1/2} \quad (19)$$

$$\hat{\Phi} = \text{diag}[\hat{\Omega} - \hat{F}' \hat{\Lambda}^2 \hat{F}]. \quad (20)$$

The system is only iteratively defined since the estimation of \hat{F} in (19) requires $\hat{\Phi}$ from (20) and vice-versa. We begin with $\hat{\Phi} = I_T$ in (19) and iterate between the two estimation problems, as suggested in Jones (2001).⁴

The estimation of the common asset-specific variances $\Phi = (\phi_1, \dots, \phi_T)'$ in (20) assumes balanced panels; the extension to unbalanced panels is straightforward. We regress each asset's excess return on a constant and the previous-iteration \hat{F} and keep the time-series regression residuals $\hat{\epsilon}_{it}$. For this

⁴In the balanced panel case, this procedure is equivalent to minimizing the criterion

$$\sum_{t=1}^T \sum_{i=1}^n (r_{it} - b'_i f_t)^2 \phi_t^{-1} + \sum_{t=1}^T \log \phi_t$$

with respect to $(\phi_1, \dots, \phi_T, f_1, \dots, f_T)$ and (b_1, \dots, b_n) subject to the identifying restrictions that $F'F/T = I_T$; see Bai and Ng (2002) for the homoskedastic special case.

set of regressions we use all assets with at least 36 months of returns within the relevant subperiod. The cross-sectional mean square of $\widehat{\varepsilon}_{it}$ at time t is $\widehat{\phi}_t$.

4 Empirical Results

4.1 Factor Estimation

We first report on the convergence of our iterative factor extraction method. We calculate the cross-sectional average of adjusted R^2 from the time-series regressions of each asset's excess return on the five factor returns. Note that the first iteration is the uncorrected APC procedure and the final iteration is the Jones procedure. For most of the subperiods (14 out of 15, not shown), the Jones procedure has a lower average R^2 than the uncorrected procedure, but the difference tends to be small. For each iteration, Table 2, columns two to four, show the minimum, maximum and average values of the average individual-asset R^2 statistics, across the fifteen subperiods. The minimum and average values are slightly higher with the uncorrected procedure and the maximum is (very slightly!) higher with the Jones procedure.⁵

To test the convergence of the iterative algorithm, we run a time-series regression of each factor return on all of the five factors from the previous iteration, and calculate the minimum of the adjusted R^2 's across these five regressions. We then take a minimum across the fifteen subperiods and use this as the convergence criterion. This minimum R^2 is displayed in the last column of Table 2. In all subperiods the algorithm converges with high precision after no more than nine iterations.

4.2 Volatility Patterns and Trends

We next report the results of our time-series model for ϕ_t , the common asset-specific variance. Figure 1a shows the time-series of ϕ_t (the dots in Figure 1a). The presence of strong secular trends seems obvious from the graph. However there are no obvious structural breaks associated with the 60-month factor model estimation subperiods, and so we use the full 900-month sample for this and later estimation stages.⁶ In view of the potential large influence of the late 1920's and early 1930's we also recomputed our estimators for the sub period 1946:01 onwards for the mean-identified model.

⁵Scott (1988) finds that the modified APC method gives results similar to APC while Jones (2001) finds that the modification improves empirical fit of the extracted factors. ITG (2002, Section 7.3) discusses the out-of-sample performance of the alternative estimators.

⁶We have also re-estimated our main time series equations with dummy variables that are 1 at the start of each of the subperiods and zero elsewhere. The substantive results are not affected by the inclusion of the dummy variables, and only one or two dummy variables themselves are significant at the 5% level in any given equation.

Also for robustness reasons we estimated the quantile-identified model using the full sample.

Our local trend model (10) has quite different implications from say a unit root process, and we first seek to establish which of these two model types is more consistent with the data. We report the results of some standard unit root tests in Table 3. We examine the level and logs using the Augmented Dickey-Fuller (ADF) test and Phillips-Perron (PP) tests, including either a constant or a constant and linear trend, as well as a number of lags to represent the short run dynamics. There seems to be very little evidence of a unit root in any of the specifications, especially when using the PP test. Therefore, we feel that our deterministic trend model is well supported by the data.

We estimate (10) by two different methods. First, we used Hildreth-Lu OLS time-series regression. In this approach we assume that the deterministic trend function $g_\phi(\cdot)$ belongs to a class of time-series polynomials, in particular, the Chebychev polynomial class. The number of terms was selected by the Akaike criterion. In the full sample case this yielded 5 terms. In the post-war subsample 4 polynomials were selected. We experimented also with the order of the autoregressive process and found that 2 and 4 lags, respectively, were sufficient to induce whiteness in the residual processes.⁷

Table 4 Panel A shows the coefficient estimates and their t-statistics, computed with White's heteroskedasticity consistent standard errors. The secular trend function $g_\phi(t)$ is shown as the solid line in Figure 1a. The main feature is the upward trend beginning in the mid-sixties. This trend is still evident in the subsample estimation beginning 1946:01. The R^2 were 0.502 for the full sample and 0.526 for the subsample.

To estimate (10) under the quantile restriction, we used nearest-neighbor quantile estimation as described in Härdle and Linton (1994) with $k = 61$ (two-sided) nearest neighbors, i.e., the window is a rolling 5 years. In Figure 2a we show the results of a nonparametric quantile regression of ψ_t on time. The graph shows the median along with the 75% and 25% quantiles. [It also shows the estimated conditional mean using the same nearest neighbor method]. The advantage of the nearest-neighbor smoothing method is that it is local, so that there is no influence from the Great Depression period to the post war period. In addition, even during a period with large outliers, the quantile smoothers should be influenced only by a substantial part of the data, not a few rogue outcomes. The main feature of this graph agrees with the results of the previous mean regressions, namely there is a strong upward trend in this component of volatility after the mid-sixties.

Next we analyze the time series heteroskedasticity in factor returns using

$$\hat{\gamma}_t = \frac{1}{n} (r_t - \hat{\varepsilon}_t)' (r_t - \hat{\varepsilon}_t).$$

⁷Specifically, we computed the Ljung-Box Q statistics for the residuals and found very high p-values uniformly across the first 36 lags. Specifically, nearly all p-values exceeded 0.5, and the lowest p-value was 0.171 out of the 36 lags for each of the four residual series.

We use the same cross-section of assets for calculating $\hat{\gamma}_t$ as used earlier for the estimation of ϕ_t . Figure 1b shows the time series of $\hat{\gamma}_t$.

As was the case for ϕ_t there is visual evidence of secular trends in $E_{t-1}[\log \gamma_t]$, and we use the same approach as in fitting the trend in $\log \phi_t$. The coefficient estimates and t-statistics are shown in Table 4, Panel B. The R^2 were 0.221 for the full sample and 0.164 for the subsample.

The secular trend function $g_\gamma(t/T)$ from the polynomial fit is shown as a black line in Figure 1b, while in Figure 2b we show the corresponding nearest neighbor fits. The series is dominated by the Great Depression part of the data, and there is much less evidence of an upward trend in γ_t in the postwar period.

The initial impression from Figures 1a and 2a is that the stock market has become increasingly volatile over the last 40 years. As Campbell *et al.* (2001) have noted and Figures 1b and 2b confirm, this initial impression is misleading. During this period, market-wide index volatility has not shown an upward trend (see Schwert (1989)). Except for the Great Depression, five-year market index volatility is fairly flat,⁸ as is factor volatility, $\hat{\gamma}_t$. It is the cross-sectional dispersion of returns, in terms of asset-specific returns that has trended upward. As pointed out by Campbell *et al.* (2001), this implies an increase in the number of assets required to achieve a given level of diversification.

4.3 Economic Forces and Volatility

Officer (1973), Schwert (1989), Hamilton and Lin (1996), and Campbell et al. (2001) find that return volatility (using various measures) moves with the business cycle with higher volatility in recessions. We look at the relation between ϕ_t , γ_t , ψ_t , (from Eq. (10)) and ξ_t (from Eq. (17)) with six macroeconomic indicators. The first macroeconomic variable is a dummy equal to 1 during NBER dated economic expansions and 0 during economic contractions. The same macroeconomic variable is used by Campbell et al (2001, Table VIII). The second macroeconomic variable that we use is the yield spread between long-term U.S. government bonds and one-month Treasury bills (from Ibbotson Associates (2003)), which we call the Term Spread. The third macroeconomic series is the yield spread between Moody's Aaa and Baa rated bonds, which we call the Default Spread. Our fourth macroeconomic series is the dividend yield on the S&P portfolio over the trailing three months (from Ibbotson Associates (2003)). Our remaining two macroeconomic series are available only for a shorter time period (February 1959 to December 2000). They are the experimental leading (XLI) and coincident (XCI) indicators of Stock and Watson (1989).⁹

⁸This result has been widely documented elsewhere. We have estimated trend models in a variety of other index volatility measures including equal weighted and value weighted, S&P500 etc, and have found no significant trend in any of them.

⁹Updated series are available at <http://ksghome.harvard.edu/~J.Stock.Academic.Ksg/xri/0206/xindex.asc>.

Both ϕ_t and γ_t have statistically significant cross-correlations (whether in levels or in logarithms) with the economic variables (except for the correlations between ϕ_t and the NBER variable). However, much of this correlation for ϕ_t is due to the predictable component since the cross correlations between the innovations, ψ_t , and the macroeconomic series (discussed in detail below) are not significant. The innovation in γ_t , ξ_t , are significantly cross-correlated with a number of the macroeconomic series.

In Table 5 we show the t-statistics for the cross correlations (for lags of 12 months to leads of 12 months) between several macroeconomic series and shocks to volatility components ψ_t , (from Eq. (10)) and ξ_t (from Eq. (17)). Shocks to the common volatility of asset-specific returns, ψ_t , are not significantly related to the levels of the macroeconomic series. However, shocks to the common factor volatility, ξ_t , are significantly positively correlated with the default spread and significantly negatively related to the levels of an NBER expansion dummy variable, the U.S. Treasury term spread, and the experimental leading index.

We find that shocks to common asset-specific volatility, ψ_t , are not significantly correlated with leads or lags of the NBER indicator. The factor volatility shock, ξ_t , is significantly negatively correlated with lags of the NBER dummy and with leads of up to five months. The results are generally consistent with the negative correlations between volatility and the NBER dummy found by Campbell et al (2001). Our results show that this relationship is much stronger for factor volatility than for common asset-specific volatility (Campbell et al do not separate the components). We calculate Ljung-Box Q-statistics (see Greene (2000, p. 542)) to test hypotheses that correlations are jointly zero for lags and leads 1-3, 1-6, 1-9 and 1-12. These are shown in Table 6. They are all insignificant for the correlations of the NBER dummy with ψ_t , and are all significant for correlations with ξ_t .

Table 5 shows that there is almost uniformly positive correlation at all leads and lags between the Term Spread and ψ_t . The correlations are generally insignificant as are the Ljung-Box statistics. In contrast, shocks to factor volatility, ξ_t , are negatively correlated with lagged values of the Term Spread (and leads up to two months). Many of these negative correlations are significant. The Q-statistics reject the hypothesis that correlations are jointly zero for all of the lag combinations.

All of the correlations between ψ_t and ξ_t and the Default Spread, except one, are positive. There is uniformly positive correlation at all leads and lags between the Default Spread and ψ_t . The individual and joint test for correlations are insignificant. Shocks to factor volatility, ξ_t , are generally positively correlated with lags and leads of the Default Spread. The correlations are individually significant for lags 1 and 0 as well as for leads 6 and 7. The Q-statistics reject the hypothesis that correlations are jointly zero for of the lag combinations of 1 to 3 months and 1 to 6 months. All lead combinations tested are significantly different from zero.

The correlations between ψ_t and ξ_t and the S&P dividend yield are individually and jointly insignificant. There is positive, but statistically insignificant, correlation between ψ_t and XLI at all leads and lags. There is negative correlation between ξ_t and XLI at all leads and lags. These are individually significant from lags of eight months to leads of four months and jointly significant for all combinations tested in Table 6. Neither ψ_t nor ξ_t are significantly correlated with the coincident index (XCI).

4.4 Purely Asset Specific Volatility Dynamics

To analyze the purely asset-specific component of dynamic heteroskedasticity, we estimate individual GARCH, GJR and EGARCH models on scaled asset specific returns, $\eta_{it} = \phi_t^{-1/2} \varepsilon_{it}$. We limit this part of the analysis to assets with at least 90 months of continuous return observations, which decreases the total number of assets to 7054. In view of the large number of assets we cannot do an asset by asset model specification search, so instead just estimate each of the individual models for each asset $i = 1, \dots, n$ using asset-by-asset maximum likelihood and then report goodness of fit and other outcomes. We estimate the parameters assuming normally distributed innovations, and then alternatively assuming a t-density where the degrees of freedom of the t-density appears as an additional parameter (Bollerslev (1987)).

For the GARCH and GJR models we use the Engle and Sheppard (2001) target variance approach to estimate the intercept coefficient. That is, we transform the GARCH model (14) into $h_{it} = \omega_{0i} + \omega_{1i}(h_{it-1} - \omega_{0i}) + \omega_{2i}(\eta_{it-1}^2 - \omega_{0i})$, which leaves the model unchanged except for the definition of the intercept coefficient. Note that in this form the intercept is the unconditional variance. The transform of the GJR model (15) is straightforward, by analogy. We then use the sample variance as the estimate of ω_{0i} . This removes the intercept coefficient from the nonlinear estimation routine. We find that using this target variance approach improves estimation reliability. There is no obvious way to implement the target variance approach for the EGARCH model so we do not attempt it. Each model is estimated using the Berndt-Hall-Hall-Hausman maximization routine. In the maximization routine we impose the non-negativity constraints required to ensure non-negative variance; for GARCH, this is $\omega_{0i}, \omega_{1i}, \omega_{2i} \geq 0$; for GJR, $\omega_{0i}, \omega_{1i}, \omega_{2i}, \omega_{3i} \geq 0$, there is no restriction needed for EGARCH. We do not impose the additional conditions for weak stationarity.

Table 7 compares the performance of the models by calculating, for each model and for each asset, the mean absolute error (MAE). This is found by treating the implied volatility \widehat{h}_{it} as a prediction of the squared residual, η_{it}^2 , taking the absolute value of the difference, and then taking the mean over the sample period (the sample periods differ across assets but are always at least 90 months long). In

Table 7 we show the median and interquartile range of these MAE's across assets.¹⁰ For each asset we also calculate which of the six model choices had the lowest and highest MAE. In some cases the estimation routine did not converge, or gave estimates which imply covariance nonstationarity; the percentages of these two cases are also shown in Table 7. The MAE calculations are done for the intersection of assets where the estimation routine converged for all six models. The best fitting of the six models is the EGARCH model with a t-density. It has the lowest interquartile range, lowest median, best fit for the largest percentage of assets, and worst fit for the smallest percentage of assets. The GARCH model with a t-density is the worst fitting by all criteria except the upper border of the interquartile range, where the EGARCH with a normal density has a slightly higher value.

Table 8 Panel A shows the median and interquartile range of all the coefficient estimates, and Panel B does the same for the t-statistics of these coefficients. For all three models, the estimated degrees of freedom of the t-density is low, indicating considerable positive excess kurtosis. (Note that the t-statistic of the degrees of freedom parameter shown in Panel B is not intended as a test for the degrees of freedom equal to zero, since this is not an allowable value for the degrees of freedom; the t-statistic in this case should be viewed simply as the coefficient estimate divided by its standard error.)

The GJR model only differs from a GARCH model if the coefficient ω_{3i} differs from ω_{2i} . The usual empirical finding is a larger volatility effect from a negative return than from a positive return of equal magnitude, which implies $\omega_{2i} > \omega_{3i}$. There is some evidence for this asymmetry, although the difference is not usually significant on an asset-by-asset basis. In the EGARCH model, larger volatility effects associated with negative returns is implied by $\omega_{4i} > 0$. Again, there is some evidence for this type of asymmetry.

5 Conclusion

In an approximate factor model, the random return of each asset is divided into factor-related return and asset-specific return. Given a factor model, the time-series heteroskedasticity of individual asset returns can be represented with three components: a factor-related, common asset-specific, and purely asset-specific component. We develop and estimate a heteroskedastic factor model for monthly US equity returns with this three-component structure.

Each of the three components is modeled separately. For the common asset-specific component, we develop a univariate stochastic volatility model. For the factor component we develop a new multivariate GARCH model that reduces in estimation to a univariate GARCH-type model and

¹⁰We report median values where possible so as to mitigate any problem with survivorship sample selection.

is invariant to factor rotations. For the purely asset-specific component we use a cross-section of GARCH-type models.

We find that all three components contribute to the heteroskedasticity of individual asset returns. We find that both factor variance and common asset-specific variance have secular trends. Both components are very high in the Great Depression and then decline until the mid-1950's. After that, common asset-specific variance shows a long steady increase.

Previous authors have shown that return volatility (using various measures) moves with the business cycle with higher volatility in recessions. Using our factor model decomposition, we extend these existing results. The innovations in factor-related volatility have negative correlation with an expansion indicator (a dummy variable which has value one in NBER-dated expansions and zero elsewhere), positive correlation with the term premium and default premium in bond yields, and positive correlation with average dividend yield. The innovations in common asset-specific volatility do not have any clearly strong links to the business cycle or yield variates. A theoretical explanation for these observed patterns, whether a rational-choice-based theory or a behavioural theory, would be a notable contribution.

Our asset-by-asset analysis of scaled asset-specific return shows that there is also an empirically identifiable purely asset-specific component to dynamic volatility. It might be interesting to examine whether the dynamic process of purely asset-specific volatility is related to dynamic information releases and/or trading volume of each particular asset.

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Table 1
Number of return observations for factor estimation

| Subperiod | Number of assets per month: n_{tt} | | | Number of assets in cross-products: $n_{t\tau}$ $t \neq \tau$ | | |
|----------------|--------------------------------------|---------|---------|---|---------|---------|
| | Minimum | Average | Maximum | Minimum | Average | Maximum |
| 1926/1-1930/12 | 494 | 610 | 730 | 429 | 539 | 722 |
| 1931/1-1935/12 | 688 | 703 | 725 | 636 | 676 | 721 |
| 1936/1-1940/12 | 708 | 757 | 780 | 642 | 723 | 778 |
| 1941/1-1945/12 | 781 | 804 | 842 | 749 | 783 | 838 |
| 1946/1-1950/12 | 847 | 939 | 1004 | 826 | 907 | 1003 |
| 1951/1-1955/12 | 1003 | 1031 | 1044 | 944 | 1006 | 1043 |
| 1956/1-1960/12 | 1041 | 1060 | 1101 | 939 | 1013 | 1099 |
| 1961/1-1965/12 | 1100 | 1756 | 2118 | 961 | 1463 | 2109 |
| 1966/1-1970/12 | 2111 | 2188 | 2344 | 1625 | 1953 | 2328 |
| 1971/1-1975/12 | 2345 | 3958 | 5482 | 2007 | 3131 | 5422 |
| 1976/1-1980/12 | 4510 | 4683 | 4826 | 3461 | 4232 | 4800 |
| 1981/1-1985/12 | 4771 | 5364 | 5880 | 3335 | 4575 | 5830 |
| 1986/1-1990/12 | 5471 | 5913 | 6266 | 3423 | 5007 | 6214 |
| 1991/1-1995/12 | 5686 | 6304 | 7087 | 3915 | 5324 | 7002 |
| 1996/1-2000/12 | 6664 | 7293 | 7687 | 4012 | 6072 | 7628 |

Notes to Table 1: Within each 60-month subperiod, the Table shows the minimum, average and maximum number of assets each month (columns two to four) and the minimum, average and maximum number of assets with returns in months t and τ for all $t \neq \tau$.

Table 2
 Cross-sectional average individual-asset adjusted R^2 s, and convergence criteria for
 iterative factor estimates*

| Iteration number | Cross-sectional average individual-asset adjusted R^2 s | | | Convergence criteria |
|------------------|---|---------|---------|----------------------|
| | Minimum | Average | Maximum | |
| 1 | .137 | .365 | .649 | NA |
| 2 | .132 | .361 | .650 | .252 |
| 3 | .131 | .360 | .650 | .847 |
| 4 | .131 | .359 | .650 | .933 |
| 5 | .131 | .358 | .650 | .964 |
| 6 | .131 | .358 | .650 | .971 |
| 7 | .131 | .358 | .650 | .988 |
| 8 | .131 | .358 | .650 | .999 |
| 9 | .131 | .358 | .650 | 1.000 |
| 10 | .131 | .358 | .650 | 1.000 |

Notes to Table 2: Time-series regression of each asset against all five factors is applied to each asset with at least 36 observations within each sixty-month factor estimation subperiod. The cross-sectional average R^2 is calculated across all the regressions within each sixty-month subperiod. The minimum, average and maximum values of these cross-sectional average R^2 s are taken across the fifteen subperiods. The convergence criteria is the minimum adjusted R^2 from the regression of each factor on the five factors from the previous iteration. The minimum is taken across all five factors and all fifteen subperiods. To save space, only the first ten iterations are shown in the table since iterations 11-15 have identical values to iteration 10.

Table 3. Unit Root Tests

| Included | no. lags | Levels | | Logs | |
|------------|----------|---------|---------|---------|---------|
| | | ADF | PP | ADF | PP |
| cons only | 0 | -13.801 | -13.801 | -15.371 | -15.371 |
| | 1 | -8.869 | -12.819 | -9.869 | -14.547 |
| | 2 | -6.658 | -12.808 | -7.818 | -14.722 |
| | 3 | -5.134 | -13.043 | -6.438 | -15.045 |
| | 4 | -4.544 | -13.528 | -5.520 | -15.480 |
| | 5 | -4.212 | -14.011 | -4.800 | -15.941 |
| | 6 | -3.434 | -14.408 | -4.274 | -16.432 |
| | 7 | -3.218 | -14.913 | -3.827 | -16.919 |
| | 8 | -2.711 | -15.346 | -3.569 | -17.420 |
| cons&trend | 0 | -14.758 | -16.225 | -16.225 | -16.225 |
| | 1 | -9.577 | -13.895 | -10.512 | -15.507 |
| | 2 | -7.251 | -13.944 | -8.384 | -15.717 |
| | 3 | -5.653 | -14.221 | -6.957 | -16.059 |
| | 4 | -5.034 | -14.733 | -6.002 | -16.503 |
| | 5 | -4.703 | -15.240 | -5.267 | -16.971 |
| | 6 | -3.898 | -15.662 | -4.720 | -17.466 |
| | 7 | -3.681 | -16.187 | -4.268 | -17.957 |
| | 8 | -3.158 | -16.642 | -4.000 | -18.461 |

Notes to Table 3. This gives the values of the Augmented Dickey Fuller (ADF) and Phillips Perron (PP) test statistics for the null hypothesis of a unit root. The test statistics are computed for the case with a constant (cons only) included and the case with both a constant and linear trend (cons and trend) included, and for a number of choices of short run dynamics (i.e., lags k). The critical values are given below:

Critical Value Without Trend: 1% is -3.440, 5% is -2.865, 10% is -2.569

Critical Value With Trend: 1% is -3.973, 5% is -3.417, 10% is -3.131

Table 4

Time-series models of common asset-specific variance and common factor variance

Panel A: Common asset-specific variance

January 1926 – December 2000

| | b ₀ | b ₁ | b ₂ | b ₃ |
|-------------|----------------|----------------|----------------|----------------|
| coefficient | -4.67 | .439 | .670 | -.052 |
| t-statistic | -148.5 | 8.32 | 14.0 | -0.99 |
| | b ₄ | b ₅ | a ₁ | a ₂ |
| coefficient | -.170 | .361 | .191 | .100 |
| t-statistic | -3.52 | 7.63 | 2.92 | 1.76 |

January 1946 – December 2000

| | b ₀ | b ₁ | b ₂ | b ₃ |
|-------------|----------------|----------------|----------------|----------------|
| Coefficient | -2.97 | -2.75 | 3.18 | -1.77 |
| t-statistic | -1.63 | -.81 | 1.23 | -1.07 |
| | b ₄ | b ₅ | a ₁ | a ₂ |
| Coefficient | .723 | -.025 | .160 | .139 |
| t-statistic | .90 | -.09 | 1.89 | 1.95 |

Panel B: Common factor variance

January 1926 – December 2000

| | b ₀ | b ₁ | b ₂ | b ₃ | b ₄ |
|-------------|----------------|----------------|----------------|----------------|----------------|
| Coefficient | -5.85 | .002 | .449 | .396 | -.184 |
| t-statistic | -72.1 | 0.01 | 3.53 | 3.09 | -1.61 |
| 6.10 | b ₅ | a ₁ | a ₂ | a ₃ | a ₄ |
| Coefficient | .702 | .136 | .106 | .106 | .041 |
| t-statistic | 6.10 | 3.73 | 3.04 | 2.96 | 1.24 |

January 1946 – December 2000

| | b ₀ | b ₁ | b ₂ | b ₃ | b ₄ |
|-------------|----------------|----------------|----------------|----------------|----------------|
| Coefficient | -1.54 | -7.92 | 6.45 | -3.31 | 1.52 |
| t-statistic | -.40 | -1.12 | 1.19 | -.95 | .90 |
| | b ₅ | a ₁ | a ₂ | a ₃ | a ₄ |
| Coefficient | .161 | .139 | .126 | .096 | .059 |
| t-statistic | .27 | 3.33 | 3.00 | 2.35 | 1.48 |

Notes to Table 4.

$$\phi_t = g_\phi(t) + u_t, \quad u_t = a_1 u_{t-1} + a_2 u_{t-2} + \varepsilon_t$$

$g_\phi(t)$ = 5th-order Chebychev polynomial in time with coefficients b_0, \dots, b_5

$$\gamma_t = g_\gamma(t) + u_t, \quad u_t = a_1 u_{t-1} + a_2 u_{t-2} + a_3 u_{t-3} + a_4 u_{t-4} + \varepsilon_t$$

$g_\gamma(t)$ = 5th-order Chebychev polynomial in time with coefficients b_0, \dots, b_5

Table 5: T-statistics of correlations between shocks to volatility components and macroeconomic series.

| lag | NBER expansion | | Term Spread | | Default Spread | | S&P Dividend Yield | | Experimental Leading | | Experimental Coincident | |
|-----|----------------|--------------|-------------|--------------|----------------|-------------|--------------------|---------|----------------------|--------------|-------------------------|---------|
| | ψ_t | ξ_t | ψ_t | ξ_t | ψ_t | ξ_t | ψ_t | ξ_t | ψ_t | ξ_t | ψ_t | ξ_t |
| -12 | -0.21 | 0.83 | 0.98 | 0.24 | 0.18 | 1.52 | -0.27 | 0.86 | -0.71 | -0.20 | -0.13 | -0.04 |
| -11 | -0.27 | 0.81 | 1.67 | 0.42 | 0.60 | 1.22 | -0.66 | -0.18 | -0.44 | -0.16 | 0.00 | -0.02 |
| -10 | -0.30 | 0.48 | 1.16 | 1.25 | 0.89 | 1.31 | -0.27 | -1.16 | -0.24 | -0.64 | -0.27 | 0.16 |
| -9 | -0.57 | 0.21 | 1.40 | 0.81 | 0.84 | 1.43 | 0.03 | -0.98 | -0.60 | -0.91 | -0.18 | 0.07 |
| -8 | -0.66 | 0.03 | 0.96 | 0.87 | 0.87 | 1.91 | 0.15 | 0.12 | -0.29 | -0.64 | -0.20 | 0.09 |
| -7 | -0.45 | -0.57 | 0.36 | 1.28 | 0.93 | 2.36 | 0.21 | 0.78 | -0.11 | -1.20 | -0.24 | 0.09 |
| -6 | -0.21 | -1.32 | 0.24 | 1.08 | 1.02 | 2.12 | -0.93 | 0.99 | -0.53 | -1.14 | -0.20 | 0.13 |
| -5 | -0.30 | -2.42 | 0.36 | 0.18 | 0.48 | 1.77 | -1.37 | 0.24 | -0.49 | -1.67 | -0.25 | 0.09 |
| -4 | 0.06 | -2.60 | 0.54 | 0.66 | 0.54 | 1.62 | -1.61 | 0.39 | -0.47 | -2.63 | -0.22 | 0.07 |
| -3 | 0.24 | -3.17 | 0.96 | 0.24 | 0.78 | 1.80 | -0.57 | 0.15 | 0.00 | -3.26 | -0.25 | 0.04 |
| -2 | 0.93 | -3.66 | 0.69 | -0.57 | 1.26 | 1.80 | 0.18 | 1.44 | 0.29 | -3.56 | -0.27 | 0.09 |
| -1 | 1.53 | -4.41 | 0.48 | -0.18 | 1.41 | 1.98 | 0.54 | 1.68 | 0.47 | -3.92 | -0.34 | 0.20 |
| 0 | 1.08 | -4.29 | -0.24 | -1.47 | 1.26 | 2.34 | 0.30 | 1.80 | 0.67 | -4.12 | -0.38 | 0.27 |
| 1 | 1.32 | -4.44 | 1.50 | -1.32 | 0.84 | 2.43 | -0.12 | 0.57 | 0.98 | -4.45 | -0.38 | 0.29 |
| 2 | 0.06 | -3.72 | 0.60 | -0.66 | 0.63 | 1.92 | -0.03 | -0.27 | 1.14 | -4.74 | -0.38 | 0.25 |
| 3 | 0.81 | -4.46 | 1.17 | -2.61 | 0.81 | 1.44 | 0.39 | 0.03 | 0.69 | -4.11 | -0.34 | 0.29 |
| 4 | 0.42 | -3.95 | 0.54 | -2.75 | 0.99 | 1.08 | 0.69 | 1.20 | 0.04 | -3.66 | -0.36 | 0.27 |
| 5 | -0.06 | -3.83 | 0.51 | -1.41 | 1.20 | 0.51 | 0.75 | 1.70 | 0.18 | -2.97 | -0.33 | 0.29 |
| 6 | -0.57 | -3.14 | 2.15 | -2.24 | 0.99 | 0.18 | -0.18 | 1.67 | 0.16 | -2.34 | -0.29 | 0.31 |
| 7 | -0.21 | -3.32 | 1.11 | -2.00 | 0.99 | 0.24 | -0.81 | 0.84 | 0.22 | -2.20 | -0.29 | 0.29 |
| 8 | -0.15 | -3.29 | 1.70 | -1.91 | 1.25 | -0.24 | -0.81 | 0.30 | 0.29 | -2.11 | -0.33 | 0.36 |
| 9 | -0.15 | -2.54 | 1.67 | -1.70 | 1.31 | -0.06 | -0.09 | -0.42 | -0.02 | -1.40 | -0.29 | 0.38 |
| 10 | -0.48 | -2.48 | 0.24 | -1.43 | 1.04 | 0.15 | 0.63 | 0.12 | -0.89 | -0.47 | -0.27 | 0.35 |
| 11 | -0.72 | -2.39 | 0.36 | -2.18 | 0.83 | 0.21 | 0.54 | 0.15 | -0.58 | -0.22 | -0.22 | 0.29 |
| 12 | -0.72 | -2.47 | 1.13 | -3.16 | 0.66 | -0.18 | 0.27 | 0.24 | -0.73 | -0.35 | -0.24 | 0.35 |

Notes to Table 5. $\text{Corr}(X_t, \text{Macro}_{t-\text{lag}})$ where $X_t = \psi_t$ (shocks to ϕ_t from Eq. (10)) and ξ_t (shocks to γ_t from Eq. (16)). $\text{Macro}_{t-\text{lag}}$ = an NBER business cycle dummy variable; the Term Spread between the yields on long-term and short-term Government bonds; the Default Spread between the yields on Baa and Aaa Corporate bonds; the S&P composite dividend yield; the Stock and Watson experimental leading indicator - XLI; and the Stock and Watson experimental coincident indicator - XCI. Boldface entries have t-statistics greater than 2.0.

Table 6: Ljung-Box statistics for joint significance across lags of Correlations between volatility components and macroeconomic series.

| lag | Correlations with NBER | | | Correlations with Term Spread | | | Correlations with Default Spread | | |
|-----------|------------------------|----------------|---------|-------------------------------|----------------|---------|----------------------------------|----------------|---------|
| | ψ_t L-B | ξ_t L-B | p-value | ψ_t L-B | ξ_t L-B | p-value | ψ_t L-B | ξ_t L-B | p-value |
| -1 to -12 | 4.558 | 59.471 | 0.000 | 10.444 | 6.978 | 0.577 | 9.303 | 37.634 | 0.677 |
| -1 to -9 | 4.357 | 57.898 | 0.000 | 5.183 | 5.130 | 0.818 | 8.091 | 32.023 | 0.525 |
| -1 to -6 | 3.376 | 57.509 | 0.000 | 2.087 | 2.017 | 0.912 | 5.776 | 20.514 | 0.449 |
| -1 to -3 | 3.244 | 42.905 | 0.000 | 1.600 | 0.416 | 0.660 | 4.226 | 10.297 | 0.238 |
| 1 to 3 | 2.447 | 53.357 | 0.000 | 4.011 | 9.052 | 0.260 | 1.726 | 11.617 | 0.631 |
| 1 to 6 | 2.936 | 93.644 | 0.000 | 9.296 | 23.830 | 0.158 | 5.176 | 13.089 | 0.521 |
| 1 to 9 | 3.031 | 122.327 | 0.000 | 16.430 | 34.537 | 0.058 | 9.494 | 13.209 | 0.393 |
| 1 to 12 | 4.317 | 140.716 | 0.000 | 17.928 | 51.684 | 0.118 | 11.761 | 13.308 | 0.465 |

Correlations with Experimental Coincident Index

Correlations with Experimental Leading Index

| lag | Correlations with S&P Dividend Yield | | | Correlations with Experimental Leading Index | | | Correlations with Experimental Coincident Index | | |
|-----------|--------------------------------------|----------------|---------|--|----------------|---------|---|----------------|---------|
| | ψ_t L-B | ξ_t L-B | p-value | ψ_t L-B | ξ_t L-B | p-value | ψ_t L-B | ξ_t L-B | p-value |
| -1 to -12 | 6.730 | 9.987 | 0.617 | 2.360 | 53.580 | 0.000 | 0.632 | 0.144 | 1.000 |
| -1 to -9 | 6.165 | 7.796 | 0.555 | 1.542 | 53.090 | 0.000 | 0.543 | 0.112 | 1.000 |
| -1 to -6 | 6.097 | 6.191 | 0.402 | 1.086 | 50.328 | 0.000 | 0.405 | 0.090 | 1.000 |
| -1 to -3 | 0.648 | 4.964 | 0.174 | 0.316 | 39.080 | 0.000 | 0.253 | 0.057 | 0.996 |
| 1 to 3 | 0.169 | 0.408 | 0.939 | 2.844 | 59.951 | 0.000 | 0.404 | 0.240 | 0.971 |
| 1 to 6 | 1.256 | 7.670 | 0.263 | 2.903 | 88.330 | 0.000 | 0.728 | 0.500 | 0.998 |
| 1 to 9 | 2.576 | 8.618 | 0.473 | 3.041 | 100.020 | 0.000 | 1.012 | 0.870 | 1.000 |
| 1 to 12 | 3.345 | 8.715 | 0.727 | 4.769 | 100.451 | 0.000 | 1.201 | 1.224 | 1.000 |

Notes to Table 6. $\text{Corr}(X_t, \text{Macro}_{t-\text{lag}})$ $X_t = \psi_t$ (shocks to ϕ_t from Eq. (10)) and ξ_t (shocks to γ_t from Eq. (16)). $\text{Macro}_{t-\text{lag}}$ = an NBER business cycle dummy variable; the Term Spread between the yields on long-term and short-term Government bonds; the Default Spread between the yields on Baa and Aaa Corporate bonds; the S&P composite dividend yield; the Stock and Watson experimental leading indicator - XLI; and the Stock and Watson experimental coincident indicator - XCI.

Table 7

Comparative performance of GARCH, Glosten-Jagannathan-Runkle and EGARCH models, using both normal and t-density for the innovations

| | %converged | %stationary | Mean Absolute Error (MAE) | | | % assets lowest MAE | % assets highest MAE |
|------------------|------------|-------------|---------------------------|--------|----------------|---------------------|----------------------|
| | | | lower quartile | median | upper quartile | | |
| GARCH | 99.9 | 97.1 | 1.58 | 2.97 | 5.92 | 0.1 | 5.8 |
| t density-GARCH | 99.6 | 98.0 | 1.62 | 3.11 | 6.19 | 0.1 | 38.7 |
| GJR | 89.4 | 89.1 | 1.41 | 2.61 | 5.09 | 21.7 | 21.5 |
| t density-GJR | 93.3 | 92.3 | 1.33 | 2.45 | 4.69 | 16.8 | 12.5 |
| EGARCH | 96.0 | 91.0 | 1.58 | 3.03 | 6.43 | 0.5 | 30.0 |
| t density-EGARCH | 95.4 | 93.6 | 1.18 | 2.16 | 4.04 | 61.0 | 0.3 |

Table 8
Coefficient estimates for GARCH, Glosten-Jagannathan-Runkle and EGARCH models,
using both normal and t-density for the innovations

Panel A: Coefficients

| | | ω_0 | ω_1 | ω_2 | ω_3 | $\omega_2 - \omega_3$ | DF |
|-----------------|----------------|------------|------------|------------|------------|-----------------------|------|
| GARCH | Lower quartile | 1.29 | .070 | .043 | - | - | - |
| | Median | 2.35 | .428 | .161 | - | - | - |
| | Upper quartile | 4.40 | .702 | .336 | - | - | - |
| Tdensity-GARCH | Lower quartile | 1.29 | .041 | .376 | - | - | 3.34 |
| | Median | 2.35 | .221 | .615 | - | - | 4.15 |
| | Upper quartile | 4.38 | .471 | .781 | - | - | 5.60 |
| GJR | Lower quartile | 1.28 | .187 | .000 | .000 | -.062 | - |
| | Median | 2.33 | .454 | .076 | .048 | .008 | - |
| | Upper quartile | 4.35 | .565 | .203 | .132 | .134 | - |
| Tdensity-GJR | Lower quartile | 1.28 | .168 | .131 | .002 | .048 | .673 |
| | Median | 2.33 | .433 | .277 | .028 | .200 | 1.49 |
| | Upper quartile | 4.34 | .660 | .479 | .132 | .425 | 3.58 |
| EGARCH | Lower quartile | .001 | -.233 | .000 | -.269 | - | - |
| | Median | .173 | .495 | .263 | .261 | - | - |
| | Upper quartile | .813 | .854 | .574 | .735 | - | - |
| Tdensity-EGARCH | Lower quartile | -.438 | .088 | .000 | -.204 | - | 2.43 |
| | Median | -.137 | .687 | .116 | .251 | - | 2.86 |
| | Upper quartile | -.012 | .883 | .222 | .684 | - | 3.49 |

Notes to Table 8.

GARCH $h_t = \omega_0 + \omega_1(h_{t-1} - \omega_0) + \omega_2(\varepsilon_{t-1}^2 - \omega_0)$

GJR $h_t = \omega_0 + \omega_1(h_{t-1} - \omega_0) + \omega_2(\varepsilon_{t-1}^2 - \omega_0)1\{\varepsilon_{t-1} < 0\} + \omega_3(\varepsilon_{t-1}^2 - \omega_0)1\{\varepsilon_{t-1} \geq 0\}$

EGARCH $\log(h_t) = \omega_0 + \omega_1 \log(h_{t-1}) + \omega_2(|\varepsilon_{t-1}/h_{t-1}| - \omega_3(\varepsilon_{t-1}/h_{t-1}))$

DF = estimated degrees of freedom of the t-density of the innovations

Panel B: t-statistics

| | | ω_0 | ω_1 | ω_2 | ω_3 | $\omega_2 - \omega_3$ | DF |
|-----------------|----------------|------------|------------|------------|------------|-----------------------|------|
| GARCH | Lower quartile | - | 0.79 | 0.37 | - | - | - |
| | Median | - | 2.47 | 1.27 | - | - | - |
| | Upper quartile | - | 6.83 | 2.53 | - | - | - |
| Tdensity-GARCH | Lower quartile | - | 0.20 | 1.30 | - | - | 3.46 |
| | Median | - | 0.68 | 2.27 | - | - | 3.92 |
| | Upper quartile | - | 1.67 | 3.73 | - | - | 4.47 |
| GJR | Lower quartile | - | 0.50 | 0.00 | 0.04 | -0.84 | - |
| | Median | - | 1.63 | 0.40 | 0.30 | -0.14 | - |
| | Upper quartile | - | 8.45 | 1.08 | 0.74 | 0.39 | - |
| Tdensity-GJR | Lower quartile | - | 0.42 | 0.45 | 0.03 | 0.67 | 0.20 |
| | Median | - | 1.06 | 0.83 | 0.16 | 0.99 | 0.92 |
| | Upper quartile | - | 2.09 | 1.46 | 0.43 | 3.09 | 4.36 |
| EGARCH | Lower quartile | 0.01 | -0.57 | 0.00 | -0.83 | - | - |
| | Median | 0.74 | 1.42 | 1.15 | 0.84 | - | - |
| | Upper quartile | 2.18 | 4.39 | 2.65 | 2.61 | - | - |
| Tdensity-EGARCH | Lower quartile | -0.96 | 0.16 | 0.00 | -0.66 | - | 2.31 |
| | Median | -0.49 | 1.71 | 0.35 | 0.84 | - | 4.23 |
| | Upper quartile | -0.13 | 3.41 | 0.67 | 2.60 | - | 6.14 |

Chebyshev Polynomial Fit Phi against Time

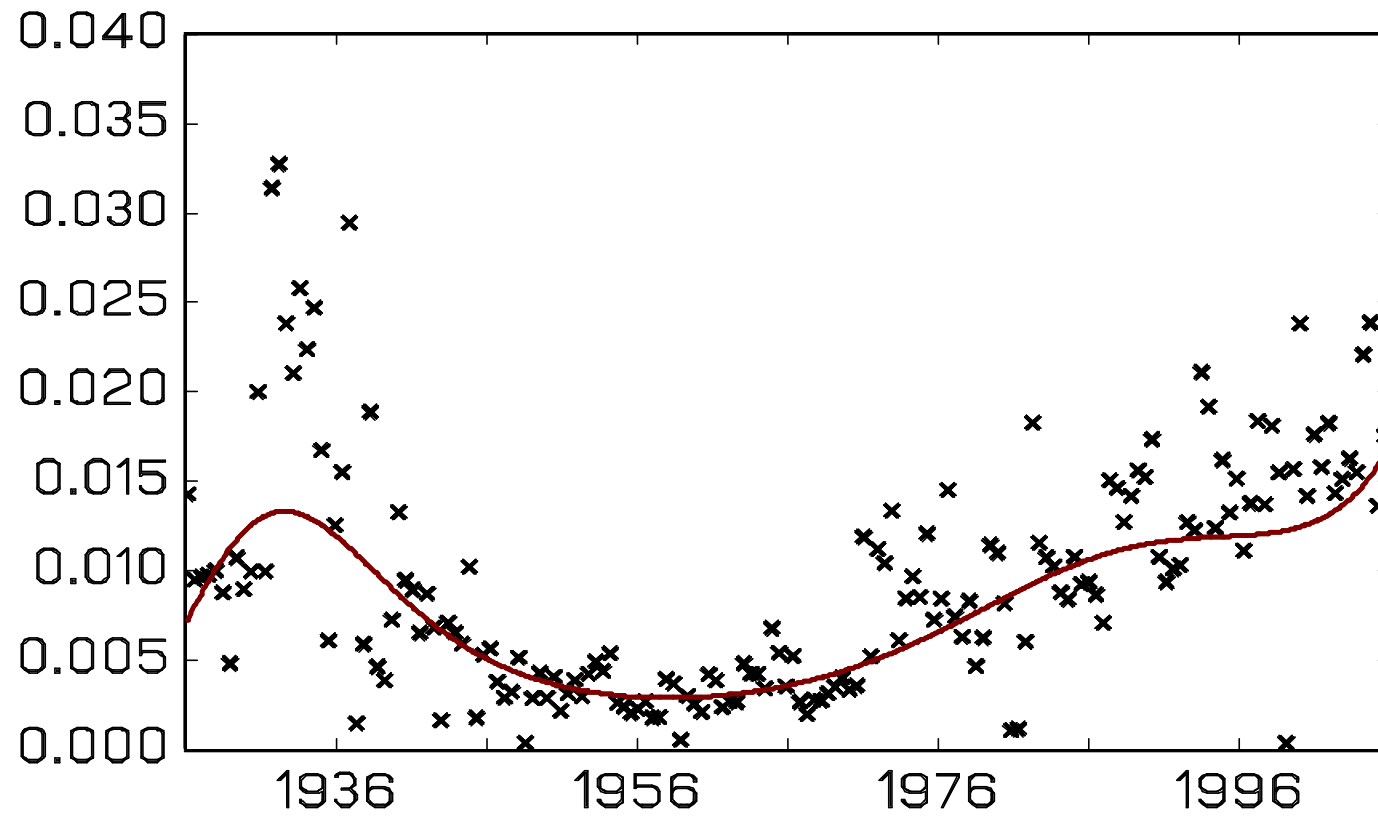


Figure 1a. Shows Chebyshev Polynomial fit of ϕ_t against time along with data

points.

Chebyshev Polynomial Fit Gamma against Time

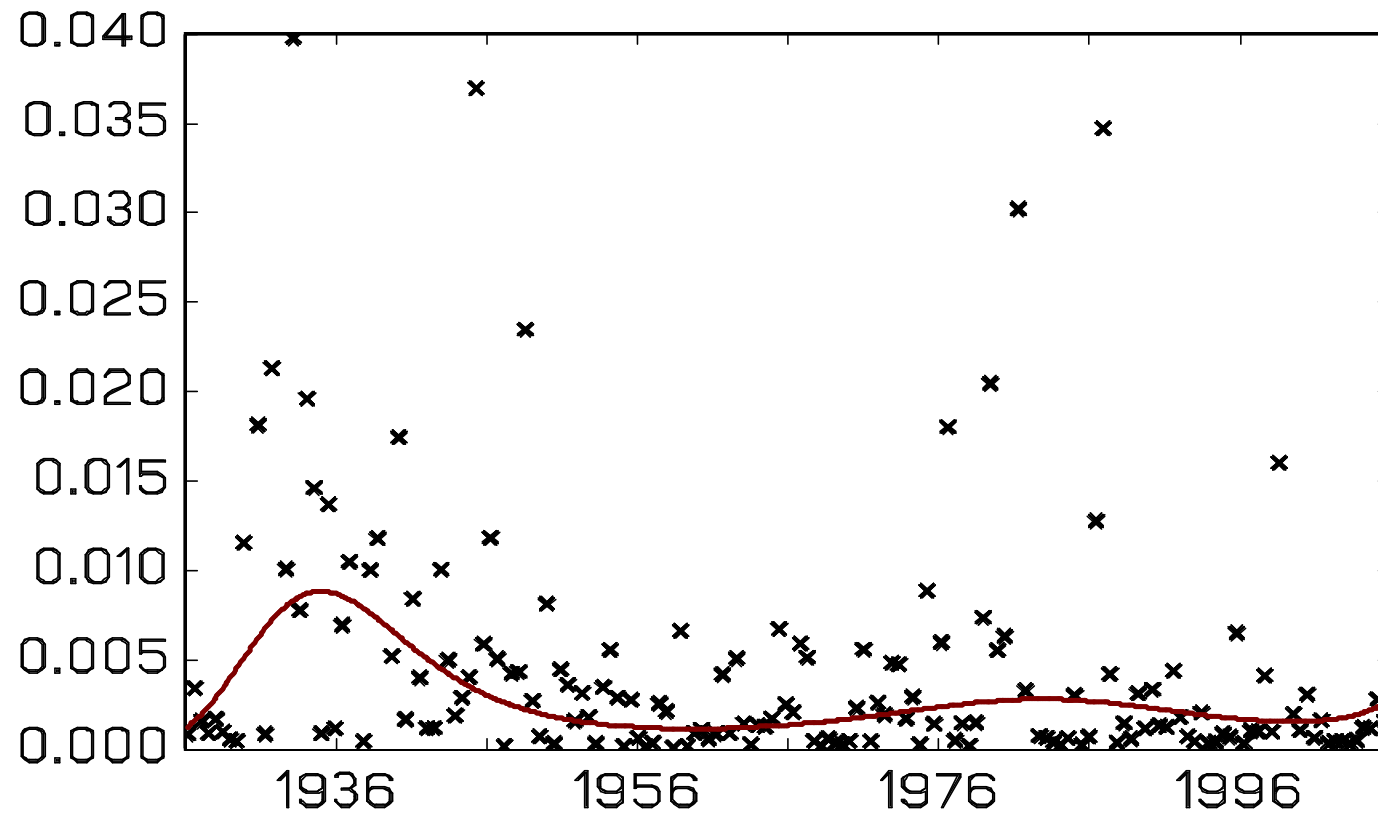


Figure 1b. Shows Chebyshev Polynomial fit of γ_t against time along with data

points.

Nearest Neighbor Quantile Regression Phi against time

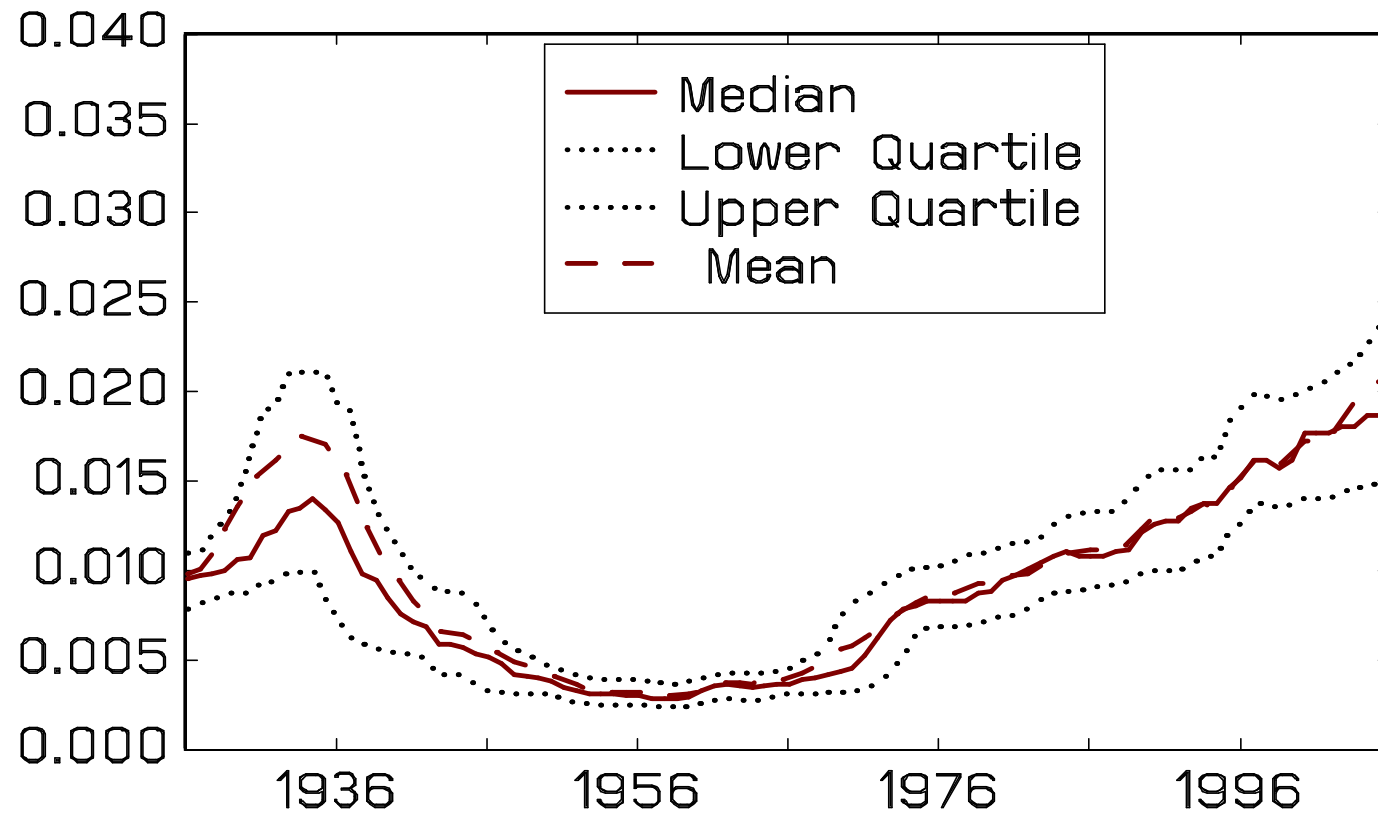


Figure 2a. Shows the nearest neighbor estimated median, lower quartile, upper quartile, and mean regressions of ϕ_t on t/T . The procedure uses $k = 60$ nearest neighbors.

Nearest Neighbor Quantile Regression Gamma against time

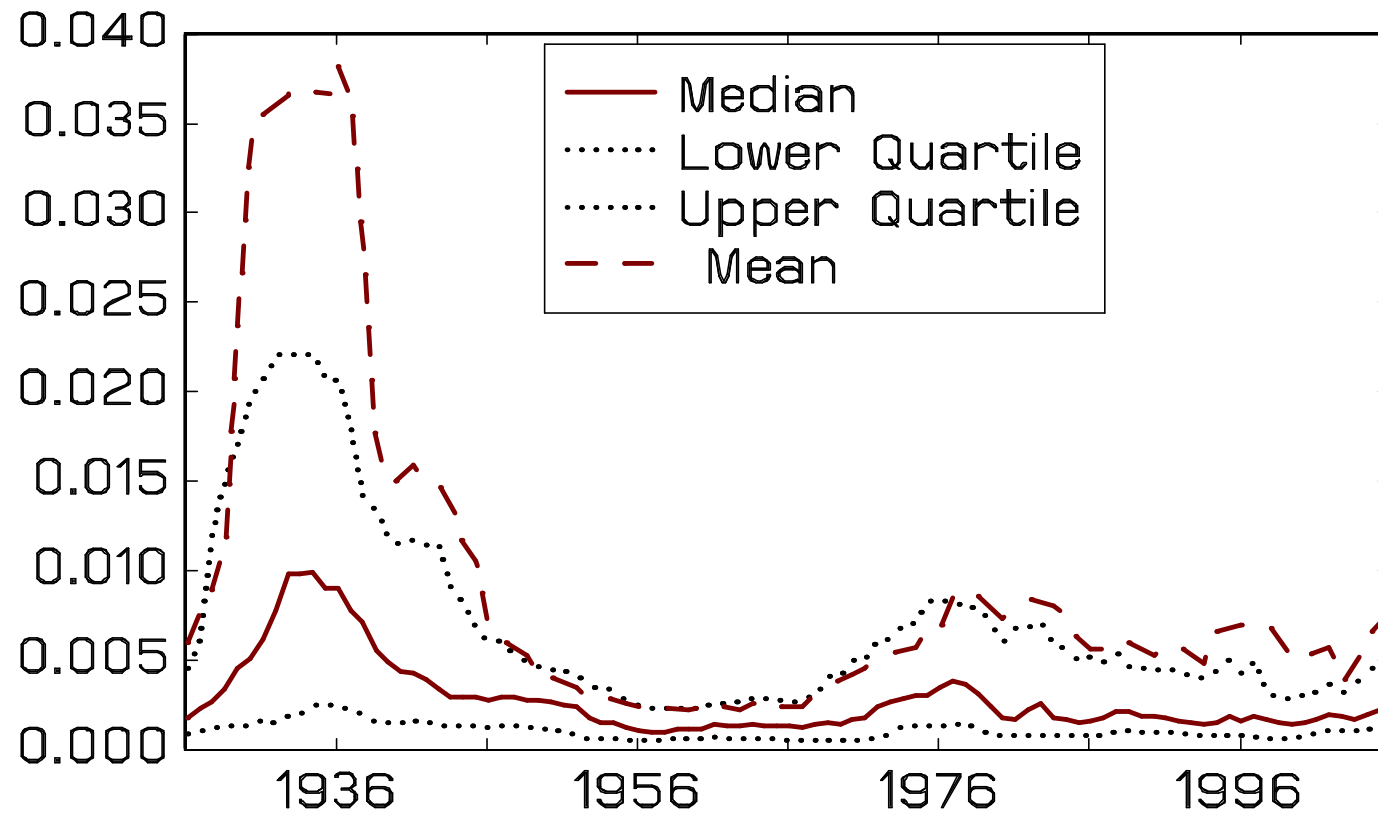


Figure 2b. Shows the nearest neighbor estimated median, lower quartile, upper quartile, and mean regressions of γ_t on t/T . The procedure uses $k = 60$ nearest neighbors.